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L^p-boundedness of Bergman projections for α-parabolic operators

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Abstract.

We consider the α-parabolic Bergman spaces on strip domains. The Bergman kernel is given by a series of derivatives of the fundamental solution. We prove the L^p-boundedness of the projection defined by the Bergman kernel and obtain the duality theorem for $1 < p < \infty$. At the same time, we give a new proof of the Huygens property, which enable us to verify all the results in [3] also for $n = 1$.

§1. Introduction

For $1 \leq p \leq \infty$, we denote by $\boldsymbol{b}_\alpha^p$ the set of all $L^{(\alpha)}$-harmonic functions which are p-th integrable with respect to $(n+1)$-dimensional Lebesgue measure on the upper half space H of the Euclidean space $\mathbf{R}^{n+1}$ and call it the α-parabolic Bergman space. In [3], we showed that $\boldsymbol{b}_\alpha^p$ is a Banach space and discussed its dual space and the explicit formula of the Bergman kernel, where the Huygens property plays an important role.

In this note, we consider an α-parabolic Bergman space $\boldsymbol{b}_\alpha^p(H_T)$ on the strip domain $H_T = \mathbf{R}^n \times (0,T)$ $(0 < T \leq \infty)$ where $H_\infty = H$. The main purpose of this note is to give an explicit form of the α-parabolic Bergman kernel and to show its boundedness on $L^p(H_T)$ by using an interpolation theory. The α-parabolic Bergman kernel has a reproducing property for $\boldsymbol{b}_\alpha^p(H_T)$. As an application, we obtain the duality $\boldsymbol{b}_\alpha^p(H_T)' \simeq \boldsymbol{b}_\alpha^q(H_T)$ for $1 < p < \infty$. Here and in the following,

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q always denotes the conjugate exponent of p. At the same time we show the Huygens property of α-parabolic Bergman functions for $n \geq 1$. This enables us to remove from [3] the restriction $n \geq 2$ on the space dimension.

§2. Preliminary

We denote the $(n+1)$-dimensional Euclidean space by $\mathbf{R}^{n+1}$ $(n \geq 1)$, and its point by (x,t) $(x \in \mathbf{R}^n,\ t \in \mathbf{R})$. For $0 < \alpha \leq 1$, we consider a parabolic operator $L^{(\alpha)}$ and its adjoint $\tilde{L}^{(\alpha)}$

$$L^{(\alpha)} = \frac{\partial}{\partial t} + (-\Delta)^\alpha, \qquad \tilde{L}^{(\alpha)} = -\frac{\partial}{\partial t} + (-\Delta)^\alpha$$

on $\mathbf{R}^{n+1}$. We remark that if $0 < \alpha < 1$, $(-\Delta)^\alpha$ is the convolution operator in the x-space $\mathbf{R}^n$ defined by $-c_{n,\alpha}\mathrm{p.f.}|x|^{-n-2\alpha}$, where $c_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$. Then for $\varphi \in C_c^\infty(\mathbf{R}^{n+1})$,

$$\begin{aligned}(\tilde{L}^{(\alpha)}\varphi)(x,t) &= -\frac{\partial}{\partial t}\varphi(x,t) + ((-\Delta)^\alpha\varphi)(x,t) \\ &= -\frac{\partial}{\partial t}\varphi(x,t) - c_{n,\alpha}\lim_{\delta\downarrow 0}\int_{|y-x|>\delta}(\varphi(y,t)-\varphi(x,t))|x-y|^{-n-2\alpha}\,dy,\end{aligned}$$

where we denote by $C_c^\infty(\mathbf{R}^{n+1})$ the totality of infinitely differentiable functions with compact support.

Lemma 2.1. *Let* $\varphi \in C_c^\infty(\mathbf{R}^{n+1})$ *with* $\mathrm{supp}(\varphi) \subset \{(x,t) | t_1 < t < t_2, |x| < r\}$. *Then* $\mathrm{supp}(\tilde{L}^{(\alpha)}\varphi) \subset \mathbf{R}^n \times (t_1,t_2)$ *and when* $0 < \alpha < 1$,

$$|(\tilde{L}^{(\alpha)}\varphi)(x,t)| \leq 2^{n+2\alpha}c_{n,\alpha}\Big(\sup_{t_1<s<t_2}\int_{\mathbf{R}^n}|\varphi(y,s)|\,dy\Big)\cdot|x|^{-n-2\alpha}$$

for (x,t) *with* $|x| \geq 2r$.

Now we define $L^{(\alpha)}$-harmonic functions.

Definition 2.1. Let D be an open set in $\mathbf{R}^{n+1}$. We put

$$s(D) := \{(x,t) | (y,t) \in D \text{ for some } y \in \mathbf{R}^n\}.$$

A Borel measurable function u on $s(D)$ is said to be $L^{(\alpha)}$-harmonic on D if it satisfies the following conditions:

(a) u is continuous on D,

(b) $\iint_{s(D)} |u \cdot \tilde{L}^{(\alpha)}\varphi|\,dxdt < \infty$ and $\iint_{s(D)} u \cdot \tilde{L}^{(\alpha)}\varphi\,dxdt = 0$ holds for every $\varphi \in C_c^\infty(D)$.

Note that each component of $s(D)$ is a strip domain.

The fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ has the form :

$$W^{(\alpha)}(x,t) = \begin{cases} (2\pi)^{-n} \int_{\mathbf{R}^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1}\, x\cdot\xi)d\xi & t>0 \\ 0 & t\le 0, \end{cases}$$

where $x\cdot\xi$ is the inner product of x and ξ, and $|\xi| = (\xi\cdot\xi)^{1/2}$. Then $\tilde{W}^{(\alpha)}(x,t) := W^{(\alpha)}(x,-t)$ is the fundamental solution of $\tilde{L}^{(\alpha)}$. Note that $W^{(1)}(x,t)$ is equal to the Gauss kernel, and $W^{(1/2)}(x,t)$ is equal to the Poisson kernel.

The following estimates will be needed later.

Lemma 2.2. *Let (β,k) be a multi-index, $1\le q\le\infty$ and $0<t_1<t_2<\infty$. Then there exists a constant C such that*

$$\partial_x^\beta\partial_t^k W^{(\alpha)}(x,t) = t^{-\frac{n+|\beta|}{2\alpha}-k}\partial_x^\beta\partial_t^k W^{(\alpha)}(t^{-1/2\alpha}x,1), \tag{2.1}$$

$$|\partial_x^\beta\partial_t^k W^{(\alpha)}(x,t)| \le Ct^{1-k}\left(t+|x|^{2\alpha}\right)^{-\frac{n+|\beta|}{2\alpha}-1} \tag{2.2}$$

and

$$\|\partial_x^\beta\partial_t^k W^{(\alpha)}\|_{L^q(\mathbf{R}^n\times(t_1,t_2))} \le C(t_2-t_1)^{\frac{1}{q}} t_1^{-\frac{n(1-1/q)+|\beta|}{2\alpha}-k}. \tag{2.3}$$

Proof. The assertions (2.1) and (2.2) are remarked in section 3 in [3]. Then we have

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbf{R}^n} |\partial_x^\beta\partial_t^k W^{(\alpha)}(x,t)|^q dxdt \\
&= \int_{t_1}^{t_2}\int_{\mathbf{R}^n} \left(t^{-\frac{n+|\beta|}{2\alpha}-k}\right)^q |\partial_x^\beta\partial_t^k W^{(\alpha)}(t^{-\frac{1}{2\alpha}}x,1)|^q dxdt \\
&= \int_{t_1}^{t_2} \left(t^{-\frac{n+|\beta|}{2\alpha}-k}\right)^q \int_{\mathbf{R}^n} |\partial_x^\beta\partial_t^k W^{(\alpha)}(y,1)|^q t^{\frac{n}{2\alpha}} dydt \\
&\le (t_2-t_1)\left(t_1^{-\frac{n(1-1/q)+|\beta|}{2\alpha}-k}\right)^q \|\partial_x^\beta\partial_t^k W^{(\alpha)}(\cdot,1)\|_{L^q(\mathbf{R}^n)}^q,
\end{aligned}$$

which shows (2.3) when $1\le q<\infty$. In the case of $q=\infty$, (2.3) follows from (2.1) immediately, because $\partial_x^\beta\partial_t^k W^{(\alpha)}(y,1)$ is bounded on $\mathbf{R}^n$. □

§3. Huygens property

In our previous paper [3], we proved the Huygens property under the condition $n\ge 2$. The condition $n\ge 2$ was not able to drop because the proof of the key lemma [3, Lemma 4.3] relied on α-harmonic function theory ([1]). In this section, we shall give another proof of the Huygens property, which is valid for all $n \geq 1$. Here we shall use the α-parabolic dilation to estimate $L^{(\alpha)}$-harmonic measures. In [2] and [4], the notion of the $L^{(\alpha)}$-harmonic measure is introduced and discussed by using the fundamental solutions $W^{(\alpha)}$ and $\tilde{W}^{(\alpha)}$ of $L^{(\alpha)}$ and $\tilde{L}^{(\alpha)}$, respectively. We handle infinite cylinders and use the following notation.

C_r $:= \{(x,t) | t \in \mathbf{R}, |x| < r\}$: infinite cylinder.

ε : the Dirac measure at the origin $(0,0)$.

ν_r^α : the $L^{(\alpha)}$-harmonic measure at the origin of C_r.

ω_r^α : the projection of ν_r^α to the x-space $\mathbf{R}^n$.

$\tilde{\omega}_r^\alpha$ $:= \int_1^2 \omega_{\lambda r}^\alpha d\lambda$, a modified measure of ω_r^α.

$\tilde{W}_r^{(\alpha)} := \tilde{W}^{(\alpha)} * (\varepsilon - \nu_r^\alpha)$.

We list the properties of ν_r^α in the following proposition.

Proposition 3.1. (1) *$0 \leq \tilde{W}_r^{(\alpha)} \leq \tilde{W}^{(\alpha)}$ and the support of $\tilde{W}_r^{(\alpha)}$ is in the closure of the cylinder C_r.*

(2) *ν_r^α is rotationally invariant with respect to the space variable.*

(3) *$\int d\nu_r^\alpha \leq 1$.*

(4) *If $0 < \alpha < 1$, ν_r^α is supported by $\{(x,t) | t \leq 0, |x| \geq r\}$ and absolutely continuous with respect to the $(n+1)$-dimensional Lebesgue measure on the exterior of C_r. The density of ν_r^α is given by*

$$c_{n,\alpha} \int_{|y| \leq r} \tilde{W}_r^{(\alpha)}(y,t) |x-y|^{-n-2\alpha} dy.$$

(5) *If $\alpha = 1$, $\operatorname{supp}(\nu_r^1) \subset \{(x,t) | t \leq 0, |x| = r\}$.*

Next lemma was the key in the proof of the Huygens property ([3, Lemma 4.3]). Now we give a new proof which is valid for all $n \geq 1$.

Lemma 3.1. *The modified measure $\tilde{\omega}_r^\alpha$ is absolutely continuous with respect to the n-dimensional Lebesgue measure, whose density $\tilde{w}_r^\alpha$ satisfies*

$$\tilde{w}_r^\alpha(x) \leq C r^{2\alpha} |x|^{-n-2\alpha} \quad \textit{and} \quad \|\tilde{w}_r^\alpha\|_{L^q(\mathbf{R}^n)} \leq C r^{-n(1-1/q)},$$

where the constant C is independent of $r > 0$ and $1 \leq q \leq \infty$.

Proof. By Proposition 3.1, we can express ω_r^α as

$$\omega_r^\alpha = w_r^\alpha(x) dx + C(r) \sigma_r, \tag{3.1}$$

where σ_r is the surface measure of the sphere $\{|x| = r\}$, $C(r)$ is a non-negative function of $r > 0$ and

$$w_r^\alpha(x) = \begin{cases} \int_{-\infty}^0 \left[c_{n,\alpha} \int_{|y|\le r} \tilde{W}_r^{(\alpha)}(y,t)|x-y|^{-n-2\alpha}dy\right]dt, & 0<\alpha<1, \\ 0, & \alpha = 1. \end{cases}$$

Then $\tilde{\omega}_r^\alpha$ is absolutely continuous and its density is given by

$$\tilde{w}_r^\alpha(x) = \int_1^2 w_{\lambda r}^\alpha(x)d\lambda + \frac{C(|x|)}{r} 1_{\{r\le|x|\le 2r\}}(x),$$

where $1_{\{r\le|x|\le 2r\}}$ denotes the characteristic function. Considering α-parabolic dilations $\tau_r^\alpha : (x,t) \mapsto (rx, r^{2\alpha}t)$, we have

$$W^{(\alpha)}(x,t) = r^n W^{(\alpha)}(\tau_r^\alpha(x,t)),$$

which shows that ν_r^α is the image measure of ν_1^α by τ_r^α. Thus we obtain $w_r^\alpha(x) = r^{-n}w_1^\alpha(x/r)$, $C(r)\int d\sigma_r = C(1)\int d\sigma_1$ and

$$\tilde{w}_r^\alpha(x) = r^{-n}\tilde{w}_1^\alpha(x/r).$$

In this way, we have only to estimate $\tilde{w}_1^\alpha$. First, we shall show the boundedness. For every $s \ge 1$,

$$\begin{aligned} \int \tilde{w}_1^\alpha(x)d\sigma_s(x) &\le \int\int_1^2 w_\lambda^\alpha(x)d\lambda d\sigma_s(x) + C(s)\int d\sigma_s \\ &= \int\int_1^2 \lambda^{-n}w_1^\alpha(x/\lambda)d\lambda d\sigma_s(x) + C(1)\int d\sigma_1 \\ &\le \frac{2}{s}\int_{s/2}^s \int w_1^\alpha(x)d\sigma_\lambda(x)d\lambda + C(1)\int d\sigma_1 \\ &\le 2\int d\omega_1^\alpha \le 2. \end{aligned}$$

Since $\tilde{w}_1^\alpha$ is rotationally invariant, we have the boundedness of $\tilde{w}_1^\alpha$. Next, we remark that $\tilde{w}_1^\alpha(x) \le C|x|^{-n-2\alpha}$. In fact, from (3) and (4) of Proposition 3.1, follows

$$\begin{aligned} 1 &\ge \int d\nu_1^\alpha \ge \int_{|x|>1}\int_{-\infty}^0 c_{n,\alpha}\int_{|y|\le 1} \tilde{W}_1^{(\alpha)}(y,t)|x-y|^{-n-2\alpha}dydtdx \\ &\ge c_{n,\alpha}\int_{-\infty}^0\int_{|y|\le 1}\tilde{W}_1^{(\alpha)}(y,t)\int_{|x-y|>2}|x-y|^{-n-2\alpha}dxdydt \\ &\ge c_{n,\alpha}\left(\int_{|x|>2}|x|^{-n-2\alpha}dx\right)\iint \tilde{W}_1^{(\alpha)}(y,t)dydt, \end{aligned}$$

which shows that $\tilde{W}_1^{(\alpha)}$ is integrable. Then taking x with $|x| \geq 2$, we have $|x| \leq |x-y| + |y| \leq 2|x-y|$ and

$$w_1^\alpha(x) = c_{n,\alpha} \int_{-\infty}^0 \int_{|y|\leq 1} \tilde{W}_1^{(\alpha)}(y,t)|x-y|^{-n-2\alpha} dydt$$
$$\leq 2^{n+2\alpha} c_{n,\alpha} \|\tilde{W}_1^{(\alpha)}\|_{L^1(\mathbf{R}^{n+1})} |x|^{-n-2\alpha}.$$

Thus taking x with $|x| \geq 4$, we have

$$\tilde{w}_1^\alpha(x) = \int_1^2 w_\lambda^\alpha(x) d\lambda = \int_1^2 \lambda^{-n} w_1^\alpha(x/\lambda) d\lambda$$
$$\leq 2^{n+2\alpha} c_{n,\alpha} \|\tilde{W}_1^{(\alpha)}\|_{L^1(\mathbf{R}^{n+1})} \Big(\int_1^2 \lambda^{2\alpha} d\lambda \Big) |x|^{-n-2\alpha}.$$

Since $\tilde{w}_1^\alpha(x)$ is bounded, we obtain

$$\tilde{w}_1^\alpha(x) \leq C|x|^{-n-2\alpha}$$

for all $x \in \mathbf{R}^n$. Therefore

$$\tilde{w}_r^\alpha(x) = r^{-n} \tilde{w}_1^\alpha(x/r) \leq C r^{2\alpha} |x|^{-n-2\alpha},$$

which also shows the norm inequality

$$\|\tilde{w}_r^\alpha\|_{L^q(\mathbf{R}^n)} \leq C r^{-n(1-1/q)},$$

because

$$\int_{|x|\geq r} \left(|x|^{-n-2\alpha}\right)^q dx = \frac{r^{-(q-1)n-2\alpha q}}{(q-1)n + 2\alpha q} \int d\sigma_1.$$

□

Using the above lemma, in the quite same manner as in the proof of Theorem 4.1 in [3], we obtain the following Huygens property. For the completeness, we give an outline of the proof.

Theorem 3.1. *If an $L^{(\alpha)}$-harmonic function u on H_T belongs to $L^p(H_T)$, then u satisfies the Huygens property:*

$$(3.2) \quad u(x,t) = \int_{\mathbf{R}^n} u(y,s) W^{(\alpha)}(x-y, t-s) dy \quad \text{for} \quad 0 < s < t < T.$$

Proof. Let $u \in L^p(H_T)$ be an arbitrary $L^{(\alpha)}$-harmonic function with $1 \leq p \leq \infty$. Take $\delta > 0$ such that $u(\cdot, \delta) \in L^p(\mathbf{R}^n)$, and put

$$v(x,t) = u(x, t+\delta) - \int_{\mathbf{R}^n} W^{(\alpha)}(x-y, t) u(y, \delta) dy$$

and $V(x,t) = \int_0^t v(x,\tau)d\tau$. Here we remark that $\|v\|_{L^p(H_{T-\delta})} \le \|u\|_{L^p(H_T)}$ and that V is $L^{(\alpha)}$-harmonic(see [3, Lemma 2.3]). For any fixed $(x,t) \in H_{T-\delta}$, taking a cylinder $\{(\xi,\tau)|0 < \tau < t, |\xi - x| < r\}$ with $r > 0$ and using the mean value property (cf. [4]), we have

$$\begin{aligned}|V(x,t)| &= \left| \int_{|\xi|\ge r, -t\le\tau\le 0} V(\xi + x, \tau + t) d\nu_r^\alpha(\xi,\tau) \right| \\ &\le \int_{|\xi|\ge r, -t\le\tau\le 0} \int_0^{\tau+t} |v(\xi + x, s)| ds d\nu_r^\alpha(\xi,\tau) \\ &= \int_0^t \int_{|\xi|\ge r, s-t\le\tau\le 0} |v(\xi + x, s)| d\nu_r^\alpha(\xi,\tau) ds \\ &\le \int_0^{T-\delta} \int |v(\xi + x, s)| d\omega_r^\alpha(\xi) ds.\end{aligned}$$

Thus we obtain

$$\begin{aligned}|V(x,t)| &\le \int_0^{T-\delta} \int |v(\xi + x, s)| \tilde{w}_r^\alpha(\xi) d\xi ds \\ &\le T^{1/q} \|v\|_{L^p(H_{T-\delta})} \|\tilde{w}_r^\alpha\|_{L^q(\mathbf{R}^n)} \\ &\le C T^{1/q} r^{-n/p} \|u\|_{L^p(H_T)},\end{aligned}$$

which shows $V(x,t) = 0$ for $1 \le p < \infty$, because $r > 0$ is arbitrary. In this way, for $\delta < s < t < T$ and $x \in \mathbf{R}^n$, we have

$$\begin{aligned}u(x,t) &= \int_{\mathbf{R}^n} W^{(\alpha)}(x - y, t - \delta) u(y,\delta) dy \\ &= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} W^{(\alpha)}(x - z, t - s) W^{(\alpha)}(z - y, s - \delta) dz\, u(y,\delta) dy \\ &= \int_{\mathbf{R}^n} W^{(\alpha)}(x - z, t - s) u(z,s) dz.\end{aligned}$$

Since $\delta > 0$ is arbitrary, we have (3.2) in the case of $1 \le p < \infty$. When $p = \infty$, (3.2) follows from [4, Proposition 11] immediately. □

§4. Some basic properties of α-parabolic Bergman functions

In this section, for $0 < T < \infty$, we define an α-parabolic Bergman space on H_T.

Definition 4.1. Let $1 \le p \le \infty$. We put

$$\boldsymbol{b}_\alpha^p(H_T) := \{u \in L^p(H_T) | L^{(\alpha)}\text{-harmonic on } H_T\},$$

which is a closed subspace of $L^p(H_T)$ (by (4.1) below) and called the α-parabolic Bergman space on the strip domain.

Remark 4.1. For any $u \in \boldsymbol{b}^p_\alpha(H_T)$, the estimate

$$|u(x,t)| \le C\|u\|_{L^p(H_T)} t^{-(\frac{n}{2\alpha}+1)\frac{1}{p}} \tag{4.1}$$

holds for $(x,t) \in H_T$ in the similar way to [3, Proposition 5.2]. Therefore u can be extended to an $L^{(\alpha)}$-harmonic function on the upper half space H by using the Huygens property. In this paper, every $u \in \boldsymbol{b}^p_\alpha(H_T)$ is considered to be extended to the upper half space as

$$u(x, t+jT) := \int_{\mathbf{R}^n} u(y,t) W^{(\alpha)}(x-y, jT)dy \tag{4.2}$$

for $(x,t) \in H_T$ and $j \in \mathbf{N}$. We remark that the extension u also satisfies the Huygens property on the whole upper half space H.

Remark 4.2. For each fixed p, $\boldsymbol{b}^p_\alpha(H_T)$ are the same and the $L^p(H_T)$-norm is equivalent to one another for all $0 < T < \infty$. In fact, by the Minkowski inequality, for $0 < s < t < \infty$,

$$\|u(\cdot,t)\|_{L^p(\mathbf{R}^n)} \le \|u(\cdot,s)\|_{L^p(\mathbf{R}^n)},$$

which shows the equivalence of the norms.

The Huygens property also yields the following estimate.

Proposition 4.1. *Let $1 \le p \le \infty$ and (β, k) be a multi-index. Then there exists a constant $C > 0$ such that*

$$|\partial_x^\beta \partial_t^k u(x,t)| \le \begin{cases} C\|u\|_{L^p(H_T)} t^{-\left(\frac{|\beta|}{2\alpha}+k\right)-\left(\frac{n}{2\alpha}+1\right)\frac{1}{p}}, & t < T, \\ CT^{-1/p}\|u\|_{L^p(H_T)} t^{-\left(\frac{|\beta|}{2\alpha}+k\right)-\frac{n}{2\alpha}\frac{1}{p}}, & t \ge T, \end{cases}$$

for any $u \in \boldsymbol{b}^p_\alpha(H_T)$ and $(x,t) \in H$. In particular, if $1 \le p < \infty$, $t^{\frac{|\beta|}{2\alpha}+k}\partial_x^\beta \partial_t^k u(\cdot,t)$ converges uniformly to 0 as $t \to \infty$.

Proof. If $0 < t < 2T$, we can show

$$|\partial_x^\beta \partial_t^k u(x,t)| \le C\|u\|_{L^p(H_T)} t^{-\left(\frac{|\beta|}{2\alpha}+k\right)-\left(\frac{n}{2\alpha}+1\right)\frac{1}{p}}$$

in the quite same manner as in [3, Proposition 5.4]. Next we assume $t \ge 2T$. By the Huygens property, we have

$$u(x,t) = \frac{1}{T}\iint_{H_T} u(y,s) W^{(\alpha)}(x-y, t-s)dyds$$

and hence

$$\partial_x^\beta \partial_t^k u(x,t) = \frac{1}{T} \iint_{H_T} u(y,s) \partial_x^\beta \partial_t^k W^{(\alpha)}(x-y, t-s) dyds.$$

Then by (2.3) in Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned} |\partial_x^\beta \partial_t^k u(x,t)| &= \frac{1}{T} \|u\|_{L^p(H_T)} \|\partial_x^\beta \partial_t^k W^{(\alpha)}\|_{L^q(\mathbf{R}^n \times (t-T,t))} \\ &\leq C T^{-1/p} \|u\|_{L^p(H_T)} t^{-\left(\frac{|\beta|}{2\alpha}+k\right) - \frac{n}{2\alpha}\frac{1}{p}}. \end{aligned}$$

□

In the same manner as in [3, Proposition 5.5], we have the following norm inequality.

Proposition 4.2. *Let $1 \leq p \leq \infty$ and (β, k) be a multi-index. Then there exists a constant $C > 0$ such that for every $u \in \boldsymbol{b}_\alpha^p(H_T)$,*

$$\|t^{\frac{|\beta|}{2\alpha}+k} \partial_x^\beta \partial_t^k u\|_{L^p(H_T)} \leq C \|u\|_{L^p(H_T)}.$$

§5. Reproducing property of the Bergman kernel

In [3, Theorem 6.3], we have shown that the α-parabolic Bergman kernel

$$R_\alpha(x,t;y,s) := -2\partial_t W^{(\alpha)}(x-y, t+s)$$

has a reproducing property for $\boldsymbol{b}_\alpha^p$ with $1 \leq p < \infty$.

In the case of the strip domain H_T $(0 < T < \infty)$, we consider the following kernel: for $(x,t), (y,s) \in H_T$,

$$\begin{aligned} R_{\alpha,T}(x,t;y,s) :&= \sum_{j=0}^{\infty} R_\alpha(x, t+jT; y, s+jT) \\ &= -2 \sum_{j=0}^{\infty} \partial_t W^{(\alpha)}(x-y, s+t+2jT), \end{aligned}$$

which turns out to be the α-parabolic Bergman kernel on H_T.

Lemma 5.1. *Let $(x,t) \in H_T$ be fixed. Then $R_{\alpha,T}(x,t;\cdot,\cdot) \in L^q(H_T)$ for $1 < q \leq \infty$.*

Proof. Let $j \geq 1$. Then by (2.3) in Lemma 2.2, we have

$$\begin{aligned} \|R_\alpha(x, t+jT; \cdot, \cdot)\|_{L^q(\mathbf{R}^n \times (jT, jT+T))} &= 2\|\partial_t W^{(\alpha)}\|_{L^q(\mathbf{R}^n \times (t+2jT, t+2jT+T))} \\ &\leq C T^{1/q} (jT)^{-\frac{n(1-1/q)}{2\alpha}-1}. \end{aligned}$$

Thus, by [3, Lemma 6.1],

$$\|R_{\alpha,T}(x,t;\cdot,\cdot)\|_{L^q(H_T)}$$
$$\leq \|R_\alpha(x,t;\cdot,\cdot)\|_{L^q(H)} + CT^{1/q}\sum_{j=1}^{\infty}(jT)^{-\frac{n(1-1/q)}{2\alpha}-1} < \infty.$$

□

Thus we can define the integral operator

$$R_{\alpha,T}u(x,t) := \iint_{H_T} R_{\alpha,T}(x,t;y,s)u(y,s)dyds$$

for every $u \in L^p(H_T)$ with $1 \leq p < \infty$. Next proposition shows that the kernel $R_{\alpha,T}$ has a reproducing property for $\boldsymbol{b}^p_\alpha(H_T)$.

Proposition 5.1. *Let $1 \leq p < \infty$. Then we have*

$$R_{\alpha,T}u(x,t) = u(x,t) \tag{5.1}$$

for every $u \in \boldsymbol{b}^p_\alpha(H_T)$ and $(x,t) \in H_T$.

Proof. Let $u \in \boldsymbol{b}^p_\alpha(H_T)$ be considered to be extended to H as in (4.2). For $\delta > 0$, we put $u_\delta(x,t) := u(x,t+\delta)$. Then using the Huygens property, we have

$$\begin{aligned}
&\iint_{H_T} u_\delta(y,s)(-2)\partial_t W^{(\alpha)}(x-y,t+s+2jT)dyds \\
&= \int_{\mathbf{R}^n} \Big\{ \big[u_\delta(y,s)(-2)W^{(\alpha)}(x-y,t+s+2jT)\big]_{s=0}^{T} \\
&\qquad - \int_0^T \partial_t u_\delta(y,s)(-2)W^{(\alpha)}(x-y,t+s+2jT)ds \Big\} dy \\
&=2u_\delta(x,t+2jT) - 2u_\delta(x,t+2(j+1)T) \\
&\qquad + \int_0^T \frac{\partial}{\partial s}\Big\{u_\delta(x,t+2s+2jT)\Big\}ds \\
&=u_\delta(x,t+2jT) - u_\delta(x,t+2(j+1)T).
\end{aligned}$$

Hence, by Proposition 4.1, we obtain

$$\iint_{H_T} R_{\alpha,T}(x,t;y,s)u_\delta(y,s)dyds$$
$$= \sum_{j=0}^{\infty}\big[u_\delta(x,t+2jT) - u_\delta(x,t+2(j+1)T)\big] = u_\delta(x,t).$$

Letting $\delta \to 0$, we have (5.1). □

Since the kernel $R_{\alpha,T}$ is symmetric and real-valued, the integral operator $R_{\alpha,T}$ is the orthogonal projection on $L^2(H_T)$ to $\boldsymbol{b}^2_\alpha(H_T)$. Therefore in particular, the operator $R_{\alpha,T}$ is bounded on $L^2(H_T)$. We call $R_{\alpha,T}$ the Bergman projection. In the next section, we discuss the boundedness for other exponents $1 < p < \infty$.

§6. L^p-boundedness of the Bergman projection

In this last section, we shall prove the boundedness of the integral operator $R_{\alpha,T}$ on $L^p(H_T)$.

Theorem 6.1. *Let $1 < p < \infty$. Then $R_{\alpha,T}$ is a bounded operator from $L^p(H_T)$ onto $\boldsymbol{b}^p_\alpha(H_T)$.*

To prove the theorem, we introduce the following theorem from the interpolation theory. We quote the theorem from [5].

Theorem 6.2. [5, p.29, Theorem 1]. *Let $K \in L^2(\mathbf{R}^n)$ such that*

(a) $\|\hat{K}\|_{L^\infty(\mathbf{R}^n)} \le B$,
(b) $K \in C^1(\mathbf{R}^n \setminus \{0\})$ *and* $|\nabla K(x)| \le B|x|^{-n-1}$

for some $B > 0$, where $\hat{K}$ denotes the Fourier transform of K. Then for $1 < p < \infty$, there exists a constant A_p, depending only on p, B and n, such that

$$\|K * f\|_{L^p(\mathbf{R}^n)} \le A_p \|f\|_{L^p(\mathbf{R}^n)} \tag{6.1}$$

for every $f \in L^p(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$.

Remark 6.1. In the above theorem, if in addition $K \in L^q(\mathbf{R}^n)$, the inequality (6.1) holds for every $f \in L^p(\mathbf{R}^n)$.

Now we return to the proof. For $t > 0$, we put

$$K_{T,t}(x) := -2\sum_{j=1}^{\infty} \partial_t W^{(\alpha)}(x, t + 2jT).$$

Lemma 6.1. *The kernel $K_{T,t}$ satisfies the condition in Theorem 6.2 with a constant B independent of $t > 0$.*

Proof. By the definition of $W^{(\alpha)}$, the Fourier transform of $W^{(\alpha)}$ satisfies

$$\hat{W}^{(\alpha)}(\xi, t) = (2\pi)^{-n/2} e^{-t|\xi|^{2\alpha}}, \quad \partial_t \hat{W}^{(\alpha)}(\xi, t) = -(2\pi)^{-n/2} |\xi|^{2\alpha} e^{-t|\xi|^{2\alpha}}.$$

Hence

$$\begin{aligned}\hat{K}_{T,t}(\xi) &= 2(2\pi)^{-n/2}|\xi|^{2\alpha}e^{-t|\xi|^{2\alpha}}\sum_{j=1}^{\infty}e^{-2jT|\xi|^{2\alpha}}\\ &= 2(2\pi)^{-n/2}e^{-t|\xi|^{2\alpha}}\frac{|\xi|^{2\alpha}e^{-2T|\xi|^{2\alpha}}}{1-e^{-2T|\xi|^{2\alpha}}}.\end{aligned}$$

This implies that $\hat{K}_{T,t} \in L^2(\mathbf{R}^n)$, i.e., $K_{T,t} \in L^2(\mathbf{R}^n)$, and

$$|\hat{K}_{T,t}(\xi)| \leq \frac{(2\pi)^{-n/2}}{T}\sup_{s>0}\frac{se^{-s}}{1-e^{-s}} =: B < \infty.$$

Clearly, $K_{T,t}$ is of class C^1 and by (2.2) in Lemma 2.2,

$$\begin{aligned}|\nabla K_{T,t}(x)| &\leq C\sum_{j=1}^{\infty}\left((t+2jT)+|x|^{2\alpha}\right)^{-\frac{n+1}{2\alpha}-1}\\ &\leq \frac{C}{2T}\int_0^{\infty}\left((t+s)+|x|^{2\alpha}\right)^{-\frac{n+1}{2\alpha}-1}ds\\ &\leq \frac{C\alpha}{T(n+1)}\left(t+|x|^{2\alpha}\right)^{-\frac{n+1}{2\alpha}} \leq \frac{C\alpha}{T(n+1)}|x|^{-n-1}.\end{aligned}$$

□

Proof of Theorem 6.1. We decompose $R_{\alpha,T}$ as

$$R_{\alpha,T}(x,t;y,s) = R_\alpha(x,t;y,s) + K_{T,t+s}(x-y).$$

For $f \in L^p(H_T) \cap L^1(H_T)$, we put $f_s(y) := f(y,s)$ and

$$\tilde{f}(y,s) := \begin{cases} f(y,s), & 0 < s < T,\\ 0, & s \geq T.\end{cases}$$

In our previous paper [3], we have shown that the integral operator R_α is bounded on $L^p(H)$. Then

$$\|R_\alpha\tilde{f}\|_{L^p(H_T)} \leq \|R_\alpha\|\cdot\|f\|_{L^p(H_T)}.$$

Since

$$R_{\alpha,T}f(x,t) = R_\alpha\tilde{f}(x,t) + \int_0^T K_{T,t+s} * f_s(x)ds,$$

the Minkowski inequality implies

$$\|R_{\alpha,T}f(\cdot,t)\|_{L^p(\mathbf{R}^n)} \leq \|R_\alpha\tilde{f}(\cdot,t)\|_{L^p(\mathbf{R}^n)} + \int_0^T \|K_{T,t+s} * f_s\|_{L^p(\mathbf{R}^n)}ds.$$

Here by Theorem 6.2, we have

$$\begin{aligned}\int_0^T \|K_{T,t+s} * f_s\|_{L^p(\mathbf{R}^n)}ds &\le A_p \int_0^T \|f_s\|_{L^p(\mathbf{R}^n)}ds \\ &\le A_p\Big(\int_0^T \|f_s\|^p_{L^p(\mathbf{R}^n)}ds\Big)^{1/p}\Big(\int_0^T ds\Big)^{1/q} \\ &\le A_p\|f\|_{L^p(H_T)}T^{1/q}.\end{aligned}$$

Taking the $L^p(0,T)$-norm, again by the Minkowski inequality, we obtain

$$\begin{aligned}\|R_{\alpha,T}f\|_{L^p(H_T)} &\le \|R_\alpha \tilde{f}\|_{L^p(H_T)} + TA_p\|f\|_{L^p(H_T)} \\ &\le (\|R_\alpha\| + TA_p)\|f\|_{L^p(H_T)}.\end{aligned}$$

This completes the proof. □

As an application, we have the following duality (cf. [3, Theorem 8.1]).

Corollary 6.1. *For $1 < p < \infty$, the following duality holds;*

$$\boldsymbol{b}^p_\alpha(H_T)' \simeq \boldsymbol{b}^q_\alpha(H_T),$$

where the pairing is given by

$$\langle f, g\rangle = \iint_{H_T} f(x,t)g(x,t)dxdt$$

for $f \in \boldsymbol{b}^p_\alpha(H_T)$ and $g \in \boldsymbol{b}^q_\alpha(H_T)$.

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