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Quasisymmetric extension, smoothing and applications

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Abstract.

We discuss quasisymmetric extension of embeddings that are close to similarities, due to Tukia and Väisälä, and smooth quasiconformal approximation of such extensions. The smoothing is done by convolution with a variable kernel in conjunction with the Tukia-Väisälä extension procedure. We can apply these to the study of branch sets of smooth quasiregular maps, and quasiconformal dimension of self-similar fractals.

§1. Branch sets of smooth quasiregular maps

A continuous mapping $f : D \to \mathbf{R}^n$ in the Sobolev space $W^{1,n}_{\mathrm{loc}}(D, \mathbf{R}^n)$ is *K-quasiregular*, $K \geq 1$, if

$$|f'(x)|^n \leq K J_f(x), \qquad \text{a.e. } x \in D.$$

Here $n \geq 2$, $D \subset \mathbf{R}^n$ is a domain, $|f'(x)|$ is the operator norm of the differential of f, and $J_f(x) = \det f'(x)$ is the Jacobian determinant. In the plane, 1-quasiregular maps are precisely analytic functions of a single complex variable. Quasiregular mappings were introduced by Yu. G. Reshetnyak [25] under the name "mappings of bounded distortion". A deep theorem of Reshetnyak states that nonconstant quasiregular maps are discrete and open. See [26] for historical accounts.

The branch set B_f of a continuous, discrete, and open mapping $f : D \to \mathbf{R}^n$ is the closed set of points in D where f does not define

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a local homeomorphism. Černavskiĭ [9], [10] proved that the *topological dimensions* of the branch set and its image satisfy

$$\dim B_f = \dim f(B_f) \le n - 2.$$

The possible values of the topological dimension of branch sets of quasiregular maps are unknown.

On the other hand, if B_f is not empty, then $\Lambda^{n-2}(f(B_f)) > 0$ by a theorem of Martio, Rickman and Väisälä [23], and $\Lambda^{n-2}(B_f) > 0$ when $n = 3$ by a result of Martio and Rickman [22]. Here Λ^r is the r-dimensional Hausdorff measure.

Branch sets of quasiregular mappings may exhibit complicated topological structure and may contain, for example, many wild Cantor sets of classical geometric topology. For recent developments and many interesting *open questions*, see [14], [15], [16], [27].

Quasiregular mappings of $\mathbf{R}^2$ can be smooth without being locally homeomorphic, for example, $f(z) = z^2$. When $n \ge 3$, sufficiently smooth nonconstant quasiregular mappings are locally homeomorphic.

Theorem 1.1. *Every nonconstant $C^{n/(n-2)}$-smooth quasiregular mapping must be locally homeomorphic when $n \ge 3$.*

Theorem 1.1 is due to Martio, Rickman and Väisälä [26, p. 12]; the exponent $n/(n-2)$ is derived from Morse-Sard Theorem and the theorem on $\Lambda^{n-2}(f(B_f))$ mentioned earlier. Church [11] has proved Theorem 1.1 for C^n mappings.

In [34], Väisälä asked whether C^1-smoothness implies local homeomorphism in Theorem 1.1. Work of Bonk and Heinonen [6] showed that the exponent $n/(n-2)$ in Theorem 1.1 is sharp when $n = 3$.

Theorem 1.2. *For every $\epsilon > 0$, there exists a $C^{3-\epsilon}$-smooth quasiregular mapping $F : \mathbf{R}^3 \to \mathbf{R}^3$ whose branch set B_F is homeomorphic to $\mathbf{R}^1$ and has Hausdorff dimension $3 - \delta(\epsilon)$ with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.*

It is proved in [19] that the exponent $n/(n-2)$ in Theorem 1.1 is sharp when $n = 4$; and the authors answered Väisälä's question in the negative for all dimensions.

Theorem 1.3. *For every $\epsilon > 0$, there exists a $C^{2-\epsilon}$-smooth quasiregular mapping $F : \mathbf{R}^4 \to \mathbf{R}^4$ whose branch set B_F is homeomorphic to $\mathbf{R}^2$ and has Hausdorff dimension $4 - 2\epsilon$. For any $n \ge 5$, there exists $\epsilon(n) > 0$ and a $C^{1+\epsilon(n)}$-smooth quasiregular map $F : \mathbf{R}^n \to \mathbf{R}^n$ whose branch set B_F is homeomorphic to $\mathbf{R}^{n-2}$.*

Bonk and Heinonen first constructed a quasiconformal mapping g in $\mathbf{R}^3$ with uniformly expanding behavior on a line L. Then g is approximated outside L by a C^∞-smooth quasiconformal mapping G by applying a theorem of Kiikka on smoothing [21]. The map G^{-1} has the correct order of smoothness on $\mathbf{R}^3$; postcomposing G^{-1} with a winding map produces the desired quasiregular map F. As explained in [6], it is unclear how to construct a quasiconformal mapping g in $\mathbf{R}^n$, $n \geq 4$, which is uniformly expanding on codimension two subspaces. Moreover, the smoothing procedure of Kiikka works in dimensions 2 and 3 only.

We discuss these issues and related topics on quasisymmetric extension, smooth approximation, existence of snowflake surfaces and quasiconformal deformation of self-similar fractals in the following sections.

§2. Quasiconformal extensions

One of the most important results on quasiconformal mappings is the Extension Theorem.

Theorem 2.1. *Every quasiconformal mapping $f : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ (quasisymmetric if $n = 2$) has a quasiconformal extension in $\mathbf{R}^n_+ = \mathbf{R}^{n-1} \times [0, \infty)$.*

This was proved by Beurling and Ahlfors [2] for $n = 2$, later by Ahlfors [1] for $n = 3$ and by Carleson [8] for $n \leq 4$. Finally, Tukia and Väisälä [30] proved the Extension Theorem for all $n \geq 2$.

Ahlfors showed that every planar quasiconformal map is a composition of mappings with dilatation arbitrarily close to 1, and that mappings in the plane with small dilatation can be extended to quasiconformal homeomorphisms of $\mathbf{R}^3$. *Whether any quasiconformal map in dimension 3 or higher can be decomposed into mappings of dilatation arbitrarily close to 1 remains unanswered.* Carleson constructed a piecewise linear approximation g of f, extended g to $\mathbf{R}^n_+$, and performed a limiting process. The approximation of Moise used is valid for dimensions 2 and 3 only. Tukia and Väisälä extended the given map on $\mathbf{R}^{n-1}$ to a homeomorphism in $\mathbf{R}^n_+$, then applied an approximation procedure of D. Sullivan to obtain the quasiconformality.

Extending a quasisymmetric homeomorphism defined on a subset of $\mathbf{R}^n$ to a quasiconformal homeomorphism of $\mathbf{R}^n$ can be difficult and is not always possible. For example, a smooth homeomorphism from a circle onto a knotted curve in $\mathbf{R}^3$ can not be extended to a homeomorphism of $\mathbf{R}^3$ for a topological reason; certain smooth homeomorphisms between a Jordan curve with two inward spikes and a Jordan curve with one inward spike and one outward spike can not be extended to be quasiconformal on $\mathbf{R}^2$ for an analytical reason.

To study extension of quasisymmetric maps on subsets of $\mathbf{R}^n$, Tukia and Väisälä [31], [35], introduced the notion of s-quasisymmetric maps, a restricted class of quasisymmetric maps which are locally uniformly close to similarities, and the notion of quasisymmetric extension property .

An embedding $f : X \to Y$ of metric spaces is called *s-quasisymmetric (s-QS)*, if f is quasisymmetric and satisfies

$$|f(a) - f(x)| \leq (t + s)|f(b) - f(x)|$$

whenever $a, b, x \in X$ with $|a - x| \leq t|b - x|$ and $t \leq 1/s$ for some $s > 0$. When X is a connected compact subset of $\mathbf{R}^p$ and $Y = \mathbf{R}^n$ with $1 \leq p \leq n$, the above definition is equivalent to the existence of a small $\varkappa > 0$ so that for every bounded $S \subset X$, there is a similarity $h : \mathbf{R}^p \to \mathbf{R}^n$ so that

$$||h - f||_S \leq \varkappa L(h) \operatorname{diam} S, \tag{2.2}$$

where $L(h)$ is the similarity ratio.

A subset A of $\mathbf{R}^n$ has the *quasisymmetric extension property (QSEP)* in $\mathbf{R}^n$ if every s-QS $f : A \to \mathbf{R}^n$ has an s_1-QS extension $g : \mathbf{R}^n \to \mathbf{R}^n$ whenever $0 < s \leq s_0(n, A)$, where $s_1 = s_1(s, n, A) \to 0$ as $s \to 0$. See [35, p. 239]. Since the extended map is quasisymmetric, it is necessarily quasiconformal.

It is not easy to determine whether a given set possesses the extension property. Tukia and Väisälä proved the following.

Theorem 2.3. *Let A be a subset of $\mathbf{R}^n$, $n \geq 2$, belonging to one the following classes:*

(a) $\mathbf{R}^p$ *or* S^p, *with* $1 \leq p \leq n - 1$,

(b) *A a closed thick set in $\mathbf{R}^p$,$1 \leq p \leq n$ such that either A or $\mathbf{R}^p \setminus A$ is bounded,*

(c) *a compact $(n - 1)$-dimensional C^1-manifold with or without boundary,*

(d) *a finite union of simplices of dimensions n and $n - 1$.*

Then A has the quasisymmetric extension property in $\mathbf{R}^n$.

A set $A \subset \mathbf{R}^p$ is *thick in* $\mathbf{R}^p$ if there are constants $r_0 > 0$ and $\beta > 0$ so that if $0 < r \leq r_0$ and $y \in A$, then there is a simplex Δ in $\mathbf{R}^p$ with $\Delta^0 \subset A \cap B(y, r)$ and $\Lambda^p(\Delta) \geq \beta r^p$.

We outline the Tukia-Väisälä extension procedures in the case when A is a thick set satisfying (b) in Theorem 2.3 with $p = n$. Let f be an s-quasisymmetric map defined on A and $\mathcal{W}$ be a fixed Whitney triangulation of $\mathbf{R}^n \setminus A$. At each vertex P of a simplex in $\mathcal{W}$, choose h_P, a similarity that approximates the mapping f on the ball $B(P, C \operatorname{dist}(P, A))$ $\cap A$, for some fixed $C > 1$, uniformly in the sense of (2.2); then define $f(P)$ to be $h_P(P)$. After f has been defined at all vertices in $\mathcal{W}$, extend f by the unique affine extension in each simplex in $\mathcal{W}$. Since A is thick, information of f on A is abundant and is sufficient to show the consistency of the affine maps associated with neighboring simplices. Since f is locally uniformly close to similarities and A or $\mathbf{R}^n \setminus A$ is relatively compact, degree theory can then be applied to prove f is injective, surjective and sense preserving when s is small. Intricate estimates coupled with the thickness condition guarantee that the extension is indeed s_1–quasisymmetric.

The extension procedure and the estimates are sensitive to the nature of the sets; for each class of the sets in Theorem 2.3, the proof has to be somewhat altered. Examples of sets which do not have the extension property are give in [35].

It would be interesting to know to what extent the thickness condition can be weakened. And it was asked in [35], *whether the manifold in (c) and the simplices in (d) can have dimension $p \leq n-2$, and whether every compact polyhedron in $\mathbf{R}^n$ has QSEP.*

Tukia-Väisälä extension procedure is especially useful in extending quasisymmetric maps on fractals, when the mappings in question are more likely to be compositions of close-to-similarities. We shall apply Theorem 2.3 to study Theorem 1.3 in section 5, and quasiconformal dimension of Sierpinski gaskets in section 6.

§3. Smoothing

Quasiconformal mappings in $\mathbf{R}^2$ or $\mathbf{R}^3$ can be approximated by C^∞-diffeomorphisms. Kiikka [21] proved the following.

Theorem 3.1. *Let $g : \Omega \to \Omega'$ be a K-quasiconformal mapping between domains in $\mathbf{R}^n$, $n = 2$ or 3. Then for any positive continuous function ϵ on Ω, there exists a $\tilde{K}$-quasiconformal C^∞-diffeomorphism $\tilde{g}$ such that $|\tilde{g}(x) - g(x)| < \epsilon(x)$ for all $x \in \Omega$. The constant $\tilde{K}$ depends only on K.*

In the proof, Kiikka used difficult work of Moise and of Munkres on smooth approximation of piecewise differentiable homeomorphisms, when dimension is 2 or 3. *This kind of approximation for general quasiconformal maps can not exist for dimension higher than 5* [29], *and is a long standing open question in dimension 4* [13].

Let A be a set in a class described in Theorem 2.3, and g be a s-quasisymmetric map on A with a very small s. Tukia-Väisälä 's construction guarantees a quasiconformal extension, again called g, to $\mathbf{R}^n$.

Sometimes it is desirable to have a smooth extension outside A. To this end, we convolve g with a variable kernel. Let δ_A be a regularized

C^∞ distance function to A, see for example, [28, p. 170]. Fix a C^∞ function φ on $\mathbf{R}^n$ which is nonnegative, radial, supported in $B(0,1)$, and satisfies $\int_{\mathbf{R}^n} \varphi(x)\,dx = 1$, $\sup_{\mathbf{R}^n} \left|\frac{\partial \varphi}{\partial x_i}\right| \le C$, $\sup_{\mathbf{R}^n} \left|\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right| \le C$. Then the map

$$G(x) = \begin{cases} \dfrac{1}{\delta_A^n(x)} \displaystyle\int_{\mathbf{R}^n} g(y)\varphi\Big(\frac{x-y}{\delta_A(x)}\Big)\,dy, & x \in \mathbf{R}^n \setminus A, \\ g(x), \quad x \in A \end{cases}$$

is C^∞-smooth outside A.

Smoothing by convolution, in general, does not preserve injectivity or quasiconformality. To obtain injectivity, quasiconformality, and the correct order of smoothness, convolution must be applied in conjunction with the Tukia-Väisälä construction. Closeness to similarities uniformly at all points on A and in all scales assures the essential inequalities required more or less preserved after convolution, whence the quasiconformality. See [19] for details.

Sometimes it is further necessary to know that an extension or its inverse is smooth in the entire $\mathbf{R}^n$. While this is not always possible, we discuss one particular situation when this can be done. Let g be the restriction of the quasiconformal mapping in Theorem 4.1 to the hyperplane $\mathbf{R}^{n-1}$ on which the snowflake property holds, and A be its image. When ϵ is very small, g is s-QS for a very small s. We can re-extend g to a global quasiconformal map on $\mathbf{R}^n$ following Tukia-Väisälä method; then apply convolution to this newly extended map to obtain a map $G : \mathbf{R}^n \to \mathbf{R}^n$ that agrees with g on $\mathbf{R}^{n-1}$ and is C^∞ outside. The snowflake property of g on $\mathbf{R}^{n-1}$ ensures that A is a thick set and that the gradient of g^{-1} is Hölder continuous on A. The function G^{-1} can then be shown to be $C^{1+\delta}$ in the entire $\mathbf{R}^n$ for some $\delta > 0$. Again see [19] for details.

§4. Snowflake Embeddings

Existence of quasisymmetric embedding f of $\mathbf{R}^{n-1}$ in $\mathbf{R}^n$ that has the snowflake property:

$$C^{-1}|x-y|\phi(|x-y|) \le |g(x)-g(y)| \le C|x-y|\phi(|x-y|),$$

for some $\phi(t) \to \infty$ as $t \to 0$, was raised in [17]. Existence of snowflake embeddings that can be further extended to become quasiconformal on $\mathbf{R}^n$ has been proved by Bishop [5], and David and Toro [12]. A special case of a theorem on embedding Reifenberg flat metric spaces into Euclidean spaces due to David and Toro can be stated as follows.

Theorem 4.1. *For each $n \geq 2$ and $0 < \epsilon < \epsilon_0(n)$, there exists a K-quasiconformal map $g : \mathbf{R}^n \to \mathbf{R}^n$ with*

$$C^{-1}|x-y|^{1/(1+\epsilon)} \leq |g(x) - g(y)| \leq C|x-y|^{1/(1+\epsilon)}$$

for all $x, y \in \mathbf{R}^{n-1}$, $|x-y| \leq 1$ and some $C = C(n) > 1$. Furthermore, $K \to 1$ as $\epsilon \to 0$.

The exponent in Theorem 4.1 necessarily satisfies $1/(1+\epsilon) > (n-1)/n$. The method of David and Toro is incisive, however does not give estimates of the number $\epsilon_0(n)$. It is not clear whether the exponent can be made arbitrarily close to $(n-1)/n$; equivalently, whether there is a snowflake embedding of $\mathbf{R}^{n-1}$ to a surface in $\mathbf{R}^n$ having Hausdorff dimension arbitrarily close to n.

It is generally believed that in order to show the order of smoothness is sharp in Theorem 1.1, a snowflake embedding from $\mathbf{R}^{n-2}$ to a surface in $\mathbf{R}^n$ having Hausdorff dimension arbitrarily close to n must be found. In $\mathbf{R}^4$, product of two planar snowflake curves is the image of a snowflake embedding from $\mathbf{R}^2$. The method of taking products breaks down for $n \geq 5$.

Therefore, it is not only intrinsically interesting but also useful to know whether there is a nearly space filling snowflake embedding from $\mathbf{R}^p$ into $\mathbf{R}^n$ for every p, $1 \leq p < n$. Paradoxically, this might be more easily achieved by subspaces $\mathbf{R}^p$ of a smaller dimension. Method of Bonk and Heinonen in [6] gives an affirmative answer for the case $p = 1$.

§5. Theorem 1.3

When $n = 4$, let Γ be a standard infinite snowflake curve of Hausdorff dimension $2 - \epsilon$. The product set $\Gamma \times \Gamma$ is to be the branch set of a $C^{2-\epsilon}$-smooth quasiregular map F. Note that there is a canonical map g from $\mathbf{R}^2$ to $\Gamma \times \Gamma$. This map can be written as a composition $g = g_{m-1} \circ \cdots \circ g_0$ such that each g_j satisfies a snowflake property, has a product of snowflake curves as its image, and is s-quasisymmetric for a small s. Construction of Tukia and Väisälä for part (a) and (b) of Theorem 2.3 can be adapted to extend g_j to be quasiconformal on $\mathbf{R}^4$. Smoothing outside products of snowflake curves via convolution with a variable kernel produces new quasiconformal maps G_j. The inverses G_j^{-1} can be shown to be $C^{1+\epsilon_j}$ for some $\epsilon_j > 0$, in the entire $\mathbf{R}^4$, following the reasoning in Section 3. Postcompose the inverse of $G_{m-1} \circ G_{m-2} \circ \cdots \circ G_0$ with a winding map $\omega : \mathbf{R}^4 \to \mathbf{R}^4$, $\omega(x_1, x_2, r, \theta) = (x_1, x_2, r\cos 2\theta, r\sin 2\theta)$, yields the desired $C^{2-\epsilon}$–quasiregular map F, having $\Gamma \times \Gamma$ as its branch set.

The method of taking products does not work in $\mathbf{R}^n$, $n \geq 5$, unless there exists an appropriate embedding of the $(n-2)$-fold product $\Gamma \times \cdots \times \Gamma \to \mathbf{R}^n$. For $n \geq 5$, Theorem 4.1 of David–Toro [12] provides a snowflake-type embedding $g : \mathbf{R}^{n-2} \hookrightarrow \Sigma \subset \mathbf{R}^{n-1}$. Embed both $\mathbf{R}^{n-2}$ and the image Σ in $\mathbf{R}^n$, and extend g directly to a global quasiconformal map on $\mathbf{R}^n$ by applying part (a) of Theorem 2.3. Smooth the extension outside $\mathbf{R}^{n-2}$ by a convolution, then postcompose the inverse with a winding map. The codimension two snowflake-type surfaces $\Sigma \subset \mathbf{R}^n$ can then be realized as the branch set of a $C^{1+\epsilon(n)}$-smooth branched quasiregular map in $\mathbf{R}^n$, $n \geq 5$. This answers Väisälä's question in the negative for all dimensions.

See [19] for details.

§6. Quasiconformal dimension of some self-similar sets

Problems on raising or lowering Hausdorff dimension of sets in $\mathbf{R}^n$ through quasiconformal homeomorphism of $\mathbf{R}^n$ have been studied for some time. Bishop [3] showed that for sets of positive dimension there is never an obstruction to raising dimension by quasiconformal maps. In fact, for any compact set E in $\mathbf{R}^n$ with $\dim(E) > 0$ and any $0 < \gamma < n$ there is a quasisymmetric map $h\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\dim(h(E)) > \gamma$. On the other hand, examples of Bishop and Tyson [4] [32] showed that the corresponding statement for lowering dimension can fail.

Given a metric space (X, d), the notion of conformal dimension was introduced by Pansu [24]:

$$C \dim X \equiv \inf\{\dim Y : (Y, \tilde{d}) \text{ quasisymmetrically equivalent to} (X, d)\}.$$

A variety of problems on conformal dimension has been studied; some have applications to geometric group theory. See, for example, work of Bonk-Kleiner [7] and Keith-Laakso [20] . Less studied is the quasiconformal dimension of a set E in $\mathbf{R}^n$ defined as follows [33]:

$$QC \dim E \equiv \inf\{\dim f(E) : f \;\text{ quasiconformal homeomorphism of } \mathbf{R}^n\}.$$

Clearly,

$$\text{topological-}\dim E \leq C \dim E \leq QC \dim E \leq \text{Hausdorff-}\dim E.$$

Analysis on self-similar fractals has been actively pursued in recent years. Sierpinski gasket due to its simplicity, and Sierpinski carpet due to its appearance in the boundary of Gromov hyperbolic groups [18] are particularly intriguing. *One of the most challenging questions in this area is to determine the conformal dimension and the quasiconformal* *dimension of the Sierpinski carpet in* $\mathbf{R}^n$, *for any* $n \geq 2$. The analogous problem on the Sierpinski gasket SG^n in $\mathbf{R}^n$ is easier [33].

Theorem 6.1. *For each* $n \geq 2$, $QC \dim SG^n = 1$.

Recall that topological dimension of SG^n is 1 and Hausdorff dimension of SG^n is $\frac{\log(n+1)}{\log 2}$. Theorem 6.1 says that SG^n can be mapped by quasiconformal self-maps of $\mathbf{R}^n$ onto sets of Hausdorff dimension arbitrarily close to its topological dimension.

The conclusion of Theorem 6.1 remains true for the invariant sets of a large class of postcritically finite iterated function systems satisfying a so-called gasket type property [33].

We describe the role of Tukia-Väisälä extension in studying quasiconformal dimension of fractals. Depending on the nature of the invariant set S in $\mathbf{R}^n$, a quasisymmetric map f is selected to map S onto the invariant set of an isomorphic function system having a smaller Hausdorff dimension. Selection of f is largely based on intuition; this step gives an upper bound of the conformal dimension of S. To obtain an upper bound for the quasiconformal dimension, f needs to be extended to be quasiconformal on $\mathbf{R}^n$. Imagining extending a map from the Sierpinski gasket in $\mathbf{R}^3$ to $\mathbf{R}^3$ by hand, it can be quite a task; Tukia-Väisälä extension procedure makes this process manageable. Under some extra conditions, the canonical map between invariant sets of two isomorphic systems can be decomposed into s-quasisymmetric maps, for a very small s. To do this a flow of function systems has to be produced so that the corresponding invariant sets are isotopic. Sums of orthogonal maps are not orthogonal. Therefore the flow can not be expressed algebraically as linear combinations; it has to be built geometrically and combinatorially. The construction of the flow can be quite daunting even for the Sierpinski gasket in $\mathbf{R}^3$, or a polygasket in $\mathbf{R}^2$ [33]. Finally the Tukia-Väisälä procedure is applied to each of the maps in the decomposition, then the extensions are recomposed.

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