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Johnson's homomorphisms and the rational cohomology of subgroups of the mapping class group

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Abstract.

The Torelli group and the Johnson kernel are important subgroups of the mapping class group of a surface. Using Johnson's homomorphisms as their free abelian quotients together with the representation theory of the symplectic groups, we will give some descriptions of cup products of rational cohomology classes of degree one obtained from the dual of Johnson's homomorphisms.

§1. Introduction

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and let $\mathrm{Diff}_+\Sigma_g$ be the topological group of orientation preserving diffeomorphisms of Σ_g with the C^∞-topology. The mapping class group $\mathcal{M}_g$ is defined as the group of all connected components of $\mathrm{Diff}_+\Sigma_g$. By a result of Earle-Eells [7], we have $\mathrm{BDiff}_+\Sigma_g = K(\mathcal{M}_g, 1)$, so that cohomology classes of $\mathcal{M}_g$ give characteristic classes of oriented Σ_g-bundles and we can study the theory of oriented Σ_g-bundles from the algebraic point of view. We refer [20] for generalities of characteristic classes of oriented Σ_g-bundles.

To understand the structure of $\mathcal{M}_g$, we consider the following two subgroups. The first one is the *Torelli group* $\mathcal{I}_g$ consisting of all elements which act trivially on the first homology group $H := H_1(\Sigma_g; \mathbb{Z})$ of Σ_g, namely it is the kernel of the classical representation $\rho_0 : \mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z})$, where $\mathrm{Sp}(2g, \mathbb{Z})$ denotes the integral symplectic group. The second one is the group $\mathcal{K}_g$ generated by Dehn twists along bounding simple closed curves. Johnson [14] showed that $\mathcal{K}_g$ coincides with the kernel of what is now called the first Johnson homomorphism $\tau_g(1) : \mathcal{I}_g \to (\wedge^3 H)/H$ of $\mathcal{I}_g$ (see [11]). For this reason, $\mathcal{K}_g$ is often called the

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Johnson kernel. Summarizing, we have the exact sequences

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathcal{M}_g \xrightarrow{\rho_0} \mathrm{Sp}(2g,\mathbb{Z}) \longrightarrow 1,$$
$$1 \longrightarrow \mathcal{K}_g \longrightarrow \mathcal{I}_g \xrightarrow{\tau_g(1)} (\wedge^3 H)/H \longrightarrow 1.$$

Generally, the structures of $\mathcal{I}_g$ and $\mathcal{K}_g$ are more difficult to understand than the structure of $\mathcal{M}_g$. In fact, it is known that $\mathcal{M}_g$ is finitely presentable. On the other hand, it is not known whether $\mathcal{I}_g$ is finitely presentable or not, while Johnson [13] showed that it is finitely generated for $g \geq 3$. As for $\mathcal{K}_g$, Biss-Farb [5] showed that it is not even finitely generated for $g \geq 2$. However, we can use Johnson's homomorphisms $\tau_g(1)$ and $\tau_g(2)$, defined by Johnson [11, 12] and Morita [22, 27], to obtain non-trivial finitely generated free abelian quotients of $\mathcal{I}_g$ and $\mathcal{K}_g$. They will play important roles as primary approximations of $\mathcal{I}_g$ and $\mathcal{K}_g$, from which we can extract several pieces of information. This situation should be compared with the fact that $\mathcal{M}_g$ is perfect for $g \geq 3$, namely it has no abelian quotients except the trivial one.

Now we are particularly interested in the rational cohomology of $\mathcal{I}_g$ and $\mathcal{K}_g$, which give characteristic classes of oriented Σ_g-bundles with restricted holonomies. By passing to the dual over $\mathbb{Q}$ of $\tau_g(k)$ $(k = 1,2)$, we can regard the rational images $\mathrm{Im}\,\tau_g^{\mathbb{Q}}(k)$ of $\tau_g(k)$ as subspaces of $H^1(\mathcal{I}_g;\mathbb{Q})$ and $H^1(\mathcal{K}_g;\mathbb{Q})$. Next we consider cup products of them. In [10], Hain determined the kernel of the cup product map

$$\cup : \wedge^2(\mathrm{Im}\,\tau_g^{\mathbb{Q}}(1)) \longrightarrow H^2(\mathcal{I}_g;\mathbb{Q}),$$

which is stable under the natural action of the rational symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$ on $\wedge^2(\mathrm{Im}\,\tau_g^{\mathbb{Q}}(1))$, in terms of the representation theory of $\mathrm{Sp}(2g,\mathbb{Q})$. Continuing his result, in Sections 3 and 4, we will summarize two previous results of the author:

- ([30]) the kernel of the cup product map

$$\cup : \wedge^2(\mathrm{Im}\,\tau_g^{\mathbb{Q}}(2)) \longrightarrow H^2(\mathcal{K}_g;\mathbb{Q}),$$

- ([29]) the kernel (modulo one irreducible summand) of the triple cup product map

$$\cup^3 : \wedge^3(\mathrm{Im}\,\tau_g^{\mathbb{Q}}(1)) \longrightarrow H^3(\mathcal{I}_g;\mathbb{Q}).$$

Note that the collections of the cup product maps $\wedge^*(\mathrm{Im}\,\tau_g^{\mathbb{Q}}(1)) \to H^*(\mathcal{I}_g;\mathbb{Q})$ and $\wedge^*(\mathrm{Im}\,\tau_g^{\mathbb{Q}}(2)) \to H^*(\mathcal{K}_g;\mathbb{Q})$ are far from surjective, since Akita [1] showed that $H^*(\mathcal{I}_g;\mathbb{Q})$ and $H^*(\mathcal{K}_g;\mathbb{Q})$ are infinitely generated vector space (while $\mathcal{I}_g$ and $\mathcal{K}_g$ have finite cohomological dimensions). It is an interesting (and difficult) problem to find cohomology classes of $\mathcal{I}_g$ and $\mathcal{K}_g$ not obtained from Johnson's homomorphisms.

We also mention some similar research. Brendle-Farb [6] studied the second cohomology of $\mathcal{I}_g$ and $\mathcal{K}_g$ by using the Birman-Craggs-Johnson homomorphism, and Pettet [28] studied the second cohomology of the (outer-)automorphism group of a free group by using its first Johnson homomorphism.

§2. Preliminaries

2.1. Surfaces and their mapping class groups

Let $H_1(\Sigma_g)$ be the first integral homology group of a closed oriented surface Σ_g of genus g. $H_1(\Sigma_g)$ has a natural intersection form $\mu : H_1(\Sigma_g) \otimes H_1(\Sigma_g) \to \mathbb{Z}$ which is non-degenerate and skew symmetric. We fix a symplectic basis $\langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle$ of $H_1(\Sigma_g)$ with respect to μ as in Figure 1.

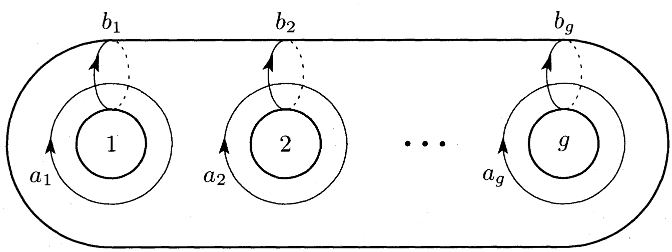

Fig. 1. A symplectic basis of $H_1(\Sigma_g)$

Then we have

$$\mu(a_i, a_j) = 0, \quad \mu(b_i, b_j) = 0, \quad \mu(a_i, b_j) = \delta_{ij}.$$

Poincaré duality gives a canonical isomorphism between $H_1(\Sigma_g)$ and its dual $\mathrm{Hom}(H_1(\Sigma_g), \mathbb{Z}) = H^1(\Sigma_g)$, the first integral cohomology group of Σ_g. In this isomorphism, a_i (resp. b_i) $\in H_1(\Sigma_g)$ corresponds to $-b_i^*$ (resp. a_i^*) $\in H^1(\Sigma_g)$ where $\langle a_1^*, \ldots, a_g^*, b_1^*, \ldots, b_g^* \rangle$ is the dual basis of $H^1(\Sigma_g)$. We use the same symbol H for these canonically isomorphic abelian groups.

We also use a compact oriented surface $\Sigma_{g,1}$ of genus g with a connected boundary. $H_1(\Sigma_{g,1})$ and $H^1(\Sigma_{g,1})$ can be naturally identified with H. The fundamental group $\pi_1\Sigma_{g,1}$ of $\Sigma_{g,1}$, where we take a base point of $\Sigma_{g,1}$ on $\partial\Sigma_{g,1}$, is known to be a free group of rank $2g$. We write

$\zeta \in \pi_1\Sigma_{g,1}$ for the boundary loop of $\Sigma_{g,1}$. Then the fundamental group $\pi_1\Sigma_g$ of Σ_g is given by $\pi_1\Sigma_{g,1}/\langle\zeta\rangle$ where $\langle\zeta\rangle$ is the normal closure of the subgroup generated by ζ.

Let $\mathcal{M}_g, \mathcal{M}_{g,*}, \mathcal{M}_{g,1}$ be the mapping class group of Σ_g, of Σ_g relative to the base point, of $\Sigma_{g,1}$, respectively. They are related by the exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,*} \longrightarrow 1, \tag{1}$$

$$1 \longrightarrow \pi_1\Sigma_g \longrightarrow \mathcal{M}_{g,*} \longrightarrow \mathcal{M}_g \longrightarrow 1, \tag{2}$$

where $\mathbb{Z}$ corresponds to the Dehn twist along a loop which is parallel to $\partial\Sigma_{g,1}$, and $\pi_1\Sigma_g$ is embedded in $\mathcal{M}_{g,*}$ as spin-maps (see Theorem 4.3 in [4]). The former sequence is a central extension.

The natural action of $\mathcal{M}_g$ on H gives the classical representation

$$\rho_0 : \mathcal{M}_g \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}),$$

and we also have similar ones for $\mathcal{M}_{g,*}$ and $\mathcal{M}_{g,1}$. The kernels of these representations are denoted by $\mathcal{I}_g$, $\mathcal{I}_{g,*}$ and $\mathcal{I}_{g,1}$, respectively and called the *Torelli group* for each case. Note that among $\mathcal{I}_g$, $\mathcal{I}_{g,*}$ and $\mathcal{I}_{g,1}$, we have exact sequences similar to (1) and (2). Indeed the Dehn twist along $\partial\Sigma_{g,1}$ and spin-maps act on H trivially.

Let $\mathcal{K}_g$ (resp. $\mathcal{K}_{g,1}$) be the subgroup of $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,1}$) generated by Dehn twists along bounding simple closed curves on Σ_g (resp. $\Sigma_{g,1}$). We define $\mathcal{K}_{g,*} \subset \mathcal{M}_{g,*}$ to be the image of $\mathcal{K}_{g,1}$ by the map $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*}$. Then we have the exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{K}_{g,1} \longrightarrow \mathcal{K}_{g,*} \longrightarrow 1,$$

$$1 \longrightarrow [\pi_1\Sigma_g, \pi_1\Sigma_g] \longrightarrow \mathcal{K}_{g,*} \longrightarrow \mathcal{K}_g \longrightarrow 1,$$

where the former sequence is the pull-back of the central extension of $\mathcal{M}_{g,*}$, and the latter one follows from a result of Asada-Kaneko [2].

2.2. Johnson's homomorphisms

Here, we recall what we call Johnson's homomorphisms defined by Johnson [11, 12] and Morita [22, 24, 27].

By results of Dehn, Nielsen and many people, we have

$$\mathcal{M}_g \cong \mathrm{Out}_+\, \pi_1\Sigma_g, \tag{3}$$

$$\mathcal{M}_{g,*} \cong \mathrm{Aut}_+\, \pi_1\Sigma_g, \tag{4}$$

$$\mathcal{M}_{g,1} \cong \{\varphi \in \mathrm{Aut}\, \pi_1\Sigma_{g,1} \mid \varphi(\zeta) = \zeta\}, \tag{5}$$

where $\mathrm{Out}_+\, \pi_1\Sigma_g := \mathrm{Ker}(\mathrm{Out}\, \pi_1\Sigma_g \to \mathrm{Aut}\, H_2(\pi_1\Sigma_g))$, and $\mathrm{Aut}_+\, \pi_1\Sigma_g$ is similar.

For a group G, let $\{\Gamma^k G\}_{k\geq 1}$ be the lower central series of G inductively defined by $\Gamma^1 G = G$ and $\Gamma^i G = [\Gamma^{i-1}G, G]$ for $i \geq 2$. The collection $\{(\Gamma^k G)/(\Gamma^{k+1}G)\}_{k\geq 1}$ forms a graded Lie algebra whose bracket map is induced from taking commutators. It is well known that the Lie algebra $\{(\Gamma^k \pi_1\Sigma_{g,1})/(\Gamma^{k+1}\pi_1\Sigma_{g,1})\}_{k\geq 1}$ is isomorphic to the free Lie algebra $\mathcal{L}_{g,1} = \{\mathcal{L}_{g,1}(k)\}_{k\geq 1}$ generated by H. Furthermore, by a result of Labute [17], the Lie algebra $\{(\Gamma^k \pi_1\Sigma_g)/(\Gamma^{k+1}\pi_1\Sigma_g)\}_{k\geq 1}$ is given by $\mathcal{L}_g := \mathcal{L}_{g,1}/I$ where I is the ideal of $\mathcal{L}_{g,1}$ generated by $\omega_0 := \sum_{i=1}^{g}[a_i, b_i]$.

The isomorphisms (3), (4) and (5) induce the homomorphisms

$$\begin{aligned}
\sigma_k \ &: \mathcal{M}_g \longrightarrow \mathrm{Out}\big(\pi_1\Sigma_g/(\Gamma^k\pi_1\Sigma_g)\big),\\
\sigma_{k,*} &: \mathcal{M}_{g,*} \longrightarrow \mathrm{Aut}\big(\pi_1\Sigma_g/(\Gamma^k\pi_1\Sigma_g)\big),\\
\sigma_{k,1} &: \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}\big(\pi_1\Sigma_{g,1}/(\Gamma^k\pi_1\Sigma_{g,1})\big)
\end{aligned}$$

for each $k \geq 2$, and we define filtrations of $\mathcal{M}_g, \mathcal{M}_{g,*}, \mathcal{M}_{g,1}$ by

$$\begin{aligned}
\mathcal{M}_g[1] &:= \mathcal{M}_g, & \mathcal{M}_g[k] &:= \mathrm{Ker}\,\sigma_k \quad (k \geq 2),\\
\mathcal{M}_{g,*}[1] &:= \mathcal{M}_{g,*}, & \mathcal{M}_{g,*}[k] &:= \mathrm{Ker}\,\sigma_{k,*} \quad (k \geq 2),\\
\mathcal{M}_{g,1}[1] &:= \mathcal{M}_{g,1}, & \mathcal{M}_{g,1}[k] &:= \mathrm{Ker}\,\sigma_{k,1} \quad (k \geq 2).
\end{aligned}$$

For each $\varphi \in \mathcal{M}_{g,1}[k+1]$ and $\gamma \in \pi_1\Sigma_{g,1}$, we have $\varphi(\gamma)\gamma^{-1} \in \Gamma^{k+1}\pi_1\Sigma_{g,1}$. This induces a map

$$\begin{aligned}
\tau_{g,1}(k) : \mathcal{M}_{g,1}[k+1] &\to \mathrm{Hom}(\pi_1\Sigma_{g,1}, (\Gamma^{k+1}\pi_1\Sigma_{g,1})/(\Gamma^{k+2}\pi_1\Sigma_{g,1}))\\
&= \mathrm{Hom}(H, \mathcal{L}_{g,1}(k+1)),
\end{aligned}$$

and it is in fact a homomorphism.

In [23, 24], Morita showed the following. By taking commutators, we can endow $\{\mathcal{M}_{g,1}[k+1]/\mathcal{M}_{g,1}[k+2]\}_{k\geq 1} = \{\mathrm{Im}\,\tau_{g,1}(k)\}_{k\geq 1} =: \mathrm{Im}\,\tau_{g,1}$ with a Lie algebra structure. We can also endow $\mathrm{Hom}(H, \mathcal{L}_{g,1}) := \{\mathrm{Hom}(H, \mathcal{L}_{g,1}(k+1))\}_{k\geq 1}$ with a Lie algebra structure, so that $\tau_{g,1} := \{\tau_{g,1}(k)\}_{k\geq 1}$ becomes a Lie algebra inclusion of $\mathrm{Im}\,\tau_{g,1}$ into $\mathrm{Hom}(H, \mathcal{L}_{g,1})$. Moreover, he showed that $\mathrm{Im}\,\tau_{g,1}$ is contained in the Lie subalgebra $\mathfrak{h}_{g,1} = \{\mathfrak{h}_{g,1}(k)\}_{k\geq 1}$ defined by

$$\mathfrak{h}_{g,1}(k) := \mathrm{Ker}\left(\mathrm{Hom}(H, \mathcal{L}_{g,1}(k+1)) \xrightarrow{\cong} H \otimes \mathcal{L}_{g,1}(k+1) \xrightarrow{[\cdot,\cdot]} \mathcal{L}_{g,1}(k+2)\right),$$

where the maps in the right hand side are given by Poincaré duality and the bracket operation.

A similar argument gives a homomorphism $\tau_{g,*}(k) : \mathcal{M}_{g,*}[k+1] \to \mathfrak{h}_{g,*}(k) \subset \mathrm{Hom}(H,\ \mathcal{L}_g(k+1))$, where

$$\mathfrak{h}_{g,*}(k) := \mathrm{Ker}\left(\mathrm{Hom}(H, \mathcal{L}_g(k+1)) \xrightarrow{\cong} H \otimes \mathcal{L}_g(k+1) \xrightarrow{[\cdot,\cdot]} \mathcal{L}_g(k+2)\right),$$

and the corresponding Lie algebra inclusion. Moreover, Asada-Kaneko [2] showed that $\pi_1\Sigma_g \cap \mathcal{M}_{g,*}[k+1] = \Gamma^k\pi_1\Sigma_g$. Hence we have an inclusion

$$\Psi_k : (\Gamma^k\pi_1\Sigma_g)/(\Gamma^{k+1}\pi_1\Sigma_g) \cong \mathcal{L}_g(k) \hookrightarrow \mathfrak{h}_{g,*}(k).$$

If we set $\mathfrak{h}_g(k) := \mathfrak{h}_{g,*}(k)/\mathcal{L}_g(k)$, we obtain a homomorphism $\tau_g(k) : \mathcal{M}_g[k+1] \to \mathfrak{h}_g(k)$ and the corresponding Lie algebra inclusion. We call the homomorphisms $\tau_g(k)$, $\tau_{g,*}(k)$, $\tau_{g,1}(k)$ the k-th Johnson homomorphism for each case. Note that $\tau_g(k)$ is $\mathcal{M}_g$-equivariant, where $\mathcal{M}_g$ acts on $\mathcal{M}_g[k+1]$ by conjugation and acts on the target through the classical representation $\rho_0 : \mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z})$. Similar results hold for $\tau_{g,*}(k)$ and $\tau_{g,1}(k)$.

We have $\mathcal{M}_{g,1}[2] = \mathcal{I}_{g,1}$, $\mathcal{M}_{g,*}[2] = \mathcal{I}_{g,*}$ and $\mathcal{M}_g[2] = \mathcal{I}_g$ by definition. Johnson [14] showed that $\mathcal{M}_{g,1}[3] = \mathcal{K}_{g,1}$ and $\mathcal{M}_g[3] = \mathcal{K}_g$. Combining the fact that the first Johnson homomorphisms for $\mathcal{I}_{g,1}$ and $\mathcal{I}_{g,*}$ have the same target $\wedge^3 H$, we can see that $\mathcal{M}_{g,*}[3] = \mathcal{K}_{g,*}$.

2.3. The representation theory of $\mathrm{Sp}(2g, \mathbb{Q})$

Here we summarize the notation and general facts concerning the representation theory of $\mathrm{Sp}(2g, \mathbb{Q})$ from [8], [10] and [27]. First we consider the Lie group $\mathrm{Sp}(2g, \mathbb{C})$ and its Lie algebra $\mathfrak{sp}(2g, \mathbb{C})$. It is known that finite dimensional representations of $\mathrm{Sp}(2g, \mathbb{C})$ coincide with those of $\mathfrak{sp}(2g, \mathbb{C})$, and their common irreducible representations (up to isomorphisms) are parameterized by Young diagrams whose numbers of rows are less than or equal to g. These representations are all defined over $\mathbb{Q}$ so that we can consider them as irreducible representations of $\mathrm{Sp}(2g, \mathbb{Q})$ and $\mathfrak{sp}(2g, \mathbb{Q})$. We follow the notation in [27] to describe Young diagrams as in Figure 2.

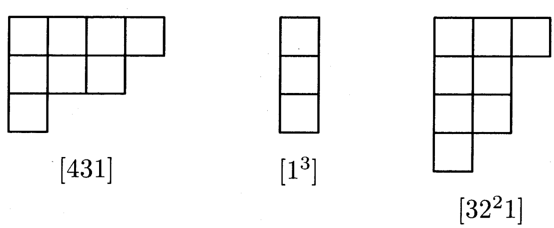

Fig. 2. Notation for Young diagrams

For example, the trivial representation $\mathbb{Q}$ is denoted by $[0]$ and the fundamental representation $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$ is denoted by $[1]$. We fix a symplectic basis $\langle a_1, \ldots, a_g, b_1, \ldots, b_g\rangle$ of $H_{\mathbb{Q}}$ with respect to the non-degenerate skew symmetric bilinear form, denoted by μ again, on $H_{\mathbb{Q}}$ induced from the intersection form μ on H. In general, the Young diagram $[n_1 n_2 \cdots n_l]$, where n_i are integers satisfying $n_1 \geq n_2 \geq \cdots \geq n_l \geq 1$ and $l \leq g$, corresponds to the $\mathrm{Sp}(2g, \mathbb{Q})$-vector space V given as follows. Let $[m_1 m_2 \cdots m_k]$ be the Young diagram obtained by transposing $[n_1 n_2 \cdots n_l]$. Then V is explicitly defined to be the irreducible $\mathrm{Sp}(2g, \mathbb{Q})$-subspace of

$$(\wedge^{m_1} H_{\mathbb{Q}}) \otimes (\wedge^{m_2} H_{\mathbb{Q}}) \otimes \cdots \otimes (\wedge^{m_k} H_{\mathbb{Q}})$$

containing the vector

$$(a_1 \wedge a_2 \wedge \cdots \wedge a_{m_1}) \otimes (a_1 \wedge a_2 \wedge \cdots \wedge a_{m_2}) \otimes \cdots \otimes (a_1 \wedge a_2 \wedge \cdots \wedge a_{m_k}),$$

which is called the *highest weight vector of* $[n_1 n_2 \cdots n_l]$.

2.4. Johnson's homomorphisms from the view point of the representation theory of $\mathrm{Sp}(2g, \mathbb{Q})$

Let $\tau_g^{\mathbb{Q}}(k)$ denote the Johnson homomorphism $\tau_g(k)$ tensored by $\mathbb{Q}$, namely

$$\tau_g^{\mathbb{Q}}(k) : (\mathcal{M}_g[k+1]/(\Gamma^2 \mathcal{M}_g[k+1])) \otimes \mathbb{Q} \longrightarrow \mathfrak{h}_g^{\mathbb{Q}}(k) := \mathfrak{h}_g(k) \otimes \mathbb{Q}.$$

As mentioned in Section 2.2, $\tau_g^{\mathbb{Q}}(k)$ is $\mathcal{M}_g$-equivariant, so that $\operatorname{Im} \tau_g^{\mathbb{Q}}(k)$ is an $\mathrm{Sp}(2g, \mathbb{Z})$-vector space. Moreover, $\operatorname{Im} \tau_g^{\mathbb{Q}}(k)$ turns out to be an $\mathrm{Sp}(2g, \mathbb{Q})$-vector space by Lemma 2.2.8 of Asada-Nakamura [3]. In particular, $\operatorname{Im} \tau_g^{\mathbb{Q}}(k)$ and $\mathfrak{h}_g^{\mathbb{Q}}(k)$ can be written in terms of the representation theory of $\mathrm{Sp}(2g, \mathbb{Q})$. Similar results hold for $\tau_{g,*}^{\mathbb{Q}}(k) := \tau_{g,*}(k) \otimes \mathbb{Q}$ and $\tau_{g,1}^{\mathbb{Q}}(k) := \tau_{g,1}(k) \otimes \mathbb{Q}$. By results of Johnson [11] for $k = 1$, Hain [10] and Morita [22] for $k = 2$, Hain [10] and Asada-Nakamura [3] for $k = 3$, we have the following.

$$\begin{aligned}
\operatorname{Im} \tau_{g,1}^{\mathbb{Q}}(1) &= \operatorname{Im} \tau_{g,*}^{\mathbb{Q}}(1) = \mathfrak{h}_{g,1}^{\mathbb{Q}}(1) = \mathfrak{h}_{g,*}^{\mathbb{Q}}(1) = [1^3] + [1] = \wedge^3 H_{\mathbb{Q}}, \\
\operatorname{Im} \tau_g^{\mathbb{Q}}(1) &= \mathfrak{h}_g^{\mathbb{Q}}(1) = [1^3] = (\wedge^3 H_{\mathbb{Q}})/H_{\mathbb{Q}}, \\
\operatorname{Im} \tau_{g,1}^{\mathbb{Q}}(2) &= \mathfrak{h}_{g,1}^{\mathbb{Q}}(2) = [2^2] + [1^2] + [0], \\
\operatorname{Im} \tau_{g,*}^{\mathbb{Q}}(2) &= \mathfrak{h}_{g,*}^{\mathbb{Q}}(2) = [2^2] + [1^2], \\
\operatorname{Im} \tau_g^{\mathbb{Q}}(2) &= \mathfrak{h}_g^{\mathbb{Q}}(2) = [2^2], \\
\operatorname{Im} \tau_{g,1}^{\mathbb{Q}}(3) &= \operatorname{Im} \tau_{g,*}^{\mathbb{Q}}(3) = [31^2] + [21] \\
&\quad \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(3) = \mathfrak{h}_{g,*}^{\mathbb{Q}}(3) = [31^2] + [21] + [3], \\
\operatorname{Im} \tau_g^{\mathbb{Q}}(3) &= [31^2] \subset \mathfrak{h}_g^{\mathbb{Q}}(3) = [31^2] + [3]
\end{aligned}$$

for $g \geq 3$, where we write $+$ for the direct sum. Moreover, in [27], Morita announced that

$$\begin{aligned} \operatorname{Im} \tau_{g,1}^{\mathbb{Q}}(4) &= \operatorname{Im} \tau_{g,*}^{\mathbb{Q}}(4) = [42] + [31^3] + 2[31] + [2^3] + [21^2] + 2[2], \\ \operatorname{Im} \tau_{g}^{\mathbb{Q}}(4) &= [42] + [31^3] + [31] + [2^3] + [2] \end{aligned}$$

in

$$\begin{aligned} \mathfrak{h}_{g,1}^{\mathbb{Q}}(4) &= [42] + [31^3] + 2[31] + [2^3] + 2[21^2] + 3[2], \\ \mathfrak{h}_{g,*}^{\mathbb{Q}}(4) &= [42] + [31^3] + 2[31] + [2^3] + 2[21^2] + 2[2], \\ \mathfrak{h}_{g}^{\mathbb{Q}}(4) &= [42] + [31^3] + [31] + [2^3] + [21^2] + [2] \end{aligned}$$

for $g \geq 4$.

Remark 2.1. Hain showed in [10] that as Lie algebras, $\operatorname{Im} \tau_{g,1}^{\mathbb{Q}}$, $\operatorname{Im} \tau_{g,*}^{\mathbb{Q}}$, $\operatorname{Im} \tau_{g}^{\mathbb{Q}}$ are generated by their degree one part for $g \geq 3$, and that $\operatorname{Im} \tau_{g,1}^{\mathbb{Q}}(k) \cong \operatorname{Im} \tau_{g,*}^{\mathbb{Q}}(k)$ for $k \geq 3$. The proof, on which Hain kindly informed the author, uses Lemmas and Propositions 4.5–4.8 in [10] with some general facts about mixed Hodge structures.

The bracket operation of $\mathfrak{h}_{g,1}$ is explicitly given in [23, 24]. However, here we use an alternative description given by Garoufalidis-Levine [9], Levine [18, 19], which will be easier to handle. We recall the following Lie algebra of labeled unitrivalent trees. Let $\mathcal{A}_k^t(H)$ be the abelian group generated by unitrivalent trees with $k+2$ univalent vertices labeled by elements of H and a cyclic order of each trivalent vertex modulo relations of AS and IHX together with linearity of labels. We can endow $\mathcal{A}^t(H) := \{\mathcal{A}_k^t(H)\}_{k \geq 1}$ with a bracket operation

$$[\cdot,\cdot] : \mathcal{A}_k^t(H) \otimes \mathcal{A}_l^t(H) \longrightarrow \mathcal{A}_{k+l}^t(H)$$

given below, and $\mathcal{A}^t(H)$ becomes a quasi Lie algebra. For labeled trees $T_1, T_2 \in \mathcal{A}^t(H)$, we define

$$[T_1, T_2] := \sum_{i,j} \mu(c_i, d_j) T_1 *_{i,j} T_2$$

where μ is the intersection form on H and the sum is taken over all pairs of a univalent vertex of T_1, labeled by c_i, and one of T_2, labeled by d_j, and $T_1 *_{i,j} T_2$ is the tree given by welding T_1 and T_2 at the pair. We define a map $\eta_k : \mathcal{A}_k^t(H) \to H \otimes \mathcal{L}_{g,1}(k+1)$ by

$$\eta_k(T) := \sum_v c_v \otimes T_v,$$

where the sum is over all univalent vertices of T, and for each univalent vertex v, c_v denotes the label of v and T_v denotes the rooted labeled planar binary tree obtained from T by removing the label c_v and considering v to be an unlabeled root, which can be regarded as an element of $\mathcal{L}_{g,1}(k+1)$ by a standard method. It is shown that $\eta := \{\eta_k\}_{k\geq 1} : \mathcal{A}^t(H) \to H \otimes \mathcal{L}_{g,1}$ is a quasi Lie algebra homomorphism and $\operatorname{Im}\eta \subset \mathfrak{h}_{g,1}$. Moreover $\eta \otimes \mathbb{Q} : \mathcal{A}^t(H) \otimes \mathbb{Q} \to \mathfrak{h}_{g,1}^{\mathbb{Q}}$ becomes an isomorphism of Lie algebras. In what follows, we identify $\mathcal{A}^t(H) \otimes \mathbb{Q}$ with $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ by $\eta \otimes \mathbb{Q}$.

Using $\mathcal{A}^t(H) \otimes \mathbb{Q}$, we now give a graphical description of the map

$$\Psi_k : \mathcal{L}_g^{\mathbb{Q}}(k) := \mathcal{L}_g(k) \otimes \mathbb{Q} \hookrightarrow \operatorname{Im}\tau_{g,*}^{\mathbb{Q}}(k) \subset \mathfrak{h}_{g,*}^{\mathbb{Q}}(k),$$

which was mentioned in Section 2.2 and is explicitly given by

$$\mathcal{L}_g^{\mathbb{Q}}(k) \ni X \mapsto \sum_{i=1}^{g} \big(a_i \otimes [b_i, X] - b_i \otimes [a_i, X]\big) \in H \otimes \mathcal{L}_g^{\mathbb{Q}}(k+1).$$

For each rooted labeled planar binary tree T as an element of $\mathcal{L}_{g,1}^{\mathbb{Q}}(k)$, we can construct an element of $\mathcal{A}_k^t(H) \otimes \mathbb{Q} \cong \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ by gluing T to the rooted labeled planar binary tree $T_{\omega_0} \in \mathcal{L}_{g,1}^{\mathbb{Q}}(2) \cong \wedge^2 H_{\mathbb{Q}}$ corresponding to $\omega_0 = \sum_{i=1}^{g} a_i \wedge b_i$ at their roots as depicted in Figure 3.

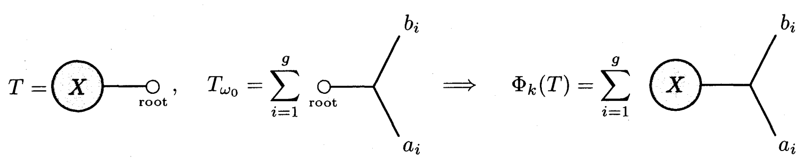

Fig. 3. The map $\Phi_k : \mathcal{L}_{g,1}^{\mathbb{Q}}(k) \to \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$

We can easily check that this construction gives an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphism $\Phi_k : \mathcal{L}_{g,1}^{\mathbb{Q}}(k) \to \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$, and it induces the desired map $\Psi_k : \mathcal{L}_g^{\mathbb{Q}}(k) \hookrightarrow \mathfrak{h}_{g,*}^{\mathbb{Q}}(k)$ by using Labute's result [17].

§3. The second cohomology of the Torelli group and the Johnson kernel

We start our investigation of cup products of cohomology classes of degree one obtained from Johnson's homomorphisms.

By passing to the dual over $\mathbb{Q}$ of $\tau_g(1) : \mathcal{I}_g \to [1^3]$, we obtain an injection

$$\tau_g^{\mathbb{Q}}(1)^* : [1^3] \hookrightarrow H^1(\mathcal{I}_g;\mathbb{Q})$$

and more generally, we have the cup product map

$$\cup^n : \wedge^n[1^3] \longrightarrow H^n(\mathcal{I}_g;\mathbb{Q})$$

for each $n \geq 2$. In [15], Johnson showed that $\tau_g^{\mathbb{Q}}(1)$ is an isomorphism, so that studying the map $\cup^n$ is equivalent to determining the subring of $H^*(\mathcal{I}_g;\mathbb{Q})$ generated by its degree one part.

The map $\cup^n$ is $\mathcal{M}_g$-equivariant, since $\tau_g(1)$ is so. Hence $\operatorname{Ker}\cup^n$ is an $\mathcal{M}_g$-submodule (in fact, an $\mathrm{Sp}(2g,\mathbb{Z})$-submodule) of $\wedge^n[1^3]$. Moreover, by Asada-Nakamura's argument [3] mentioned before, it is stable under the natural action of $\mathrm{Sp}(2g,\mathbb{Q})$ extending that of $\mathrm{Sp}(2g,\mathbb{Z})$.

The kernel of the cup product map $\cup : \wedge^2[1^3] \to H^2(\mathcal{I}_g;\mathbb{Q})$ was determined by Hain as follows. (We only mention the case in the stable range $g \geq 6$.)

Lemma 3.1 (Hain [10]). *For $g \geq 6$, the irreducible decomposition of $\wedge^2[1^3]$ is given by*

$$\wedge^2[1^3] = [2^2 1^2] + [2^2] + [1^6] + [1^4] + [1^2] + [0].$$

Theorem 3.2 (Hain [10]). *For $g \geq 6$, the kernel of the cup product map $\cup : \wedge^2[1^3] \to H^2(\mathcal{I}_g;\mathbb{Q})$ is $[2^2] + [0]$.*

This theorem is obtained as a corollary of his argument about the finite presentation of Torelli Lie algebra and it is not mentioned directly in [10]. Here we sketch the proof of the theorem, which is divided into the following two steps. Note that the idea of this proof will be applicable to general cases.

Step 1. (Lower bound of $\operatorname{Ker}\cup$) Since $\wedge^2[1^3] = H^2([1^3];\mathbb{Q})$, the map $\cup : \wedge^2[1^3] \to H^2(\mathcal{I}_g;\mathbb{Q})$ coincides with the homomorphism

$$\tau_g(1)^* : H^2([1^3];\mathbb{Q}) \longrightarrow H^2(\mathcal{I}_g;\mathbb{Q}).$$

Hence, by passing to the dual, our task is equivalent to observing the cokernel of the map

$$\tau_g(1)_* : H_2(\mathcal{I}_g;\mathbb{Q}) \longrightarrow H_2([1^3];\mathbb{Q}).$$

By applying Stallings' exact sequence [31] to the group extension

$$1 \longrightarrow \Gamma^2\mathcal{I}_g \longrightarrow \mathcal{I}_g \longrightarrow H_1(\mathcal{I}_g) \longrightarrow 1$$

and observing the homomorphisms, we obtain the exact sequence

$$H_2(\mathcal{I}_g;\mathbb{Q}) \longrightarrow \wedge^2(H_1(\mathcal{I}_g;\mathbb{Q})) = \wedge^2[1^3] \xrightarrow{[\cdot,\cdot]} (\Gamma^2\mathcal{I}_g/\Gamma^3\mathcal{I}_g)\otimes\mathbb{Q} \longrightarrow 0$$

where the first map is the coproduct on the rational homology and the second one is the Lie bracket

$$[\cdot,\cdot] : \wedge^2(H_1(\mathcal{I}_g)) = \wedge^2(\Gamma^1\mathcal{I}_g/\Gamma^2\mathcal{I}_g) \longrightarrow \Gamma^2\mathcal{I}_g/\Gamma^3\mathcal{I}_g.$$

By observing the image of this map, we obtain $[2^2] \subset \operatorname{Ker}\cup$.

We can see $[0] \subset \operatorname{Ker}\cup$ from the fact that the first Morita-Miller-Mumford class vanishes on $H^2(\mathcal{I}_g;\mathbb{Q})$ (see [26] for details).

Step 2. (Upper bound of $\operatorname{Ker}\cup$) We can obtain summands in $\wedge^2[1^3]$ which survive in $H^2(\mathcal{I}_g;\mathbb{Q})$ when we take Kronecker products with elements in $H_2(\mathcal{I}_g;\mathbb{Q})$ and the results are non-trivial. We can construct an element of $H_2(\mathcal{I}_g;\mathbb{Q})$ by constructing a homomorphism $\mathbb{Z}^2 \to \mathcal{I}_g$ and considering the image of $1 \in \mathbb{Z} \cong H_2(\mathbb{Z}^2)$ in $H_2(\mathcal{I}_g)$. Such classes are called *abelian cycles*. Note that what we need to construct a homomorphism $\mathbb{Z}^2 \to \mathcal{I}_g$ is only a choice of a pair of commuting elements in $\mathcal{I}_g$. By an explicit computation, we can see that $[2^21^2]+[1^6]+[1^4]+[1^2]$ is not in $\operatorname{Ker}\cup$.

Since the upper bound and the lower one coincides, Theorem 3.2 is proved. □

Remark 3.3. While some additional arguments are needed, the cases of $\mathcal{I}_{g,*}$ and $\mathcal{I}_{g,1}$ will be settled similarly. See Section 6 of [26] for details.)

Next we consider the case of $\mathcal{K}_g$. By passing to the dual of $\tau_g(2) : \mathcal{K}_g \to [2^2]$, we obtain an injection

$$\tau_g^{\mathbb{Q}}(2)^* : [2^2] \hookrightarrow H^1(\mathcal{K}_g;\mathbb{Q})$$

and the cup product map

$$\cup^n : \wedge^n[2^2] \longrightarrow H^n(\mathcal{K}_g;\mathbb{Q})$$

for each $n \geq 2$. We now observe the map $\cup : \wedge^2[2^2] \to H^2(\mathcal{K}_g;\mathbb{Q})$. In this case, the stability range is given by $g \geq 4$.

Lemma 3.4 ([30]). *For $g \geq 4$, the irreducible decomposition of $\wedge^2[2^2]$ is given by*

$$\wedge^2[2^2] = [431]+[42]+[32^21]+[321]+[31^3]+[31]+[2^3]+[21^2]+[2].$$

Theorem 3.5 ([30]). *For $g \geq 4$, the kernel of the cup product map $\cup : \wedge^2[2^2] \to H^2(\mathcal{K}_g;\mathbb{Q})$ is*

$$[42] + [31^3] + [31] + [2^3] + [2],$$

which is, as an $\mathrm{Sp}(2g,\mathbb{Q})$*-vector space, isomorphic to the rational image of the fourth Johnson homomorphism* $\tau_g(4)$.

(*Sketch of Proof*) The proof goes parallel to that of Theorem 3.2. The point is the relationship to the fourth Johnson homomorphism.

Step 1. (Lower bound of $\operatorname{Ker}\cup$) A lower bound of $\operatorname{Ker}\cup$ is obtained by observing the map

$$[\cdot,\cdot] : \wedge^2(H_1(\mathcal{K}_g)) = \wedge^2(\Gamma^1\mathcal{K}_g/\Gamma^2\mathcal{K}_g) \longrightarrow \Gamma^2\mathcal{K}_g/\Gamma^3\mathcal{K}_g.$$

To see the image of this map, we can use the diagram

$$\begin{tikzcd}
\wedge^2(H_1(\mathcal{K}_g)) \arrow[r, "{[\cdot,\cdot]}"] \arrow[d, "\wedge^2\tau_g(2)"'] & \Gamma^2\mathcal{K}_g/\Gamma^3\mathcal{K}_g \arrow[r] & \mathcal{M}_g[5]/\mathcal{M}_g[6] \arrow[d, "\tau_g(4)", "\cong"'] \\
\wedge^2(\operatorname{Im}\tau_g(2)) \arrow[r, "{[\cdot,\cdot]}"] & \operatorname{Im}\tau_g(4) \arrow[r, equal] & \operatorname{Im}\tau_g(4),
\end{tikzcd}$$

whose commutativity follows from the fact that the collection $\{\tau_g(k)\}_{k\geq 1}$ forms a Lie algebra homomorphism. By direct computations, we can show that $[42] + [31^3] + [31] + [2^3] + [2] \subset \operatorname{Ker}\cup$.

Step 2. (Upper bound of $\operatorname{Ker}\cup$) By computations using abelian cycles in $H_2(\mathcal{K}_g)$, we can see that $[431] + [32^21] + [321] + [21^2]$ is not in $\operatorname{Ker}\cup$.

Since the upper bound and the lower one coincides, Theorem 3.5 is proved. $\square$

Remark 3.6. From Theorems 3.2 and 3.5, we can give lower bounds of the ranks of $H^2(\mathcal{I}_g)$ and $H^2(\mathcal{K}_g)$, which may be infinite, by using Weyl's character formula (see Section 24.2 of [8]). For example, the summand $[32^21] \subset \wedge^2[2^2]$ survives in $H^2(\mathcal{K}_g;\mathbb{Q})$, so that the rational dimension

$$\frac{1}{36}(g-3)(g-2)(g-1)(g+2)(2g-1)(2g+1)^2(2g+3)$$

of this summand gives a lower bound of the rank of $H^2(\mathcal{K}_g)$.

§4. The third cohomology of the Torelli group

Finally, we consider triple cup products of $[1^3] = H^1(\mathcal{I}_g; \mathbb{Q})$. In this case the stability range is given by $g \geq 9$.

Lemma 4.1 ([29]). *For $g \geq 9$, the irreducible decomposition of $\wedge^3[1^3]$ is given by*

$$\begin{aligned}\wedge^3[1^3] = \quad & [3^2 1^3] + [3^2 1] + [32^3] + [321^2] + [32] \\ & + [2^3 1^3] + [2^3 1] + [2^2 1^5] + 2[2^2 1^3] + 2[2^2 1] + [21^5] + 2[21^3] + [21] \\ & + [1^9] + [1^7] + 2[1^5] + 3[1^3] + [1].\end{aligned}$$

Theorem 4.2 ([29]). *For $g \geq 9$, the kernel of the cup product map $\cup^3 : \wedge^3[1^3] \to H^3(\mathcal{I}_g; \mathbb{Q})$ contains the direct sum*

$$[3^2 1] + [321^2] + [32] + [2^2 1^3] + [2^2 1] + [21^3] + [21] + 2[1^3]$$

which is equal to $\mathrm{Im}(\cup : [1^3] \otimes ([2^2] + [0]) \to \wedge^3[1^3])$. *Moreover, one of the following two possibilities holds:*

a) $\mathrm{Ker}\cup^3$ *coincides with the above.*

b) $\mathrm{Ker}\cup^3$ *coincides with the direct sum of the above summands and one more summand* $[1]$.

(*Sketch of Proof*) Recall that $\mathrm{Ker}(\cup : \wedge^2[1^3] \to H^2(\mathcal{I}_g; \mathbb{Q})) = [2^2] + [0]$. Hence a lower bound of $\mathrm{Ker}\cup^3$ is given by observing the image of the cup product map $\cup : [1^3] \otimes ([2^2] + [0]) \to \wedge^3[1^3]$. On the other hand, an upper bound of $\mathrm{Ker}\cup^3$ is given by computations using abelian cycles in $H_3(\mathcal{I}_g)$. By these computations, we can give the bounds which coincide modulo the summand $[1]$. $\square$

As for the summand $[1]$, we can relate it with the Euler class $e \in H^2(\mathcal{M}_{g,*}; \mathbb{Q})$ and the pull-back of the second Morita-Miller-Mumford class $e_2 \in H^4(\mathcal{M}_g; \mathbb{Q})$ to $\mathcal{M}_{g,*}$ (see [20] for the definitions). We have the following.

Theorem 4.3 ([29]). *For $g \geq 5$,*

$$[1] \subset \mathrm{Ker}\cup^3 \iff e_2 - (2-2g)e^2 = 0 \in H^4(\mathcal{I}_{g,*}; \mathbb{Q}).$$

(*Sketch of Proof*) First we can see that $[1] \subset \mathrm{Ker}\cup^3$ if and only if the pull-back $p \circ \tau_g(1) : H_3(\mathcal{I}_g) \to [1]$ of the unique (up to scalar) $\mathrm{Sp}(2g, \mathbb{Q})$-equivariant projection $p : \wedge^3[1^3] \to [1]$ to $H_3(\mathcal{I}_g)$ is trivial. The map $p \circ \tau_g(1)$ belongs to $\mathrm{Hom}(H_3(\mathcal{I}_g), [1]) \cong H^3(\mathcal{I}_g; [1])$, which can be embedded in $H^4(\mathcal{I}_{g,*}; \mathbb{Q})$ by using a canonical inclusion given by Kawazumi-Morita [16]. On the other hand, using the natural embedding

$$\mathrm{Hom}(\wedge^3[1^3], [1]) \cong \wedge^3[1^3] \otimes [1] \hookrightarrow \wedge^4(\wedge^3[1])$$

and the first Johnson homomorphism $\tau_{g,*} : \mathcal{I}_{g,*} \to \wedge^3[1] = [1^3] + [1]$ for $\mathcal{I}_{g,*}$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\wedge^3[1^3],[1]) & \xrightarrow{\tau_g(1)^*} & H^3(\mathcal{I}_g;[1]) \\ \downarrow & & \downarrow \\ \wedge^4(\wedge^3[1]) & \xrightarrow{\tau_{g,*}(1)^*} & H^4(\mathcal{I}_{g,*};\mathbb{Q}). \end{array}$$

Therefore we see that $[1] \subset \mathrm{Ker} \cup^3$ if and only if $\tau_{g,*}(1)^*(p) \in H^4(\mathcal{I}_{g,*};\mathbb{Q})$ is trivial.

Since $p : \wedge^3[1^3] \to [1]$ is $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant, as an element of $\wedge^4(\wedge^3[1])$, it belongs to the $\mathrm{Sp}(2g,\mathbb{Q})$-invariant part $(\wedge^4(\wedge^3[1]))^{\mathrm{Sp}}$. In [26], Morita constructed a commutative diagram

$$\begin{array}{ccc} (\wedge^4(\wedge^3[1]))^{\mathrm{Sp}} & \xrightarrow{\rho_1^*} & H^4(\mathcal{M}_{g,*};\mathbb{Q}) \\ \downarrow & & \downarrow \\ \wedge^4(\wedge^3[1]) & \xrightarrow{\tau_{g,*}(1)^*} & H^4(\mathcal{I}_{g,*};\mathbb{Q}), \end{array}$$

where the upper horizontal map is induced from the extended Johnson homomorphism $\rho_1 : \mathcal{M}_{g,*} \to \wedge^3[1^3] \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ (see [25]). Finally, using an explicit description of ρ_1^* given by Kawazumi-Morita [16], we can compute that $\tau_{g,*}(1)^*(p) = e_2 - (2-2g)e^2$ up to scalar. Theorem 4.3 follows from this. □

At present, it is not known whether powers of the Euler class e^i $(i \geq 2)$ and even Morita-Miller-Mumford classes e_{2i} $(i \geq 1)$ are trivial or not when they are restricted to the Torelli group, and we have little information about it. (As for odd Morita-Miller-Mumford classes e_{2i-1} $(i \geq 1)$, it is known that they are trivial in $H^*(\mathcal{I}_g,\mathbb{Q})$. See [20] for details.) Theorem 4.3 can be read that there is a method to attack the non-triviality problem for $e^2, e_2 \in H^4(\mathcal{I}_{g,*};\mathbb{Q})$ from the theory of *three*-dimensional manifolds. That is, by the well-known fact about the realization of a homology class of degree three, it follows that if $\cup^3([1])$ is non-trivial, it must be evaluated non-trivially by the fundamental class of an oriented closed three-dimensional manifold. However this approach has the following difficulty. The condition in Theorem 4.3 is compatible with the pull-back of the universal Σ_g-bundle. Therefore comparing the result of Morita in [21] that the pull-back of $e^2 \in H^4(\mathcal{M}_{g,*};\mathbb{Q})$ to an amenable group always vanishes, we obtain the following.

Corollary 4.4 ([29]). *For every amenable group G and every group homomorphism $f : G \to \mathcal{I}_g$,*

$$f^* \circ \cup^3([1]) = \{0\} \subset H^3(G;\mathbb{Q}).$$

Since abelian groups are amenable, this corollary implies that we cannot evaluate $\cup^3([1])$ by using abelian cycles even if $\cup^3([1])$ is non-trivial in $H^3(\mathcal{I}_g;\mathbb{Q})$. Therefore the first step to determine whether $\cup^3([1])$ is trivial or not is to study the following problem.

Problem 4.5. *Construct a non-trivial homomorphism $G \to \mathcal{I}_g$ where G is not amenable and is given as the fundamental group of an oriented closed three-dimensional manifold.*

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