Advanced Studies in Pure Mathematics 41, 2004
Stochastic Analysis and Related Topics
pp. 257–278

Risk-sensitive Portfolio Optimization with Full and Partial Information

Hideo Nagai

Abstract.

We discuss an application of risk-sensitive control to portfolio optimization problems for a general factor model, which is considered a variation of Merton's intertemporal capital asset pricing model ([18]). In the model the instantaneous mean returns as well as volatilities of the security prices are affected by economic factors and the security prices. The economic factors are assumed to satisfy stocahstic differential equations whose coefficients depend on the security prices as well as themselves. In such general incomplete market models under Markovian setting we consider constructing optimal strategies for risk-sensitive portfolio optimization problems on a finite time horizon. We study the Bellman equations of parabolic type corresponding to the optimization problems. Through analysis of the Bellman equations we construct optimal strategies from the solution of the equation. We further discuss the problem with partial information. We shall obtain a necessary condition for optimality using backward stochastic partial differential equations.

§1. Introduction

Let us consider a market model with $m+1$ securities $(S_t^0, S_t) := (S_t^0, S_t^1, ..., S_t^m)^*$ and n factors $X_t = (X_t^1, X_t^2, ..., X_t^n)^*$. Here S^* stands for transposed matrix of S. We assume that the set of securities includes one bond, whose price is defined by the ordinary differential equation:

$$dS^0(t) = r(X_t, S_t)S^0(t)dt, \quad S^0(0) = s^0, \tag{1.1}$$

where $r(x,s)$ is a nonnegative function on R^{n+m}. The other secutity prices S_t^i, $i = 1, 2, ..., m$, and the factors X_t are assumed to satsfy the following stochastic differential equations:

Received February 28, 2003.

$$
(1.2) \quad \begin{aligned} dS^i(t) &= S^i(t)\{g^i(X_t, S_t)dt + \textstyle\sum_{k=1}^{n+m} \sigma_k^i(X_t, S_t)dW_t^k\}, \\ S^i(0) &= s^i, \ \ i = 1, ..., m \end{aligned}
$$

and

$$
(1.3) \quad \begin{aligned} dX_t &= b(X_t, S_t)dt + \lambda(X_t, S_t)dW_t, \\ X(0) &= x \in R^n, \end{aligned}
$$

where $W_t = (W_t^k)_{k=1,..,(n+m)}$ is an $m+n$ dimensional standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. Here σ and λ are respectively $m \times (m+n)$, $n \times (m+n)$ matrix valued functions. Set

$$
\mathcal{G}_t = \sigma(S(u), X(u); u \le t)
$$

and let us denote investment strategy to i-th security $S^i(t)$ by $h^i(t)$, $(i = 0, 1, ..., m)$ representing portfolio proportion of the amount of the i-th security to the total wealth V_t that the investor possesses, which is defined as follows:

Definition 1.1. $(h^0(t), h(t)) \equiv (h^0(t), (h^1(t), h^2(t), ...,, h^m(t))^*)$ *is said to be an invetment strategy if the following conditions are satisfied*

i) *$h(t)$ is an R^m valued $\mathcal{G}_t$ progressively measurable stochastic process such that*

$$
\sum_{i=1}^m h^i(t) + h^0(t) = 1
$$

ii) *and that*

$$
P(\int_0^T |h(s)|^2 ds < \infty) = 1.
$$

The set of all investment strategies will be denoted by $\mathcal{H}(T)$. When $(h^0(t), h(t)^*)_{0 \le t \le T} \in \mathcal{H}(T)$ we will often write $h \in \mathcal{H}(T)$ for simplicity. In what follows we always assume that

$$
(1.4) \quad \sigma\sigma^* > 0.
$$

For given $h \in \mathcal{H}(T)$ the wealth process $V_t = V_t(h)$ satisfies

$$
\begin{aligned} \frac{dV_t}{V_t} &= \textstyle\sum_{i=0}^m h^i(t) \frac{dS^i(t)}{S^i(t)} \\ &= h^0(t) r(X_t, S_t)dt + \textstyle\sum_{i=1}^m h^i(t)\{g^i(X_t, S_t)dt \\ &\qquad + \textstyle\sum_{k=1}^{m+n} \sigma_k^i(X_t, S_t)dW_t^k\} \\ V_0 &= v \end{aligned}
$$

under the assumption of the self-financing condition. Then, taking i) above into account it turns out to be the solution of

$$\frac{dV_t}{V_t} = r(X_t, S_t)dt + h(t)^*(g(X_t, S_t) - r(X_t, S_t)\mathbf{1})dt + h(t)^*\sigma(X_t, S_t)dW_t,$$

$$V_0 = v,$$

where $\mathbf{1} = (1, 1, ..., 1)^*$.

We first consider the following problem. For a given constant $\mu < 1$, $\mu \neq 0$ maximize the following risk-sensitized expected growth rate up to time horizon T:

$$J(v, x; h; T) = \frac{1}{\mu} \log E[e^{\mu \log V_T(h)}] = \frac{1}{\mu} \log E[V_T(h)^\mu], \tag{1.5}$$

where h ranges over the set $\mathcal{A}(T)$ of all admissible strategies defined later. The meaning of the maximization is well understood by looking at the asymptotics of the criterion as $\mu \to 0$:

$$\frac{1}{\mu} \log E[e^{\mu \log V_T(h)}] \sim E[\log V_T(h)] + \frac{\mu}{2} \mathrm{Var}[\log V_T(h)] + O(\mu^2).$$

Maximizing (1.5) is a risk-sensitive counterpart of the problem maximizing the expected growth rate of the investor's wealth. The case where $\mu < 0$ is called risk averse and $\mu > 0$ risk seeking. Concerning this problem we introduce the Bellman equation corresponding to the value function and we present the results constructing an optimal strategy from the solution to the equation through its analysis in section 2. Note that the problem maximizing the criterion $J(v, x; h; T)$ is equivalent to HARA utility maximization:

$$\sup_h E[\frac{1}{\mu} V_T(h)^\mu] = \sup_h E[\frac{1}{\mu} e^{\mu \log V_T(h)}], \quad \mu < 1.$$

The problems on infinite time horizon maximizing

$$\liminf_{T \to \infty} \frac{1}{\mu T} \log E[e^{\mu \log V_T(h)}] \tag{1.6}$$

have been considered by several authors e.g. in [8],[9],[10],[14], in the case of linear Gaussian factor models since the work by Fleming [7], or under more general setting in [5] with the assumption that randomness of security price processes and that of factor processes are independent. In [23] we have discussed the problem under rather general setting for so called Merton's ICAPM ([18]), and the results in section 2 of the present paper are its genaralization in the case of finite time horizon. Here by Merton's ICAPM, we mean the case that

$$r(x,s) = r_1(x), \;\; g(x,s) = g_1(x), \;\; b(x,s) = b_1(x),$$
$$\sigma(x,s) = \sigma_1(x), \;\; \lambda(x,s) = \lambda_1(x).$$

If r_1, g_1 and b_1 are linear functions and σ_1 and λ_1 are constant matrices the models are said to be linear Gaussian. In that case the solutions of the Bellman equations are expressed explicitly as the quadratic functions of x whose coefficients are determined as the solutions of the matrix Riccati differential equations and linear differential equations.

We then consider the maximization problem with partial infromation. In the above investment strategies are defined as $\mathcal{G}_t$ progressively measurable processes. However, it is not always realistic since economic factors X_t are to be considered implicit and so it might be better to select our strategies without using all past informations of securities S_t and factors X_t. Our strategies may be well selected by using only informations of security prices. Rishel [24] has considered the problem on a finite time horizon in such a way in a particular case, namely for a linear Gaussian model of one factor and one risky and one riskless assets under the assumtion that randomness of the factor process and that of the risky asset are independent. We have also considered the problem for general linear Gaussian factor models [21] on a finite time horizon and, by solving two kinds of Riccati differential equations, constructed an optimal strategy. The results are extended to the case of infinite time horizon in [22] by studying asymptotics of the solutions of inhomogeneous (time dependent) Riccati differential equations as time horizon goes to infinity. In the present paper we shall consider the maximization problem in section 3 under more general setting, namely the case where coefficients of security prices are nonlinearly depend on economic factors. In that case we don't have explicit expression of the optimal strategies but study necessity of optimality. We introduce backward stochastic partial differential equations (BSPDEs), which are considered to be adjoint equations of the problems, and find the necessary condition for optimality by using the solutions of the BSPDEs under suitable conditions. Such necessary condition is a kind of maximum principle and it has been studied by A. Bensoussan for stochastic control problems for partially observed diffusion processes (cf. [1],[2], [26]).

§2. Full information case

Let us set

$$Y_t^i = \log S_t^i, \quad i = 0, 1, 2, ..., m,$$

$Y_t = (Y_t^1, Y_t^2, ..., Y_t^m)^*$ and $\mathbf{e}^Y = (e^{Y^1}, ..., e^{Y^m})^*$. Then

$$dY_t^0 = r(X_t, \mathbf{e}^{Y_t})dt$$

and

$$dY_t = F(X_t, Y_t)dt + \Sigma(X_t, Y_t)dW_t, \tag{2.1}$$

where

$$\begin{aligned} F^i(x,y) &= g^i(x,\mathbf{e}^y) - \tfrac{1}{2}(\sigma\sigma^*)^{ii}(x,\mathbf{e}^y), \\ \Sigma_k^i(x,y) &= \sigma_k^i(x,\mathbf{e}^y). \end{aligned}$$

In the same way, set

$$B(x,y) = b(x,\mathbf{e}^y), \quad \Lambda(x,y) = \lambda(x,\mathbf{e}^y).$$

Then the factor prosess is described as

$$dX_t = B(X_t, Y_t)dt + \Lambda(X_t, Y_t)dW_t \tag{2.2}$$

So, by setting $Z_t = (X_t, Y_t)^*$ and

$$\beta(z) = (B(x,y), F(x,y))^*, \quad \alpha(z) = (\Lambda(x,y), \Sigma(x,y))^*,$$

we have

$$dZ_t = \beta(Z_t)dt + \alpha(Z_t)dW_t \tag{2.3}$$

Furthermore, by setting $\tilde{g}(z) = g(x, \mathbf{e}^y)$, $\tilde{r}(z) = r(x, \mathbf{e}^y)$ for simplicity we have

$$\frac{dV_t}{V_t} = \tilde{r}(Z_t)dt + h_t^*(\tilde{g}(Z_t) - \tilde{r}(Z_t)\mathbf{1})dt + h_t^*\Sigma(Z_t)dW_t$$

and so,

$$\begin{aligned} V_t^\mu = v^\mu \exp\{ -\mu \int_0^t \eta(Z_s, h_s)ds + \mu \int_0^t h_s^*\Sigma(Z_s)dW_s \\ -\frac{\mu^2}{2} \int_0^t h_s^*\Sigma\Sigma^*(Z_s)h_s ds\}, \end{aligned}$$

where

$$\eta(z,h) = \frac{1-\mu}{2} h^*\Sigma\Sigma^*(z)h - \tilde{r}(z) - h^*(\tilde{g}(z) - \tilde{r}(z)\mathbf{1})$$

If a given investment strategy h satisfies

$$(2.4)\qquad E[e^{\mu\int_0^T h_s^*\Sigma^*(Z_s)dW_s-\frac{\mu^2}{2}\int_0^T h_s^*\Sigma\Sigma^*(Z_s)h_s ds}]=1,$$

then we can introduce a probability measure P^h given by

$$P^h(A)=E[e^{\mu\int_0^T h_s^*\Sigma(Z_s)dW_s-\frac{\mu^2}{2}\int_0^T h_s^*\Sigma\Sigma^*(Z_s)h_s ds};A]$$

for $A\in\mathcal{F}_T$, $T>0$. By the probability measure P^h our criterion $J(v,x;h;T)$ can be written as follows:

$$(2.5)\qquad J(v,x;h,T)=\log v+\frac{1}{\mu}\log E^h[e^{-\mu\int_0^T\eta(Z_s,h_s)ds}].$$

On the other hand, under the probability measure

$$\begin{aligned} W_t^h &= W_t-\langle W_\cdot,\mu\textstyle\int_0^\cdot h^*(s)\Sigma(Z_s)dW_s\rangle_t \\ &= W_t-\mu\textstyle\int_0^t\Sigma^*(Z_s)h(s)ds \end{aligned}$$

is a standard Brownian motion process, and therefore the factor process X_t satisfies the following stochastic differential equation

$$(2.6)\qquad \begin{aligned} dX_s &= (B(X_s,Y_s)+\mu\Lambda\Sigma^*(X_s,Y_s)h_s)ds+\Lambda(X_s,Y_s)dW_s^h \\ dY_s &= (F(X_s,Y_s)+\mu\Sigma\Sigma^*(X_s,Y_s)h_s)ds+\Sigma(X_s,Y_s)dW_s^h. \end{aligned}$$

And so,

$$(2.7)\qquad dZ_t=\beta_\mu(Z_t,h_t)dt+\alpha(Z_t)dW_t^h,$$

where

$$\beta_\mu(z,h)=\beta(z)+\mu\alpha\Sigma^*(z)h.$$

We regard (2.7) as a stochastic differential equation controlled by h and the criterion function is written by P^h as follows:

$$(2.8)\qquad J(v,x;h;T-t)=\log v+\frac{1}{\mu}\log E^h[e^{-\mu\int_0^{T-t}\eta(Z_s,h(s))ds}]$$

and the value function

$$(2.9)\qquad u(t,z)=\sup_{h\in\mathcal{A}(T-t)}J(v,z;h;T-t),\ 0\le t\le T.$$

Here we denote by $\mathcal{A}(T)$ the set of all investment strategies satisfying (2.4). Then, according to Bellman's dynamic programming principle, it should satisfy the following Bellman equation

$$\begin{aligned} &\frac{\partial u}{\partial t} + \sup_{h\in R^m} L^h u = 0, \\ &u(T,z) = \log v, \end{aligned} \tag{2.10}$$

where L^h is defined by

$$L^h u(t,z) = \frac{1}{2}\mathrm{tr}(\alpha\alpha^*(z)D^2u) + \beta_\mu(z,h)Du + \frac{\mu}{2}(Du)^*\alpha\alpha^*(z)Du - \eta(z,h).$$

Note that $\sup_{h\in R^m} L^h u$ can be written as

$$\begin{aligned} \sup_{h\in R^m} L^h u(t,z) &= \frac{1}{2}\mathrm{tr}(\alpha\alpha^*(z)D^2u) + \beta(z)^*Du + \frac{\mu}{2}(Du)^*\alpha\alpha^*Du + \tilde{r} \\ &+ \sup_h\{\mu h^*\Sigma\alpha^*Du + h^*(\tilde{g} - \tilde{r}\mathbf{1}) - \tfrac{1-\mu}{2}h^*\Sigma\Sigma^*h\} \\ &= \tfrac{1}{2}\mathrm{tr}(\alpha\alpha^*(z)D^2u) + \beta(z)^*Du + \tfrac{\mu}{1-\mu}(\tilde{g} - \tilde{r}\mathbf{1})^*(\Sigma\Sigma^*)^{-1}\Sigma\alpha^*Du \\ &+ \tfrac{\mu}{2}(Du)^*\alpha(I + \tfrac{\mu}{1-\mu}\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\alpha^*Du \\ &+ \tfrac{1}{2(1-\mu)}(\tilde{g} - \tilde{r}\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(\tilde{g} - \tilde{r}\mathbf{1}) + \tilde{r} \end{aligned}$$

Therefore our Bellman equation (2.10) is written as follows:

$$\begin{aligned} &\tfrac{\partial u}{\partial t} + \tfrac{1}{2}\mathrm{tr}(\alpha\alpha^*D^2u) + \hat{\beta}_\mu^*Du + (Du)^*\alpha N^{-1}\alpha^*Du + U(z) = 0, \\ &u(T,z) = \log v, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} \hat{\beta}_\mu(z) &= \beta(z) + \tfrac{\mu}{1-\mu}\alpha\Sigma^*(\Sigma\Sigma^*)^{-1}(\tilde{g} - \tilde{r}\mathbf{1}) \\ N^{-1}(z) &= \tfrac{\mu}{2}(I + \tfrac{\mu}{1-\mu}\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma(z)) \\ U(z) &= \tfrac{1}{2(1-\mu)}(\tilde{g} - \tilde{r}\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(\tilde{g} - \tilde{r}\mathbf{1}) + \tilde{r}(z). \end{aligned} \tag{2.12}$$

As for (2.11) we note that if $\mu < 0$, then

$$\frac{\mu}{2(1-\mu)} I \leq N^{-1} \leq \frac{\mu}{2} I$$

and therefore we have

$$\frac{\mu}{2(1-\mu)} \alpha\alpha^* \leq \alpha N^{-1} \alpha^* \leq \frac{\mu}{2} \alpha\alpha^* < 0.$$

On the other hand if $0 < \mu < 1$, then

$$\frac{\mu}{2} I \leq N^{-1} \leq \frac{\mu}{2(1-\mu)} I$$

and therefore we have

$$0 < \frac{\mu}{2} \alpha\alpha^* \leq \alpha N^{-1} \alpha^* \leq \frac{\mu}{2(1-\mu)} \alpha\alpha^*.$$

In what follows we assume that

$$\text{(2.13)} \qquad \begin{array}{l} B, F, \Lambda, \Sigma \text{ are locally Lipshitz and that} \\ \frac{1}{2} \parallel \Sigma\Sigma^* \parallel + \frac{1}{2} \parallel \Lambda\Lambda^* \parallel + \beta^* z \leq c(1 + |z|^2), \end{array}$$

then we have a solution (X_t, Y_t) of (2.1) and (2.2), and so setting

$$S_t^i = e^{Y_t^i}, \quad i = 1, 2, ..., m, \quad S_t^0 = e^{Y_t^0} = e^{\log s^0 + \int_0^t r(X_s, e^{Y_s}) ds}$$

we have a market model (S_t^0, S_t) satisfying (1.2) and (1.3). Then we have the following theorem.

Theorem 2.1. *Let $u \in C^{1,2}([0,T) \times R^N)$ be a solution of (2.11). Define*

$$\hat{h}_t = \hat{h}(t, Z_t)$$

$$\hat{h}(t, z) = \frac{1}{1-\mu} (\Sigma\Sigma^*)^{-1} (\tilde{g} - \tilde{r}\mathbf{1} + \mu\Sigma\alpha^* Du)(t, z),$$

where Z_t is the solution of (2.3), then, under the assumption that

(2.14)

$$E[e^{-\int_0^T (2N^{-1}\alpha^* Du + 2\mu K)^* dW_s - \frac{1}{2} \int_0^T (2N^{-1}\alpha^* Du + 2\mu K)^* (2N^{-1}\alpha^* Du + 2\mu K) ds}] = 1,$$

with

$$K = \frac{1}{2(1-\mu)}\Sigma^*(\Sigma\Sigma^*)^{-1}(\tilde{g} - \tilde{r}\mathbf{1})$$

$\hat{h}_t \in \mathcal{A}_T$ is an optimal strategy for the portfolio optimization problem of maximizing the criterion (1.5).

The proof of this theorem is similar to that of Proposition 2.1 in [23] and we omit it here.

We then consider equation (2.11). Such kinds of equations have been studied in [20], or [3] in relation to risk-sensitive control problems under more general settings in the case of $\mu < 0$ and in [4] in the case where $\mu > 0$. Here we consider the case where $\mu < 0$ and obtain the following result along the line [3], Theorem 5.1 with refinement on estimate (2.16). It is a generalization of Theorem 2.1 in [23].

Theorem 2.2. *i) If, in addition to (2.13), $\mu < 0$ and*

$$\nu_r|\xi|^2 \leq \xi^*\alpha\alpha^*(z)\xi \leq \nu_r'|\xi|^2, \quad r = |z|, \quad \nu_r, \ \nu_r' > 0, \tag{2.15}$$

then we have a solution of (2.11) such that

$$u, \ \tfrac{\partial u}{\partial t}, \ D_k u, \ D_{kj}u \in L^p(0,T; L^p_{loc}(R^{n+m})),$$

$$\tfrac{\partial^2 u}{\partial t^2}, \ \tfrac{\partial D_k u}{\partial t}, \ \tfrac{\partial D_{kj} u}{\partial t}, D_{kjl}u \in L^p(0,T; L^p_{loc}(R^{n+m})),$$

$$u \geq \log v, \ \tfrac{\partial u}{\partial t} \leq 0.$$

$$1 < \forall p < \infty$$

Furthermore we have the estimate

$$\begin{aligned} |\nabla u|^2(t,z) - \frac{c_0}{\nu_r}\frac{\partial u}{\partial t}(t,z) &\leq c_r(|\nabla Q|^2_{2r} + |Q|^2_{2r} \\ &+|\nabla(\alpha\alpha^*)|^2_{2r} + |\nabla\beta_\mu|_{2r} + |\beta_\mu|^2_{2r} \\ &+ |U|_{2r} + |\nabla U|^2_{2r} + 1), \qquad z \in B_r, \ t \in [0,T) \end{aligned} \tag{2.16}$$

where

$$Q = \alpha N^{-1}\alpha^*, \ c_0 = \tfrac{4(1+c)(1-\mu)}{-\mu}, \ c > 0$$

$$|\cdot|_{2r} = \|\cdot\|_{L^\infty(B_{2r})}$$

and c_r is a positive constant depending on n, r, ν_r, ν_r' and c.

ii) If, in addition to the above conditions,

$$\inf_{|z|\geq r} U(z),\ r^2 \frac{1}{\nu_r'} \inf_{|z|\geq r} U(z),\ r \inf_{|z|\geq r} \frac{U(z)}{|\beta_\mu(z)|} \to \infty$$

as $r \to \infty$, then the above solution u satisfies

$$\inf_{|z|\geq r, t\in(0,T)} u(z,t) \to \infty, \quad as\ \ r \to \infty.$$

Moreover, there exists at most one such solution in $L^\infty(0,T; W^{1,\infty}_{loc}(R^{n+m}))$

Remark. If

$$\frac{1}{\nu_r},\ \nu_r' \leq M(1+r^m), \quad \exists m > 0, \tag{2.17}$$

then we have

$$c_r \leq M'(1+r^{m'}),\ \ \exists m'$$

in estimete (2.16). In particular, if $m = 0$, then c_r can be taken independent of r.

Corollary 2.1. *Condition (2.14) is valid if*

$$\begin{aligned} &c_1|\xi|^2 \leq \xi^* \alpha\alpha^*(x)\xi \leq c_2|\xi|^2,\ \ c_1, c_2 > 0 \\ &B,\ F,\ \Lambda,\ \Sigma\ \ \textit{are globally Lipshitz.} \end{aligned} \tag{2.18}$$

The proofs of Theorem 2.2 and Corollary 2.1 are similar to those of Theorem 2.1 and Proposition 2.1 (ii) in [23] and we omit them here. Instead, we illustrate an example.

Example (Generalized linear Gaussian factor model)
Let us consider the case where B, $\tilde{g}$, and $\tilde{r}$ are all linear functions of z and Λ and Σ are constant matrices, namely

$$\beta(z) = \begin{pmatrix} B(x,y) \\ F(x,y) \end{pmatrix} = \begin{pmatrix} B_1 x + B_2 y + b \\ A_1 x + A_2 y + a - \frac{1}{2}\widehat{(\Sigma\Sigma^*)} \end{pmatrix}$$

$$\tilde{g}(x,y) = A_1 x + A_2 y + a, \quad \tilde{r}(x,y) = R_1^* x + R_2^* y + r,$$

where $\widehat{(\Sigma\Sigma^*)} = ((\Sigma\Sigma^*)^{ii}) \in R^m$. Then

$$\begin{aligned}\hat{\beta}_\mu(z) &= \beta(z) + \tfrac{\mu}{1-\mu}\begin{pmatrix} \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(\tilde{g} - r\mathbf{1}) \\ \tilde{g} - r\mathbf{1} \end{pmatrix} \\ &= K_1 z + L_1\end{aligned}$$

where

$K_1 =$

$$\begin{pmatrix} B_1 + \frac{\mu}{1-\mu}\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(A_1 - \mathbf{1}R_1^*) & B_2 + \frac{\mu}{1-\mu}\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(A_2 - \mathbf{1}R_2^*) \\ \frac{1}{1-\mu}A_1 - \frac{\mu}{1-\mu}\mathbf{1}R_1^* & \frac{1}{1-\mu}A_2 - \frac{\mu}{1-\mu}\mathbf{1}R_2^* \end{pmatrix}$$

$$L_1 = \begin{pmatrix} b + \frac{\mu}{1-\mu}\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}) \\ a - \frac{1}{2}\widehat{(\Sigma\Sigma^*)} + \frac{\mu}{1-\mu}(a - r\mathbf{1}) \end{pmatrix}.$$

Furthermore

$$\alpha N^{-1}\alpha^* = \frac{\mu}{2}\begin{pmatrix} \Lambda(I + \frac{\mu}{1-\mu}\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\Lambda^* & \frac{1}{1-\mu}\Lambda\Sigma^* \\ \frac{1}{1-\mu}\Sigma\Lambda^* & \frac{1}{1-\mu}\Sigma\Sigma^* \end{pmatrix} \equiv \frac{1}{2}K_0,$$

$$U(z) = \frac{1}{2}z^*K_2 z + L_2 z + r + \frac{1}{2(1-\mu)}(a - r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}),$$

where

$K_2 =$

$$\frac{1}{1-\mu}\begin{pmatrix} (A_1 - \mathbf{1}R_1^*)^*(\Sigma\Sigma^*)^{-1}(A_1 - \mathbf{1}R_1^*) & (A_1 - \mathbf{1}R_1^*)^*(\Sigma\Sigma^*)^{-1}(A_2 - \mathbf{1}R_2^*) \\ (A_2 - \mathbf{1}R_2^*)^*(\Sigma\Sigma^*)^{-1}(A_1 - \mathbf{1}R_1^*) & (A_2 - \mathbf{1}R_2^*)^*(\Sigma\Sigma^*)^{-1}(A_2 - \mathbf{1}R_2^*) \end{pmatrix}$$

and

$$L_2 = \frac{1}{1-\mu}\begin{pmatrix} (A_1 - \mathbf{1}R_1^*)^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}) \\ (A_2 - \mathbf{1}R_2^*)^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}) \end{pmatrix}.$$

In this case the solution to (2.11) has an explicit form such that

$$u(t,z) = \frac{1}{2}z^*P(t)z + q(t)^*z + k(t),$$

provided that equation (2.19) below has a solution. Here $P(t)$, $q(t)$ and $k(t)$ are the solutions to the following ordinary differential equations:

(2.19) $\dot{P}(t) + K_1^*P(t) + P(t)K_1 + P(t)K_0P(t) + K_2 = 0, \quad P(T) = 0,$

(2.20)

$$\dot{q}(t) + K_1 q(t) + P(t)L_1 + P(t)K_0 q(t) + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + L_2 = 0, \quad q(T) = 0$$

and
(2.21)

$$\dot{k}(t) + \frac{1}{2}\mathrm{tr}(\alpha\alpha^* P(t)) + L_1^* q(t) + r + \frac{1}{2(1-\mu)}(a-r\mathbf{1})^*(\Sigma\Sigma^*)^{-1}(a-r\mathbf{1})=0,$$

$$k(T) = \log v.$$

Note that if $\mu < 0$, then (2.19) has a unique solution and so do (2.20) and (2.21).

§3. Partial information case

Now we consider a partial information case. Namely, the case where portfolio strategies are selected by using only past information of security prices. In this case the economic factor process X_t is considered unobservable and so we cannot use the information about it to choose our strategies. Thus the factor process X_t defined before by (2.2) may be reformulated as the solution with the initial condition $X_0 = x_0$, where x_0 is a random variable having a disrtibution density $\pi(x)$ on R^n. We then introduce

$$\tilde{\mathcal{G}}_t = \sigma(S(u); u \leq t)$$

and the admissible strategies are assumed to be $\tilde{\mathcal{G}}_t$ measurable. In this case we consider more specific one than the above, namely we assume (2.18) and that $\sigma(x, S) = \sigma(S)$.

Then we consider the problem maximising the criterion (1.5) by selecting portfolio stratgies which are $\tilde{\mathcal{G}}_t$ measurable.

Let us set

$$I(v; h; T) = E[e^{\mu \log V_T(h)}], \tag{3.1}$$

and reformulate the problem as the one of partially observable stochastic control. Recall that Y_t is a solution of

$$dY_t = F(X_t, Y_t)dt + \Sigma(Y_t)dW_t, \tag{2.1$'$}$$

in the present case and we regard it as the SDE defining the observation process. On the other hand, X_t defined by (2.2) with the initial condition $X_0 = x_0$ is regarded as a system process. System noise $\Lambda(X_t, Y_t)dW_t$ and observation noise $\Sigma(Y_t)dW_t$ are correlated in general. $\sigma(Y_u,; u \leq t) = \sigma(S(u); u \leq t)$ holds since log is a strictly increasing function, so our problem is to minimize (or maximize) the criterion (3.1) while looking at the observation process Y_t and choosing a $\sigma(Y_u,; u \leq t) = \tilde{\mathcal{G}}_t$ measurable strategy $h(t)$. Though there is no control in SDE (2.2) defining the system process X_t the criterion $I(v; h; T)$ is defined as a functional of the strategy $h(t)$ measurable with respect to observation and the problem is the one of stochstic control with partial observation.

In what follows we consider the case where $F(x, y) = F(x)$, $\Sigma(y) = \Sigma \equiv$ constant, $B(x, y) = B(x)$, $\Lambda(x, y) = \Lambda(x)$, $\tilde{r}(x, y) = r(x)$ for simplicity. Similar arguments are possible for general case as long as Σ does not depend on x. Now let us introduce a new probability measure $\hat{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{F}_T} = \rho_T,$$

where

$$\rho_t = \exp\{-\textstyle\int_0^t F(X_s)^*(\Sigma\Sigma^*)^{-1}\Sigma dW_s \\ -\frac{1}{2}\int_0^t F(X_s)^*(\Sigma\Sigma^*)^{-1}F(X_s)ds\}. \tag{3.2}$$

We see that $\hat{P}$ is a probability measure since it can be seen by standard arguments (cf. [1]) that ρ_t is a martingale and $E[\rho_T] = 1$ under assumption (2.18). Moreover, according to Girsanov theorem,

$$\hat{W}_t = W_t + \int_0^t \Sigma^*(\Sigma\Sigma^*)^{-1}F(X_s)ds \tag{3.3}$$

turns out to be a standard Brownian motion process under the probability measure $\hat{P}$ and we have

$$dY_t = \Sigma d\hat{W}_t \tag{3.4}$$

$$dX_t = \{B(X_t) - \Lambda(X_t)\Sigma^*(\Sigma\Sigma^*)^{-1}F(X_t)\}dt + \Lambda(X_t)d\hat{W}_t. \tag{3.5}$$

We rewrite our criterion $I(v; h; T)$ by new probability measure $\hat{P}$.

$$I(v; h; T) = v^\mu \hat{E}[\hat{E}[\exp\{-\mu \int_0^T \eta(X_s, h_s)ds\}\Psi_T|\tilde{\mathcal{G}}_t]], \tag{3.6}$$

where

$$\Psi_t = \exp\{\int_0^t Q(X_s, h_s)^* dY_s - \frac{1}{2}\int_0^t Q(X_s, h_s)^*(\Sigma\Sigma^*)Q(X_s, h_s)ds\}$$

and

$$\begin{aligned} Q(x, h)^* &= (\Sigma\Sigma^*)^{-1}F(x) + \mu h \\ &= (\Sigma\Sigma^*)^{-1}\{F(x) + \mu(\Sigma\Sigma^*)h\}. \end{aligned}$$

Set

$$(3.7) \qquad q^h(t)(\varphi(t)) = \hat{E}[\exp\{-\mu \int_0^t \eta(X_s, h_s))ds\}\Psi_t\varphi(t, X_t)|\mathcal{G}_t].$$

Then (3.6) reads

$$(3.8) \qquad I(v; h; T) = v^\mu \hat{E}[q^h(T)(1)]$$

Hence, if $\mu < 0$ (resp. $1 > \mu > 0$) our problem is reduced to minimize (resp. maximize) I of (3.8) when taking h over $\mathcal{H}(T)$. Let us set

$$(3.9) \qquad L\varphi = \frac{1}{2}(\Lambda\Lambda^*)^{ij}(x)D_{ij}\varphi + B(x)^i D_i\varphi.$$

Here and in what follows we utilize summation convention. Then, we can see that $q^h(t)$ satisfies a so called modified Zakai equation in a similar way to deducing Zakai equations as for conditional expectations of diffusion processes with respect to unnormalized conditional probabilities (cf. [2], [13], [21]). We actually have the following proposition.

Proposition 3.1. *Assume (2.18), then $q(t)(\varphi(t)) \equiv q^h(t)(\varphi(t))$ satisfies the following stochastic partial differential equation (SPDE):*
(3.10)

$$q(t)(\varphi(t)) = q(0)(\varphi(0)) + \int_0^t q(s)(\tfrac{\partial\varphi}{\partial t}(s,\cdot) + L\varphi(s,\cdot) + \mu h_s^*\Sigma\Lambda^*(\cdot)D\varphi(s,\cdot)$$

$$- \mu\eta_s(\cdot)\varphi(s,\cdot))ds + \int_0^t q(s)(\varphi(s,\cdot)Q(\cdot, h_s))dY_s$$

$$+ \int_0^t q(s)((D\varphi)^*(s,\cdot)\Lambda(\cdot)\Sigma^*(\Sigma\Sigma^*)^{-1})dY_s,$$

where $\eta_s(\cdot) = \eta(\cdot, h_s))$.

Let us introduce some notations and describe a strong form of stochstic partial differential equation (3.10). Set

$$L^0\varphi = \frac{1}{2}D_i(\Lambda\Lambda^*(x)^{ij}D_j\varphi),$$

$$\tilde{B}(x)^i = B(x)^i - \frac{1}{2}D_j(\Lambda\Lambda^*)^{ji}.$$

Then $L\varphi = L^0\varphi + \tilde{B}(x)^*D\varphi$ and its formal adjoint L^* is written as

$$L^*\varphi = L^0\varphi - D_i(\tilde{B}(x)^i\phi).$$

We set

$$G(h)\varphi = -D_i(\tilde{B}^i(\cdot)\varphi) - \mu h_s^i D_j((\Sigma\Lambda^*)^{ij}\varphi) - \mu\eta(\cdot, h_s)\varphi$$

and

$$M(h)_j\varphi = Q_j(\cdot, h_s)\varphi - D_i([\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}]^i_j\varphi)$$

We define

$$\begin{aligned}\mathcal{L}^2_Y(0,T;H^1(R^n)) &= \{v \in L^2(\Omega,\mathcal{F},\hat{P};L^2(0,T;H^1(R^n)),\\ &\qquad v(t) \in L^2(\Omega,\tilde{\mathcal{G}}_t,\hat{P};H^1(R^n)) \text{ a.e. } t\}\end{aligned}$$

Then we consider the following stochastic partial differential equation which has a solution $q(t)$ such that $q_t e^{\delta\sqrt{1+|x|^2}} \in \mathcal{L}_Y([0,T];H^1)$.

$$dq_t = (L^0 q_t + G(h)q_t)dt + M(h)_j q_t dY^j_t. \tag{3.11}$$

Furthermore we assume

$$\Lambda,\ D\Lambda, B,\ DB,\ F, \quad \text{are bounded} \tag{3.12}$$

and the set of admissible strategies $\mathcal{A}_T$ is defined as the totality of $\tilde{\mathcal{G}}_t$ measurable strategy h satisfying the condition i) of definition 2.1 and $h_t \in \Gamma$, $\forall t$ for some convex compact $\Gamma \subset R^m$. Take a positive constant $\delta > 0$. Then we have the following theorem.

Proposition 3.2. *Let us assume (2.18), (3.12), and* $\pi e^{\delta\sqrt{1+|x|^2}} \in H^1$. *Then for each addmissible strategy h (3.11) has a unique solution* $q_t = q(t,x)$ *such that* $q_t e^{\delta\sqrt{1+|x|^2}} \in \mathcal{L}^2_Y(0,T;H^1(R^n)) \cap L^2(\Omega,\mathcal{F},\hat{P}; C(0,T;L^2(R^n))$ *and that* $q_0 = \pi$. *Furthermore we have* $\int q(T,x)\psi(x)dx = q(T)(\psi)$ *for all bounded Borel function* ψ.

For the proof of this proposition we prepare the following lemma.

Lemma 3.1. *Under assumption (2.18)*

$$\Lambda(I_{n+m} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\Lambda^* \geq c_1 I_n$$

Proof. Note that

$$(\xi_1^*, \xi_2^*)\begin{pmatrix} \Lambda\Lambda^* & \Lambda\Sigma^* \\ \Sigma\Lambda^* & \Sigma\Sigma^* \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \geq c_1|\xi|^2, \quad \forall\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{3.13}$$

under assumtion (2.18). Therefore, setting $\zeta = -(\Sigma\Sigma^*)^{-1}\Sigma\Lambda^*\xi_1$ for $\xi_1 \in R^n$, we see that

$$\begin{aligned}&\xi_1^*\Lambda(I_{n+m} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\Lambda^*\xi_1\\ &= \xi_1^*\Lambda\Lambda^*\xi_1 + \xi_1^*\Lambda\Sigma^*\zeta + \zeta\Sigma\Lambda^*\xi_1 + \zeta\Sigma\Sigma^*\zeta\\ &\geq c_1(|\xi_1|^2 + |\zeta|^2) \geq c_1|\xi_1|^2\end{aligned}$$

□

Proof of Proposition 3.2. Set $\tilde{q}_t = q_t e^{\delta\sqrt{1+|x|^2}}$, $\delta > 0$ and $\nu(x) = \delta\sqrt{1+|x|^2}$. Then (3.11) can be written as

$$d\tilde{q}_t = (L^0\tilde{q}_t + \tilde{G}(h)\tilde{q}_t)dt + \tilde{M}(h)_j\tilde{q}_t dY_t^j, \tag{3.14}$$

where

$$\tilde{G}(h)\tilde{q} = G(h)\tilde{q} + (\Lambda\Lambda)^{ij}D_j\nu D_i\tilde{q}$$
$$+\{\tfrac{1}{2}D_i((\Lambda\Lambda)^{ij}D_j\nu)+\tfrac{1}{2}(\Lambda\Lambda^*)^{ij}D_i\nu D_j\nu-(\tilde{B}^i D_i\nu+\mu h^i(\Sigma\Lambda^*)^{ij}D_j\nu)\}\tilde{q},$$

$$\tilde{M}(h)_j\tilde{q} = \tilde{Q}_j\tilde{q} - D_i([\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}]_j^i\tilde{q})$$

and

$$\tilde{Q}_j = Q_j - D_i\nu[\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}]_j^i.$$

It suffices to check the coercivity condition for (3.14) because of general theory of stochastic partial differential equarions ([2],[13], [25]):

$$-2\langle L^0 q, q\rangle - 2\langle\tilde{G}(h)q, q\rangle + c_1\|q\|_{L^2}^2 \geq c_0\|q\|_{H^1}^2 + \langle\tilde{M}(h)_j q, (\Sigma\Sigma^*)^{jk}\tilde{M}(h)_k q\rangle$$

Indeed, setting $\hat{Q}_j = \tilde{Q}_j - D_i([\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}]_j^i)$ we see that

$$\begin{aligned}
&-2\langle L^0 q, q\rangle - \langle\tilde{M}(h)_j q, (\Sigma\Sigma^*)^{jk}\tilde{M}(h)_k q\rangle \\
&= \int (Dq)^*\Lambda\Lambda^* Dq dx - \int (Dq)^*\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\Lambda^* Dq dx \\
&\quad - \int (\hat{Q}^*\Sigma\Sigma^*\hat{Q})q^2 dx - 2\int (Dq)^*\Lambda\Sigma^*\hat{Q}q dx \\
&\geq \int (Dq)^*\Lambda(I - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\Lambda^* Dq dx - \int \hat{Q}(\Sigma\Sigma^*)\hat{Q}q^2 dx \\
&\quad - \epsilon\int |Dq|^2 dx - \tfrac{1}{\epsilon}\int |\Lambda\Sigma^*\hat{Q}|^2 q^2 dx \\
&\geq c_2\int |Dq|^2 dx - \epsilon\int |Dq|^2 dx - c_3\int |q|^2 dx
\end{aligned}$$

for some c_2, $c_3 > 0$ and sufficiently small $\epsilon > 0$ by using the above lemma. Since

$$|\langle\tilde{G}q, q\rangle| \leq \epsilon_1\int |Dq|^2 dx + (\frac{c_4}{\epsilon_1} + c_5)\int |q|^2 dx$$

we can easily see the coercivity condition holds for (3.14).

□

Lemma 3.2. *Let us assume the assumptions of the above proposition and h, k be admissible strategies, then*

$$\lim_{\theta\to 0}\frac{I(v;h_\cdot+\theta k_\cdot;T)-I(v;h_\cdot;T)}{\theta}=v^\mu\hat{E}[<\zeta_T(k),1>], \tag{3.15}$$

where $\zeta_t=\zeta_t(k)$ *is a solution of the following stochastic partial differential equation*
(3.16)

$$d\zeta_t=(L^0\zeta_t+G(h)\zeta_t+k_t^iG_{h^i}(h)q_t)dt+(M(h)_j\zeta_t+k_t^iM_{h^i}(h)_jq_t)dY_t^j$$
$$\zeta_0=0,$$

where q_t *is a solution to (3.11),*

$$\begin{aligned} G_{h^i}(h)q &= -\mu D_j((\Sigma\Lambda^*)^{ij}q)-\mu\frac{\partial\eta}{\partial h^i}q \\ &= -\mu D_j((\Sigma\Lambda^*)^{ij}q)-\mu[(1-\mu)(\Sigma\Sigma^*h)^i-(g-r\mathbf{1})^i] \end{aligned}$$

and

$$M_{h^i}(h)_jq=\frac{\partial Q_j(\cdot,h)}{\partial h^i}q=\mu\delta_{ij}q.$$

Proof. Note that we can see that (3.14) has a unique solution such that $q_te^{\delta\sqrt{1+|x|^2}}\in\mathcal{L}_Y^2(0,T;H^1(R^n))\cap L^2(\Omega,\mathcal{F},\hat{P};C(0,T;L^2(R^n))$ in a similar way to the proof of the above proposition. Let us set

$$\bar{q}_\theta(t)=\frac{q_\theta(t)-q(t)}{\theta}-\zeta,$$

where $q_\theta(t)$ is the solution to :
(3.17)

$$dq_\theta(t)=\{L^0q_\theta(t)+G(h+\theta k)q_\theta(t)\}dt+M(h+\theta k)_jq_\theta(t)dY_t^j$$
$$q_\theta(0)=\pi$$

We define in the same way as above

$$\tilde{q}_\theta(t)=q_\theta(t)e^{\nu(x)},\quad \tilde{\zeta}=\zeta e^{\nu(x)}.$$

Then in a similar way to getting (3.14), we have stochastic partial differential equations for $\tilde{q}_\theta(t)$ and $\tilde{\zeta}$. We set

$$\tilde{\bar{q}}_\theta(t)=\frac{\tilde{q}_\theta(t)-\tilde{q}(t)}{\theta}-\tilde{\zeta}.$$

Then we can see that

$$\sup_{0\le t\le T}E[\|\tilde{\bar{q}}_\theta(t)\|_{L^2}^2]\to 0$$

as $\theta \to 0$ by using the energy equality for $\tilde{\bar{q}}_\theta(t)$. Since

$$\frac{I(v;h_\cdot+\theta k_\cdot;T)-I(v;h_\cdot;T)}{\theta} - v^\mu \hat{E}[\langle \zeta_T, 1\rangle] = v^\mu \hat{E}[\langle \tilde{\bar{q}}_\theta(T), e^{-\nu(x)}\rangle]$$

we obtain the present proposition.

□

Let us introduce the following backward stochastic partial differential equation.

$$(3.18)\quad \begin{array}{rcl} -d\gamma_t &=& (L^0\gamma_t + \hat{G}(h)\gamma_t + \hat{M}(h)R_t)dt - R_t^*(\Sigma\Sigma^*)^{-1}dY_t \\ \gamma_T &=& 1 \end{array}$$

where

$$\hat{G}(h)\varphi = \tilde{B}^* D_\varphi + \mu h^*\Sigma\Lambda^* D\varphi - \mu\eta(\cdot,h)\varphi$$

$$\hat{M}(h)R = R^j Q_j(\cdot,h) + [\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}]^i_j D_i R^j.$$

Set

$$\check{\gamma}_t = e^{-\nu(x)}\gamma_t, \quad \check{R}_t = e^{-\nu(x)}R_t.$$

Then we have the following backward SPDE

$$(3.19)\quad \begin{array}{rcl} -d\check{\gamma}_t &=& (L^0\check{\gamma}_t + \check{G}(h)\check{\gamma}_t + \check{M}(h)\check{R}_t)dt - \check{R}_t^*(\Sigma\Sigma^*)^{-1}dY_t \\ \check{\gamma}_T &=& e^{-\nu(x)}, \end{array}$$

where

$$\begin{aligned} \check{G}(h)\varphi &= \{(D\nu)^*\Lambda\Lambda^* + \tilde{B}^* + \mu h^*\Sigma\Lambda^*\}D\varphi \\ &\quad + \{L^0\nu + \tfrac{1}{2}(D\nu)^*\Lambda\Lambda^* D\nu + (\tilde{B}^* + \mu h^*\Sigma\Lambda^*)D\nu - \mu\eta(\cdot,h)\}\varphi \\ &\equiv G_1^* D\varphi + G_2\varphi \end{aligned}$$

and

$$\begin{aligned} \check{M}(h)U &= \sum_{i,j}[\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}]^i_j D_i U^j \\ &\quad + \textstyle\sum_j \{Q_j(\cdot,h) + \sum_i [\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}]^i_j D_i\nu\} U^j \\ &\equiv \textstyle\sum_{i,j}(M_1)^i_j D_i U^j + \sum_j (M_2)_j U^j. \end{aligned}$$

Let $\check{\gamma}_t$ be a solution to (3.19) with the terminal condition $\check{\gamma}_T = 0$ and set $(M_2')_j = (M_2)_j + D_j(G_1)^j$. We have by Itô's formula
(3.20)

$$\begin{aligned}
&\hat{E}[\| \check{\gamma}_t \|_{L^2}^2] \\
&= \hat{E}[\int_t^T \{2\langle L^0\check{\gamma}_s + \check{G}(h)\check{\gamma}_s + \check{M}(h)\check{R}_s, \check{\gamma}_s\rangle - (\check{R}_s, (\Sigma\Sigma^*)^{-1}\check{R}_s)\}ds] \\
&= \hat{E}[\int_t^T \int \{-(D\check{\gamma}_s)^*\Lambda\Lambda^* D\check{\gamma}_s + G_1^* D(\check{\gamma}_s^2) + 2G_2\check{\gamma}_s^2 \\
&\quad + 2(M_2')_j \check{R}_s^j \check{\gamma}_s - 2\check{R}_s^j (M_1)_j^i D_i\check{\gamma}_s - \check{R}_s^*(\Sigma\Sigma^*)^{-1}\check{R}_s\}dx\, ds \\
&= \hat{E}[\int_t^T \int \{-(D\check{\gamma}_s)\Lambda(I - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\Lambda^* D\check{\gamma}_s \\
&\quad - [(\Sigma\Sigma^*)^{-1}\check{R}_s + M_1^* D\check{\gamma}_s - (M_2')^*\gamma_s]^*(\Sigma\Sigma^*)[(\Sigma\Sigma^*)^{-1}\check{R}_s \\
&\qquad + M_1^* D\check{\gamma}_s - (M_2')^*\gamma_s] \\
&\quad + [M_2'\Sigma\Sigma^*(M_2')^* + 2G_2 - \textstyle\sum_j D_j(G_1 + M_1\Sigma\Sigma^*(M_2')^*)^j]\check{\gamma}_s^2\}dx\, ds \\
&\leq C\int_t^T \hat{E}[\| \check{\gamma}_s \|_{L^2}^2]ds
\end{aligned}$$

for some constant $C > 0$. By using (3.20) we can obtain the following lemma.

Lemma 3.3. *Under the assumptions of Proposition 3.2 the solution* (γ_t, R_t) *to (3.18) such that* $e^{-\delta\sqrt{1+|x|^2}}\gamma_t \in \mathcal{L}_Y^2(0,T;H^1(R^n)) \cap L^2(\Omega, \mathcal{F}, \hat{P}; C(0,T;L^2(R^n))$ *and* $e^{-\delta\sqrt{1+|x|^2}}R^i \in \mathcal{L}_Y^2(0,T;H^1(R^n)) \cap L^2(\Omega, \mathcal{F}, \hat{P}; C(0,T;L^2(R^n))$, $i = 1, 2, ..., m$ *is unique.*

We can also see the existence of the solution to (3.18) in a similar way to Theorem 8.2.3 [2] through aproximation procedure, or directly thanks to Chapter 5, Theorem 2.2 in [16].

Lemma 3.4. *Under the assumptions of Proposition 3.2*

$$\hat{E}[< \zeta_T, 1 >] = \hat{E}[\int_0^T \{\langle\gamma_t, k_t^i G_{h^i}(h)q_t > + < R_t^j, k_t^i M_{h^i}(h)_j q_t >\}dt].$$

Proof. From (3.16) and (3.18) we obtain

$$\begin{aligned}
d\langle\zeta_t, \gamma_t\rangle &= \{\langle k_t^i G_{h^i}(h)q_t, \gamma_t\rangle + \langle k_t^i M_{h^i}(h)_j q_t, R_t^j\rangle\}dt \\
&\quad + \{\langle M(h)_j\zeta_t, \gamma_t\rangle + \langle k_t^i M_{h^i}(h)_j q_t, \gamma_t\rangle + \langle M(h)_j\zeta_t, \gamma_t\rangle\}dY_t^j
\end{aligned}$$

and we have the present lemma.

□

Finally we have the following theorem

Theorem 3.1. *We assume the assumptions of Proposition 3.2. If h is optimal, then it satisfies*
(3.21)
$$(k-h_t)^*\{-(1-\mu)(\Sigma\Sigma^*)h_t < \gamma_t, q_t > + < \Sigma\Lambda^* D\gamma_t, q_t) > \\ + < (g-r\mathbf{1})\gamma_t, q_t > + < R_t, q_t >\} \leq 0,$$

a.e. t *a.s.* $\forall k \in \Gamma$.

Proof. Let h_t, k_t be admissible atrategies and h_t is an optimal one. Since Γ is convex $h + \theta(k-h) = (1-\theta)h + \theta k \in \Gamma$, for $h, k \in \Gamma$. Thus we have

$$I(v; h_.; T) \geq I(v; h_. + \theta(k_. - h_.); T), \quad 0 \leq \forall\theta \leq 1$$

if $\mu > 0$. Therefore, because of Lemma 3.2

$$\hat{E}[\langle\zeta_T(k-h), 1\rangle] \leq 0,$$

which implies that

$$(3.22)\quad \hat{E}[\int_0^T \{\langle\gamma_t, (k_t^i - h_t^i)G_{h^i}(h)q_t\rangle + \langle R_t^j, (k_t^i - h_t^i)M_{h^i}(h)_j q_t\rangle\}dt] \leq 0$$

for all admissible strategy k_t by Lemma 3.4. Set

$$(U_t)_i = \langle\gamma_t, G_{h^i}(h)q_t\rangle + \langle R_t^j, M_{h^i}(h)_j q_t\rangle.$$

For each $t_0 \in [0, T]$, $\epsilon > 0$, $M > 0$ and $\tilde{G}_{t_0}$ measurable random variable k_{t_0}, we define

$$k_t = \begin{cases} k_{t_0}1_{\{|U_t| \leq M\}} + h_t 1_{\{|U_t| > M\}}, & t_0 \leq t \leq t_0 + \epsilon \\ h_t, & t \in [t_0, t_0 + \epsilon]^c. \end{cases}$$

Then, through limiting procedure as $\epsilon \to 0$ and $M \to \infty$ after multilying (3.22) by $\frac{1}{\epsilon}$, we see that

$$\hat{E}[(k_{t_0} - h_{t_0})^i (U_{t_0})_i] \leq 0$$

for each t_0 and $\tilde{G}_{t_0}$ measurable random variable k_{t_0}, which implies that

$$(k - h_t)^i\{\langle\gamma_t, G_{h^i}(h)q_t\rangle + \langle R_t^j, M_{h^i}(h)_j q\rangle\} \leq 0 \quad \text{a.e. } t \text{ a.s.}$$

for each $k \in \Gamma$. Hence

(3.23)

$$\mu(k-h_t)^*\{-(1-\mu)(\Sigma\Sigma^*)h_t < \gamma_t, q_t > + < \Sigma\Lambda^* D\gamma_t, q_t) >$$
$$+ < (g - r\mathbf{1})\gamma_t, q_t > + < R_t, q_t >\} \le 0.$$

Since $\mu > 0$ we have (3.21).

If $\mu < 0$ we obtain the converse inquality of (3.23) and we conclude the present theorem.

□

References

[1] A. Bensoussan, *Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions*, Stochastics 9 (1983) 169-222

[2] A. Bensoussan, *Stochastic Control of Partially Observable Systems*, Cambridge (1992)

[3] A. Bensoussan, J. Frehse and H. Nagai, *Some Results on Risk-sensitive with full observation*, Appl. Math. and its Optimization, vol.37 (1998)1-41

[4] A. Bensoussan and H. Nagai, *Condition for no breakdown and Bellman equations of risk-sensitive control*, Appl. Math. and its Optimization, vol.42 (2000)91-101

[5] T.R. Bielecki and S.R. Pliska, *Risk-Sensitive Dynamic Asset Management*, Appl. Math. Optim. vol. 39 (1999) 337-360

[6] T.R. Bielecki and S.R. Pliska, *Risk-Sensitive Intertemporal CAPM, With Application to Fixed Income Management*, preprint

[7] W.H. Fleming, *Optimal investment models and risk sensitive stochastic control*, Mathematical Finance, edited by M. Davis et al., Springer-Verlag (1995)75-88

[8] W.H. Fleming and S.J. Sheu, *Optimal Long Term Growth Rate of Expected Utility of Wealth*, Ann. Appl. Prob.9(1999)871-903

[9] W.H. Fleming and S.J. Sheu, *Risk-sensitive control and an optimal investment model*, Mathematical Finance, 10 (2000)197-213

[10] W.H. Fleming and S.J. Sheu, *Risk-sensitive control and an optimal investment model (II)*, Annals of Prob.,12(2002)730-767

[11] H. Kaise and H. Nagai, *Bellman-Isaacs equations of ergodic type related to risk-sensitive control and their singular limits*, Asymptotic Analysis, 16 (1998) 347-362

[12] H. Kaise and H. Nagai Ergodic type Bellman equations of risk-sensitive control with large parameters and their singular limits. Asymptotic Analysis, 20 (1999) 279-299

[13] N.V. Krylov and B.L. Rozovski, *On conditional distributions of diffusion processes*, Math USSR Izvestya 12 (1978) 336-356

[14] K. Kuroda and H. Nagai *Risk-sensitive portfolio optimization on infinite time horizon*, Stochastics and Stochastics Reports, 73 (2002) 309-331

[15] J. Ma and J. Yong, *On linear, degenerate backward stochastic partial differential equations*, Prob. Theory Rel. Fields, 113 (1999) 135-170

[16] J. Ma and J. Yong, *Forward-Backward Stochastic Differential equations and Their Applications*, Lect. Notes in Math. 1702, Springer(1999)

[17] R.C. Merton, *Optimum Consumption and Portfolio Rules in a Continuous-Time Model*, Journal of Economic Theory 3 (1971) 373-413

[18] R.C. Merton, *An Intertemporal Capital Asset Pricing Model*, Econometrica (1973) 867-887

[19] R.C. Merton, *Continuous Time Finance*, Blackwell, Malden (1990)

[20] H. Nagai, *Bellman equations of risk-sensitive control*, SIAM J. Control and Optimization. Vol. 34, No. 1 (1996) 74-101.

[21] H. Nagai, *Risk-senstive dynamic asset management with partial information*, Stochastics in finite and infinite dimension, a volume in honor of G. Kallianpur, Eds. Rajput et al., Birkhäuser (2000) 321-340

[22] H. Nagai and S. Peng, *Risk-sensitive dynamic portfolio optimization with partial information on infinite time horizon*, Annals of Applied Probability, vol.12, No.1(2002)173-195

[23] H. Nagai, *Optimal strategies for risk-sensitive portfolo optimization problems for general factor models*, to appear in SIAM J. Control and Optim.

[24] R. Rishel, *Optimal Portfolio Management with Partial Observations and Power Utility Function*, Stochastic Analysis, Control, Optimization and Applications, Birkhäuser (1999) 605-620

[25] B.L. Rozovski, *Stochastic Evolution Systems*, Kluwer Academic Publishers (1990)

[26] X.Y. Zhou, *On the necessary conditions of optimal controls for stochstic partial differential equations*, SIAM J. Control Optim. 31,(1993) 1462-1478

Department of Mathematical Science
Osaka University
Toyonaka, 560-8531
Japan