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The Isaacs character correspondence and isotypies between blocks of finite groups

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§1. Introduction

Let S and G be finite groups such that S acts on G via automorphism and $(|S|, |G|) = 1$. It is well known that in this situation there is a natural bijection $\pi(G, S)$ from the set $\mathrm{Irr}_S(G)$ of S-invariant irreducible characters of G onto the set $\mathrm{Irr}(C_G(S))$ of irreducible characters of $C_G(S)$. When S is solvable, this is obtained by G. Glauberman and when $|G|$ is odd this is obtained by I. M. Isaacs. Moreover it is shown in [Wo1] that when S is solvable and $|G|$ is odd these are equal. Let p be a prime. In [Wa] we showed that the Glauberman character correspondence gives an isotypy between an S-invariant p-block B of G and a p-block of $C_G(S)$ if a defect group of B is centralized by S. In [H] H. Horimoto proved that the Isaacs character correspondence gives a perfect isometry between an S-invariant p-block B of G and a p-block of $C_G(S)$ under the same assumption as in the Glauberman correspondence case (see Theorem 3.2 for the detail). The purpose of this paper is to show that the perfect isometry is an isotypy (Theorem 3.6).

Let $(\mathcal{K}, \mathcal{R}, \mathcal{F})$ be a p-modular system such that $\mathcal{K}$ is algebraically closed. Here we state the definition of isotypies between blocks, where a block means a p-block. Let B be a block of G with defect group D and (D, B_D) be a maximal B-subpair of G. We denote by $\mathbf{Br}_B(G)$ the Brauer category of B. $\mathbf{Br}_B(G)$ is the category whose objects are B-subpairs of G and whose morphisms are defined in the following way: For B-subpairs (Q, b) and (R, b') $\mathrm{Mor}((Q, b), (R, b'))$ is the set of all cosets $gC_G(Q)$ of G such that ${}^g(Q, b) \subseteq (R, b')$ (see [B-O], §1). We denote by $\mathbf{Br}_{B,D}(G)$ the full subcategory of $\mathbf{Br}_B(G)$ whose objects are the B-subpairs (Q, b) such that $(Q, b) \subseteq (D, B_D)$. We note that for any $Q \leq D$ there exists a unique block b such that $(Q, b) \subseteq (D, B_D)$, and we set $b =$

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B_Q. Let $\mathrm{CF}(G,\mathcal{K})$ be the $\mathcal{K}$-vector space of $\mathcal{K}$-valued class functions on G and let $\mathrm{CF}(G,B,\mathcal{K})$ be the subspace of $\mathrm{CF}(G,\mathcal{K})$ of class functions α such that α is a $\mathcal{K}$-linear combination of χ's in $\mathrm{Irr}(B)$. Let $\mathrm{CF}_{p'}(G,B,\mathcal{K})$ be the subspace of $\mathrm{CF}(G,B,\mathcal{K})$ of class functions vanishing on the p-singular elements of G. Let $(x,\mathbf{b})$ be a B-Brauer element of G. The decomposition map

$$d_G^{(x,\mathbf{b})} : \mathrm{CF}(G,B,\mathcal{K}) \to \mathrm{CF}_{p'}(C_G(x),\mathbf{b},\mathcal{K})$$

is defined by $d_G^{(x,\mathbf{b})}(\alpha)(y) = \alpha(xye_{\mathbf{b}})$ for any p'-element y of $C_G(x)$, where $e_{\mathbf{b}}$ is the block idempotents of $\mathcal{R}C_G(x)$ corresponding to $\mathbf{b}$. Finally let H be a second finite group and B' be a block of H with D as a defect group. Let (D,B'_D) be a maximal B'-subpair of H and for any subgroup Q of D let (Q,B'_Q) be the B'-subpair of H such that $(Q,B'_Q)\subseteq(D,B'_D)$.

Definition ([B, 4.6]). With the above notations (G,B) and (H,B') are *isotypic* if the following conditions hold :

(i) The inclusion of D into G and H induces an equivalence of the Brauer categories $\mathbf{Br}_{B,D}(G)$ and $\mathbf{Br}_{B',D}(H)$.

(ii) There exists a family of perfect isometries

$$\{R^Q : \mathcal{R}_\mathcal{K}(C_G(Q),B_Q) \to \mathcal{R}_\mathcal{K}(C_H(Q),B'_Q)\}_{\{Q(cyclic)\leq D\}}$$

such that for any $x\in D$

$$(*)\qquad d_H^{(x,B'_{\langle x\rangle})}\circ R^{\langle 1\rangle} = (R^{\langle x\rangle})_{p'}\circ d_G^{(x,B_{\langle x\rangle})},$$

where $(R^{\langle x\rangle})_{p'}$ is the $\mathcal{K}$-linear map from $\mathrm{CF}_{p'}(C_G(x),B_{\langle x\rangle},\mathcal{K})$ onto $\mathrm{CF}_{p'}(C_H(x),B'_{\langle x\rangle},\mathcal{K})$ induced by $R^{\langle x\rangle}$ and we regard $R^{\langle 1\rangle}$ as a $\mathcal{K}$-linear map from $\mathrm{CF}(G,B,\mathcal{K})$ onto $\mathrm{CF}(H,B',\mathcal{K})$. In the above $R^{\langle 1\rangle}$ is called an *isotypy* between B and B', and $(R^Q)_{\{Q(cyclic)\leq D\}}$ is called the *local system* of $R^{\langle 1\rangle}$. See [B, §1] for the definition of perfect isometries between blocks.

§2. Isaacs character correspondence

In this section we recall the definition of the Isaacs correspondence and state some results in [I1], [Wo1, 2] which are used in the next section. Let G be a finite group. For normal subgroups $L\leq K$ of G such that K/L is abelian, and for G-invariant irreducible characters θ of K and ϕ of L, if θ is fully ramified with respect to K/L and ϕ is an irreducible constituent of θ_L, then (G,K,L,θ,ϕ) is called a character five.

Theorem 2.1 ([I1, Theorem 9.1 ; Corollary 6.4]). *Let (G, K, L, θ, ϕ) be a character five. Assume that either $|G : K|$ or $|K : L|$ is odd. Let $\Psi^{(K/L)}$ be the character of G/K defined with respect to the form $\ll, \gg_\phi$ on K/L, and view $\Psi^{(K/L)}$ as a character of G. Then there exists a conjugacy class $\mathcal{U}$ of subgroups $U \leq G$ such that*

(a) $(\Psi^{(K/L)}(x))^2 = \pm|C_{K/L}(x)|$ *for* $x \in G$;
(b) $UK = G$ *and* $U \cap K = L$;
(c) U^a *is G-conjugate to U for all $a \in \mathrm{Aut}(G)$ such that* $K^a = K$, $L^a = L$ *and* $\phi^a = \phi$;
(d) *the equation* $\chi_U = (\Psi^{(K/L)})_U \xi$, *for* $\chi \in \mathrm{Irr}(G|\theta)$ *and* $\xi \in \mathrm{Irr}(U|\phi)$ *defines a* 1-1 *correspondence between these sets of characters, and*
(e) *if $|G : K|$ is odd,* $\chi \in \mathrm{Irr}(G|\theta)$ *and* $\xi \in \mathrm{Irr}(U|\phi)$, *then* $\chi_U = (\Psi^{(K/L)})_U \xi$ *if and only if* (χ_U, ξ) *is odd.*

For the definition of $\Psi^{(K/L)}$ in the above, see [I1], page 619, Theorem 6.3 and the above of Theorem 9.1. $\Psi^{(K/L)}$ is determined by the form $\ll, \gg_\phi$ on K/L and the action of G/K on K/L.

Hypothesis 2.2. Let S act on G via automorphism such that $(|S|, |G|) = 1$. Let $C = C_G(S)$ and let Γ be the semi-direct product GS.

Lemma 2.3 ([I1, Corollary 10.7]; [Wo1, Corollary 4.3]). *Assume Hypothesis* 2.2 *with $|G|$ odd. Let $[G, S]'C \leq H \leq G$ such that H is S-invariant. Then there exists a bijection* $\sigma(G, H, S) : \mathrm{Irr}_S(G) \to \mathrm{Irr}_S(H)$ *such that for* $\chi \in \mathrm{Irr}_S(G)$, $\sigma(G, H, S)(\chi)$ *is the unique S-invariant irreducible character α of H with (χ_H, α) odd.*

Definition ([I1, §10]). Assume Hypothesis 2.2 with $|G|$ odd. If $C < G$, then let

$$G = G_0 > G_1 > G_2 > G_3 > \cdots > G_n = C$$

by $G_{i+1} = [G_i, S]'C$, for $i \geq 0$. The Isaacs character correspondence $\pi(G, S) : \mathrm{Irr}_S(G) \to \mathrm{Irr}(C)$ is the composition map

$$\sigma(G_{n-1}, C, S)\sigma(G_{n-2}, G_{n-1}, S) \cdots \sigma(G_2, G_1, S)\sigma(G, G_1, S)$$

if $C < G$, otherwise $\pi(G, S)$ is the identity map.

The following lemmas play big roles in this paper.

Lemma 2.4 ([Wo1, Theorem 4.6]). *Assume Hypothesis* 2.2 *with $|G|$ odd. Let $K = [G, S]$, $L = K'$, and $U = LC$. Assume that $U \leq H \leq G$ is S-invariant. Let $\chi \in \mathrm{Irr}_S(G)$ and $\psi = \sigma(G, H, S)(\chi)$. Then*

(a) $\sigma(G, U, S)(\chi) = \sigma(H, U, S)(\psi)$, *and*

(b) $\pi(G,S)(\chi) = \pi(H,S)(\psi)$.

Lemma 2.5 ([Wo2, Lemma 2.5]). *Assume Hypothesis* 2.2 *and* $N \triangleleft \Gamma$ *and* $N \leq G$. *Let* $\chi \in \mathrm{Irr}_S(G)$, $\theta \in \mathrm{Irr}_S(N)$, $T = T_G(\theta)$ *the inertial subgroup of* θ *in* G, $\mu = \pi(G,S)(\chi)$, *and* $\nu = \pi(N,S)(\theta)$. *Then*

(a) $(\chi_N, \theta) \neq 0$ *if and only if* $(\mu_{N\cap C}, \nu) \neq 0$,
(b) $T \cap C = T_C(\nu)$ *and* $\pi(G,S)(\psi^G) = (\pi(T,S)(\psi))^C$ *for* $\psi \in \mathrm{Irr}_S(T|\theta)$.

Lemma 2.6 ([Wo1, Lemma 4.9]). *Assume Hypothesis* 2.2 *with* $|G|$ *odd. Let* U *be a normal subgroup of* S *and* $H = C_G(U)$. *Then* $\pi(G,U)$ *maps* $\mathrm{Irr}_S(G)$ *onto* $Irr_S(H)$ *and* $\pi(G,S) = \pi(H,S/U)\pi(G,U)$.

§3. Isotypies obtained from Isaacs character correspondences

Since the Isaacs character correspondence is defined in the case $|G|$ is odd, we set the following hypothesis. Then G is solvable by the Feit-Thompson's theorem.

Hypothesis 3.1. Let S and G be finite groups such that S acts on G, $(|S|, |G|) = 1$ and that $|G|$ is odd. Put $C = C_G(S)$.

Theorem 3.2 ([H, Theorem 1, (a)]). *Under the above hypothesis, let* B *be an* S*-invariant block of* G *such that a defect group* D *of* B *is centralized by* S. *Then there exists a block* b *of* C *such that* $\mathrm{Irr}(b) = \{\pi(G,S)(\chi) | \chi \in \mathrm{Irr}(B)\}$ *and* $\pi(G,S)$ *gives a perfect isometry* R *between* B *and* b. *Moreover* D *is a defect group of* b.

In the above theorem the assumption for B implies that $\chi \in \mathrm{Irr}(B)$ is S-invariant by [Wa, Proposition 1]. We call b the *Isaacs correspondent of* B. We will show that the perfect isometry R in the above theorem is an isotypy .

Lemma 3.3. *Let* (G, K, L, θ, ϕ) *be a character five such that* K *is a* p'*-group and* $|G|$ *is odd. Let* $\Psi^{(K/L)}$ *be the character of* G/K *defined with respect to the form* $\ll, \gg_\phi$ *on* K/L. *Let* Q *be a* p*-subgroup of* C, $\theta^* = \pi(K,Q)(\theta)$ *and* $\phi^* = \pi(L,Q)(\phi)$. *Then the following hold.*

(i) $(C_G(Q), C_K(Q), C_L(Q), \theta^*, \phi^*)$ *is a character five.*
(ii) *Suppose that* Q *is a cyclic group generated by* x *and let* $\Psi^{(C_K(x)/C_L(x))}$ *be the character of* $C_G(x)/C_K(x)$ *defined with respect to the form* $\ll, \gg_{\phi^*}$ *on* $C_K(x)/C_L(x)$. *If* K/L *is a* q*-group for a prime* q, *then there exists a sign* $\epsilon_x = \pm 1$ *such that*

$$\Psi^{(K/L)}(x\rho) = \epsilon_x \Psi^{(C_K(x)/C_L(x))}(\rho)$$

for all $\rho \in C_G(x)_{p'}$.

Proof. (i) Since θ is G-invariant, by [I2, Theorem 13.14] and [Wo1, Theorem 5.1] θ^* the unique constituent of $\theta_{C_K(Q)}$ such that p does not divide $(\theta_{C_K(Q)}, \theta^*)$. Therefore θ^* is $C_G(Q)$-invariant. Similarly ϕ^* is $C_G(Q)$-invariant. Since θ is a unique constituent of ϕ^K as ϕ is fully ramified with respect to K/L, θ^* is a unique constituent of $(\phi^*)^{C_K(Q)}$ by Lemma 2.5. Hence $(C_G(Q), C_K(Q), C_L(Q), \theta^*, \phi^*)$ is a character five. Here we show $\ll, \gg_\phi = \ll, \gg_{\phi^*}$ on $C_K(Q)/C_L(Q)$ without the assumption ϕ is fully ramified with respect to K/L, where $C_K(Q)/C_L(Q)$ is identified with a subgroup of K/L. Let $y \in C_K(Q)$ and $\hat{\phi}$ be an extension of ϕ to $\langle L, y\rangle$. We can show that $\hat{\phi}$ is Q-invariant by a theorem of Glauberman([I2, Lemma 13.8]). Then $\pi(\langle L, y\rangle, Q)(\hat{\phi})$ is an extension of ϕ^* to $\langle C_L(Q), y\rangle$ by Lemma 2.5 because $\langle C_L(Q), y\rangle / C_L(Q)$ is cyclic. For $z \in C_K(Q)$ let $(\hat{\phi})^z = \lambda\hat{\phi}$ where λ is a linear character of $\langle L, y\rangle$ so that $L \subseteq \mathrm{Ker}\lambda$. Then we see easily $\pi(\langle L, y\rangle, Q)((\hat{\phi})^z) = \pi(\langle L, y\rangle, Q)(\lambda\hat{\phi}) = \lambda\pi(\langle L, y\rangle, Q)(\hat{\phi})$ where λ is regarded as a character of $\langle C_L(Q), y\rangle$. So we have $(\pi(\langle L, y\rangle, Q)(\hat{\phi}))^z = \pi(\langle L, y\rangle, Q)((\hat{\phi})^z) = \lambda\pi(\langle L, y\rangle, Q)(\hat{\phi})$. Hence $\ll y, z \gg_\phi = \lambda(y) = \ll y, z \gg_{\phi^*}$.

(ii) Let $E = K/L$, $E_1 = C_E(x)$ and $E_2 = [E, x]$. Then $E = E_1 \times E_2$. Since $E_1 = C_K(x)L/L$, E_1 and $C_K(x)/C_L(x)$ are $C_G(x)/C_K(x)$-isomorphic when $C_G(x)/C_K(x)$ acts on them. Suppose that $1 < E_1 < E$. Then by the algorithm for computation of $\Psi^{(E)}$,

$$\Psi^{(E)}(x\rho) = \Psi^{(E_1)}(x\rho)\Psi^{(E_2)}(x\rho)$$

for all $\rho \in C_G(x)_{p'}$. Since $C_{E_2}(x\rho)$ is the identity group, by [I1, Corollary 6.4], $\Psi^{(E_2)}(x\rho) = \pm 1$ and hence we have $\Psi^{(E_2)}(x\rho) = \Psi^{(E_2)}(x)$ for $\rho \in C_G(x)_{p'}$ because $x\rho$ is a $2'$-element. On the other hand since $\ll, \gg_\phi = \ll, \gg_{\phi^*}$ on $E_1 \cong C_K(x)/C_L(x)$, by [I1, Theorem 6.3] and by the algorithm for computation of $\Psi^{(E)}$ we have $\Psi^{(E_1)} = \Psi^{(C_K(x)/C_L(x))}$ as characters of $C_G(x)/C_K(x)$. Moreover $x \in \mathrm{Ker}\Psi^{(C_K(x)/C_L(x))}$ by [I1, Corollary 6.4] because x is a $2'$-element. So if we put $\epsilon_x = \Psi^{(E_2)}(x)$, we have $\Psi^{(E)}(x\rho) = \epsilon_x\Psi^{(C_K(x)/C_L(x))}(\rho)$ for all $\rho \in C_G(x)_{p'}$. Next suppose that $E_1 = E$. Then we have $\Psi^{(E)}(x\rho) = \Psi^{(E)}(\rho) = \Psi^{(C_K(x)/C_L(x))}(\rho)$ for all $\rho \in C_G(x)_{p'}$ by the same argument as in the above. So we may assume E_1 is the identity group. Then by [I1, Corollary 6.4] again, $\Psi^{(E)}(x\rho) = \Psi^{(E)}(x) = \pm 1$ for all $\rho \in C_G(x)_{p'}$. On the other hand $\Psi^{(C_K(x)/C_L(x))}(\rho) = 1$ for all $\rho \in C_G(x)_{p'}$. So if we put $\epsilon_x = \Psi^{(E)}(x)$, then we have $\Psi^{(E)}(x\rho) = \epsilon_x\Psi^{(C_K(x)/C_L(x))}(\rho)$ for all $\rho \in C_G(x)_{p'}$. This completes the proof of (ii). Q.E.D.

Lemma 3.4. *Assume Hypothesis 3.1. Let B be an S-invariant block of G such that a defect group D of B is centralized by S and let*

b be the Isaacs correspondent of B. Let (Q, B_Q) be an S-invariant B-subpair of G such that $Q \subseteq D$ and a defect group of B_Q is centralized by S and let b_Q be the Isaacs correspondent of B_Q. Then b_Q is associated with b in the sense of Brauer.

Proof. We prove by induction on $|G|$. Let $K = O_{p'}(G)$ and ζ^* be an irreducible character of $C_K(S)$ covered by b. We may assume that ζ^* is Q-invariant because Q is contained in a defect group D of b. Let $\zeta \in \mathrm{Irr}_S(K)$ have the Isaacs correspondent ζ^*, and let $H = T_G(\zeta)$, $T_G(\zeta)$ is the stabilizer of ζ in G. Then B covers ζ by Lemma 2.5 and H is S-invariant. By Lemma 2.5 again, $Q \leq T_C(\zeta^*) = H \cap C$, *i.e.*, ζ is SQ-invariant. Let $\zeta_1 = \pi(K, Q)(\zeta)$ and $\zeta_2 = \pi(C_K(Q), S)(\zeta_1)$. Then we have $\zeta_2 = \pi(K, SQ)(\zeta) = \pi(C_K(S), Q)(\zeta^*)$ by Lemma 2.6. And $T_{C_G(Q)}(\zeta_1) = H \cap C_G(Q)$ and $T_{C_C(Q)}(\zeta_2) = H \cap C_G(Q) \cap C_C(Q) = C \cap H \cap C_G(Q)$. Moreover by the assumption, B_Q covers ζ_1 because B covers ζ. Hence b_Q covers ζ_2 by Lemma 2.5 since b_Q is the Isaacs correspondent B_Q. Let $\tilde{b}_Q$ be a block of $C \cap H \cap C_G(Q)$ such that $\tilde{b}_Q$ is the Clifford correspondent of b_Q and similarly let $\tilde{B}_Q$ be a block of $H \cap C_G(Q)$ such that $\tilde{B}_Q$ is the Clifford correspondent of B_Q. Since ζ and hence ζ_1 is S-invariant and B_Q is S-invariant, $\tilde{B}_Q$ is S-invariant. Let $\tilde{B} = (\tilde{B}_Q)^H$. Then $\tilde{B}$ covers ζ and we have $\tilde{B}^G = ((\tilde{B}_Q)^{C_G(Q)})^G = B$. Hence B is the Clifford correspondent of $\tilde{B}$. Moreover since ζ, and B is S-invariant, $\tilde{B}$ is S-invariant. Here we show that a defect group of $\tilde{B}$ is centralized by S. S acts on the defect groups of $\tilde{B}$. By a theorem of Glauberman there exists a defect group $\tilde{D}$ of $\tilde{B}$ which is S-invariant. So when S acts on the defect groups of B, D and $\tilde{D}$ are fixed elements. So D are $\tilde{D}$ are C-conjugate by a theorem of Glauberman. So $\tilde{D}$ is centralized by S. Similarly a defect group of $\tilde{B}_Q$ is centralized by S. By Lemma 2.5 and the assumption, $\tilde{b}_Q$ is the Isaacs correspondent of $\tilde{B}_Q$. Now let $\tilde{b}$ be the Isaacs correspondent of $\tilde{B}$. $\tilde{b}$ covers ζ^* by Lemma 2.5. Here assume $H < G$. By the induction hypothesis $\tilde{b}_Q$ is associated with $\tilde{b}$. On the other hand since b is the Isaacs correspondent B, b is the Clifford correspondent of $\tilde{b}$. These imply $(b_Q)^C = (\tilde{b}_Q)^C = ((\tilde{b}_Q)^{H \cap C})^C = (\tilde{b})^C = b$. Thus we assume ζ is G-invariant. Hence B is of maximum defect and D is a Sylow p-subgroup of G because G is solvable. Now we can show that a p-complement of G is S-invariant by using a theorem of Glauberman. So $[G, S]$ is a p'-group. Hence $G = KC$. From this $K \cap C$ is the maximal normal p'-subgroup of C. Since ζ^* is C-invariant, b is the unique p-block of C which covers ζ^*. Now $(b_Q)^C$ covers ζ^* because b_Q covers ζ_2 and $\zeta_2 = \pi(C_K(S), Q)(\zeta^*)$. So $b = (b_Q)^C$. This completes the proof. Q.E.D.

Under Hypothesis 3.1 let B be an S-invariant block of G with the Isaacs correspondent b. Let D be a common defect group of B and b and (D, B_D) be an S-invariant maximal B-subpair. Let (Q, B_Q) be a B-subpair contained in (D, B_D). Then B_Q is S-invariant and a defect group of B_Q is centralized by S as we proved in [Wa, §3]. We prove it again for the self-containedness. Let $(Q, B_Q) \not\trianglelefteq (R, B_R)$ be B-subpairs contained in (D, B_D). If B_R is S-invariant, then B_Q is S-invariant. So we can show that B_Q is S-invariant by the induction on $|D : Q|$. Next we show that a defect group of B_Q is centralized by S for any $Q \leq D$. In fact we show that a defect group of $(B_Q)^T$ is centralized by S where T is the inertial group of B_Q in $N_G(Q)$. Let U be a defect group of $(B_Q)^T$. Since $(B_Q)^T$ is associated with B, $Q^v \leq U^v \leq D$ for some $v \in G$. So we have $C_\Gamma(Q) \geq S^{v^{-1}}$ and $C_\Gamma(Q) \geq S$. Since $C_\Gamma(Q) = SC_G(Q)$, by the Schur-Zassenhaus theorem there exists an element $u \in C_G(Q)$ such that $S^{v^{-1}} = S^u$. Then $v^{-1}u^{-1} \in C$. Hence we have $U^{u^{-1}} \leq D^{v^{-1}u^{-1}} \subseteq C$. Thus $U^{u^{-1}}$ is a defect group of $(B_Q)^T$ centralized by S. Now let b_Q be the Isaacs correspondent of B_Q. By Lemma 3.4 (Q, b_Q) is a b-subpair of C.

Proposition 3.5. *With the above notations we have the following.*

(i) *(D, b_D) is a maximal b-subpair of C and $(Q, b_Q) \subseteq (D, b_D)$ for any $Q \leq D$.*

(ii) *The Brauer categories $\mathbf{Br}_{B,D}(G)$ and $\mathbf{Br}_{b,D}(C)$ are equivalent.*

Proof. (i) By Lemma 3.4 and Theorem 3.2, it is evident that (D, b_D) is a maximal b-subpair of C. We prove the latter of (i) by the induction on $|D : Q|$. Assume $Q \triangleleft R \leq D$ and $(R, b_R) \subseteq (D, b_D)$. Then $(B_R)^{RC_G(Q)} = B_Q$ and R fixes B_Q. So we see that R stabilizes b_Q because the map $\pi(C_G(Q), S)$ is an $N_C(Q)$-map. Now B_Q as a block of $RC_G(Q)$ is S-invariant and a defect group of B_Q is centralized by S as we saw in the above. So by Lemma 2.5, b_Q as a block of $RC_C(Q)$ is the Isaacs correspondent of B_Q. Hence by Lemma 3.4, we have $(b_R)^{RC_C(Q)} = b_Q$ and hence $(Q, b_Q) \subseteq (R, b_R) \subseteq (D, b_D)$.

(ii) Let $Q \leq D$ and let $(Q, B_Q)^x \subseteq (D, B_D)$ for $x \in G$. Then $(B_Q)^x = B_{Q^x}$. Since $Q^x \leq D \leq C$, we can show $x \in C_G(Q)C$ by the Schur-Zassenhaus theorem. So we may assume $x \in C$. Then $(b_Q)^x$ is the Isaacs correspondent of $(B_Q)^x$ and hence we have $(b_Q)^x = b_{Q^x}$. Conversely if $(Q, b_Q)^y \leq (D, b_D)$ for $y \in C$ then we have $(b_Q)^y = b_{Q^y}$, hence $(B_Q)^y = B_{Q^y}$ because $(b_Q)^y$ is the Isaacs correspondent of $(B_Q)^y$. This implies that $\mathbf{Br}_{B,D}(G)$ and $\mathbf{Br}_{b,D}(C)$ are equivalent. This completes the proof of the proposition. Q.E.D.

With the notations in the just above of Proposition 3.5, let R^Q be the perfect isometry from $\mathcal{R}_{\mathcal{K}}(C_G(Q), B_Q)$ onto $\mathcal{R}_{\mathcal{K}}(C_C(Q), b_Q)$ for $Q \leq D$ and let $R = R^{\langle 1 \rangle}$. We are now in a position to prove our main theorem.

Theorem 3.6. *Assume Hypothesis 3.1 and let B be an S- invariant block of G such that a defect group D of B is centralized by S and b be the Isaacs correspondent of B. Then R is an isotypy between B and b with local system $(\pm R^Q)_{\{Q(\mathrm{cyclic}) \leq D\}}$, where R^Q is as in the just above.*

Proof. We prove by induction on $|G|$. Since the Brauer categories $\mathbf{Br}_{B,D}(G)$ and $\mathbf{Br}_{b,D}(C)$ are equivalent by Proposition 3.5, it suffices to prove

$$\pm (R^{\langle x \rangle})_{p'} \circ d_G^{(x,B_x)} = d_C^{(x,b_x)} \circ R \tag{1}$$

for any $x \in D$, where $B_x = B_{\langle x \rangle}$ and $b_x = b_{\langle x \rangle}$. Let H be a normal p'-subgroup of G and let ζ be an S-invariant irreducible character of H covered by B. We put $\pi(H,S)(\zeta) = \zeta^*$. By Lemma 2.5, b covers ζ^*. Let $T = T_G(\zeta)$ and $\tilde{B}$ be a block of T such that $\tilde{B}$ covers ζ and that $\tilde{B}$ corresponds to B by the Clifford theorem (then we say that$\tilde{B}$ and B are *Clifford induction equivalent.*) From the argument in the proof of Lemma 3.4, $\tilde{B}$ is S-invariant and a defect group of $\tilde{B}$ is centralized by S. Let $\tilde{b}$ be the Isaacs correspondent of $\tilde{B}$. By Lemma 2.5 $T_C(\zeta^*) = T \cap C = C_T(S)$ and $\tilde{b}$ covers ζ^*. Moreover we see $\tilde{b}$ and b are Clifford induction equivalent by Lemma 2.5 again because b is the Isaacs correspondent of B. Let $\tilde{D}$ be a defect group of $\tilde{b}$. Then $\tilde{D}$ is a defect group of B. Since $\tilde{D}$ and D are S-invariant, they are conjugate by an element of C by a theorem of Glauberman. So we may assume $\tilde{D} = D$. In fact let $g \in C$. $(D^g, (B_D)^g)$ is an S-invariant maximal B-subpair of G with $D^g \subseteq C$ and $(Q^g, (B_Q)^g) \subseteq (D^g, (B_D)^g)$ for any $Q \leq D$. On the other hand by the definition of Isaacs correspondence, $(b_Q)^g$ is the Isaacs correspondent of $(B_Q)^g$, and $(R^Q)^g$ is the perfect isometry from $\mathcal{R}_{\mathcal{K}}(C_G(Q^g), (B_Q)^g)$ onto $\mathcal{R}_{\mathcal{K}}(C_C(Q^g), (b_Q)^g)$. Moreover we can see that (1) holds for (x, B_x) if and only if (1) holds for $(x, B_x)^g$, that is, $\pm((R^{\langle x \rangle})^g)_{p'} \circ d_G^{(x^g,(B_x)^g)} = d_C^{(x^g,(b_x)^g)} \circ R$ for all $x \in D$. Thus we may assume $\tilde{D} = D$.

Let $(D, \tilde{B}_D)$ be an S-invariant maximal $\tilde{B}$-subpair of T. By [F-H], p 3471, Remark, $(\tilde{B}_D)^{C_G(D)}$ is defined and it is Clifford induction equivalent to $\tilde{B}_D$. And $(\tilde{B}_D)^{C_G(D)}$ is S-invariant because $\tilde{B}_D$ is S-invariant. Hence $(\tilde{B}_D)^{C_G(D)}$ and B_D are $N_C(D)$-conjugate by a theorem of Glauberman. So we may assume $(\tilde{B}_D)^{C_G(D)} = B_D$ if necessary by replacing ζ with $N_C(C)$-conjugate of it. Now let $Q \leq D$ and $\zeta_1 =$

$\pi(H,Q)(\zeta)$. Then $T_{C_G(Q)}(\zeta_1) = T \cap C_G(Q)$. Let $\zeta_2 = \pi(C_H(Q),S)(\zeta_1)$. By Lemma 2.6 we have $\zeta_2 = \pi(C_H(S),Q)(\zeta^*)$, and hence we have also $T_{C_C(Q)}(\zeta_2) = T \cap C \cap C_G(Q)$. Let $(Q,\tilde{B}_Q) \subseteq (D,\tilde{B}_D)$ and $\tilde{b}_Q$ be the Isaacs correspondent of $\tilde{B}_Q$. Then $\tilde{B}_Q$ covers ζ_1, and $\tilde{b}_Q$ covers ζ_2 by Lemma 2.5. Therefore $(\tilde{B}_Q)^{C_G(Q)}$ is defined and this is Clifford induction equivalent to $\tilde{B}_Q$. By [F-H], p 3471, Remark, $(Q,(\tilde{B}_Q)^{C_G(Q)}) \subseteq (D,(\tilde{B}_D)^{C_G(D)}) = (D,B_D)$ because $(Q,\tilde{B}_Q) \subseteq (D,\tilde{B}_D)$, and hence we have $B_Q = (\tilde{B}_Q)^{C_G(Q)}$. So Lemma 2.5 implies $b_Q = (\tilde{b}_Q)^{C_C(Q)}$, that is, $\tilde{b}_Q$ and b_Q are Clifford induction equivalent. Here we assume $T < G$. By the induction hypothesis (1) holds for $\tilde{B}$. On the other hand $\tilde{B}$ and B are isotypic by the induction of characters, similarly $\tilde{b}$ and b are also isotypic by [F-H], p 3471, Remark. So combining these facts with Lemma 3.5 we can see that (1) holds for B. Hence we may assume $T = G$. In particular we may assume that B is of maximum defect and hence a Sylow p-subgroup of G is centralized by S.

Let $K = [G,S]$ and θ be an S-invariant irreducible character of K covered by B. By a theorem of Glauberman we have $G = CK$ and we have also $C \cap K \subseteq K'$. From the above arguments K is a p'-group and θ is G-invariant. Moreover we may assume $C < G$. Let $\Gamma = SG$ the semi direct product of G by S, K/L be a chief factor group of Γ and $X = LC$. Then $G = XK$, $X \cap K = L$ and $X < G$. Besides a Sylow p-subgroup of X also is centralized by S. So the Isaacs correspondence gives a bijection between $\mathrm{Bl}_S(X)$ and $\mathrm{Bl}(C)$ by Theorem 3.2. Let B_X be an S-invariant block of X with Isaacs correspondent b. We note D is a defect group of B_X. On the other hand, since $X \supseteq CK'$, by Lemma 3.4, there exists a perfect isometry R' from $\mathcal{R}_{\mathcal{K}}(G,B)$ onto $\mathcal{R}_{\mathcal{K}}(X,B_X)$ such that for $\chi \in \mathrm{Irr}(B)$ $R'(\chi)$ is the unique S-invariant irreducible character α of X with (α,χ_X) odd. Moreover R is the composition of R' and the perfect isometry from $\mathcal{R}_{\mathcal{K}}(X,B_X)$ onto $\mathcal{R}_{K}(C,b)$. Now let $Q \le D$. $C_X(Q)$ is S-invariant and a Sylow p-subgroup of $N_X(Q)$, and hence that of $C_X(Q)$ is centralized by S from the Schur-Zassenhaus theorem. Moreover $C_X(Q) = C_C(Q)[C_X(Q),S] = C_C(Q)C_L(Q) \ge C_C(Q)[C_G(Q),S]'$. Let B'_Q be an S-invariant block of $C_X(Q)$ with Isaacs correspondent b_Q. By the same reason as in the above there exists a perfect isometry R'^Q from $\mathcal{R}_{\mathcal{K}}(C_G(Q),B_Q)$ onto $\mathcal{R}_{\mathcal{K}}(C_X(Q),B'_Q)$ such that for $\mu \in \mathrm{Irr}(B_Q)$, $R'^Q(\mu)$ is the unique S-invariant irreducible character β of $C_X(Q)$ with $(\beta,\mu_{C_X(Q)})$ odd. And R^Q is the composition of R'^Q and the perfect isometry from $\mathcal{R}_{\mathcal{K}}(C_X(Q),B'_Q)$ onto $\mathcal{R}_{\mathcal{K}}(C_C(Q),b_Q)$. Since a Sylow p-subgroup of X is centralized by S and $b = (b_Q)^C$, $(B'_Q)^X$ has b as the Isaacs correspondent by Lemma 3.4. So we have $(B'_Q)^X = B_X$. Let

$(Q, \mathbf{b})$ be a B_X-subpair contained in (D, B'_D). Since $(Q, b_Q) \subseteq (D, b_D)$ by Proposition 3.5, $\mathbf{b}$ has b_Q as the Isaacs correspondent. So $\mathbf{b} = B'_Q$. Thus by the induction hypothesis for X and B_X, it suffices to show

$$\pm({R'}^{\langle x\rangle})_{p'} \circ d_G^{(x,B_x)} = d_X^{(x,B'_x)} \circ R' \tag{2}$$

for all $x \in D$ where $B'_x = B'_{\langle x\rangle}$,

Now let ϕ be an S-invariant irreducible character of L covered by B_X. Then it is clear ϕ is a constituent of θ_L. Moreover since θ is G-invariant and K/L is abelian, $T_K(\phi)$ is normal in Γ. Hence $T_K(\phi) = L$ or $T_K(\phi) = K$ because K/L is a chief factor of Γ. We assume $T_K(\phi) = L$ for a while. At first we show $X = T_G(\phi)$ as follows. Since θ is G-invariant, we have $G = T_G(\phi)K$ and $T_G(\phi) \cap K = L$. Since $T_G(\phi)$ is S-invariant, we have $T_G(\phi) = T_C(\phi)[T_G(\phi), S] \leq XT_K(\phi) = X$. The fact that $|T_G(\phi)| = |X|$ implies $X = T_G(\phi)$. From this $\xi \leftrightarrow \xi^G$ defines a one-to-one correspondence between $\mathrm{Irr}(X|\phi)$ and $\mathrm{Irr}(G|\theta)$ preserving the actions of S on them. Therefore in particular B_X and B are Clifford induction equivalent and $R'(\xi^G) = \xi$ for $\xi \in \mathrm{Irr}(B_X)$. Let $Q \leq D$, $\theta^* = \pi(K, Q)(\theta)$ and $\phi^* = \pi(L, Q)(\phi)$. Then B_Q covers θ^* and B'_Q covers ϕ^*. Besides θ^* is $C_G(Q)$-invariant and $T_{C_K(Q)}(\phi^*) = C_L(Q)$. Since $C_G(Q) = C_X(Q)C_K(Q)$, from the same argument as for B_X and B, B'_Q and B_Q are Clifford induction equivalent and ${R'}^Q(\eta^{C_G(Q)}) = \eta$ for $\eta \in \mathrm{Irr}(B'_Q)$. So we have $({R'}^{\langle x\rangle})_{p'} \circ d_G^{(x,B_x)} = d_X^{(x,B'_x)} \circ R'$ from [F-H], p 3471, Remark since $(Q, B'_Q) \subseteq (D, B'_D)$ for any $Q \leq D$. Thus (2) holds.

Next suppose $T_K(\phi) = K$. Then $G = T_G(\phi)$ and $T_{C_K(Q)}(\phi^*) = C_K(Q)$ for any $Q \leq D$. Since $K^\perp = \{c \in K | \ll c,\ y \gg_\phi = 1\ \forall\ y \in K\}$ is normal in Γ by [I1, Lemma 2.1], $K^\perp = K$ or $K^\perp = L$. At first we discuss the case $K^\perp = K$. Then ϕ is extendible to K, by [I1, Theorem 2.7]. Moreover B_X and B are isomorphic by [I1, Lemma 10.5] and $R'(\chi) = \chi_X$ for $\chi \in \mathrm{Irr}(B)$. In the proof of Lemma 3.3, (i) we proved that $\ll,\gg_\phi = \ll,\gg_{\phi^*}$ on $C_K(Q)/C_L(Q) \subseteq K/L$. So $C_K(Q)^\perp = \{c \in C_K(Q) | \ll c,\ y \gg_\phi = 1\ \forall\, y \in C_K(Q)\} = C_K(Q)$. Hence by [I1, Theorem 2.7] again, ϕ^* is extendible to $C_K(Q)$. Since $C_G(Q) = C_X(Q)C_K(Q)$ and $C_L(Q) = C_X(Q) \cap C_K(Q)$, by applying [I1, Theorem 10.5] for $C_G(Q)$ and B_Q, we see B_Q and B'_Q are isomorphic, and ${R'}^Q(\gamma) = \gamma_{C_X(Q)}$ for $\gamma \in \mathrm{Irr}(B_Q)$. On the other hand $(Q, B'_Q) \subseteq (D, B'_D)$ for any $Q \leq D$ and by Proposition 3.5 the inclusion of D into G and X induces an equivalence of the Brauer categories $\mathbf{Br}_{B,D}(G)$ and $\mathbf{Br}_{B_X,D}(X)$. The proof of Proposition 3.5, (ii) implies also that for any $x, y \in D$, B-Brauer pairs (x, B_x) and (y, B_y) are G-conjugate if and only if (x, B'_x)

and (y, B'_y) are C-conjugate. Moreover if $x \in D$ and $\mathbf{b}$ is a block of $C_G(x)$ associated with B, then $(x, \mathbf{b})$ is C-conjugate to (y, B_y) for some $y \in D$. So we can see that, $(R'^{\langle x \rangle})_{p'} \circ d_G^{(x,B_x)} = d_X^{(x,B'_x)} \circ R'$ for all $x \in D$. Thus (2) holds for all $x \in D$.

Thus our proof is reduced to the case $K^\perp = L$. Then (G, K, L, θ, ϕ) is a character five by [I1, Theorem 2.7]. This time we will use Theorem 2.1 for this character five. Since $K = [G, S]$ and S fixes ϕ, we can see $X \in \mathcal{U}$ in Theorem 2.1. In fact an S-invariant member of $\mathcal{U}$ coincides with X. By Theorem 2.1, (d) and (e), we have

$$\chi_X = (\Psi^{(K/L)})_X R'(\chi) \tag{3}$$

for $\chi \in \mathrm{Irr}(B)$. Then we say that B and B_X are *fully ramified equivalent* with respect to (G, K, L, θ, ϕ). Let $Q \leq D$, $\theta^* = \pi(K, Q)(\theta)$ and $\phi^* = \pi(L, Q)(\phi)$. By lemma 3.3, (i), $(C_G(Q), C_K(Q), C_L(Q), \theta^*, \phi^*)$ is an S-invariant character five. So by [H, Proposition 4, (a)], the equation

$$\psi_{C_X(Q)} = (\Psi^{(C_K(Q)/C_L(Q))})_{C_X(Q)} \psi' \tag{4}$$

for $\psi \in \mathrm{IBr}(C_G(Q)|\theta^*)$ and $\psi' \in \mathrm{IBr}(C_X(Q)|\phi^*)$ defines a 1-1 correspondence between these sets. Since B_Q covers θ^* and B'_Q covers ϕ^*, and B_Q and B'_Q have the same Isaacs correspondent b_Q, by Theorem 2.1, (d) and (e), we see that B_Q and B'_Q are fully ramified equivalent with respect to $(C_G(Q), C_K(Q), C_L(Q), \theta^*, \phi^*)$. We note $\varphi' = R'^Q(\varphi)$ for $\varphi \in \mathrm{IBr}(B_Q)$. Now let $x \in D$ and let $\chi \in \mathrm{Irr}(B)$ and $\xi = R'(\chi)$. Putting $Q = \langle x \rangle$ from (4) we have

$$\chi(x\rho) = \sum_{\psi \in \mathrm{IBr}(C_G(x)|\theta^*)} d^x_{\chi\psi} \psi(\rho) = \sum_{\psi} d^x_{\chi\psi} \Psi^{(C_K(x)/C_L(x))}(\rho) \psi'(\rho)$$

for $\rho \in C_X(x)_{p'}$ where $d^x_{\chi\psi}$ is the generalized decomposition number. Recalling that K/L is a chief factor of Γ and hence K/L is a q-group for a prime number q, we have the following from (3) and Lemma 3.3, (ii)

$$\begin{aligned}
\chi(x\rho) &= \Psi^{(K/L)}(x\rho)\xi(x\rho) \\
&= \epsilon_x \Psi^{(C_K(x)/C_L(x))}(\rho) \sum_{\psi \in \mathrm{IBr}(C_G(x)|\theta^*)} d^x_{\xi\psi'} \psi'(\rho) \\
&= \sum_{\psi} \epsilon_x \Psi^{(C_K(x)/C_L(x))}(\rho) d^x_{\xi\psi'} \psi'(\rho),
\end{aligned}$$

where $\epsilon_x = \pm 1$. From this and the fact $\Psi^{(C_K(x)/C_L(x))}(\rho) \neq 0$ by Theorem 2.1, (a), we have

$$\sum_{\varphi \in \mathrm{IBr}(B_x)} d^x_{\chi\varphi} R'^{\langle x \rangle}(\varphi)(\rho) = \epsilon_x \sum_{\nu \in \mathrm{IBr}(B'_x)} d^x_{\xi\nu} \nu(\rho)$$

for all $\rho \in C_X(x)_{p'}$. Thus we have $(R'^{\langle x \rangle})_{p'} \circ d_G^{(x,B_x)} = \epsilon_x d_X^{(x,B'_x)} \circ R'$. This completes the proof of the theorem. Q.E.D.

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