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Kato class functions of Markov processes under ultracontractivity

Kazuhiro Kuwae and Masayuki Takahashi

Dedicated to Professor Shintaro Nakao on his Sixtieth Birthday

Abstract.

We show that $f \in L^p(X;m)$ implies $|f|dm \in S^1_K$ for $p > D$ with $D > 0$, where S^1_K is a subfamily of Kato class measures relative to a semigroup kernel $p_t(x,y)$ of a Markov process associated with a (non-symmetric) Dirichlet form on $L^2(X;m)$. We only assume that $p_t(x,y)$ satisfies the Nash type estimate of small time depending on D. No concrete expression of $p_t(x,y)$ is needed for the result.

§1. Introduction

A measurable function f on $\mathbb{R}^d$ is said to be in the *Kato class* K_d if

$$\begin{aligned} \lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<r}\frac{|f(y)|}{|x-y|^{d-2}}dy &= 0 \text{ for } d\geq 3, \\ \lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<r}(\log|x-y|^{-1})|f(y)|dy &= 0 \text{ for } d=2, \\ \sup_{x\in\mathbb{R}^d}\int_{|x-y|<1}|f(y)|dy &< \infty \text{ for } d=1. \end{aligned}$$

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Let $\mathbf{M}^{\mathrm{w}} = (\Omega, B_t, P_x)_{x\in\mathbb{R}^d}$ be a d-dimensional Brownian motion on $\mathbb{R}^d$. The following theorem is shown in Aizenman and Simon [1]:

Theorem 1.1 (Theorem 1.3(ii) in [1]). *$f \in K_d$ if and only if*

$$\sup_{x\in\mathbb{R}^d} E_x\left[\int_0^t |f(B_s)|ds\right] = \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_0^t p_s(x,y)ds\right)|f(y)|dy \xrightarrow{t\to 0} 0,$$

where $p_t(x,y) := \frac{1}{(2\pi t)^{d/2}}\exp[-\frac{|x-y|^2}{2t}]$ is the heat kernel of $\mathbf{M}^{\mathrm{w}}$.

Zhao [13] extends this in more general setting including a subclass of Lévy processes, but his result does not assure the low dimensional case even if the process is $\mathbf{M}^{\mathrm{w}}$. The following is also shown in [1]:

Theorem 1.2 (cf. Theorem 1.4(iii) in [1]). *$L^p(\mathbb{R}^d) \subset K_d$ holds if $p > d/2$ with $d \geq 2$, or $p \geq 1$ with $d = 1$.*

Note that there is an $f \in L^{d/2}(\mathbb{R}^d) \setminus K_d$ for $d \geq 2$. Indeed, taking $g \in C_0([0, 2/e[\to [0,\infty])$ with $g(r) := 1/(r^2 \log r^{-1})$ if $d \geq 3$, $:= 1/(r^2(\log r^{-1})^{1+\varepsilon})$, $\varepsilon \in]0,1[$ if $d = 2$ for $r \in [0, 1/e]$, $f(x) := g(|x|)$ does the job through the proof of Proposition 4.10 in [1]. Here (4.10) in [1] should be changed to $\int_0^{1/e} r(\log r^{-1})|V(r)|dr < \infty$ if $d = 2$.

In the framework of strongly local regular Dirichlet forms with the notions of volume doubling and weak Poincaré inequality, Biroli and Mosco [3] gave a similar result with Theorem 1.2 (see Proposition 3.7 in [3]). Their definition of Kato class depends on the volume growth of balls. The purpose of this note is to show that Theorem 1.2 holds true in more general context replacing K_d with S_K^1 the family of Kato class smooth measures in the strict sense in terms of semigroup kernel of Markov processes associated with (non-symmetric) Dirichlet forms (see Theorem 2.1 below).

Finally we will announce the content of [10]. In [10], we extend Theorem 1.1, that is, under some conditions, we establish $K_{d,\beta} = S_K^1$ in the framework of symmetric Markov processes which admits a semigroup kernel possessing upper and lower estimates, which includes the low dimensional case. Here $K_{d,\beta}$ is the family of Kato class measures in terms of a Green kernel depending on $d, \beta > 0$. In particular, Theorem 2.1 below can be strengthened by replacing $L^p(X;m)$ with $L^p_{\mathrm{unif}}(X;m)$.

§2. Result

Let X be a locally compact separable metric space and m a positive Radon measure with full support. Let $X_\Delta := X \cup \{\Delta\}$ be a one point compactification of X. We consider and fix a (non-symmetric) regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$. Then there exists a pair of Hunt processes $(\mathbf{M},\widehat{\mathbf{M}})$, $\mathbf{M}=(\Omega,X_t,\zeta,P_x)$, $\widehat{\mathbf{M}}=(\hat{\Omega},\hat{X}_t,\hat{\zeta},\hat{P}_x)$ such that for each Borel $u\in L^2(X;m)$, $T_tu(x)=E_x[u(X_t)]$ m-a.e. $x\in X$ and $\hat{T}_tu(x)=\hat{E}_x[u(\hat{X}_t)]$ m-a.e. $x\in X$ for all $t>0$, where $(T_t)_{t>0}$ (resp. $(\hat{T}_t)_{t>0}$) is the semigroup associated with $(\mathcal{E},\mathcal{F})$ (resp. $(\hat{\mathcal{E}},\mathcal{F})$), where $\hat{\mathcal{E}}(u,v):=\mathcal{E}(v,u)$ for $u,v\in\mathcal{F}$ is the dual form of $(\mathcal{E},\mathcal{F})$. Here $\zeta:=\inf\{t\geq 0\mid X_t=\Delta\}$ (resp. $\hat{\zeta}:=\inf\{t\geq 0\mid \hat{X}_t=\Delta\}$) denotes the life time of $\mathbf{M}$ (resp. $\widehat{\mathbf{M}}$). Further, we assume that there exists a kernel $p_t(x,y)$ defined for all $(t,x,y)\in]0,\infty[\times X\times X$ such that $E_x[u(X_t)]=P_tu(x):=\int_X p_t(x,y)u(y)m(dy)$ and $\hat{E}_x[u(\hat{X}_t)]=\hat{P}u(x):=\int_X \hat{p}_t(x,y)u(y)m(dy)$ for any $x\in X$, bounded Borel function u and $t>0$, where $\hat{p}_t(x,y):=p_t(y,x)$. $p_t(x,y)$ is said to be a *semigroup kernel*, or sometimes called a *heat kernel* of $\mathbf{M}$ on the analogy of heat kernel of diffusions. Then P_t and $\hat{P}_t$ can be extended to contractive semigroups on $L^p(X;m)$ for $p\geq 1$. The following are well-known:

(a) $p_{t+s}(x,y)=\displaystyle\int_X p_s(x,z)p_t(z,y)m(dz),\quad {}^\forall x,y\in X,\ {}^\forall t,s>0.$

(b) $p_t(x,dy)=p_t(x,y)m(dy),\quad {}^\forall x\in X,\ {}^\forall t>0.$

(c) $\displaystyle\int_X p_t(x,y)m(dy)\leq 1,\quad {}^\forall x\in X,\ {}^\forall t>0.$

The same properties also hold for $\hat{p}_t(x,y)$.

Definition 2.1 (Kato class S_K^0, Dynkin class S_D^0)**.** For a positive Borel measure μ on X, μ is said to be in *Kato class relative to the semigroup kernel* $p_t(x,y)$ (write $\mu\in S_K^0$) if

$$\lim_{t\to 0}\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy)=0 \tag{2.1}$$

and μ is said to be in *Dynkin class relative to the semigroup kernel* $p_t(x,y)$ (write $\mu\in S_D^0$) if

$$\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy)<\infty\quad\text{for } {}^\exists t>0. \tag{2.2}$$

Clearly, $S_K^0\subset S_D^0$. The notions $\hat{S}_K^0$ and $\hat{S}_D^0$ are similarly defined by replacing $p_t(x,y)$ with $\hat{p}_t(x,y)$.

Definition 2.2 (Measures of finite energy integrals: S_0, S_{00}, cf. [6])**.** A Borel measure μ on X is said to be *of finite energy integral* with respect to $(\mathcal{E},\mathcal{F})$ (write $\mu\in S_0$) if there exists $C>0$ such that

$$\int_X |v|d\mu\leq C\sqrt{\mathcal{E}_1(v,v)},\quad {}^\forall v\in\mathcal{F}\cap C_0(X).$$

In that case, for every $\alpha > 0$, there exist $U_\alpha\mu, \hat{U}_\alpha\mu \in \mathcal{F}$ such that

$$\mathcal{E}_\alpha(U_\alpha\mu, v) = \mathcal{E}_\alpha(v, \hat{U}_\alpha\mu) = \int_X v(x)\mu(dx), \quad {}^\forall v \in \mathcal{F} \cap C_0(X).$$

Moreover we write $\mu \in S_{00}$ (resp. $\mu \in \hat{S}_{00}$) if $\mu(X) < \infty$ and $U_\alpha\mu \in \mathcal{F} \cap L^\infty(X;m)$ (resp. $\hat{U}_\alpha\mu \in \mathcal{F} \cap L^\infty(X;m)$) for some/all $\alpha > 0$.

Definition 2.3 (Smooth measures in the strict sense: S_1, cf. [6]). A Borel measure μ on X is said to be a *smooth measure in the strict sense* with respect to $(\mathcal{E}, \mathcal{F})$ (write $\mu \in S_1$) if there exists an increasing sequence $\{E_n\}$ of Borel sets such that $X = \bigcup_{n=1}^\infty E_n$, ${}^\forall n \in \mathbb{N}$, $I_{E_n}\mu \in S_{00}$ and $P_x(\lim_{n\to\infty} \sigma_{X\setminus E_n} \geq \zeta) = 1$, ${}^\forall x \in X$. Here ζ is the life time of $\mathbf{M}$. The family of *smooth measure in the strict sense* with respect to $(\hat{\mathcal{E}}, \mathcal{F})$ (write $\hat{S}_1$) can be similarly defined.

Definition 2.4. We define $S_K^1 := S_K^0 \cap S_1$, $S_D^1 := S_D^0 \cap S_1$, $\hat{S}_K^1 := \hat{S}_K^0 \cap \hat{S}_1$ and $\hat{S}_D^1 := \hat{S}_D^0 \cap \hat{S}_1$.

We fix $D > 0$ and assume the Nash type estimate: for each $t_0 > 0$ we have

$$\text{(2.3)} \quad {}^\exists C_{D,t_0} > 0 \text{ s.t. } \sup_{x,y\in X} p_t(x,y) \leq C_{D,t_0} t^{-D}, \quad {}^\forall t \in]0, t_0[.$$

Remark 2.1. The condition (2.3) implies the following:

(a) ${}^\exists C_{D,t_0} > 0$ s.t. $||P_t||_{1\to\infty} \leq C_{D,t_0} t^{-D}$ for any $t \in]0, t_0[$.

(b) For each $p \geq 1$, ${}^\exists C_{D,p,t_0} > 0$ s.t. $||P_t||_{p\to\infty} \leq C_{D,p,t_0} t^{-D/p}$ for any $t \in]0, t_0[$.

If $(\mathcal{E}, \mathcal{F})$ is a symmetric Dirichlet form, (2.3) is equivalent to one (hence all) of (a),(b). If further $D > 1$, (2.3) is also equivalent to the Sobolev inequality (see [5]): there exists $C_D^* > 0$ and $\gamma > 0$ such that

(c) $||u||_{\frac{2D}{D-1}} \leq C_D^* \mathcal{E}_\gamma(u,u)$ for all $u \in \mathcal{F}$.

Next theorem extends Theorem 1.2 and the lower estimate of p in this theorem is best possible as remarked after Theorem 1.2.

Theorem 2.1. *Suppose (2.3) and $p > D$ with $D \in [1, \infty[$ or $p \geq 1$ with $D \in]0, 1[$. Then $f \in L^p(X;m)$ implies $|f|dm \in S_K^1 \cap \hat{S}_K^1$.*

§3. Proof of Theorem 2.1

We set $r_\alpha(x,y) := \int_0^\infty e^{-\alpha t} p_t(x,y)dt$. First we show the following:

Lemma 3.1. *$\mu \in S_K^0$ is equivalent to*

$$\lim_{\alpha\to\infty}\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy)=0 \tag{3.1}$$

and $\mu \in S_D^0$ is equivalent to

$$\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy)<\infty, \quad {}^\exists\alpha>0. \tag{3.2}$$

Proof. We first show (2.1)⇒(3.1). Take $\alpha_0>0$ with $\alpha\ge\alpha_0$,

$$\begin{aligned}
&\int_X r_\alpha(x,y)\mu(dy)\\
&=\int_X\int_0^t e^{-\alpha s}p_s(x,y)ds\mu(dy)+\int_X\int_t^\infty e^{-\alpha s}p_s(x,y)ds\mu(dy)\\
&\le\int_X\int_0^t p_s(x,y)ds\mu(dy)+e^{-(\alpha-\alpha_0)t}\int_X\int_t^\infty e^{-\alpha_0 s}p_s(x,y)ds\mu(dy).
\end{aligned}$$

Here

$$\begin{aligned}
\int_X\int_t^\infty e^{-\alpha_0 s}p_s(x,y)ds\mu(dy)&=\int_X\sum_{k=1}^\infty\int_{kt}^{(k+1)t}e^{-\alpha_0 s}p_s(x,y)ds\mu(dy)\\
&=\sum_{k=1}^\infty\int_X\int_0^t e^{-\alpha_0(u+kt)}p_{u+kt}(x,y)du\mu(dy).
\end{aligned}$$

Since $p_{u+kt}(x,y)=\int_X p_{kt}(x,z)p_u(z,y)m(dz)$,

$$\begin{aligned}
&\int_X\int_t^\infty e^{-\alpha_0 s}p_s(x,y)ds\mu(dy)\\
&\quad=\sum_{k=1}^\infty e^{-\alpha_0 kt}\int_X p_{kt}(x,z)\int_X\int_0^t e^{-\alpha_0 u}p_u(z,y)du\mu(dy)m(dz)\\
&\quad\le\sum_{k=1}^\infty e^{-\alpha_0 kt}\int_X p_{kt}(x,z)\int_X\int_0^t p_u(z,y)du\mu(dy)m(dz).
\end{aligned}$$

From (2.1), $N_t:=\sup_{z\in X}\int_X\int_0^t p_u(z,y)du\mu(dy)<\infty$. Then

$$\begin{aligned}
&\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy)\\
&\qquad\le\sup_{x\in X}\int_X\int_0^t p_s(x,y)ds\mu(dy)+\frac{e^{-\alpha t}}{1-e^{-\alpha_0 t}}N_t.
\end{aligned} \tag{3.3}$$

Therefore

$$\lim_{\alpha\to\infty}\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy)\le\sup_{x\in X}\int_X\int_0^t p_s(x,y)ds\mu(dy)\xrightarrow{t\to 0}0.$$

Next we show (3.1)⇒(2.1). We have

$$\sup_{x\in X}\int_X\int_0^t p_s(x,y)ds\mu(dy)\le e^{\alpha t}\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy). \tag{3.4}$$

Therefore

$$\lim_{t\to 0}\sup_{x\in X}\int_X\int_0^t p_s(x,y)ds\mu(dy)\le\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy)\xrightarrow{\alpha\to\infty}0.$$

The implications (3.2)⟺(2.2) are clear from (3.3) and (3.4). □

Lemma 3.2. *The following are equivalent to each other.*

(a) $\mu\in S_D^0$.

(b) $\displaystyle\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy)<\infty$ *for* $^\forall t>0$.

(c) $\displaystyle\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy)<\infty$ *for* $^\forall\alpha>0$.

Proof. We first show (a)⟹(b). Suppose that (a) holds for some $t_0>0$. For any $t>0$, we take $n\in\mathbb{N}$ with $t\le nt_0$. We have

$$\begin{aligned}&\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy)\\&\qquad\le\sup_{x\in X}\sum_{k=1}^n\int_X p_{kt_0}(x,z)\Big(\int_0^{t_0}\int_X p_s(z,y)\mu(dy)ds\Big)m(dz)\\&\qquad\le n\sup_{x\in X}\int_0^{t_0}\Big(\int_X p_s(x,y)\mu(dy)\Big)ds<\infty.\end{aligned}$$

(b)⟹(c) is clear from (3.3) and (c)⟹(a) is clear. □

Proposition 3.1. *Suppose that* $\mu\in S_D^0$ *is a positive Radon measure on* X. *Then* $\mu\in S_1$.

Proof. It suffices to show that for a positive Radon measure $\mu\in S_D^0$, $I_K\mu\in S_0$ for any compact set K. Indeed, there exists an increasing sequence $\{G_n\}$ of relatively compact open set with $\bigcup_{n=1}^\infty G_n=X$. Then we see $I_{G_n}\mu\in S_{00}$ for each $n\in\mathbb{N}$, which implies $\mu\in S_1$ by Thoerem 5.1.7(iii) in [6]. Though the framework of Thoerem 5.1.7(iii) in [6] is symmetric, its proof only depends on the quasi-left-continuity of $\mathbf{M}$ and remains valid in the present context. We show $I_K\mu \in S_0$ for a compact set K. Fix $\alpha > 0$ and set $R_\alpha\mu(x) := \int_X r_\alpha(x,y)\mu(dy)$. First we show $R_\alpha(I_K\mu) \in L^2(X;m)$.

$$\begin{aligned}\|R_\alpha(I_K\mu)\|_2^2 &\le \|R_\alpha(I_K\mu)\|_\infty \|R_\alpha(I_K\mu)\|_1 \\ &= \|R_\alpha(I_K\mu)\|_\infty \langle I_K\mu, \hat{R}_\alpha 1\rangle \\ &= \frac{1}{\alpha}\|R_\alpha(I_K\mu)\|_\infty \mu(K) < \infty.\end{aligned}$$

Next we prove $R_\alpha(I_K\mu) \in \mathcal{F}$. It suffices to show

$$\sup_{\beta>0} \mathcal{E}_\alpha^{(\beta)}(R_\alpha(I_K\mu), R_\alpha(I_K\mu)) < \infty,$$

where $\mathcal{E}_\alpha^{(\beta)}(u,v) := \beta(u - \beta R_{\beta+\alpha}u, v)_m$ for $u, v \in L^2(X;m)$. Then

$$\begin{aligned}\sup_{\beta>0} \mathcal{E}_\alpha^{(\beta)}(R_\alpha(I_K\mu), R_\alpha(I_K\mu)) &= \sup_{\beta>0} \beta(R_{\beta+\alpha}(I_K\mu), R_\alpha(I_K\mu))_m \\ &= \|R_\alpha(I_K\mu)\|_\infty \sup_{\beta>0} \beta\langle I_K\mu, \hat{R}_{\beta+\alpha}1\rangle \\ &\le \|R_\alpha(I_K\mu)\|_\infty \mu(K) < \infty.\end{aligned}$$

Finally we prove $I_K\mu \in S_0$ and $R_\alpha(I_K\mu) = U_\alpha(I_K\mu)$. It suffices to show that for any $v \in \mathcal{F} \cap C_0(X)$

$$\begin{aligned}\mathcal{E}_\alpha(R_\alpha(I_K\mu), v) &= \lim_{\beta\to\infty} \mathcal{E}_\alpha^{(\beta)}(R_\alpha(I_K\mu), v) \\ &= \lim_{\beta\to\infty} \beta(R_{\beta+\alpha}(I_K\mu), v)_m \\ &= \lim_{\beta\to\infty} \beta\langle I_K\mu, \hat{R}_{\beta+\alpha}v\rangle = \langle I_K\mu, v\rangle,\end{aligned}$$

where we use the right continuity of the sample paths of $\widehat{\mathbf{M}}$. □

Proof of Theorem 2.1. By duality, it suffices only to prove that $f \in L^p(X;m)$ implies $|f|dm \in S_K^1$. Take $p > D$ with $D \in [1,\infty[$ or $p \ge 1$ with $D \in]0,1[$. Since $\|P_t\|_{p\to\infty} \le C_{D,p,t_0} t^{-D/p}$ for $t \in]0, t_0[$, we have

$$\begin{aligned}&\sup_{x\in X} \int_X \Big(\int_0^t p_s(x,y)ds\Big)|f(y)|m(dy) \\ &\qquad = \sup_{x\in X} \int_0^t \Big(\int_X |f(y)|p_s(x,y)m(dy)\Big)ds \\ &\qquad \le C_{D,p,t_0}\|f\|_p \int_0^t s^{-D/p}ds \\ &\qquad = C_{D,p,t_0}\|f\|_p \frac{p}{p-D} t^{1-D/p} \xrightarrow{t\to 0} 0.\end{aligned}$$

Then $|f|dm \in S_K^0$. Since $|f|dm$ with $f \in L^p(X;m)$ is a Radon measure, we conclude $|f|dm \in S_1$ by Proposition 3.1. Therefore $|f|dm \in S_K^1$. □

§4. Examples

Example 4.1 (Symmetric α-stable process). Take $\alpha \in]0,2[$. Let $\mathbf{M}^\alpha = (\Omega, X_t, P_x)_{x\in\mathbb{R}^d}$ be the symmetric α-stable process on $\mathbb{R}^d$, that is, Lévy process satisfying $E_0[e^{\sqrt{-1}\langle\xi,X_t\rangle}] = e^{-t|\xi|^\alpha}$. It is well-known that $\mathbf{M}^\alpha$ admits a semigroup kernel $p_t(x,y)$ satisfying the following (cf. [2],[7]): $^\exists C_i = C_i(\alpha,d) > 0$, $i = 1,2$ such that for all $(t,x,y) \in]0,\infty[\times\mathbb{R}^d \times \mathbb{R}^d$

$$\frac{C_1}{t^{d/\alpha}} \frac{1}{\left(1+\frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}} \le p_t(x,y) \le \frac{C_2}{t^{d/\alpha}} \frac{1}{\left(1+\frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}}.$$

Similar estimate holds for jump type process over d-sets (see [4]). In particular, there exists $C_2 = C_2(\alpha,d) > 0$ with $p_t(x,y) \le C_2 t^{-d/\alpha}$ for $(t,x,y) \in]0,\infty[\times\mathbb{R}^d \times \mathbb{R}^d$. Then we have that $f \in L^p(\mathbb{R}^d)$ implies $|f(x)|dx \in S_K^1$ if $p > d/\alpha$ with $d \ge \alpha$, or $p \ge 1$ with $d < \alpha$.

Example 4.2 (Relativistic Hamiltonian process). Let $\mathbf{M}^{\mathrm{H}}$ be the relativistic Hamiltonian process on $\mathbb{R}^d$ with mass $m > 0$, that is, $\mathbf{M}^{\mathrm{H}} = (\Omega, X_t, P_x)_{x\in\mathbb{R}^d}$ is a Lévy process satisfying

$$E_0\left[e^{\sqrt{-1}\langle\xi,X_t\rangle}\right] = e^{-t(\sqrt{|\xi|^2+m^2}-m)}.$$

It is shown in [8], the semigroup kernel $p_t(x,y)$ of $\mathbf{M}^{\mathrm{H}}$ is given by

$$p_t(x,y) = (2\pi)^{-d} \frac{t}{\sqrt{|x-y|^2+t^2}} \int_{\mathbb{R}^d} e^{mt} e^{-\sqrt{(|x-y|^2+t^2)(|z|^2+m^2)}} dz.$$

Hence we have that for each $t_0 > 0$, there exist $C_i = C_i(d) > 0$, $i = 1,2$ independent of t_0 such that for any $t \in]0,t_0[$, $x,y \in \mathbb{R}^d$

$$\frac{C_1}{t^d} \frac{e^{-m|x-y|}}{\left(1+\frac{|x-y|^2}{t^2}\right)^{(d+1)/2}} \le p_t(x,y) \le \frac{C_2}{t^d} \frac{e^{mt_0}}{\left(1+\frac{|x-y|^2}{t^2}\right)^{(d+1)/2}}.$$

In particular, $\sup_{x,y\in\mathbb{R}^d} p_t(x,y) \le C_2 e^{mt_0}/t^d$ for $t \in]0,t_0[$. Then we have that $f \in L^p(\mathbb{R}^d)$ implies $|f(x)|dx \in S_K^1$ for $p > d$.

Example 4.3 (Brownian motion penetrating fracrals, cf. [9]). The diffusion process on $\mathbb{R}^d$ constructed in [9] admits the heat kernel $p_t(x,y)$ which has the following upper estimate: there exists $C > 0$ such that

$\sup_{x,y\in\mathbb{R}^d} p_t(x,y) \le Ct^{-d/2}$ if $t \in]0,1]$. Hence $f \in L^p(\mathbb{R}^d)$ implies $|f(x)|dx \in S^1_K$ for $p > d/2$ with $d \ge 2$ or $p \ge 1$ with $d = 1$.

Example 4.4 (Diffusions with bounded drift). Let a be the symmetric matrix valued measurable function such that $\lambda|\xi|^2 \le \langle a(x)\xi,\xi\rangle \le \Lambda|\xi|^2$, ${}^\forall x,\xi \in \mathbb{R}^d$ for $0 < {}^\exists\lambda \le {}^\exists\Lambda$. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded measurable function and assume $\operatorname{div} b \ge 0$ in the distributional sense. Consider $(\mathcal{E}^{a,b}, C_0^\infty(\mathbb{R}^d))$ defined by

$$\mathcal{E}^{a,b}(u,v) := \frac{1}{2}\int_{\mathbb{R}^d}\langle a(x)\nabla u(x),\nabla v(x)\rangle dx - \int_{\mathbb{R}^d}\langle b(x),\nabla u(x)\rangle v(x)dx$$

for $u,v \in C_0^\infty(\mathbb{R}^d)$. Then we see $\mathcal{E}^{a,b}(u,u) \ge 0$ for $u \in C_0^\infty(\mathbb{R}^d)$ and $(\mathcal{E}^{a,b}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d)$ (see Chapter II 2(d) in [11]). We denote by $(\mathcal{E}^{a,b}, H^1(\mathbb{R}^d))$ its closure on $L^2(\mathbb{R}^d)$. $(\mathcal{E}^{a,b}, H^1(\mathbb{R}^d))$ is a non-symmetric Dirichlet form on $L^2(\mathbb{R}^d)$. Let $\{T_t^{a,b}\}_{t>0}$ be the $L^2(\mathbb{R}^d)$-semigroups associated with $(\mathcal{E}^{a,b}, H^1(\mathbb{R}^d))$. Then, by §II. 2 in [12], $T_t^{a,b}$ admits a heat kernel $p_t^{a,b}(x,y)$ on $]0,\infty[\times\mathbb{R}^d\times\mathbb{R}^d$ such that $P_t^{a,b}f(x) := \int_{\mathbb{R}^d} p_t^{a,b}(x,y)f(x)dy$ is an m-version of $T_t^{a,b}f$ for $f \in L^2(\mathbb{R}^d)$ and $p_t^{a,b}(x,y)$ satisfies the Aronson's estimates: (see (II. 2.4) in [12]) there exists an $M := M(\lambda,\Lambda,d) \in [1,\infty)$ such that for all $x,y \in \mathbb{R}^d, t \in]0,1[$

$$\text{(4.1)} \quad \frac{1}{Mt^{d/2}}e^{-M(t+|x-y|^2/t)} \le p_t^{a,b}(x,y) \le \frac{M}{t^{d/2}}e^{Mt-|x-y|^2/Mt}.$$

In particular, $\sup_{x,y\in\mathbb{R}^d} p_t^{a,b}(x,y) \le Me^M/t^{d/2}$ for all $t \in]0,1[$, hence $f \in L^p(\mathbb{R}^d)$ implies $|f(x)|dx \in S^1_K \cap \hat{S}^1_K$ for $p > d/2$ with $d \ge 2$, or $p \ge 1$ with $d = 1$.

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Kazuhiro Kuwae
Department of Mathematics
Faculty of Education
Kumamoto University
Kumamoto 860–8500
Japan
E-mail address: kuwae@gpo.kumamoto-u.ac.jp

Masayuki Takahashi
The Japan Research Institute, Limited
Toei Mishuku Bldg. 1-13-1
Mishuku, Setagaya-ku
Tokyo 154–0005
Japan
E-mail address: masayuki_0116@hotmail.com