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On a covering property of rarefied sets at infnity in a cone

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Abstract.

This paper gives a quantitative property of rarefied sets at ∞ of a cone. The proof is based on the fact in which the estimations of Green potential and Poisson integral with measures are connected with a kind of densities of the measures modified from the measures.

§1. Introduction

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n$ $(n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, y)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set S in $\mathbf{R}^n$ are denoted by ∂S and $\bar{S}$, respectively.

We introduce a system of spherical coordinates $(r, \Theta), \Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, y)$ by $y = r \cos \theta_1$.

The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n;\ r \in \Lambda,\ (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half-space $\mathbf{R}_+ \times \mathbf{S}^{n-1}_+ = \{(X, y) \in \mathbf{R}^n;\ y > 0\}$ will be denoted by $\mathbf{T}_n$.

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Let Ω be a domain on $\mathbf{S}^{n-1}$ ($n \geq 2$) with smooth boundary. Consider the Dirichlet problem

$$\begin{aligned}(\Lambda_n + \tau)f = 0 \quad &\text{on } \Omega,\\ f = 0 \quad &\text{on } \partial\Omega,\end{aligned}$$

where Λ_n is the spherical part of the Laplace operator Δ_n

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by τ_Ω and the normalized positive eigenfunction corresponding to τ_Ω by $f_\Omega(\Theta)$. We denote the solutions of the equation $t^2 + (n-2)t - \tau_\Omega = 0$ by $\alpha_\Omega, -\beta_\Omega$ ($\alpha_\Omega,\ \beta_\Omega > 0$). If $\Omega = \mathbf{S}_+^{n-1}$, then $\alpha_\Omega = 1,\ \beta_\Omega = n-1$ and $f_\Omega(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1$, where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

To simplify our consideration in the following, we shall assume that if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) on $\mathbf{S}^{n-1}$ (e.g. see Gilbarg and Trudinger [7, pp.88-89] for the definition of $C^{2,\alpha}$-domain).

By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}(n \geq 2)$. We call it a cone. Then $\mathbf{T}_n$ is a special cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$.

It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, and the Martin functions at ∞ and at O with respect to a reference point chosen suitably are given by $K(P;\infty,\Omega) = r^{\alpha_\Omega} f_\Omega(\Theta)$ and $K(P;O,\Omega) = \iota r^{-\beta_\Omega} f_\Omega(\Theta)$ ($P = (r,\Theta) \in C_n(\Omega)$), respectively, where ι is a positive number.

Let E be a bounded subset of $C_n(\Omega)$. Then $\hat{R}^E_{K(\cdot;\infty,\Omega)}$ is bounded on $C_n(\Omega)$ and hence the greatest harmonic minorant of $\hat{R}^E_{K(\cdot;\infty,\Omega)}$ is zero. When by $G^\Omega(P,Q)$ ($P \in C_n(\Omega), Q \in C_n(\Omega)$) and $G^\Omega\xi(P)$ ($P \in C_n(\Omega)$) we denote the Green function of $C_n(\Omega)$ and the Green potential with a positive measure ξ on $C_n(\Omega)$, respectively, we see from the Riesz decomposition theorem that there exists a unique positive measure λ_E on $C_n(\Omega)$ such that

$$\hat{R}^E_{K(\cdot;\infty,\Omega)}(P) = G^\Omega\lambda_E(P) \quad (P \in C_n(\Omega)).$$

Let E be a subset of $C_n(\Omega)$ and $E_k = E \cap I_k$ ($k = 0,1,2,\ldots$), where $I_k = \{P = (r,\Theta) \in \mathbf{R}^n; 2^k \leq r < 2^{k+1}\}$. A subset E of $C_n(\Omega)$ is said to be *rarefied at* ∞ *with respect to* $C_n(\Omega)$, if

$$\sum_{k=0}^{\infty} 2^{-k\beta_\Omega}\lambda_{E_k}(C_n(\Omega)) < +\infty.$$

Remark 1. This definition of rarefied sets was given by Essén and Jackson [4] for sets in the half-space. This exceptional sets were originally investigated in Ahlfors and Heins [1] and Hayman [8] in connection with the regularity of value distribution of subharmonic functions in the half plane.

As in $\mathbf{T}_n$ (Essén and Jackson [4, Remark 4.4], Aikawa and Essén [2, Definition 12.4, p.74]) and in $\mathbf{T}_2$ (Hayman [9, p.474]), we proved

Theorem A (Miyamoto and Yoshida [10, Theorem 2]). *A subset E of $C_n(\Omega)$ is rarefied at ∞ with respect to $C_n(\Omega)$ if and only if there exists a positive superharmonic function $v(P)$ in $C_n(\Omega)$ such that*

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K(P;\infty,\Omega)} = 0$$

and $E \subset \{P = (r,\Theta) \in C_n(\Omega);\ v(P) \geq r^{\alpha_\Omega}\}$.

In this paper, we shall give a quantitative property of rarefied sets at ∞ with respect to $C_n(\Omega)$ (Theorem 2), which extends a result obtained by Essén, Jackson and Rippon [5] with respect to $\mathbf{T}_n$ and complements Azarin's result (Corollary 1). It follows from two results. One is another characterization of rarefied sets at ∞ with respect to $C_n(\Omega)$ (Theorem A). The other is the fact that the value distributions of Green potential and Poisson integral with respect to any positive measure on $C_n(\Omega)$ and $\partial C_n(\Omega)$ are connected with a kind of densities of the measures modified from the measures, respectively (Theorem 1). Our proof is completely different from the way used by Essén, Jackson and Rippon [5] and is essentially based on Hayman [8], Ušakova [12] and Azarin [3].

In order to avoid complexity of our proofs, we shall assume $n \geq 3$. All our results in this paper are true, even if $n = 2$.

§2. Statements of results

In the following we denote the sets $I\times\Omega$ and $I\times\partial\Omega$ with an interval I on $\mathbf{R}$ by $C_n(\Omega;I)$ and $S_n(\Omega;I)$. By $S_n(\Omega)$ we denote $S_n(\Omega;(0,+\infty))$ which is $\partial C_n(\Omega) - \{O\}$. We shall also denote a ball in $\mathbf{R}^n$ having a center P and a radius r by $B(P,r)$.

Let m be any positive measure on $\mathbf{R}^n$. Let q and ε be two positive numbers. When for each $P = (r,\Theta) \in \mathbf{R}^n - \{O\}$ we set

$$M(P;m,q) = \sup_{0<\rho\leq 2^{-1}r} \frac{m(B(P,\rho))}{\rho^q},$$

the set $\{P \in \mathbf{R}^n - \{O\};\ M(P; m, q)r^q > \varepsilon\}$ is denoted by $\Psi(\varepsilon; m, q)$.

Remark 2. If $m(\{P\}) > 0$ ($P \neq O$), then $M(P; m, q) = +\infty$ for any positive number q and hence $\{P \in \mathbf{R}^n - \{O\};\ m(\{P\}) > 0\} \subset \Psi(\varepsilon; m, q)$ for any positive number ε.

Let μ be any positive measure on $C_n(\Omega)$ such that $G^{\Omega}\mu(P) \not\equiv +\infty$ ($P \in C_n(\Omega)$). The positive measure $m_\mu^{(1)}$ on $\mathbf{R}^n$ is defined by

$$dm_\mu^{(1)}(Q) = \begin{cases} t^{-\beta_\Omega} f_\Omega(\Phi) d\mu(t, \Phi) & (Q = (t, \Phi) \in C_n(\Omega;\ (1, +\infty))) \\ 0 & (Q \in \mathbf{R}^n - C_n(\Omega;\ (1, +\infty))). \end{cases}$$

Let ν be any positive measure on $S_n(\Omega)$ such that the Poisson integral

$$\Pi^{\Omega}\nu(P) = \int_{S_n(\Omega)} \frac{\partial G^{\Omega}(P, Q)}{\partial n_Q} d\nu(Q) \not\equiv +\infty \qquad (P \in C_n(\Omega)),$$

where $\frac{\partial}{\partial n_Q}$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$. We define the positive measure $m_\nu^{(2)}$ on $\mathbf{R}^n$ by

$$dm_\nu^{(2)}(Q) = \begin{cases} t^{-\beta_\Omega - 1} \frac{\partial f_\Omega(\Phi)}{\partial n_\Phi} d\nu(Q) & (Q = (t, \Phi) \in S_n(\Omega;\ (1, +\infty))) \\ 0 & (Q \in \mathbf{R}^n - S_n(\Omega;\ (1, +\infty))). \end{cases}$$

Remark 3. We remark from Miyamoto and Yoshida [10, (i) of Lemma 1] (resp. [10, (i) of Lemma 4]) that the total mass of $m_\mu^{(1)}$ (resp. $m_\nu^{(2)}$) is finite.

The following Theorem 1 gives a way to estimate the Green potential and the Poisson integral with measures on $C_n(\Omega)$ and $S_n(\Omega)$, respectively.

Theorem 1. *Let μ and ν be two positive measures on $C_n(\Omega)$ and $S_n(\Omega)$ such that $G^{\Omega}\mu(P) \not\equiv +\infty$ and $\Pi^{\Omega}\nu(P) \not\equiv +\infty$ ($P \in C_n(\Omega)$), respectively. Then for a sufficiently large L and a sufficiently small ε we have*

$$\{P = (r, \Theta) \in C_n(\Omega; (L, +\infty));\ G^{\Omega}\mu(P) \geq r^{\alpha_\Omega}\} \subset \Psi(\varepsilon; m_\mu^{(1)}, n-1), \tag{2.1}$$

$$\{P \in C_n(\Omega; (L, +\infty));\ \Pi^{\Omega}\nu(P) \geq r^{\alpha_\Omega}\} \subset \Psi(\varepsilon; m_\nu^{(2)}, n-1). \tag{2.2}$$

As in $\mathbf{T}_n$ (Essén, Jackson and Rippon [5, p.397]) we have the following result for rarefied sets in $C_n(\Omega)$ by using Theorems A and 1.

Theorem 2. *If a subset E of $C_n(\Omega)$ is rarefied at ∞ with respect to $C_n(\Omega)$, then E is covered by a sequence of balls B_k (k=1,2,3,...) satisfying*

$$\sum_{k=1}^{\infty}(r_k/R_k)^{n-1} < +\infty, \tag{2.3}$$

where r_k is the radius of B_k and R_k is the distance between the origin and the center of B_k.

Remark 4. By giving an example we shall show that the reverse of Theorem 2 is not true. When the radius r_k of a ball B_k and the distance R_k between the origin and the center of it are given by $r_k = 3 \cdot 2^{k-1}k^{-\frac{1}{n-2}}, R_k = 3 \cdot 2^{k-1}$ $(k = 1, 2, 3, ...)$, they satisfy

$$\sum_{k=1}^{\infty}(r_k/R_k)^{n-1} = \sum_{k=1}^{\infty} k^{-(n-1)/(n-2)} < +\infty.$$

Let $C_n(\Omega')$ be a subcone of $C_n(\Omega)$ i.e. $\overline{\Omega'} \subset \Omega$. Suppose that these balls are so located: there is an integer k_0 such that $B_k \subset C_n(\Omega'), r_k/R_k < 2^{-1}$ $(k \geq k_0)$. Then the set $E = \cup_{k=k_0}^{\infty} B_k$ is not rarefied. This proof will be given at the end in the last section 4.

From this Theorem 2 and Miyamoto and Yoshida [10, Theorem 3], we immediately have the following corollary.

Corollay 1 (Azarin [3, Theorem 2]). *Let $v(P)$ be a positive superharmonic function on $C_n(\Omega)$. Then $v(P)r^{\alpha_\Omega}$ uniformly converges to $c(v)f_\Omega(\Theta)$ as $r \to +\infty$ outside a set which is covered by a sequence of balls B_k satisfying (2.3), where*

$$c(v) = \inf_{P \in C_n(\Omega)} \frac{v(P)}{K(P; \infty, \Omega)}.$$

§3. Proof of Theorem 1

All constants appearing in the expressions in the following all sections will be always written A, because we do not need to specify them.

Inclusion (2.1) is an analogous result to [11, Theorem 2]. Hence we shall prove only (2.2) of Theorem 1. To do it, we need two inequalities which follow from Azarin [3, Lemma 1] (also see Essén and Lewis [6, Lemma 2]) and Azarin [3, Lemma 4 and Remark]:

$$\frac{\partial}{\partial n_Q} G^{\Omega}(P,Q) \leq A r^{\alpha_\Omega - 1} t^{-\beta_\Omega} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) \tag{3.1}$$

$$\left(resp. \frac{\partial}{\partial n_Q} G^{\Omega}(P,Q) \leq A r^{\alpha_\Omega} t^{-\beta_\Omega - 1} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\right) \tag{3.2}$$

for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in C_n(\Omega)$ satisfying $0 < t/r \leq 4/5$ (resp. $0 < r/t \leq 4/5$);

$$\frac{\partial}{\partial n_Q} G^{\Omega}(P,Q) \leq A \frac{f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)}{t^{n-1}} + A \frac{r f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)}{|P-Q|^n} \tag{3.3}$$

for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in S_n(\Omega;\ ((4/5)r, (5/4)r])$.

Poof of Theorem 1. If we can show that for a sufficiently large L and a sufficiently small positive number ε,

$$\Pi^{\Omega}\nu(P) < r^{\alpha_\Omega} \qquad (P \in C_n(\Omega; (L, +\infty)) - \Psi(\varepsilon; m_\nu^{(2)}, n-1)), \tag{3.4}$$

then we can conclude (2.2).

For any point $P = (r, \Theta) \in C_n(\Omega)$, write $\Pi^{\Omega}\nu(P)$ as the sum

$$\Pi^{\Omega}\nu(P) = I_1(P) + I_2(P) + I_3(P), \tag{3.5}$$

where

$$I_i(P) = \int_{S_n(\Omega; J_i)} \frac{\partial}{\partial n_Q} G^{\Omega}(P,Q) d\nu(Q) \quad (i = 1, 2, 3),$$

where $J_1 = (0, (4/5)r]$, $J_2 = ((4/5)r, (5/4)r])$ and $J_3 = ((5/4)r, \infty)$.

From (3.1) and the boundedness of $f_\Omega(\Theta)$ $(\Theta \in \Omega)$ we first have

$$I_1(P) \leq A r^{\alpha_\Omega} \left(\frac{4}{5} r\right)^{-(\alpha_\Omega + \beta_\Omega)} \int_{S_n(\Omega;\ (0, \frac{4}{5} r])} t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) d\nu(Q),$$

and hence

$$I_1(P) = o(1) r^{\alpha_\Omega} \quad (r \to \infty) \tag{3.6}$$

by Miyamoto and Yoshida [10, (ii) of Lemma 4].

We similarly have

$$I_3(P) \leq Ar^{\alpha_\Omega} \int_{S_n(\Omega;\ (\frac{4}{5}r,+\infty))} t^{-\beta_\Omega-1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) d\nu(Q),$$

from (3.2) and hence

$$I_3(P) = o(1) r^{\alpha_\Omega} \quad (r \to \infty) \tag{3.7}$$

by Remark 3.

For $I_2(P)$ we have

$$I_2(P) \leq I_{2,1}(P) + I_{2,2}(P), \tag{3.8}$$

where

$$I_{2,1}(P) \leq A \int_{S_n(\Omega;\ (\frac{4}{5}r,\frac{5}{4}r])} \frac{f_\Omega(\Theta) t^{\beta_\Omega+1}}{t^{n-1}} dm_\nu^{(2)}(Q),$$

$$I_{2,2}(P) = A \int_{S_n(\Omega;\ (\frac{4}{5}r,\frac{5}{4}r])} \frac{t^{\beta_\Omega+1} r f_\Omega(\Theta)}{|P-Q|^n} dm_\nu^{(2)}(Q).$$

Since $f_\Omega(\Theta)$ is bounded on Ω, we first have

$$I_{2,1}(P) \leq Ar^{\alpha_\Omega} \int_{S_n(\Omega;\ (\frac{4}{5}r,\frac{5}{4}r])} dm_\nu^{(2)}(Q) = o(1) r^{\alpha_\Omega} \quad (r \to \infty) \tag{3.9}$$

from Remark 3.

We shall estimate $I_{2,2}(P)$. Take a sufficiently small positive number κ such that $S_n(\Omega;\ ((4/5)r, (5/4)r]) \subset B(P, 2^{-1}r)$ for any $P = (r, \Theta) \in \Lambda(\kappa)$, where

$$\Lambda(\kappa) = \{Q = (t, \Phi) \in C_n(\Omega);\ \inf_{Z \in \partial\Omega} |(1, \Phi) - (1, Z)| \leq \kappa,\ 0 < t < +\infty\}$$

and divide $C_n(\Omega)$ into two sets $\Lambda(\kappa)$ and $C_n(\Omega) - \Lambda(\kappa)$.

If $P = (r, \Theta) \in C_n(\Omega) - \Lambda(\kappa)$, then there exists a positive constant κ' such that $|P - Q| > \kappa' r$ for any $Q \in S_n(\Omega)$, and hence

$$I_{2,2}(P) \leq Ar^{\alpha_\Omega} \int_{S_n(\Omega;\ (\frac{4}{5}r,+\infty))} dm_\nu^{(2)}(Q) = o(1) r^{\alpha_\Omega} \quad (r \to +\infty) \tag{3.10}$$

from Remark 3.

We shall consider the case where $P \in \Lambda(\kappa)$. Now put

$$W_i(P) = \{Q \in S_n(\Omega;\ ((4/5)r, (5/4)r]);\ 2^{i-1}\delta(P) \leq |P - Q| < 2^i \delta(P)\},$$

where $\delta(P) = \inf_{Q \in \partial C_n(\Omega)} |P - Q|$. Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n;\ |P - Q| < \delta(P)\} = \emptyset$, we have

$$I_{2,2}(P) = \Sigma_{i=1}^{i(P)} A \int_{W_i(P)} \frac{t^{\beta_\Omega+1} r f_\Omega(\Theta)}{|P-Q|^n} dm_\nu^{(2)}(Q),$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \le r/2 < 2^{i(P)}\delta(P)$. Since $r f_\Omega(\Theta) \le A\delta(P)$ $(P = (r, \Theta) \in C_n(\Omega))$, we have

$$\int_{W_i(P)} \frac{t^{\beta_\Omega+1} r f_\Omega(\Theta)}{|P-Q|^n} dm_\nu^{(2)}(Q) \le A r^{\alpha_\Omega} 2^{n-i} \frac{m_\nu^{(2)}(W_i(P))}{\{2^i\delta(P)\}^{n-1}}$$

for $i = 0, 1, 2, ..., i(P)$. Suppose that $P \notin \Psi(\varepsilon; m_\nu^{(2)}, n-1)$ for a positive number ε. Then we have

$$\frac{m_\nu^{(2)}(W_i(P))}{\{2^i\delta(P)\}^{n-1}} \le \frac{m_\nu^{(2)}(B(P, 2^i\delta(P))}{\{2^i\delta(P)\}^{n-1}} \le M(P; m_\nu^{(2)}, n-1) \le \varepsilon r^{1-n}$$

for $i = 0, 1, 2, ..., i(P)-1$ and

$$\frac{m_\nu^{(2)}(W_{i(P)}(P))}{\{2^{i(p)}\delta(P)\}^{n-1}} \le \frac{m_\nu^{(2)}(B(P, \frac{r}{2}))}{(\frac{r}{2})^{n-1}} \le \varepsilon r^{1-n}.$$

In this case we also have

$$I_{2,2}(P) \le A\varepsilon r^{\alpha_\Omega}. \tag{3.11}$$

From (3.5),(3.6),(3.7),(3.8),(3.9),(3.10) and (3.11), we finally obtain that if L is sufficiently large and ε is sufficiently small, then $\Pi^\Omega\nu(P) < r^{\alpha_\Omega}$ for any $P \in C_n(\Omega; (L, +\infty)) - \Psi(\varepsilon; m_\nu^{(2)}, n-1)$.

§4. Proof of Theorem 2

The following Lemma 1 is a result concerning measure theory, which was proved in Miyamoto and Yoshida [11].

Lemma 1 . *Let m be any positive measure on $\mathbf{R}^n$ having the finite total mass. Let ε and q be two any positive numbers. Then $S(\varepsilon; m, q)$ is covered by a sequence of balls B_j $(j = 1, 2, ...)$ satisfying*

$$\sum_{j=1}^{\infty} (r_j/R_j)^q < +\infty,$$

where r_j is the radius of B_j and R_j is the distance between the origin and the center of B_j.

Proof of Theorem 2. Since E is rarefied at ∞ with respect to $C_n(\Omega)$, by Theorem A there exists a positive superharmonic function $v(P)$ in $C_n(\Omega)$ such that

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K(P;\infty,\Omega)} = 0 \tag{4.1}$$

and

$$E \subset \{P=(r,\Theta)\in C_n(\Omega);\ v(P)\geq r^{\alpha_\Omega}\}. \tag{4.2}$$

By Miyamoto and Yoshida [10, Lemma 3] (also see Azarin [3, Theorem 1]) and (4.1), for this $v(P)$ there exist a unique positive measure μ' on $C_n(\Omega)$ and a unique positive measure ν' on $S_n(\Omega)$ such that

$$v(P) = c_0(v)K(P;O,\Omega) + G^\Omega\mu'(P) + \Pi^\Omega\nu'(P).$$

Let us denote the sets $\{P=(r,\Theta)\in C_n(\Omega);\ c_0(v)K(P;O,\Omega)\geq 3^{-1}r^{\alpha_\Omega}\}$, $\{P=(r,\Theta)\in C_n(\Omega);\ G^\Omega\mu'(P)\geq 3^{-1}r^{\alpha_\Omega}\}$ and $\{P=(r,\Theta)\in C_n(\Omega);$ $\Pi^\Omega\nu'(P)\geq 3^{-1}r^{\alpha_\Omega}\}$ by $E^{(1)}, E^{(2)}$ and $E^{(3)}$, respectively. Then we see from (4.2) that

$$E\subset E^{(1)}\cup E^{(2)}\cup E^{(3)}. \tag{4.3}$$

For each $E^{(i)}$ $(i=1,2,3)$ we shall find a sequence of balls which covers it.

It is evident from the boundedness of $E^{(1)}$ that $E^{(1)}$ is covered by a finite ball B_1 satisfying

$$r_1/R_1 < +\infty, \tag{4.4}$$

where r_1 is the radius of B_1 and R_1 is the distance between the origin and the center of B_1.

When we apply Theorem 1 with the measures μ and ν defined by $\mu=3\mu'$ and $\nu=3\nu'$ we can find two positive constants L and ε such that $E^{(2)}\cap C_n(\Omega;(L,+\infty))\subset\Psi(\varepsilon;m_\mu^{(1)},n-1)$ and $E^{(3)}\cap C_n(\Omega;(L,+\infty))$ $\subset\Psi(\varepsilon;m_\nu^{(2)},n-1)$, respectively. By Lemma 1 these $\Psi(\varepsilon;m_\mu^{(1)},n-1)$ and $\Psi(\varepsilon;m_\nu^{(2)},n-1)$ are covered by two sequences of balls $B_j^{(2)}$ and $B_j^{(3)}$ $(j=1,2,...)$ satisfying

$$\sum_{j=1}^{\infty}(r_j^{(2)}/R_j^{(2)})^{n-1}<+\infty \quad \textit{and} \quad \sum_{j=1}^{\infty}(r_j^{(3)}/R_j^{(3)})^{n-1}<+\infty,$$

respectively, where $r_j^{(2)}$ (resp. $r_j^{(3)}$) is the radius of $B_j^{(2)}$ (resp. $B_j^{(3)}$) and $R_j^{(2)}$ (resp. $R_j^{(3)}$) is the distance between the origin and the center of $B_j^{(2)}$ (resp. $B_j^{(3)}$). Hence $E^{(2)}$ and $E^{(3)}$ are also covered by the sequences of balls $B_j^{(2)}$ and $B_j^{(3)}$ $(j = 0, 1, ...)$ with an additional finite ball $B_0^{(2)}$ covering $C_n(\Omega; (0, L])$ satisfying

$$(4.5) \qquad \sum_{j=0}^{\infty}(r_j^{(2)}/R_j^{(2)})^{n-1} < +\infty \quad \textit{and} \quad \sum_{j=1}^{\infty}(r_j^{(3)}/R_j^{(3)})^{n-1} < +\infty,$$

respectively.

Thus by rearranging $B_1, B_j^{(2)} (j = 0, 1, ...), B_j^{(3)} (j = 1, ...)$, we have a sequence of balls B_k $(k = 1, 2, ...)$ which covers E from (4.3) and satisfies (2.3) from (4.4), (4.5).

Proof of Remark 4. Since $f_\Omega(\Theta) \geq A$ for any $\Theta \in \Omega'$ and $r_k R_k^{-1} < 2^{-1}$ $(k \geq k_0)$ for a positive integer k_0, we have that $K(P; \infty, \Omega) \geq AR_k^{\alpha_\Omega}$ and hence

$$(4.6) \qquad \hat{R}^{B_k}_{K(\cdot;\infty,\Omega)}(P) \geq AR_k^{\alpha_\Omega} \quad (k \geq k_0)$$

for any $P \in \overline{B_k}$ $(k \geq k_0)$.

Take a measure τ on $C_n(\Omega)$, supp $\tau \subset \overline{B_k}$, $\tau(\overline{B_k}) = 1$ such that

$$(4.7) \qquad \int_{C_n(\Omega)} |P - Q|^{2-n} d\tau(P) = \{\mathrm{Cap}(\overline{B_k})\}^{-1},$$

for any $Q \in \overline{B_k}$, where *Cap* denotes the Newtonian capacity. Since $G^\Omega(P, Q) \leq |P - Q|^{2-n}$ $(P \in C_n(\Omega),\ Q \in C_n(\Omega))$, we have

$$\int \Big(\int G^\Omega(P, Q) d\lambda_{B_k}(Q)\Big) d\tau(P) \leq \{\mathrm{Cap}(\overline{B_k})\}^{-1} \lambda_{B_k}(C_n(\Omega))$$

from (4.7) and

$$\int \Big(\int G^\Omega(P, Q) d\lambda_{B_k}(Q)\Big) d\tau(P)$$

$$= \int (\hat{R}^{B_k}_{K(\cdot;\infty,\Omega)}(P)) d\tau(P) \geq AR_k^{\alpha_\Omega}\tau(\overline{B_k}) = AR_k^{\alpha_\Omega}$$

from (4.6). Hence we have that $\lambda_{B_k}(C_n(\Omega)) \geq A\mathrm{Cap}(\overline{B_k})R_k^{\alpha_\Omega} \geq Ar_k^{n-2}R_k^{\alpha_\Omega}$, because $\mathrm{Cap}(\overline{B_k}) = r_k^{n-2}$.

Thus if we observe $\lambda_{E_k}(C_n(\Omega)) = \lambda_{B_k}(C_n(\Omega))$, then we have

$$\sum_{k=k_0}^{\infty} 2^{-k\beta_\Omega} \lambda_{E_k}(C_n(\Omega)) \geq A \sum_{k=k_0}^{\infty} (r_k/R_k)^{n-2} = A \sum_{k=k_0}^{\infty} k^{-1} = +\infty,$$

which shows that E is not rarefied.

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