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Series and polynomial representations for weighted Rogers-Ramanujan partitions and products modulo 6

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Abstract.

Infinite series representations are now obtained for certain weighted Rogers-Ramanujan partitions which we recently showed are related to partitions into parts $\not\equiv 0, \pm i \pmod 6$, for $i = 1, 2, 3$. We also show that our series can be transformed to the series previously obtained by Bressoud which connect the partitions into parts $\not\equiv 0 \pm i \pmod 6$ with partitions satisfying certain bounds on their successive ranks. Finally, we obtain finite versions of our series representations, namely, polynomial identities which tend to the infinite series identities when certain parameters tend to infinity.

Introduction

Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod k$. Motivated by studies in mathematical physics, Andrews et-al [5] showed that $A_{k,i}(n)$ is equal to the the number of partitions $B_{k,i}(n)$ of n whose successive ranks all lie in the interval $[-i+2, k-i-2]$. In a recent paper [2] we showed that for $i = 1, 2, 3$, $B_{6,i}(n)$ and $B_{7,i}(n)$ are equal to the number of Rogers-Ramanujan partitions of n counted with certain weights. Here by a Rogers-Ramanujan (R-R) partition we mean a partition into parts differing by at least 2. The weights which yield the connection with $B_{6,i}(n)$ (and $A_{6,i}(n)$) are certain powers of 2 to be specified in the sequel (see Theorems 1, 2, and 3 below). The weights that need to be attached to connect the R-R partitions with $B_{7,i}(n)$ (and $A_{7,i}(n)$) are products of Fibonacci numbers (see Theorems, 4, 5, 6 of [2]). Traditionally the R-R partitions have been associated with

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the modulus 5. Ours was the first instance [2] when the classical R-R partitions were connected to moduli other than 5 and that is what motivated us to study these weighted partition identities. The method we used in [2] was to show that $B_{6,i}(n)$ and $B_{7,i}(n)$ are equal to the weighted count of the R-R partitions by constructing certain surjective maps, and viewing these weights as the sizes of the inverse images of the points in the codomain under these surjections.

While studying certain general classes of Rogers-Ramanujan type partition identities, Bressoud [7] obtained series representations for the generating functions of $B_{6,i}(n)$ (see (2.7), (2.8), and (2.9) below). While it is obvious that the generating function of $A_{k,i}(n)$ is a product, it is not at all easy to show that the series obtained by Bressoud are the generating functions of the partitions satisfying certain prescribed bounds on the successive ranks.

Our purpose here is three-fold: (i) First we demonstrate that the generating functions of our weighted Rogers-Ramanujan partitions can be shown directly to be the series given in (2.4), (2.5), and (2.6). (ii) Next we show that our series (2.4), (2.5), and (2.6), are equal to the Bressoud series (2.7), (2.8), and (2.9), by applying Lemmas 4 and 5 (transformation lemmas) appropriately. Thus a quicker way to go from $B_{6,i}(n)$ to the series in (2.6), (2.7) and (2.8) is to first convert $B_{6,i}(n)$ to weighted R-R partitions as in [2], then show that the generating functions of these wighted R-R partitions are the series (2.4), (2.5), and (2.6) as in §2, and then transform these to the Bressoud series (2.7), (2.8) and (2.9) as demonstrated in §3. (iii) Finally we present finite versions of our infinite series identities (2.4), (2,5) and (2.6), namely polynomial identities (5.2), (5.3) and (5.4) which tend to our infinite identities when a certain parameter L tends to infinity.

Our approach provides new connections and fresh insight into the structure of various partition functions and their generating functions. For instance, the proofs of Lemmas 4 and 5 rest on the study of certain triple series which are actually related to the series obtained by Alladi-Gordon [3] in the study of generalizations of Schur's partition theorem. Specializations of these triple series yield some of the results established here (see §4).

In §5 we will discuss finite (polynomial) versions of our generating function identities (2.4), (2.5) and (2.6). The proof and combinatorial interpretation of these finite identities is given in §6. Our finite identities lead to a better understanding of of the nature of the relations between our weighted R-R partitions, $B_{6,i}(n), A_{6,i}(n)$ and their generating functions under a limiting process. This is explained in §7.

We will use the standard notation

$$(a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$$

for a positive integer n and complex numbers a, q. Also

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n = \prod_{j=0}^{\infty}(1-aq^j), \qquad \text{when} \qquad |q|<1.$$

Finally,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q)_n}{(q)_m(q)_{n-m}}$$

is the q-binomial coefficient for integers $0 \le m \le n$.

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§1. Weighted Rogers-Ramanujan partition theorems

Every Rogers-Ramanujan partition can be decomposed uniquely into *chains*, where a chain is a maximal string of parts with difference 2 between consecutive parts. Since all parts of a chain have the same parity, we may refer to a chain as an odd (resp. even) chain if the smallest part is odd (resp. even). We denote by $\mathcal{R}$, the set of all R-R partitions, and by $\mathcal{R}_2$, the subset of $\mathcal{R}$ where the the least part is ≥ 2. Also, for any partition π, we let $\sigma(\pi)$ denote the sum of the parts of π. The following results were established in [2]:

Theorem 1. *Let $A_{6,1}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm 1 \pmod 6$. Suppose $\pi \in \mathcal{R}_2$ has k even chains with least part > 2. Let $\omega_1(\pi) = 2^k$ be the weight of π. Then*

$$\sum_{\pi\in\mathcal{R}_2, \sigma(\pi)=n} \omega_1(\pi) = A_{6,1}(n).$$

Theorem 2. *Let $A_{6,2}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm 2 \pmod 6$. Suppose $\pi \in \mathcal{R}$ has k odd chains with least part > 1. Let $\omega_2(\pi) = 2^k$ be the weight of π. Then*

$$\sum_{\pi\in\mathcal{R},\sigma(\pi)=n} \omega_2(\pi) = A_{6,2}(n).$$

Theorem 3. *Let $A_{6,3}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm 3 \pmod 6$. Suppose $\pi \in \mathcal{R}$ has k even chains. Let $\omega_3(\pi) = 2^k$ be the weight of π. Then*

$$\sum_{\pi\in\mathcal{R},\sigma(\pi)=n} \omega_3(\pi) = A_{6,3}(n).$$

Remarks:

(i) Note that $A_{6,2}(n)$ is the number of partitions of n into odd parts which is equal to the number of partitions $Q(n)$ of n into distinct parts. Actually, Theorem 2 was first established in [1] with $Q(n)$ in the place of $A_{6,2}(n)$. It was in [2] that we replaced $Q(n)$ by $A_{6,2}(n)$ and this led us to discover Theorems 1 and 3.

(ii) Since the residue classes $\pm 3 \pmod 6$ are one and the same, the function $A_{6,3}(n)$ has to be defined properly via the generating function

$$\sum_{n=0}^{\infty} A_{6,3}(n)q^n = \frac{(q^6, q^6)_\infty (q^3; q^6)_\infty (q^3; q^6)_\infty}{(q)_\infty}. \tag{1.1}$$

The product on the right in (1.1) can be written as

$$\frac{1}{(q;q^3)_\infty (q^2;q^3)_\infty (-q^3;q^3)_\infty} \tag{1.2}$$

and viewed as the generating function of unrestricted partitions counted with weight $(-1)^{\nu_3(\pi)}$, where $\nu_3(\pi)$ is the number of parts of π which are multiples of 3. Andrews and Lewis [6] have observed that the product in (1.1) can be written as

$$\prod_{m=1}^{\infty} (1 + \frac{q^{3m-1}}{1-q^{3m-1}} + \frac{q^{3m-2}}{1-q^{3m-2}}),$$

and interpreted as the generating function of an ordinary partition function (not a weighted partition function), namely, the number of partitions of n into non-multiples of 3 in which no two parts differ by exactly 1.

Next, we obtain the generating functions of the weighted R-R partitions in the above theorems.

§2. Series representations

First we consider partitions $\pi \in \mathcal{R}_2$ for which weights 2^k are attached, where k is the number of even chains with least part ≥ 4. Let such a partition have r odd parts and s even parts. Let the partition π be $m_1 + m_2 + m_3 + ... + m_{r+s}$, where the gap between the parts is ≥ 2 and $m_{r+s} \geq 2$.

Subtract 0 from m_{r+s}, 2 from m_{r+s-1},...., $2(r+s-1)$ from m_1. We call this processs as *Euler subtraction.* The resulting partition π^* after Euler subtraction is an ordinary partition having r odd parts and s even parts. Note that the number of chains of the original partition π is equal to the number of different parts of π^* because parts of π^* which repeat are derived from those belonging to the same chain in π. Also, the even chains in π with least part ≥ 4 correspond to the even parts of π^* which are ≥ 4.

Next we add 0 to the smallest even part of π^*, 2 to the second smallest even part of π^*, ..., $2(s-1)$ to the largest even part of π^*, $2s$ to the smallest odd part of π^*, $2s+2$ to the second smallest odd part of π^*, ..., $2(r+s-1)$ to the largest odd part of π^*. We call this process as *Bressoud redistribution.* We have thus created a partition π^{**} in $\mathcal{R}_2$ with the following properties: (i) The even parts of π^{**} form a partition π_e such that the even chains of π which have least part ≥ 4 correspond to the even chains in π_e with least part ≥ 4, and (ii) the odd parts of π^{**} form a partition π_o into r distinct odd parts such that the least part is $\geq 2s+2$.

To compute generating functions, we first observe that the smallest partition into r distinct odd parts $\geq 2s+2$ is $(2s+3)+(2s+5)+...+(2s+2r+1)$ of the integer $r^2+2rs+2r$. Thus the generating function of the partition π_o is clearly

$$\frac{q^{r^2+2rs+2r}}{(q^2;q^2)_r}. \tag{2.1}$$

Similarly, the generating function of π_e of partitions into s distinct even parts is

$$\frac{q^{s^2+s}}{(q^2;q^2)_s}. \tag{2.2}$$

However, we need to count the partitions π_e with weights 2^k, where k is the number of even chains in π_e with least part ≥ 4. We will describe a combinatorial process which will let us determine this generating function from the simpler expression in (2.2).

The chains of π_e with least part ≥ 4 are determined by gaps between consecutive parts which are > 2 in the partition obtained from π_e by adding a 0 to it. Consider the 2-modular Ferrers graph of π_e, where at every node we place a 2, and the rows of 2 add up to the parts of π_e. Whenever there is a gap between parts of π_e which is > 2, we can *extract* a column of twos from this Ferrers graph and still be left with a graph of a partition into s distinct even parts. We have two choices: either extract this column or not extract it. Since there are k such gaps which correspond to the chains, we have 2^k choices and this corresponds to the weight 2^k in the theorem.

The extracted columns form a partition into distinct even parts which are $\leq 2s$ and so their generating function is

$$(-q^2; q^2)_s.$$

Thus the generating function of the partitions π_e counted with the weights 2^k is

$$\frac{q^{r^2+2rs+2r}(-q^2; q^2)_s}{(q^2; q^2)_s}. \tag{2.3}$$

Finally, we need to multiply the expressions in (2.1) and (2.3) to get the generating function of the partitions π we started with having r odd parts and s even parts. If we sum this expression over all non-negative integral values of r, s, we get the following analytic form of Theorem 1:

$$\sum_{r,s} \frac{q^{(r+s)^2+2r+s}(-q^2; q^2)_s}{(q^2; q^2)_r (q^2; q^2)_s} = \frac{1}{(q^2; q^6)_\infty (q^3; q^6)_\infty (q^4; q^6)_\infty}. \tag{2.4}$$

Similarly, by using the method described above, Theorem 2 can be cast in the following analytic form:

$$\sum_{r,s} \frac{q^{(r+s)^2+r}(-q^2; q^2)_s}{(q^2; q^2)_r (q^2; q^2)_s} = \frac{1}{(q; q^6)_\infty (q^3; q^6)_\infty (q^5; q^6)_\infty}. \tag{2.5}$$

The analytic form of Theorem 3 using (1.1) is

$$\sum_{r,s} \frac{q^{(r+s)^2+s}(-1; q^2)_s}{(q^2; q^2)_r (q^2; q^2)_s} = \frac{1}{(q; q^3)_\infty (q^2; q^3)_\infty (-q^3; q^3)_\infty}. \tag{2.6}$$

By considering partitions with prescribed bounds on the successive ranks, and determining their generating functions, Bressoud [7] obtained the following analytic identities:

$$\sum_{n_1,n_2} \frac{q^{N_1^2+N_2^2+N_1+N_2}}{(q)_{n_1}(q^2;q^2)_{n_2}} = \frac{1}{(q^2;q^6)_\infty(q^3;q^6)_\infty(q^4;q^6)_\infty}, \tag{2.7}$$

$$\sum_{n_1,n_2} \frac{q^{N_1^2+N_2^2+N_2}}{(q)_{n_1}(q^2;q^2)_{n_2}} = \frac{1}{(q;q^6)_\infty(q^3;q^6)_\infty(q^5;q^6)_\infty}, \tag{2.8}$$

and

$$\sum_{n_1,n_2} \frac{q^{N_1^2+N_2^2}}{(q)_{n_1}(q^2;q^2)_{n_2}} = \frac{1}{(q;q^3)_\infty(q^2;q^3)_\infty(-q^3;q^3)_\infty}. \tag{2.9}$$

Here $N_1 = n_1 + n_2$ and $N_2 = n_2$, where n_1, n_2 run over all non negative integers.

Bressoud did not use the product form (1.2) in (2.9), but we have preferred to state Bressoud's third identity using (1.2) for uniformity with our identities.

It is quite difficult to show that the series obtained by Bressoud are indeed the generating functions of certain partitions with prescribed bounds on their successive ranks. In the next section we will establish identities (transformation lemmas) which will transform (2.4), (2.5) and (2.6) to (2.7), (2.8) and (2.9) respectively.

§3. Transformation lemmas

We begin with

Lemma 4.

$$\sum_{r,s} \frac{a^{r+s}q^{(r+s)^2+r}(-bq;q^2)_s}{(q^2;q^2)_r(q^2;q^2)_s} = \sum_{r,s} \frac{a^{r+s}q^{(r+s)^2+s}(-b;q^2)_s}{(q^2;q^2)_r(q^2;q^2)_s}.$$

Proof: If we use the q-binomial theorem to expand $(-bq;q^2)_s$ and cancel a common factor $(q^2;q^2)_s$, the left hand side can be rewritten as

$$LHS = \sum_{r,s} \frac{a^{r+s}q^{r^2+s^2+2rs+r}}{(q^2;q^2)_r} \sum_{l=0}^{s} \frac{b^l q^{l^2}}{(q^2;q^2)_s(q^2;q^2)_{s-l}} \tag{3.1}$$

This can be rewritten as a triple sum over r, s, l with the restriction $l \le s$. If we replace s by $s + l$, then this frees us of the restriction. Thus after this replacement we have

$$LHS = \sum_{r,s,l} \frac{a^{r+s+l} b^l q^{r^2+s^2+l^2+2rs+2rl+2sl+r+l^2}}{(q^2;q^2)_r (q^2;q^2)_s (q^2;q^2)_l} \tag{3.2}$$

Similarly, on the right hand side, if we use the q-binomial expansion on $(-b;q^2)_s$, convert it into a triple sum, and replace s by $s+l$, we get

$$RHS = \sum_{r,s,l} \frac{a^{r+s+l} b^l q^{r^2+s^2+l^2+2rs+2rl+2sl+s+l^2}}{(q^2;q^2)_r (q^2;q^2)_s (q^2;q^2)_l}. \tag{3.3}$$

Observe that the triple sums in (3.2) and (3.3) are equal by just interchanging r and s. This proves Lemma 4.

Lemma 5.

$$\sum_{n_1,n_2} \frac{a^{n_1}(ab)^{n_2} q^{(n_1+n_2)^2+n_2^2}}{(q)_{n_1}(q^2;q^2)_{n_2}} = \sum_{r,s} \frac{a^{r+s} q^{(r+s)^2+s}(-b;q^2)_s}{(q^2;q^2)_r (q^2;q^2)_s}.$$

In order to prove Lemma 5, we need

Lemma 6. *For any non-negative integer n, we have*

$$\frac{1}{(q)_n} = \sum_{j=0}^{n} \frac{q^j}{(q^2;q^2)_j} \frac{1}{(q^2;q^2)_{n-j}}.$$

Proof of Lemma 6: We interpret the left hand side of Lemma 6 as the generating function of partitions into no more than n parts. Among all such partitions suppose we consider those which have exactly j odd parts. Then the generating function of the odd part component of such partitions is

$$\frac{q^j}{(q^2;q^2)_j}. \tag{3.4}$$

The number of even parts in such partitions must be no more than $n-j$ and the generating function for this even component is

$$\frac{1}{(q^2;q^2)_{n-j}}. \tag{3.5}$$

If we multiply the expressions in (3.4) and (3.5) and sum over j, this must equal $1/(q)_n$. This proves Lemma 6.

Proof of Lemma 5: Using Lemma 6, rewrite the left hand side of Lemma 5 as

$$\sum_{n_1,n_2} \frac{a^{n_1}(ab)^{n_2}q^{(n_1+n_2)^2+n_2^2}}{(q)_{n_1}(q^2;q^2)_{n_2}}$$

$$= \sum_{n_1,n_2} \frac{a^{n_1}(ab)^{n_2}q^{(n_1+n_2)^2+n_2^2}}{(q^2;q^2)_{n_2}} \sum_{j=0}^{n_1} \frac{q^j}{(q^2;q^2)_j} \frac{1}{(q^2;q^2)_{n_1-j}}.$$

$$= \sum_{n_1,n_2,j\le n_1} \frac{a^{n_1}(ab)^{n_2}q^{(n_1+n_2)^2+n_2^2+j}}{(q^2;q^2)_{n_2}(q^2;q^2)_j(q^2;q^2)_{n_1-j}}$$

$$= \sum_{n_1,n_2,j} \frac{a^{n_1+j}(ab)^{n_2}q^{(n_1+n_2+j)^2+n_2^2+j}}{(q^2;q^2)_{n_2}(q^2;q^2)_j(q^2;q^2)_{n_1}}, \tag{3.6}$$

where to get the final expression in (3.6) we have replaced n_1 by $n_1 + j$ to free ourselves of the restriction $j \le n_1$.

Next replace n_2 by k, n_1 by s, and j by r, to rewrite the final expression in (3.6) as

$$= \sum_{r,s,k} \frac{a^{r+s}(ab)^k q^{(r+s+k)^2+k^2+r}}{(q^2;q^2)_k(q^2;q^2)_r(q^2;q^2)_s}, \tag{3.7}$$

In (3.7) we replace s by $s-k$ so that this forces the restriction $k \le s$ to get

$$= \sum_{r,s,k\le s} \frac{a^{r+s}b^k q^{(r+s)^2+k^2+r}}{(q^2;q^2)_k(q^2;q^2)_r(q^2;q^2)_{s-k}}$$

$$= \sum_{r,s} \frac{a^{r+s}q^{(r+s)^2+r}}{(q^2;q^2)_r} \sum_{k=0}^{s} \frac{b^k q^{k^2}}{(q^2;q^2)_k(q^2;q^2)_{s-k}}$$

which by the q-binomial theorem is

$$\sum_{r,s} \frac{a^{r+s}q^{(r+s)^2+r}(-bq;q^2)_s}{(q^2;q^2)_r(q^2;q^2)_s}. \tag{3.8}$$

The final expression in (3.8) is the left hand side of Lemma 4, which can be replaced by the right hand side of Lemma 4, yielding the right hand side of Lemma 5 thereby completing the proof.

To get the series in (2.9) from the series in (2.6) put $a = b = 1$ in Lemma 5.

Next, replace b by bq in Lemma 5 which gives the Bressoud series in (2.8) with parameters a and b. The right hand side of Lemma 5 with this choice is:

$$\sum_{r,s} \frac{a^{r+s}q^{(r+s)^2+s}(-bq;q^2)_s}{(q^2;q^2)_r(q^2;q^2)_s}. \tag{3.9}$$

Now in Lemma 4 replace b by bq to realize that the expression in (3.9) is

$$\sum_{r,s} \frac{a^{r+s}q^{(r+s)^2+r}(-bq^2;q^2)_s}{(q^2;q^2)_r(q^2;q^2)_s} \tag{3.10}$$

which is the series (2.5) with parameters a and b. Thus what we have observed is that

$$\sum_{n_1,n_2} \frac{a^{n_1}(ab)^{n_2}q^{(n_1+n_2)^2+n_2^2}}{(q)_{n_1}(q^2;q^2)_{n_2}} = \sum_{r,s} \frac{a^{r+s}q^{(r+s)^2+r}(-bq^2;q^2)_s}{(q^2;q^2)_r(q^2;q^2)_s}. \tag{3.11}$$

Finally, replace a by aq in (3.11). This gives the equality between (2.4) and (2.7) in refined form.

§4. Triple series for generalized Schur partitions

The equivalence of our series with those of Bressoud was established using the transformation lemmas 1 and 2, and the proofs of these lemmas depended on showing the the double sums on both sides could be written as triple sums, and that equality is achieved because the triple sum form is the same for both sides. We now will show that these triple sums arise from the generating functions of a generalization of Schur's partition theorem due to Alladi and Gordon [3].

Schur's celebrated partition theorem of 1926 is:

Theorem 7 (Schur). *Let $S(n)$ denote the number of partitions of n into distinct parts $\equiv 1, 2 \pmod 3$.*

Let $D(n)$ denote the number of partitions of n into parts differing by ≥ 3, where the inequality is strict if a part is a multiple of 3. Then

$$D(n) = S(n).$$

Alladi and Gordon [3] viewed Schur's theorem as emerging out of a certain series expansion of the product

$$\prod_{m=1}^{\infty}(1+aq^m)(1+bq^m) \tag{4.1}$$

under the substitutions q replaced by q^3, a replaced by aq^{-2}, b replaced by bq^{-1}. Clearly these substitutions convert the product in (4.1) to

$$\prod_{m=1}^{\infty}(1+aq^{3m-2})(1+bq^{3m-1}),$$

which is a two parameter refinement of the generating function of $S(n)$ in Theorem 7.

Now assume that integer 1 occurs in two primary colors A and B, and that integers $n \geq 2$ occur in primary colors A and B as well as in the secondary color AB. Let A_n, B_n, AB_n, and denote the integer n in colors A, B, AB, respectively. To discuss partitions, an ordering is required, and the one chosen in [3] was

$$A_1 < B_1 < AB_2 < A_2 < B_2 < AB_3 < A_3 < B_3 < \ldots \tag{4.2}$$

The reason for choosing this ordering was that under the above substitutions, the colored integers become $A_n = 3n-2, B_n = 3n-1, AB_n = 3n-3$, and so the ordering in (4.2) becomes

$$1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < \ldots$$

the natural ordering among the positive integers.

Next, Alladi-Gordon defined Type 1 partitions to be those of the form $m_{x_1} + m_{x_2} + \ldots$, where m_i is the symbol in position i in (4.1), such that the difference between the x_i is ≥ 3 with strict inequality if a part is of secondary color. Under the substitutions, Type 1 partitions become the partitions of the type enumerated by $D(n)$ in Theorem 7.

Suppose we consider Type 1 partitions in which we prescribe the number of A-parts to be i, the number of B-parts to be j, and the number of AB-parts to be k. Suppose each such Type 1 partition is counted with weight $a^i b^j c^k$. Then it is shown in [3] that the generating function of all Type 1 partitions is given by

$$\sum_{i,j,k} \frac{a^i b^j c^k q^{T_{i+j+k}+T_k}}{(q)_i (q)_j (q)_k}, \tag{4.3}$$

where $T_n = n(n+1)/2$ is the n-th triangular number. It was observed in [3] that when

$$c = ab,$$

the series in (4.3) is equal to the product in (4.1), thereby yielding a two parameter generalization and refinement of Schur's partition theorem.

The reason we described this generalized approach to Schur's theorem is because we wish to point out that the triple series considered in Section 3 are all of the type in (4.3) with q dilated to q^2, and the parameters a, b, c being suitably chosen. Just as the choice $c = ab$ in (4.3) leads to the product in (4.1), the special choices in Section 3 lead to the three mod 6 products. Thus, from this point of view, the generating function of Type 1 partitions of Schur type forms the underlying link between our three series and those of Bressoud.

§5. Polynomial versions

In this section we will obtain finite versions of our identities (2.4), (2.5) and (2.6), that is polynomial identities which will will become the identities in Section 3 when certain parameters tend to infinity. For instance

$$\lim_{n\to\infty} \begin{bmatrix} n \\ m \end{bmatrix} = \frac{1}{(q)_m}. \tag{5.1}$$

Thus the q-binomial coefficient which is a polynomial, tends to the power series generated by $1/(q)_m$.

A finite version of (2.4) is:

$$\begin{aligned} \sum_{\substack{r+s+i\le L-2\\ i\le s}} & q^{(r+s)^2+2r+s+i^2+i} \begin{bmatrix} L-2-s \\ r \end{bmatrix}_{q^2} \\ & \begin{bmatrix} L-2-r-s \\ i \end{bmatrix}_{q^2} \begin{bmatrix} L-1-r-i \\ s-i \end{bmatrix}_{q^2} + q^{L(L-1)} \\ = \sum_{j=-\infty}^{\infty} & \{q^{12j^2-4j} \begin{bmatrix} 2L \\ L-6j \end{bmatrix} - q^{12j^2-8j+1} \begin{bmatrix} 2L \\ L+1-6j \end{bmatrix}\}. \end{aligned} \tag{5.2}$$

Similarly, a finite version of (2.5) is:

(5.3)

$$\sum_{\substack{r+s+i\le L-1\\ i\le s}} q^{(r+s)^2+r+i^2+i} \begin{bmatrix} L-1-s \\ r \end{bmatrix}_{q^2} \begin{bmatrix} L-1-r-s \\ i \end{bmatrix}_{q^2} \begin{bmatrix} L-r-i \\ s-i \end{bmatrix}_{q^2}$$

$$+q^{L^2} = \sum_{j=-\infty}^{\infty} \{q^{12j^2-2j} \begin{bmatrix} 2L \\ L-6j \end{bmatrix} - q^{12j^2-10j+2} \begin{bmatrix} 2L \\ L+2-6j \end{bmatrix}\}.$$

Finally, a finite version of (2.6) is:

(5.4)

$$\sum_{\substack{r+s+i\leq L\\ r+s<L-1, i\leq s}} q^{(r+s)^2+s+i^2-i} \begin{bmatrix} L-s \\ r \end{bmatrix}_{q^2} \begin{bmatrix} L-r-s \\ i \end{bmatrix}_{q^2} \begin{bmatrix} L-1-r-i \\ s-i \end{bmatrix}_{q^2}$$

$$+q^{L^2} = \sum_{j=-\infty}^{\infty} (-1)^j q^{3j^2} \begin{bmatrix} 2L \\ L+3j \end{bmatrix}.$$

It is to be noted that the sums on the right in (5.2), (5.3), and (5.4) are finite sums because the q-binomial coefficients vanish when $|j|$ is sufficiently large. In view of (5.1), when $L \to \infty$, the finite identity (5.3) becomes

$$\sum_{r,s} \frac{q^{(r+s)^2+r}}{(q^2;q^2)_r} \sum_{i=0}^{s} \frac{q^{i^2+i}}{(q^2;q^2)_i (q^2;q^2)_{s-i}}$$

$$= \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \{q^{12j^2-2j} - q^{12j^2-10j+2}\}. \tag{5.5}$$

Now use the q-binomial theorem to evaluate the inner sum on the left in (5.5) as

$$\frac{(-q^2;q^2)_s}{(q^2;q^2)_s}.$$

Also, the series on the right can be rewritten as

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{3j^2+j} \tag{5.6}$$

which can be shown to be equal to

$$(q^2;q^6)_\infty (q^4;q^6)_\infty (q^6;q^6)_\infty \tag{5.7}$$

by Jacobi's triple product identity. Thus (5.5) yields (2.5) and so (5.3) is indeed a finite version of (2.5). In a similar manner it can be shown that (5.2) is a finite version of (2.4) and (5.4) is a finite version of (2.6), as claimed.

§6. Combinatorial interpretation of polynomial identities

In this section we will show that the combinatorial interpretation of the finite identities (5.2), (5.3) and (5.4) is the equality between the partition functions $B_{6,i}(n)$ and the weighted R-R partitions in Theorems 1, 2, and 3, but with bounds imposed on the size of the parts. This will then lead to some new insights into the nature of the results obtained by Andrews et-al [5] as explained at the end of this section.

We will now show that the left hand side of (5.2) is the generating function of the weighted R-R partitions in Theorem 1 but now with the extra conditions that all parts are $\leq 2L-2$ and that if an even chain in a partition has $2L-2$ as a part, then its weight is counted as 1. To this end, let $\pi : m_1 + m_2 + ... + m_{r+s}$ be a partition in $\mathcal{R}_2$ having r odd parts and s even parts and with largest part $m1 < 2L-1$. Let each even chain in π whose smallest part is > 2 and largest part $< 2L-2$ be counted with weight 2. As in §2, we perform *Euler subtraction*, on π. This yields an unrestricted partition $\pi^* : m_1^* + m_2^* + ... + m_{r+s}^*$ with r odd parts and s even parts in which the smallest parts of the chains of π are now identified with the parts of π that are different in value (because repeated parts belong to the same chain). Note that

$$m_1^* \leq 2L - 2 - (2r + 2s - 2) = 2L - 2r - 2s. \tag{6.1}$$

Now perform *Bressoud redistribution* on the parts of π^* where the small even numbers up to $2s-2$ are added to the even parts of π^* and the large even numbers from $2s$ to $2s+2r-2$ are added to the odd parts of π^* in succession from the smallest upwards. Thus we have constructed two partitions π_e and π_o out of π^*. Note that π_o has r distinct odd numbers in the interval $[2s+3, 2L-3]$. The smallest partition into r distinct odd parts in that interval is

(6.2)

$$(2s+3)+(2s+5)+...+(2s+2r+1) \qquad \text{of the number} \qquad r^2+2r+2rs.$$

If we represent this partition by a Ferrers graph, then to get all such partitions π_o out of this minimal partition, we need to imbed columns of twos of length no more than r and the number of columns of twos to be imbedded is

$$\leq \frac{2L-3-(2r+2s+1)}{2} = L - 2 - r - s. \tag{6.3}$$

Thus from (6.2) and (6.3) we see that the generating function of π_o is

$$q^{r^2+2r+2rs} \begin{bmatrix} L-2-s \\ r \end{bmatrix}_{q^2}. \tag{6.4}$$

The partition π_e consists of s distinct even parts $\leq 2L - 2 - 2r$ in which each chain with smallest part > 2 is counted with weight 2, and an even chain with smallest part 2 is counted with weight 1. If we subtract $2, 4, 6, ..., 2s$ from the smalllest part of π_e upwards in succession, we get a partition π_e^* into non-negative even parts $\leq 2L - 2 - 2r - 2s$ where we count the partition π_e^* with weight 2^k with k representing the number of different positive even parts of π_e^* which are $< 2L - 2 - 2r - 2s$. Thus the generating function of π_e is

$$q^{s^2+s}X, \tag{6.5}$$

where X is the generating function of π_e^*.

In order to determine X, we need the following:

Lemma 8. *Let π be a partition into into s non-negative parts $\leq M$. Let k be the number of different positive parts of π which are $< M$. Suppose π is counted with weight 2^k. Then the generating function of such weighted partitions π is*

$$\sum_{i=0}^{s} q^{T_i} \begin{bmatrix} M-1 \\ i \end{bmatrix} \begin{bmatrix} M+s-i \\ s-i \end{bmatrix}.$$

Proof of Lemma 8: First observe that

$$\frac{1+zq^j}{1-zq^j} = 1 + 2\{zq^j + z^2q^{2j} + z^3q^{3j} + ...\}. \tag{6.6}$$

In (6.6), the power of z represents the frequency of occurrence of the part j, but 2 occurs only once and does not depend on the frequency of occurrence. Thus

$$\frac{1}{(1-z)} \cdot \prod_{j=1}^{M-1} \frac{(1+zq^j)}{1-zq^j} \frac{1}{1-zq^m} = \frac{(-zq)_{M-1}}{(z)_{M-1}} \tag{6.7}$$

is the generating function of π counted with the weight 2^k as above. The power of z in (6.7) represents the total number of non-negative parts of π. this means that

$$\text{the coefficient of} \quad z^s \quad \text{in} \quad \frac{(-zq)_{M-1}}{(z)_{M+1}} \tag{6.8}$$

is the generating function that is sought in Lemma 8.

To compute the desired coefficient, we use the expansions

$$(-zq)_{M-1} = \sum_{i=0}^{M-1} z^i q^{T_i} \begin{bmatrix} M-1 \\ i \end{bmatrix} \tag{6.9}$$

and

$$\frac{1}{(z)_{M+1}} = \sum_{j=0}^{\infty} z^j \begin{bmatrix} M+j \\ j \end{bmatrix}. \tag{6.10}$$

Lemma 4 follows from (6.8), (6.9) and (6.10).

Next we observe that the generating function X in (6.5) can be obtained by replacing q by q^2 and taking $M = L-1-r-s$ in Lemma 8. This yields

$$X = \sum_{i=0}^{s} q^{i^2+i} \begin{bmatrix} L-2-r-s \\ i \end{bmatrix}_{q^2} \begin{bmatrix} L-1-r-i \\ s-i \end{bmatrix}_{q^2}. \tag{6.11}$$

Finally, from (6.4), (6.5) and (6.11) we get the sum on the left hand side of (5.2).

In the above combinatorial analysis, we have tacitly assumed that $s < L-1$. The exceptional case $s = L-1$ corresponds to the partition $2+4+6+...+(2L-2)$ of the integer $L(L-1)$. This partition being a chain starting at 2 will be counted with weight 1. Thus $q^{L(L-1)}$ needs to be added to the sum on the left in (5.2) in order to get the generating function of the weighted R-R partitions in Theorem 1 but now with the extra conditions that all parts are $\le 2L-2$ and if a chain has $2L-2$ as a part, then its weight is 1.

Andrews et-al [5] have computed the generating function of $B_{k,i}(n,L)$, the number of partitions of n into at most L parts, each $\le L$, and with successive ranks in the interval $[-i+2, k-i-2]$. By taking $k=6, i=1$ in their result, we see that the expression on the right hand side of (5.2) is the generating function of $B_{6,1}(n,L)$. Thus the combinatorial version of (5.2) is:

Theorem 9. *Suppose $\pi \in \mathcal{R}_2$ has k even chains with least part > 2 and largest part $< 2L-2$. Let $\omega_1(\pi, L) = 2^k$ and $\lambda(\pi)$ be the largest part of π. Then*

$$\sum_{\pi\in\mathcal{R}_2, \sigma(\pi)=n, \lambda(\pi)\le 2L-2} \omega_1(\pi, L) = B_{6,1}(n,L).$$

Similarly Lemma 8 can be used to show that the combinatorial interpretation of (5.3) is Theorem 10 below.

Theorem 10. *Suppose $\pi \in \mathcal{R}$ has k odd chains with least part > 2 and largest part $< 2L-1$. Let $\omega_2(\pi, L) = 2^k$ and $\lambda(\pi)$ be the largest part of π. Then*

$$\sum_{\pi\in\mathcal{R}, \sigma(\pi)=n, \lambda(\pi)\le 2L-1} \omega_2(\pi, L) = B_{6,2}(n,L).$$

There is one more result, namely

Theorem 11. *Let $\pi \in \mathcal{R}$, $\omega_3(\pi)$ as in Theorem 3 and $\lambda(\pi)$ the largest part of π. Then*

$$\sum_{\pi\in\mathcal{R},\sigma(\pi)=n,\lambda(\pi)\le 2L-1} \omega_3(\pi) = B_{6,3}(n,L).$$

From Theorems 9, 10, and 11, Theorems 1, 2, and 3 respectively follow by letting L tend to ∞. It is to be noted that in Theorem 11 alone the weight is exactly as in Theorem 3. Just as Lemma 8 was used to show that Theorems 9 and 10 are combinatorial interpretations of (5.2) and (5.3), the following lemma is to be used to show that Theorem 11 is the combinatorial interprestation of (5.4):

Lemma 12. *Let π be a partition into s non-negative parts $\le M$. Let k be the number of different parts of π. Suppose π is counted with weight 2^k. Then the generating function of such weighted partitions π is*

$$\sum_{i=0}^{s} q^{T_{i-1}} \begin{bmatrix} M+1 \\ i \end{bmatrix} \begin{bmatrix} M+s-i \\ s-i \end{bmatrix}.$$

From the methods of [5] it follows that the expression on the right in (5.2), (5.3) and (5.4) and the generating function of $B_{6,i}(n,L)$ are equal because they both satisfy the same recurrences and the same initial conditions.

In [2] it was shown combinatorially, that the partitions enumerated by $B_{6,i}(n)$ correspond to the weighted R-R partitions in Theorems 1, 2, and 3. The proof of this equality in [2] is achieved by considering the Ferrers graphs of the partitions enumerated by $B_{6,i}(n)$, for $i = 1,2,3$, and then constructing R-R partitions by counting the number of nodes along each hook of the Ferrers graph. In this correspondence, it is to be noted that imposing the bound L on both the size and number of parts of partitions enumerated by $B_{6,i}(n)$ is the same as saying that the number of nodes along the largest hook is $\le 2L-1$. Thus Theorems 9, 10, and 11, can be proved by the combinatorial method in [2] together with the following observation: for Theorem 9 (Theorem 10), if a chain has largest part $2L-2$ $(2L-1)$, then it must be counted with weight 1. What is new here is our observation that the series on the left in (5.2), (5.3), and (5.4) represent the generating functions of the weighted R-R partitions with certain prescribed bounds on the size of the parts.

§7. Combinatorial information under limits

In this section we draw attention to certain important aspects of the nature of combinatorial information that is either gained or lost in a limiting process.

It is to be noted that whereas the polynomials on the right in (5.2), (5.3) and (5.4) represent the generating functions of $B_{6,i}(n, L)$, when we let $L \to \infty$, we get a theta series as in (5.6) divided by $(q)_\infty$ which by the use of Jacobi's triple product identity for theta functions is seen to be the generating function of $A_{6,i}(n)$. Thus the advantage is that we get the equality

$$B_{6,i}(n) = A_{6,i}(n) \tag{7.1}$$

but in this process we *do not* get the generating function of $B_{6,i}(n)$! This information is somehow lost in the limiting process while gaining the proof of the equality (7.1). This is to be contrasted with polynomials on the left hand sides of (5.2), (5.3) and (5.4), which under the limiting process tend to the series (2.4), (2.5) and (2.6) respectively. Thus the generating functions of the weighted R-R partitions as infinite series are obtained under this limiting process. This difference in behavior of the left and right sides of (5.2), (5.3) and (5,4) has a very important consequence, namely, it allows us first to establish the equality of the weighted R-R partitions with those enumerated by the $B_{6,i}(n)$ with bounds on the parts, and then by letting $L \to \infty$, we achieve equality with $A_{6,i}(n)$.

In summary, the study of these weighted R-R partitions has led to a better understanding of $B_{6,i}(n)$ and their generating functions.

§8. Series for products mod 7

In this paper we have concentrated on series representations for the three mod 6 products which are the generating functions of $A_{6,i}(n)$, for $i = 1, 2, 3$. In [2], we connect both $A_{6,i}(n)$ and $A_{7,i}(n)$ to weighted R-R partitions. The weights in the case of the modulus 7 are products of Fibonacci numbers. This raises the question of series representations for these weighted R-R partitions with weights as products of Fibonacci numbers. There are the well known series of Rogers and Selberg and the double series of Andrews for these products mod 7 (see Andrews [4]), which represent the next level result beyond the celebrated Rogers-Ramanujan identities in Gordon's famous generalization [8] of the Rogers-Ramanujan identities to any odd modulus $2k+1$. In a subsequent paper we plan to investigate the series for these weighted R-R partitions which are connected to the products modulo 7 and discuss connections with the series of Rogers-Selberg and of Andrews.

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