

Proportionality of indices of 1-forms on singular varieties

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§0. Introduction

M.-H. Schwartz in [20, 21] introduced the technique of radial extension of stratified vector fields and frames on singular varieties, and used this to construct cocycles representing classes in the cohomology $H^*(M, M \setminus V)$, where V is a singular variety embedded in a complex manifold M ; these are now called *the Schwartz classes* of V . A basic property of radial extension is that the index of the vector fields (or frames) constructed in this way is the same when measured in the strata or in the ambient space; this is called the Schwartz index of the vector field (or frame). MacPherson in [15] introduced the notion of local Euler obstruction, an invariant defined at each point of a singular variety using an index of an appropriate radial 1-form, and used this (among other things) to construct the homology Chern classes of singular varieties. Brasselet and Schwartz in [3] proved that the Alexander isomorphism $H^*(M, M \setminus V) \cong H_*(V)$ carries the Schwartz classes into the MacPherson classes; a key ingredient for this proof is their *proportionality theorem* relating the Schwartz index and the local Euler obstruction.

These were the first indices of vector fields and 1-forms on singular spaces, in the literature. Later in [8] was introduced another index for vector fields on isolated hypersurface singularities, and this definition was extended in [23] to vector fields on complete intersection germs. This is known as the GSV-index and one of its main properties is that it is invariant under perturbations of both, the vector field and the functions that define the singular variety. The definition of this index was recently extended in [4] for vector fields with isolated singularities on

Received March 21, 2005

Revised August 9, 2005

* partially supported by CONACYT and DGAPA-UNAM, Mexico.

† partially supported by JSPS.

hypersurface germs with non-isolated singularities, and it was proved that this index satisfies a proportionality property analogous to the one proved in [3] for the Schwartz index and the local Euler obstruction, the proportionality factor being now the Euler-Poincaré characteristic of a local Milnor fiber.

In [5] Ebeling and Gusein-Zade observed that when dealing with singular varieties, 1-forms have certain advantages over vector fields, as for instance the fact that for a vector field on the ambient space the condition of being tangent to a (stratified) singular variety is very stringent, while every 1-form on the ambient space defines, by restriction, one on the singular variety. They adapted the definition of the GSV-index to 1-forms on complete intersection germs with isolated singularities, and proved a very nice formula for it in the case when the form is holomorphic, generalizing the well-known formula of Lê-Greuel for the Milnor number of a function.

This article is about 1-forms on complex analytic varieties and it is particularly relevant when the variety has non-isolated singularities. We show in Section 2 how the radial extension technique of M.-H. Schwartz can be adapted to 1-forms, allowing us to define *the Schwartz index* of 1-forms with isolated singularities on singular varieties. Then we see (Section 3) how MacPherson's local Euler obstruction, adapted to 1-forms in general, relates to the Schwartz index, thus obtaining a proportionality theorem for these indices analogous to the one in [3] for vector fields. We then extend (in Section 4) the definition of the GSV-index to 1-forms with isolated singularities on (local) complete intersections with non-isolated singularities that satisfy the Thom a_f -condition (which is always satisfied if the variety is a hypersurface), and we prove the corresponding proportionality theorem for this index. When the form is the differential of a holomorphic function h , this index measures the number of critical points of a generic perturbation of h on a local Milnor fiber, so it is analogous to invariants studied by a number of authors (see for instance [9, 11, 22]). Section 1 is a review of well-known facts about real and complex valued 1-forms.

The radial extension of 1-forms can be made global on compact varieties, and it can also be made for frames of differential 1-forms. One gets in this way the dual Schwartz classes of singular varieties, which equal the usual ones up to sign. One also has the dual Chern-Mather classes of V , already envisaged in [17], and the proportionality formula 3.3 can be used as in [3] to express the dual Chern-Mather classes as "weighted" dual Schwartz classes, the weights been given by the local Euler obstruction. Similarly, in analogy with Theorem 1.1 in [4], the corresponding GSV-index and the proportionality Theorem 4.4 extend

to frames and can be used to express the dual Fulton-Johnson classes of singular hypersurfaces embedded with trivial normal bundle in compact complex manifolds, as “weighted” dual Schwartz classes, the weights been now given by the Euler-Poincaré characteristic of the local Milnor fiber.

This work was done while the second and third named authors were visiting the “Institut de Mathématiques de Luminy”, France; they acknowledge the support of the CNRS, France and the “Université de la Méditerranée”.

The authors thank J. Schürmann for his comments and suggestions on the first version of the paper. In particular, he gave us an alternative proof of Theorem 3.3 in the case of the differential form associated to a Morse function, using stratified Morse theory and the micro-local index formula in [19].

§1. Some basic facts about 1-forms

In this section we study some basic facts about the geometry of 1-forms and the relation between real and complex valued 1-forms on (almost) complex manifolds, which plays an important role in the sequel. The material here is all contained in the literature; we include it for completeness and to set up our notation with no possible ambiguities. We give precise references when appropriate.

Let M be an almost complex manifold of real dimension $2m > 0$. Let TM be its complex tangent bundle. We denote by T^*M the cotangent bundle of M , dual of TM ; each fiber $(T^*M)_x$ consists of the \mathbb{C} -linear maps $TM_x \rightarrow \mathbb{C}$. Similarly, we denote by $T_{\mathbb{R}}M$ the underlying real tangent bundle of M ; it is a real vector bundle of fiber dimension $2m$, endowed with a canonical orientation. Its dual $T_{\mathbb{R}}^*M$ has as fiber the \mathbb{R} -linear maps $(T_{\mathbb{R}}M)_x \rightarrow \mathbb{R}$.

1.1 Definition. Let A be a subset of M . By a real (valued) 1-form η on A we mean the restriction to A of a continuous section of the bundle $T_{\mathbb{R}}^*M$, i.e., for each $x \in A$, η_x is an \mathbb{R} -linear map $(T_{\mathbb{R}}M)_x \rightarrow \mathbb{R}$. We usually drop the word “valued” and say simply real 1-forms on A . Similarly, a complex 1-form ω on A means the restriction to A of a continuous section of the bundle T^*M , i.e., for each $x \in A$, ω_x is a \mathbb{C} -linear map $(TM)_x \rightarrow \mathbb{C}$.

Notice that the kernel of a real form η at a point x is either the whole fiber $(T_{\mathbb{R}}M)_x$ or a real hyperplane in it. In the first case we say that x is a singular point (or zero) of η . In the second case the kernel

$\ker \eta_x$ splits $(T_{\mathbb{R}}M)_x$ in two half spaces $(T_{\mathbb{R}}M^{\pm})_x$; in one of these the form takes positive values and negative in the other.

We recall that a vector field v in \mathbb{R}^N is radial at a point x_o if it is transverse to every sufficiently small sphere around x_o in \mathbb{R}^N . The duality between real 1-forms and vector fields assigns to each tangent vector $\partial/\partial x_i$ the form dx_i (extending it by linearity to all tangent vectors). This refines the classical duality that assigns to each hyperplane in \mathbb{R}^N the line orthogonal to it and motivates the following definition (c.f. [5, 6]):

1.2 Definition. A real 1-form η on M is **radial** (outwards-pointing) at a point $x_o \in M$ if, locally, it is dual over \mathbb{R} to a radial outwards-pointing vector field at x_o . Inwards-pointing radial vector fields are defined similarly.

In other words, η is **radial** at a point x_o if it is everywhere positive when evaluated in some radial vector field at x_o .

Thus, for instance, if for a fixed $x_o \in M$ we let $\rho_{x_o}(x)$ be the function $\|x - x_o\|^2$ (for some Riemmanian metric), then its differential is a radial form.

1.3 Remark. The concept of radial forms was introduced in [5]. In [6] radial forms are defined using more relaxed conditions than we do here. However this is a concept inspired by the corresponding notion of radial vector fields, so we use Definition 1.2.

A complex 1-form ω on $A \subset M$ can be written in terms of its real and imaginary parts:

$$\omega = \operatorname{Re}(\omega) + i \operatorname{Im}(\omega).$$

Both $\operatorname{Re}(\omega)$ and $\operatorname{Im}(\omega)$ are real 1-forms, and the linearity of ω implies that for each tangent vector one has:

$$\operatorname{Im}(\omega)(v) = -\operatorname{Re}(\omega)(iv),$$

thus

$$\omega(v) = \operatorname{Re}(\omega)(v) - i \operatorname{Re}(\omega)(iv).$$

In other words the form ω is determined by its real part and one has a 1-to-1 correspondence between real and complex forms, assigning to each complex form its real part, and conversely, to a real 1-form η corresponds the complex form ω defined by:

$$\omega(v) = \eta(v) - i\eta(iv).$$

This statement (noted in [6]) refines the obvious fact that a complex hyperplane P in \mathbb{C}^m , say defined by a linear form H , is the intersection of the real hyperplanes $\widehat{H} := \{\operatorname{Re} H = 0\}$ and $i\widehat{H}$. This justifies the following definition:

1.4 Definition. A complex 1-form ω is **radial** at a point $x \in M$ if its real part is radial at x .

Recall that the Euler class of an oriented vector bundle is the primary obstruction to constructing a non-zero section [24]. In the case of the bundle $T_{\mathbb{R}}^*M$, this class equals the Euler class $\operatorname{Eu}(M)$ of the underlying real tangent bundle $T_{\mathbb{R}}M$, since they are isomorphic. Thus, if M is compact then its Euler class evaluated on the orientation cycle of M gives the Euler-Poincaré characteristic $\chi(M)$. We can say this in different words: let η be a real 1-form on M with isolated (hence finitely many) singularities x_1, \dots, x_r . At each x_i this 1-form defines a map, $\mathbb{S}_\varepsilon \xrightarrow{\eta/\|\eta\|} \mathbb{S}^{2m-1}$, from a small sphere in M around x_i into the unit sphere in the fiber $(T_{\mathbb{R}}^*M)_x$. The degree of this map is the **Poincaré-Hopf** local index of η at x_i , that we may denote by $\operatorname{Ind}_{\text{PH}}(\eta, x_i)$. Then the total index of η in M is by definition the sum of its local indices at the x_i and it equals $\chi(M)$. Its Poincaré dual class in $H^{2m}(M)$ is the Euler class of $T_{\mathbb{R}}^*M \cong T_{\mathbb{R}}M$.

More generally, if M is a compact, C^∞ manifold of real dimension $2m$ with non-empty boundary ∂M and a complex structure in its tangent bundle, one can speak of real and complex valued 1-forms as above. Elementary obstruction theory (see [24]) implies that one can always find real and complex 1-forms on M with isolated singularities, all contained in the interior of M . In fact, if a real 1-form η is defined in a neighborhood of ∂M in M and it is non-singular there, then we can always extend it to the interior of M with finitely many singularities, and its total index in M does not depend on the choice of the extension.

1.5 Definition. Let M be an almost complex manifold with boundary ∂M and let ω be a (real or complex) 1-form on M , non-singular on a neighborhood of ∂M ; let $\operatorname{Re} \omega$ be its real part if ω is a complex form, otherwise $\operatorname{Re} \omega = \omega$ for real forms. The form ω is **radial** at the boundary if for each vector $v(x) \in TM$, $x \in \partial M$, which is normal to the boundary (for some metric), pointing outwards of M , one has $\operatorname{Re} \omega(v(x)) > 0$.

By the theorem of Poincaré-Hopf for manifolds with boundary, if a real 1-form η is radial at the boundary and M is compact, then the total index of η is $\chi(M)$.

We now make similar considerations for complex 1-forms. We let M be a compact, C^∞ manifold of real dimension $2m$ (with or without boundary ∂M), with a complex structure in its tangent bundle TM . Let T^*M be as before, the cotangent bundle of M , i.e., the bundle of complex valued continuous 1-forms. The top Chern class $c^m(T^*M)$ is the primary obstruction to constructing a section of this bundle, i.e., if M has empty boundary, then $c^m(T^*M)$ is the number of points, counted with their local indices, of the zeroes of a section ω of T^*M (i.e., a complex 1-form) with isolated singularities (i.e., points where it vanishes). It is well known (see for instance [16]) that one has:

$$c^m(T^*M) = (-1)^m c^m(TM).$$

This corresponds to the fact that at each isolated singularity x_i of ω one has two local indices: one of them is the index of its real part defined as above, $\text{Ind}_{\text{PH}}(\text{Re}\omega, x_i)$; the other is the degree of the map $\mathbb{S}_\varepsilon \xrightarrow{\omega/\|\omega\|} \mathbb{S}^{2m-1}$, that we denote by $\text{Ind}_{\text{PH}}(\omega, x_i)$. These two indices are related by the equality:

$$\text{Ind}_{\text{PH}}(\omega, x_i) = (-1)^m \text{Ind}_{\text{PH}}(\text{Re}\omega, x_i),$$

and the index on the right corresponds to the local Poincaré-Hopf index of the vector field defined by duality near x_i . For instance, the form $\omega = \sum z_i dz_i$ in C^m has index 1 at 0, while its real part $\sum(x_i dx_i - y_i dy_i)$ has index $(-1)^m$.

If we take M as above, compact and with possibly non-empty boundary, and ω is a complex 1-form with isolated singularities in the interior of M and radial on the boundary, then (by the previous considerations) the total index of ω in M is $(-1)^m \chi(M)$. We summarize some of the previous discussion in the following theorem (c.f. [5, 6]):

1.6 Theorem. *Let M be a compact, C^∞ manifold of real dimension $2m$ (with or without boundary ∂M), with a complex structure in its tangent bundle TM . Let $T_{\mathbb{R}}^*M$ and T^*M be as before, the bundles of real and complex valued continuous 1-forms on M , respectively. Then:*

i) *Every real 1-form η on M determines a complex 1-form ω by the formula*

$$\omega(v) = \eta(v) - i\eta(iv);$$

so the real part of ω is η .

ii) *The local Poincaré-Hopf indices at an isolated singularity of a complex 1-form and its real part are related by:*

$$\text{Ind}_{\text{PH}}(\omega, x_i) = (-1)^m \text{Ind}_{\text{PH}}(\text{Re}\omega, x_i).$$

iii) If a real 1-form on M is radial at the boundary ∂M , then its total Poincaré-Hopf index in M is $\chi(M)$. In particular, a radial real 1-form has local index 1.

vi) If a complex 1-form on M is radial at the boundary ∂M , then its total Poincaré-Hopf index in M is $(-1)^m \chi(M)$.

1.7 Remark. One may consider frames of complex 1-forms on M instead of a single 1-form. This means considering sets of k complex 1-forms, whose singularities are the points where these forms become linearly dependent over \mathbb{C} . By definition (see [24]) the primary obstruction to constructing such a frame is the Chern class $c^{m-k+1}(T^*M)$, so these classes also have an expression similar to 1.6 but using indices of frames of 1-forms. One always has $c^i(T^*M) = (-1)^i c^i(TM)$. Thus the Chern classes, and all the Chern numbers of M , can be computed using indices of either vector fields or 1-forms.

§2. Radial extension and the Schwartz index

In the sequel we will be interested in considering forms defined on singular varieties in a complex manifold, so we introduce some standard notation. Let V be a reduced, equidimensional complex analytic space of dimension n in a complex manifold M of dimension m , endowed with a Whitney stratification $\{V_\alpha\}$ adapted to V , i.e., V is a union of strata.

The following definition is an immediate extension for 1-forms of the corresponding (standard) definition for functions on stratified spaces in terms of its differential (c.f. [6, 7, 12]).

2.1 Definition. Let ω be a (real or complex) 1-form on V , i.e., a continuous section of either $T_{\mathbb{R}}^*M|_V$ or $T^*M|_V$. A singularity of ω with respect to the Whitney stratification $\{V_\alpha\}$ means a point x where the kernel of ω contains the tangent space of the corresponding stratum.

This means that the pull back of the form to V_α vanishes at x .

In Section 1 we introduced the notion of radial forms, which is dual to the “radiality” for vector fields. We now extend this notion relaxing the condition of radiality in the directions tangent to the strata. From now on, unless it is otherwise stated explicitly, by a singularity of a 1-form on V we mean a singularity in the stratified sense, i.e., in the sense of 2.1.

2.2 Definition. Let ω be a (real or complex) 1-form on V . The form is **normally radial** at a point $x_o \in V_\alpha \subset V$ if it is radial when restricted to vectors which are not tangent to the stratum V_α that contains x_o . In other words, for every vector $v(x)$ tangent to M at a point $x \notin V_\alpha$, x sufficiently close to x_o and $v(x)$ pointing outwards a tubular

neighborhood of the stratum V_α , one has $\operatorname{Re}\omega(v) > 0$ (or $\operatorname{Re}\omega(v) < 0$ for all such vectors; if ω is real then it equals $\operatorname{Re}\omega$).

Obviously a radial 1-form is also normally radial, since it is radial in all directions.

For each point x in a stratum V_α , one has a neighborhood U_x of x in M which is diffeomorphic to the product $U_\alpha \times \mathbb{D}_\alpha$, where $U_\alpha = U_x \cap V_\alpha$ and \mathbb{D}_α is a small disc in M transverse to V_α . Let π be the projection $\pi: U_x \rightarrow U_\alpha$ and p the projection $p: U_x \rightarrow \mathbb{D}_\alpha$. One has an isomorphism:

$$T^*U_x \cong \pi^*T^*U_\alpha \oplus p^*T^*\mathbb{D}_\alpha.$$

That a (real or complex) 1-form ω be normally radial at x means that up to a local change of coordinates in M , ω is the direct sum of the pull back of a (real or complex) form on U_α , i.e., a section of the (real or complex) cotangent bundle T^*U_α , and a section of the (real or complex) cotangent bundle $T^*\mathbb{D}_\alpha$ which is a (real or complex) radial form in the disc.

It is possible to make for 1-forms the classical construction of **radial extension** introduced by M.-H. Schwartz in [20, 21] for stratified vector fields and frames. Locally, the construction can be described as follows. We consider first real 1-forms. Let η be a 1-form on U_α , denote by $\hat{\eta}$ its pull back to a section of $\pi^*T_{\mathbb{R}}^*U_\alpha$. This corresponds to the **parallel extension** of stratified vector fields done by Schwartz. Now look at the function ρ given by the square of the distance to the origin in \mathbb{D}_α . The form $p^*d\rho$ on U_x vanishes on U_α and away from U_α its kernel is transverse to the strata of V by Whitney conditions. The sum $\eta' = \hat{\eta} + p^*d\rho$ defines a normally radial 1-form on U_x which coincides with η on U_α ; away from U_α its kernel is transverse to the strata of V . Thus, if η is non-singular at x , then η' is non-singular everywhere on U_x . If η has an isolated singularity at $x \in V_\alpha$, then η' also has an isolated singularity there. In particular, if the dimension of the stratum V_α is zero then η' is a radial form in the sense of Section 1.

Following the terminology of [20, 21] we say that the form η' is obtained from η by **radial extension**. Since the index in M of a normally radial form is its index in the stratum times the index of a radial form in the disc \mathbb{D}_α , we obtain the following important property of forms constructed by radial extension.

2.3 Proposition. *Let η be a real 1-form on the stratum V_α with an isolated singularity at a point x with local Poincaré-Hopf index $\operatorname{Ind}_{\text{PH}}(\eta, V_\alpha; x)$. Let η' the 1-form on a neighborhood of x in M obtained by radial extension. Then the index of η in the stratum equals the*

index of η' in M :

$$\text{Ind}_{\text{PH}}(\eta, V_\alpha; x) = \text{Ind}_{\text{PH}}(\eta', M; x).$$

2.4 Definition. The **Schwartz index** of the continuous real 1-form η at an isolated singularity $x \in V_\alpha \subset V$, denoted $\text{Ind}_{\text{Sch}}(\eta, V; x)$, is the Poincaré-Hopf index of the 1-form η' obtained from η by radial extension; or equivalently, if the stratum of x has dimension more than 0, $\text{Ind}_{\text{Sch}}(\eta, V; x)$ is the Poincaré-Hopf index at x of η in the stratum V_α .

If x is an isolated singularity of V then every 1-form on V must be singular at x since its kernel contains the “tangent space” of the stratum. In this case the index of the form in the stratum is defined to be 1, and this is consistent with the previous definition since in this case the radial extension of η is actually radial at x , so it has index 1 in the ambient space.

The previous process is easily adapted to give radial extension for complex 1-forms. Let ω be such a form on V_α ; let η be its real part. We extend η as above, by radial extension, to obtain a real 1-form η' which is normally radial at x . Then we use statement i) in Theorem 1.6 above to obtain a complex 1-form ω' on U_x that extends ω and is also normally radial at x . If we prefer, we can make this process in a different but equivalent way: first make a parallel extension of ω to U_x as above, using the projection π ; denote by $\widehat{\omega}$ this complex 1-form. Now use 1.6.i) to define a complex 1-form $\widehat{d\rho}$ on U_x whose real part is $d\rho$, and take the direct sum of $\widehat{\omega}$ and $\widehat{d\rho}$ at each point to obtain the extension ω' . We say that ω' is obtained from ω by **radial extension**.

We have the equivalent of Proposition 2.3 for complex forms, modified with the appropriate signs:

$$(-1)^s \text{Ind}_{\text{PH}}(\omega, V_\alpha; x) = (-1)^m \text{Ind}_{\text{PH}}(\omega', M; x),$$

where $2s$ is the real dimension of V_α and $2m$ that of M .

2.5 Definition. The **Schwartz index** of the continuous complex 1-form ω at an isolated singularity $x \in V_\alpha \subset V$, denoted $\text{Ind}_{\text{Sch}}(\omega, V; x)$, is $(-1)^n$ -times the index of its real part:

$$\text{Ind}_{\text{Sch}}(\omega, V; x) = (-1)^n \text{Ind}_{\text{Sch}}(\text{Re } \omega, V; x).$$

§3. Local Euler obstruction and the Proportionality Theorem

We are now concerned only with a local situation, so we take V to be embedded in an open ball $\mathbb{B} \subset \mathbb{C}^m$ centered at the origin 0. On

the regular part of V one has the map $\sigma: V_{\text{reg}} \rightarrow G_{n,m}$ into the Grassmannian of $n(= \dim V)$ -planes in \mathbb{C}^m , that assigns to each point the corresponding tangent space of V_{reg} . The Nash blow up $\tilde{V} \xrightarrow{\nu} V$ of V is by definition the closure in $\mathbb{B} \times G_{n,m}$ of the graph of the map σ . One also has the Nash bundle $\tilde{T} \xrightarrow{p} \tilde{V}$, restriction to \tilde{V} of the tautological bundle over $\mathbb{B} \times G_{n,m}$.

The corresponding dual bundles of complex and real 1-forms are denoted by $\tilde{T}^* \xrightarrow{p} \tilde{V}$ and $\tilde{T}_{\mathbb{R}}^* \xrightarrow{p} \tilde{V}$, respectively. Observe that a point in \tilde{T}^* is a triple (x, P, ω) where x is in V , P is an n -plane in the tangent space $T_x \mathbb{B}$ which is limit of a sequence $\{(TV_{\text{reg}})_{x_i}\}$, where the x_i are points in the regular part of V converging to x , and ω is a \mathbb{C} -linear map $P \rightarrow \mathbb{C}$. (Similarly for $\tilde{T}_{\mathbb{R}}^*$.)

Let us denote by ρ the function given by the square of the distance to 0. We recall that MacPherson in [15] observed that the Whitney condition (a) implies that the pull-back of the differential $d\rho$ defines a never-zero section $\tilde{d}\rho$ of $\tilde{T}_{\mathbb{R}}^*$ over $\nu^{-1}(\mathbb{S}_{\varepsilon} \cap V) \subset \tilde{V}$, where \mathbb{S}_{ε} is the boundary of a small ball \mathbb{B}_{ε} in \mathbb{B} centered at 0. The obstruction to extending $\tilde{d}\rho$ as a never-zero section of $\tilde{T}_{\mathbb{R}}^*$ over $\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V) \subset \tilde{V}$ is a cohomology class in $H^{2n}(\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V), \nu^{-1}(\mathbb{S}_{\varepsilon} \cap V); \mathbb{Z})$, and MacPherson defined the **local Euler obstruction** $\text{Eu}_V(0)$ of V at 0 to be the integer obtained by evaluating this class on the orientation cycle $[\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V), \nu^{-1}(\mathbb{S}_{\varepsilon} \cap V)]$.

More generally, given a section η of $T_{\mathbb{R}}^* \mathbb{B}|_A$, $A \subset V$, there is a canonical way of constructing a section $\tilde{\eta}$ of $\tilde{T}_{\mathbb{R}}^*|_{\tilde{A}}$, $\tilde{A} = \nu^{-1}A$, which is described in the following. The same construction works for complex forms. First, taking the pull-back $\nu^* \eta$, we get a section of $\nu^* T_{\mathbb{R}}^* \mathbb{B}|_V$. Then $\tilde{\eta}$ is obtained by projecting $\nu^* \eta$ to a section of $\tilde{T}_{\mathbb{R}}^*$ by the canonical bundle homomorphism

$$\nu^* T_{\mathbb{R}}^* \mathbb{B}|_V \longrightarrow \tilde{T}_{\mathbb{R}}^*.$$

Thus the value of $\tilde{\eta}$ at a point (x, P) is simply the restriction of the linear map $\eta(x): (T_{\mathbb{R}} \mathbb{B})_x \rightarrow \mathbb{R}$ to P . We call $\tilde{\eta}$ the **canonical lifting** of η .

By the Whitney condition (a), if $a \in V_{\alpha}$ is the limit point of the sequence $\{x_i\} \in V_{\text{reg}}$ such that $P = \lim(TV_{\text{reg}})_{x_i}$ and if the kernel of η is transverse to V_{α} , then the linear form $\tilde{\eta}$ will be non-vanishing on P . Thus, if η has an isolated singularity at the point $0 \in V$ (in the stratified sense), then we have a never-zero section $\tilde{\eta}$ of the dual Nash bundle $\tilde{T}_{\mathbb{R}}^*$ over $\nu^{-1}(\mathbb{S}_{\varepsilon} \cap V) \subset \tilde{V}$. Let $o(\eta) \in H^{2n}(\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V), \nu^{-1}(\mathbb{S}_{\varepsilon} \cap V); \mathbb{Z})$ be the cohomology class of the obstruction cycle to extend this to a section of $\tilde{T}_{\mathbb{R}}^*$ over $\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V)$. Then define (c.f. [2, 6]):

3.1 Definition. The **local Euler obstruction** of the real differential form η at an isolated singularity is the integer $\text{Eu}_V(\eta, 0)$ obtained by evaluating the obstruction cohomology class $o(\eta)$ on the orientation cycle $[\nu^{-1}(\mathbb{B}_\varepsilon \cap V), \nu^{-1}(\mathbb{S}_\varepsilon \cap V)]$.

The local Euler obstruction $\text{Eu}_V(0)$ of MacPherson corresponds to taking the differential of the squared function distance to 0. In the complex case, one can perform the same construction, using the corresponding complex bundles. If ω is a complex differential form, section of $T^*\mathbb{B}|_A$ with an isolated singularity, one can define the local Euler obstruction $\text{Eu}_V(\omega, 0)$. Notice that it is equal to that of its real part up to sign:

$$(3.2) \quad \text{Eu}_V(\omega, 0) = (-1)^n \text{Eu}_V(\text{Re } \omega, 0).$$

This is an immediate consequence of the relation between the Chern classes of a complex vector bundle and those of its dual (see for instance [16]).

We note that the idea to consider the (complex) dual Nash bundle was already present in [17], where Sabbah introduces a local Euler obstruction $\text{E}\ddot{\text{u}}_V(0)$ that satisfies $\text{E}\ddot{\text{u}}_V(0) = (-1)^n \text{Eu}_V(0)$. See also Schürmann [18], sec. 5.2.

Just as for vector fields (see [3]), one has in this situation the following:

3.3 Theorem. *Let $V_\alpha \subset V$ be the stratum containing 0, $\text{Eu}_V(0)$ the local Euler obstruction of V at 0 and ω a (real or complex) 1-form on V_α with an isolated singularity at 0. Then the local Euler obstruction of the radial extension ω' of ω and the Schwartz index of ω at 0 are related by the following proportionality formula:*

$$\text{Eu}_V(\omega', 0) = \text{Eu}_V(0) \cdot \text{Ind}_{\text{Sch}}(\omega, V; 0).$$

Proof. By 3.2 and Theorem 1.6 above, it is enough to prove 3.3 for either real or complex 1-forms, each case implying the other. We prove it for real forms.

Let ω and ω' be as above. Also, let η_{rad} denote a real radial form at 0.

By construction and definition, we have

$$(3.4) \quad \text{Ind}_{\text{PH}}(\omega, V_\alpha; 0) = \text{Ind}_{\text{PH}}(\omega', \mathbb{B}; 0) = \text{Ind}_{\text{Sch}}(\omega, V; 0).$$

By definition of $\text{Ind}_{\text{PH}}(\omega', \mathbb{B}; 0)$, there is a homotopy

$$\Psi: [0, 1] \times \mathbb{S}_\varepsilon \longrightarrow T_{\mathbb{R}}^*\mathbb{B}|_{\mathbb{S}_\varepsilon}$$

such that its image satisfies:

$$(3.5) \quad \partial \operatorname{Im} \Psi = \omega'(\mathbb{S}_\varepsilon) - \operatorname{Ind}_{\text{PH}}(\omega', \mathbb{B}; 0) \cdot \eta_{\text{rad}}(\mathbb{S}_\varepsilon)$$

as chains in $T_{\mathbb{R}}^*\mathbb{B}|_{\mathbb{S}_\varepsilon}$. The restriction of Ψ gives a homotopy

$$\psi: [0, 1] \times (\mathbb{S}_\varepsilon \cap V) \longrightarrow T_{\mathbb{R}}^*\mathbb{B}|_{\mathbb{S}_\varepsilon \cap V}$$

such that (c.f. (3.4))

$$\partial \operatorname{Im} \psi = \omega'(\mathbb{S}_\varepsilon \cap V) - \operatorname{Ind}_{\text{Sch}}(\omega, V; 0) \cdot \eta_{\text{rad}}(\mathbb{S}_\varepsilon \cap V).$$

Now we can lift ψ , ω' and η_{rad} to sections $\nu^*\psi$, $\nu^*\omega'$ and $\nu^*\eta_{\text{rad}}$ of $\nu^*T_{\mathbb{R}}^*\mathbb{B}$ to get a homotopy

$$\nu^*\psi: [0, 1] \times \nu^{-1}(\mathbb{S}_\varepsilon \cap V) \longrightarrow \nu^*T_{\mathbb{R}}^*\mathbb{B}|_{\nu^{-1}(\mathbb{S}_\varepsilon \cap V)}$$

and, since ν is an isomorphism away from the singularity of V , we still have

$$\partial \operatorname{Im} \nu^*\psi = \nu^*\omega'(\nu^{-1}(\mathbb{S}_\varepsilon \cap V)) - \operatorname{Ind}_{\text{Sch}}(\omega, V; 0) \cdot \nu^*\eta_{\text{rad}}(\nu^{-1}(\mathbb{S}_\varepsilon \cap V))$$

as chains in $\nu^*T_{\mathbb{R}}^*\mathbb{B}|_{\nu^{-1}(\mathbb{S}_\varepsilon \cap V)}$. Recall that we get the canonical liftings $\tilde{\psi}$, $\tilde{\omega}'$ and $\tilde{\eta}_{\text{rad}}$ of ψ , ω' and η_{rad} by taking the images of $\nu^*\psi$, $\nu^*\omega'$ and $\nu^*\eta_{\text{rad}}$ by the canonical bundle homomorphism $\nu^*T_{\mathbb{R}}^*\mathbb{B} \longrightarrow \tilde{T}_{\mathbb{R}}^*$. Thus we have

$$\partial \operatorname{Im} \tilde{\psi} = \tilde{\omega}'(\nu^{-1}(\mathbb{S}_\varepsilon \cap V)) - \operatorname{Ind}_{\text{Sch}}(\omega, V; 0) \cdot \tilde{\eta}_{\text{rad}}(\nu^{-1}(\mathbb{S}_\varepsilon \cap V))$$

as chains in $\tilde{T}_{\mathbb{R}}^*|_{\nu^{-1}(\mathbb{S}_\varepsilon \cap V)}$. The sections $\tilde{\omega}'$ and $\tilde{\eta}_{\text{rad}}$ are non-vanishing on $\nu^{-1}(\mathbb{S}_\varepsilon \cap V)$, by the Whitney condition, and by definition of the Euler obstructions, we have the theorem by the Whitney condition, and by definition of the Euler obstructions, we have the theorem. Q.E.D.

§4. The GSV-index

We recall ([8, 23]) that the GSV-index of a vector field v on an isolated complete intersection germ V can be defined to be the Poincaré-Hopf index of an extension of v to a Milnor fiber F . Similarly, the GSV-index of a 1-form ω on V can be defined to be the Poincaré-Hopf index of the form on F , i.e., the number of singularities of ω in F counted with multiplicities [5]. When V has non-isolated singularities one may not have a Milnor fibration in general, but one does if V has a Whitney stratification with Thom's a_f -condition, $f = (f_1, \dots, f_k)$ being the functions that define V (c.f. [13, 14, 4]).

Let $(V, 0)$ be a complete intersection of complex dimension n defined in a ball \mathbb{B} in \mathbb{C}^{n+k} by functions $f = (f_1, \dots, f_k)$, and assume 0 is a singular point of V (not necessarily an isolated singularity). As before, we endow \mathbb{B} with a Whitney stratification adapted to V , and we assume that we can choose $\{V_\alpha\}$ so that it satisfies the a_f -condition of Thom (see for instance [14]). In particular one always has such a stratification if $k = 1$, by [10].

Let ω be as before, a (real or complex) 1-form on \mathbb{B} , and assume its restriction to V has an isolated singularity at 0 . This means that the kernel of $\omega(0)$ contains the tangent space of the stratum V_α containing 0 , but everywhere else it is transverse to each stratum $V_\alpha \subset V$. Now let $F = F_t$ be a Milnor fiber of V , i.e., $F = f^{-1}(t) \cap \mathbb{B}_\varepsilon$, where \mathbb{B}_ε is a sufficiently small ball in \mathbb{B} around 0 and $t \in \mathbb{C}^k$ is a regular value of f with $\|t\|$ sufficiently small with respect to ε . Notice that the a_f -condition implies that for every sequence t_n of regular values converging to 0 , and for every sequence $\{x_n\}$ of points in the corresponding Milnor fibers converging to a point $x_o \in V$ so that the sequence of tangent spaces $\{(TF)_{x_n}\}$ has a limit T , one has that T contains the space $(TV_\alpha)_{x_o}$, tangent to the stratum that contains x_o . By transversality this implies that choosing the regular value t sufficiently close to 0 we can assure that the kernel of ω is transverse to the Milnor fiber at every point in its boundary ∂F . Thus its pull-back to F is a 1-form on this smooth manifold, and it is never-zero on its boundary, thus ω has a well defined Poincaré-Hopf index in F as in Section 1. This index is well-defined and depends only on the restriction of ω to V and the topology of the Milnor fiber F , which is well-defined once we fix the defining function f (which is assumed to satisfy the a_f -condition for some Whitney stratification).

4.1 Definition. The GSV-index of ω at $0 \in V$ relative to f , $\text{Ind}_{\text{GSV}}(\omega, 0)$, is the Poincaré-Hopf index of ω in F .

In other words this index measures the number of points (counted with signs) in which a generic perturbation of ω is tangent to F . In fact the inclusion $F \xrightarrow{i} M$ pulls the form ω to a section of the (real or complex, as the case may be) cotangent bundle of F , which is never-zero near the boundary because ω has an isolated singularity at 0 and, by hypothesis, the map f satisfies the a_f -condition of Thom. If the form ω is real then

$$(4.2) \quad \text{Ind}_{\text{GSV}}(\omega, 0) = \text{Eu}(F; \omega)[F],$$

where $\text{Eu}(F; \omega) \in H^{2n}(F, \partial F)$ is the Euler class of the real cotangent bundle $T_{\mathbb{R}}^*F$ relative to the section defined by ω on the boundary, and

$[F]$ is the orientation cycle of the pair $(F, \partial F)$. If ω is a complex form, then one has:

$$(4.3.i) \quad \text{Ind}_{\text{GSV}}(\omega, 0) = c^n(T^*F; \omega)[F],$$

where $c^n(T^*F; \omega)$ is the top Chern class of the cotangent bundle of F relative to the form ω on its boundary. In this case one can, alternatively, express this index as the relative Chern class:

$$(4.3.ii) \quad \text{Ind}_{\text{GSV}}(\omega, 0) = c^n(T^*M|_F; \Omega)[F],$$

where Ω is the frame of $k + 1$ complex 1-forms on the boundary of F given by

$$\Omega = (\omega, df_1, df_2, \dots, df_k),$$

since the forms (df_1, \dots, df_k) are linearly independent everywhere on F . Notice that if the form ω is holomorphic, then this index is necessarily non-negative because it can be regarded as an intersection number of complex submanifolds. For every complex 1-form one has:

$$\text{Ind}_{\text{GSV}}(\omega, 0) = (-1)^n \text{Ind}_{\text{GSV}}(\text{Re } \omega, 0).$$

We remark that if V has an isolated singularity at 0, this is the index envisaged in [5], i.e., the degree of the map from the link K of V into the Stiefel manifold of complex $(k + 1)$ -frames in the dual \mathbb{C}^{n+k} given by the map $(\omega, df_1, \dots, df_k)$. Also notice that this index is somehow dual to the index defined in [4] for vector fields, which is related to the top Fulton-Johnson class of singular hypersurfaces.

So, given the (non-isolated) complete intersection singularity $(V, 0)$ and a (real or complex) 1-form ω on V with an isolated singularity at 0, one has three different indices: the Euler obstruction (Section 2), the GSV-index just defined and the index of its pull back to a 1-form on the stratum of 0. One also has the index of the form in the ambient manifold M . For forms obtained by radial extension, the index in the stratum equals its index in M , and this is by definition the Schwartz index. The following proportionality theorem is analogous to the one in [4] for vector fields.

4.4 Theorem. *Let ω be a (real or complex) 1-form on the stratum V_α of 0 with an isolated singularity at 0. Then the GSV index of its radial extension ω' is proportional to the Schwartz index, the proportionality factor being the Euler-Poincaré characteristic of the Milnor fiber F :*

$$\text{Ind}_{\text{GSV}}(\omega', 0) = \chi(F) \cdot \text{Ind}_{\text{Sch}}(\omega, V; 0).$$

Proof. It is enough to prove 4.4 either for complex forms or for real forms, each one implying the other. The proof is similar to that of 3.3. Let ω' and η_{rad} be as in the proof of Theorem 3.3. Then 4.4 is proved by taking the retraction to F of each section in (3.5) as a differential form, noting that $\text{Ind}_{\text{GSV}}(\eta_{\text{rad}}, 0) = \chi(F)$. Q.E.D.

4.5 Remark. We notice that 4.4 and 3.3 can also be proved using the stability of the index under perturbations; this works for vector fields too. More precisely, one can easily show that the Euler obstruction $\text{Eu}_V(\omega, x)$ and the GSV-index are stable when we perturb the 1-form (or the vector field) in the stratum and then extend it radially; then the sum of the indices at the singularities of the new 1-form (vector field) give the corresponding index for the original singularity. This implies the proportionality of the indices.

References

- [1] J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, Milnor classes of local complete intersections, *Trans. Amer. Math. Soc.*, **354** (2001), 1351–1371.
- [2] J.-P. Brasselet, D. B. Massey, A. J. Parameswaran and J. Seade, Euler obstruction and defects of functions on singular varieties, *J. London Math. Soc.*, **70** (2004), 59–76.
- [3] J.-P. Brasselet et M.-H. Schwartz, Sur les classes de Chern d'un ensemble analytique complexe, *Astérisque*, **82-83** (1981), 93–147.
- [4] J.-P. Brasselet, J. Seade and T. Suwa, An explicit cycle representing the Fulton-Johnson class, I, *Singularités Franco-Japonaises, Séminaires et Congrès 10*, Soc. Math. France, 2005, 21–38.
- [5] W. Ebeling and S. Gusein-Zade, Indices of 1-forms on an isolated complete intersection singularity, *Moscow. Math. J.*, **3** (2003), 439–455.
- [6] W. Ebeling and S. Gusein-Zade, Radial index and Euler obstruction of a 1-form on a singular variety, preprint, 2004.
- [7] M. Goresky and R. MacPherson, Stratified Morse theory, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 14*, Springer-Verlag 1988.
- [8] X. Gómez-Mont, J. Seade and A. Verjovsky, The index of a holomorphic flow with an isolated singularity, *Math. Ann.*, **291** (1991), 737–751.
- [9] V. Goryunov, Functions on space curves, *J. London Math. Soc.*, **61** (2000), 807–822.
- [10] H. Hironaka, Stratification and Flatness, *Real and Complex Singularities Oslo 1976*, Sijhoff en Nordhoff, 1977, 199–265.
- [11] T. Izawa and T. Suwa, Multiplicity of functions on singular varieties, *Internat. J. Math.*, **14** (2003), 541–558.

- [12] D. T. Lê, Complex analytic functions with isolated singularities, *J. Algebraic Geom.*, **1** (1992), 83–100.
- [13] D. T. Lê, Some remarks on the relative monodromy, *Real and Complex Singularities Oslo 1976*, Sijhoff en Nordhoff, 1977, 397–403.
- [14] D. T. Lê et B. Teissier, Cycles Evanescents et conditions de Whitney, In: *Proc. Symp. Pure Math.*, **40**, Part 2, Amer. Math. Soc., 1983, 65–103.
- [15] R. MacPherson, Chern classes for singular varieties, *Ann. of Math.*, **100** (1974), 423–432.
- [16] J. Milnor and J. Stasheff, *Characteristic Classes*, *Annals of Mathematics Studies*, **76**, Princeton University Press, Princeton, 1974.
- [17] C. Sabbah, Quelques remarques sur la géométrie des espaces conormaux, In: *Differential systems and singularities*, Luminy, 1983, *Astérisque*, **130** (1985), 161–192.
- [18] J. Schürmann, *Topology of Singular Spaces and Constructible Sheaves*, *Monografie Matematyczne, Intytut Matematyczny PAN*, **63**, New Series, Birkhäuser, 2003.
- [19] J. Schürmann, A general intersection formula for Lagrangian cycles, *Compos. Math.*, **140** (2004), 1037–1052.
- [20] M.-H. Schwartz, Classes caractéristiques définies par une stratification d'une variété analytique complexe, *C. R. Acad. Sci. Paris*, **260** (1965), 3262–3264, 3535–3537.
- [21] M.-H. Schwartz, *Champs radiaux sur une stratification analytique complexe*, *Travaux en Cours*, **39**, Hermann, Paris, 1991.
- [22] J. Seade, M. Tibăr and A. Verjovsky, Milnor numbers and Euler obstruction, *math. CV/0311109*, *Bol. Soc. Bras. Mat.*, to appear.
- [23] J. Seade and T. Suwa, A residue formula for the index of a holomorphic flow, *Math. Ann.*, **304** (1996), 621–634.
- [24] N. Steenrod, *The Topology of Fiber Bundles*, Princeton Univ. Press, 1951.

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