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2F-modules with quadratic offender for the finite simple groups

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Abstract.

There is a long running project due to U. Meierfrankenfeld and the author to investigate the so called small modules for the finite simple groups. These modules show up in the amalgam method which recently became important for the revision of parts of the classification of the finite simple groups. A small module either is a quadratic module or a module on which an elementary abelian group acts such that the codimension of the centralizer is small compared with its order. In this paper we determine all irreducible modules V over $GF(2)$ for the finite simple groups G such that $|V : C_V(A)| \leq |A|^2$ for some nontrivial elementary abelian subgroup A of G where in addition we have $[V, A, A] = 1$.

In this paper we are going to determine the irreducible faithful $2F$-modules for the finite quasisimple groups. Here we just concern about $2F$-modules over $GF(2)$. A module V is called $2F$-module for G, if there is a nontrivial elementary abelian subgroup A in G such that $|V/C_V(A)| \leq |A|^2$. The group A then will be called an offending subgroup or an offender. The offender is called quadratic if $[V, A, A] = 1$. More precisely we prove

Theorem *Let G be a quasisimple group and V be an irreducible faithful $2F$-module in characteristic two for G with a quadratic offender. Then one of the following holds*

(i) $G/Z(G) \cong A_n$ *and one of the following is true*
  (a) V *is the natural module*
  (b) $n \leq 8$ *and* $|V| = 16$
  (c) $n = 6$, $|Z(G)| = 3$ *and* $|V| = 64$
  (d) $n = 9$ *and* $|V| = 2^8$ *is the spin module.*

(ii) $G \cong 3M_{22}$ *or* $3U_4(3)$ *and* V *is the 12-dimensional* $SU_6(2)$*–module.*

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(iii) $G/Z(G) = G(q)$ *is a group of Lie type* , $q = 2^t$, *and one of the following is true*

(a) $G(q) \cong L_n(q)$, $Sp(2n,q)'$, $\Omega^{\pm}(2n,q)$ *or* $U_n(q)$ *and* V *is the natural or dual module.*

(b) $G(q) \cong L_n(q)$ *and* V *is the exterior square of the natural or dual module*

(c) $G(q) \cong L_6(q)$, *or* $U_6(q)$ *and* V *is the exterior cube of the natural module.*

(d) $G(q) \cong Sp(6,q)$, $Sp(8,q)$ *or* $Sp(10,q)$ *and* V *is the spin module*

(e) $G(q) \cong \Omega^{\pm}(8,q)$, $\Omega^{\pm}(10,q)$ *or* $\Omega^{+}(12,q)$ *and* V *is the half spin module*

(f) $G(q) \cong E_6(q)$ *and* $V = V(\lambda_1)$ *or* $V(\lambda_6)$.

(g) $G(q) \cong E_7(q)$ *and* $V = V(\lambda_7)$.

(h) $G(q) \cong F_4(q)$ *and* $V = V(\lambda_1)$ *or* $V(\lambda_4)$.

(i) $G(q) \cong G_2(q)'$ *and* V *is the natural module.*

(j) $G(q) \cong Sz(q)$ *and* V *is the natural module.*

The proof of the theorem will depend on two main results. First of all we will use the classification of F–modules. This can be found in an unpublished paper [MeiStr3] due to U. Meierfrankenfeld and the author. A preprint can be found on the homepage (http://coxeter.mathematik.uni-halle.de:8080/~stroth/rep_html). But there is also a classification in the literature. The F-modules for the sporadic groups, alternating groups and groups of Lie type in odd characteristic have been classified by M. Aschbacher [Asch]. The F-modules for the groups of Lie type in characteristic two have been classified by B. Cooperstein [Coop] and in an unpublished paper by B. Cooperstein and G. Mason [CM].

We further will use the classification of quadratic modules in [MeiStr1], [MeiStr2] and [Str] to end up with a very short list of modules and so it is easy to detect the $2F$-modules.

In fact there are still some open question. The module $V(\lambda_1)$ for $E_6(q)$ is a $2F$–module. But we do not know whether it allows a quadratic offender. If not then the cases (iii)(f) - (h) will not show up. Further the paper just considers irreducible modules. So for general module one has to have an overview over the possible offenders, one has to study extensions of the modules above and/or of irreducible $2F$–modules by trivial modules. This all has not been done so far but would be very useful for applications.

The main reason for the classification of these modules comes from the application in the so called amalgam method. This method basically provides us with $2F$–modules with quadratic offender for the groups involved, provided the parameter b is not 1. So we will use the results in this paper in the revision of the classification of the finite simple groups of characteristic two type.

The notations will be standard. Concerning the representations of the groups of Lie type we follow [Stei] and will use these results freely. In what follows G will always be a quasisimple group. A Chevalley group or a group of Lie type will always mean a central factor of the corresponding universal Chevalley group.

By the restriction given by the editors that we do not have more than 10 pages for the paper we had to drop all proofs. The interested reader may download a version containing proofs from the authors homepage.

§1. Preliminaries

For convenience of the reader we first state the main result of [MeiStr3].

Theorem 1.1. *Let $E(G) = F^*(G)$ be a quasisimple group and V be an irreducible faithful F-module in characteristic two for G. Then one of the following holds*

(i) $E(G)/Z(E(G)) \cong A_n$ *and one of the following is true*
  (α) V *is the natural module*
  (β) $n \leq 8$ *and* $|V| = 16$
  (γ) $n = 6$, $|Z(E(G))| = 3$ *and* $|V| = 64$

(ii) $E(G)/Z(E(G)) = G(q)$ *is a group of Lie type ,* $q = 2^t$, *and one of the following is true*
  (α) $G(q) \cong L_n(q)$, $Sp(2n, q)$, $\Omega^{\pm}(2n, q)$ *or* $U_n(q)$ *and* V *is the natural or dual module.*
  (β) $G(q) \cong L_n(q)$ *and* V *is the exterior square of the natural or dual module*
  (γ) $G(q) \cong Sp(6, q)$ *and* V *is the spin module*
  (δ) $G(q) \cong \Omega^+(8, q)$ *or* $\Omega^+(10, q)$ *and* V *is the half spin module*
  (ϵ) $G(q) \cong G_2(q)$ *and* V *is the natural module.*

Lemma 1.2. *Let $q = 2^n$, G be quasisimple with $G/Z(G) = L_2(q)$ and V be a faithful module in characteristic two. Then for any involution $a \in G$ we have $|V : C_V(a)| \geq q$. If $G/Z(G) = Sz(q)$ or $U_3(q)$, then for any involution a we get $|V : C_V(a)| \geq q^2$.*

Lemma 1.3. *Let $q = 2^n$, G be quasisimple with $G/Z(G) = Sz(q)$ or $U_3(q)$ and V be an irreducible faithful $2F$–module for G with quadratic offender A. Then V is the natural module.*

The next five lemmas are more or less well known results on the representations of groups of Lie type. We will use them freely in the sequel.

Proposition 1.4. *Let $G = G(q)$ be a group of Lie type and V be an irreducible module over $GF(q)$. Then*

$$V = V_1^{\sigma_1} \otimes \cdots \otimes V_l^{\sigma_l}$$

where the V_i are basic irreducible $GF(q)G$–modules and the σ_i are field-automorphisms. Further distinct l–tuples $(V_1, \dots V_l)$, $(V_1', \dots , V_l')$ give nonisomorphic $GF(q)G$–modules.

Proof. [Stei, Th 41 and 43] Q.E.D.

Lemma 1.5. *Let $G = G(q)$ be a Chevalley group then $GF(q)$ is a splitting field for any irreducible module.*

Proof. [Stei, (7.5)] Q.E.D.

Lemma 1.6. *Let $G/Z(G) = G(q)$ be a Chevalley group and V an absolutely irreducible KG–module for some $K \subseteq \widetilde{GF(q)}$, where $\widetilde{GF(q)}$ is the algebraic closure of $GF(q)$. Let $\{\sigma_1, \dots \sigma_r\} = Gal_{GF(p)}(K)$. Let $GF(p)(\chi)$ be the field of definition, or splitting field for V. Then $K = GF(p)(\chi)$ iff $V^{\sigma_1}, \dots , V^{\sigma_r}$ are pairwise nonisomorphic KG–modules.*

We like to consider representations of twisted Chevalley groups as well. Here we fix notation as follows. We have ${}^\sigma G(q) \leq G(q^\sigma)$, where $G(q^\sigma)$ is the corresponding untwisted group, $G \neq F_4(q)$ or $B_2(q)$. Now following [Stei, chapter 9] we see that any basic module for $G(q^\sigma)$ reduced to ${}^\sigma G(q)$ remains irreducible. Moreover by [Stei, 9.3] all irreducible modules are given by the tensor product theorem. Further by [Stei, 7.5] $GF(q^\sigma)$ is a splitting field.

It remains the cases $G = F_4(q)$ or $B_2(q)$. Then there is a duality between the long and short roots. Just take all weights which vanish on all long roots. We call these modules restricted to ${}^\sigma G(q)$ the basic modules. Then again the tensor product theorem holds. If the rank is ℓ we now get $q^{\ell/2}$ modules. By [Stei, 12.2] $GF(q)$ is the field of definition for all these modules.

Proposition 1.7. *Let $G = A_\ell(q)$, $D_\ell(q)$, $E_6(q)$ or $D_4(q)$ and V be a basic module with high weight λ for ${}^\sigma G$. Then the following holds for γ the diagram automorphism*

(1) *If $\lambda \neq \gamma(\lambda)$. Then $GF(q^\sigma)$ is the field of definition for V.*
(2) *If $\lambda = \gamma(\lambda)$ then $GF(q)$ is the field of definition for V.*

Lemma 1.8. *Let $G = G(q), q = p^f$, be a Chevalley group and V be an irreducible module in characteristic p over the splitting field. Let P be a parabolic and $V_P = C_V(O_p(P))$. Then V_P is an irreducible P − module.*

Proof. This is [Sm]. Q.E.D.

Definition 1.9. Let V be a faithful $GF(p)$-module for G. For $\epsilon \in I\!R$ define $\mathcal{P}_\epsilon(G,V)$ to be the set of all non-identity p-subgroups X of G such that $|X|^\epsilon |C_V(X)| \geq |Y|^\epsilon |C_V(Y)|$ for all $Y \leq X$ (including $Y = 1$).

If V is a $2F$–module with quadratic offender there is always some quadratic offender in $\mathcal{P}_2(G,V)$.

Lemma 1.10. *Let V be a $2F$-module or F–module for G with quadratic offender and V_1 be an invariant subspace. Then V_1 is a trivial subspace for all quadratic offenders or V_1 is a $2F$-module, F–module respectively, with quadratic offender too.*

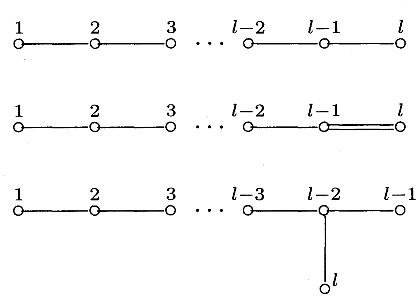
Lemma 1.11. *Let V be a $GF(p)$–module for G.*

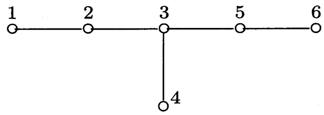
(a) *Let $A, B \in \mathcal{P}_\epsilon(G,V)$ be with $|C_V(A)||A|^\epsilon = |C_V(B)||B|^\epsilon$ be maximal. If $\langle A, B\rangle$ is a p–group then $AB \in \mathcal{P}_\epsilon(G,V)$ and $|C_V(AB)||AB|^\epsilon = |C_V(A)||A|^\epsilon$.*

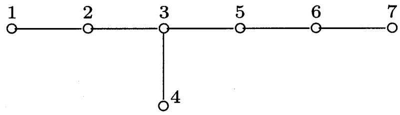
(b) *Let $A \in \mathcal{P}_\epsilon(G,V)$, $A \leq O_p(G)$. If $|C_V(A)||A|^\epsilon$ is maximal among all such A, then $\langle A^G\rangle$ is a (maybe nonabelian) ϵ –offender on V, i.e. $|V : C_V(\langle A^G\rangle)| \leq |\langle A^G\rangle|^\epsilon$.*

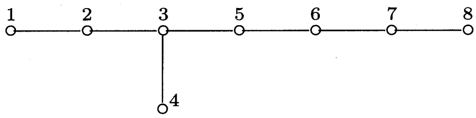
Lemma 1.12. *Let V be a $GF(2)$–module for G with quadratically acting elementary abelian 2–subgroup A. Let $g \in G$ and $a \in A$ with $a^g = az$. Then also $\langle z, C_A(g)\rangle$ acts quadratically.*

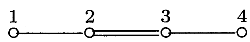
Let W be a Weyl group with root system Φ and fundamental roots Π. We assume throughout this section that the Dynkin diagram on Π is connected. Let $l = |\Pi|$, $I = \{1, 2, \ldots l\}$ and $\Pi = \{\alpha_1, \alpha_2, \ldots \alpha_l\}$ where we choose the labeling as follows:











For the remainder of this paper we fix notation according to the labeling of the diagrams above. Let G be a group of Lie type. We fix a Sylow 2–subgroup S, $B = N_G(S)$, the Borel subgroup, and let $P_1, \dots P_\ell$ be the minimal parabolics containing B. Further for $i \in I$ we denote by $G_i = \langle P_j \mid j \in I \setminus \{i\}\rangle$. These are the maximal parabolics of G. Then G_i/B_{G_i} is a group of Lie type belonging to the diagram for $I \setminus \{i\}$, where B_{G_i} is the largest normal subgroup of G_i contained in B. We assume the reader to be familiar with the structure of the G_i at least those which belong to a connected diagram.

Let now G be as before. Set $K_i = O_2(P_i)$, $i \in I$. Let V be an irreducible module for G over $GF(q)$. Then V is uniquely determined by the action of P_i on $C_V(K_i)$. If $V = V(\lambda)$, $\lambda = \sum_{i=1}^{\ell} a_i\lambda_i$, this means that whenever P_i acts nontrivially on $C_V(K_i)$, we get $a_i \neq 0$ otherwise $a_i = 0$. If all $C_V(K_i)$ are trivial up to one, which is the natural module, then $V = V(\lambda)$ for some fundamental weight λ.

Most modules occurring in this paper, will be fundamental modules. We will get them via [Str], i.e. by showing that they are strong quadratic. Here a module V for $G = G(q)$ is called strong quadratic if there is a group A acting quadratically on V, intersects a root subgroup of G nontrivially but is not contained in that root subgroup.

Lemma 1.13. *Let $q = 2^n$, G be quasisimple with $G/Z(G) = L_n(q)$ or $U_n(q)$ and $V = V(\lambda_1 + \lambda_{n-1})$. Assume $A \leq G$ with $[V, A, A] = 1$. Then $|A| \leq q$.*

Lemma 1.14. *Let V, W be a faithful $GF(q)$–modules and A be quadratic on $X = V \otimes W$. Then $|A| \leq q$.*

Lemma 1.15. *Let $X = V \otimes W$, be a faithful $GF(q)$–modules. Suppose X to be irreducible. If X is a $2F$–module with quadratic offender, then $G \cong L_2(q)$ and $X = V \otimes V^\sigma$, V the natural module and σ some field automorphism of $GF(q)$.*

Lemma 1.16. *Let G be quasisimple with $G/Z(G) = G(q)$ be of Lie type of rank at least two.*

a) *Let $A \leq G$, $|A| \leq q$ and V be an irreducible module with $|V : C_V(A)| \leq |A|^2$ and A quadratic, then V is strong quadratic.*

b) *Let R be a root group in G with $|[V, R]| \leq q^2$ for some irreducible module V, then $G/Z(G) \cong L_n(q)$, $U_n(q)$, $\Omega^{\pm}(2n, q)$, $Sp(2n, q)$ and V is the natural module, or $G \cong G_2(q)$ and V is the 6–dimensional module.*

Lemma 1.17. *Let G be quasisimple with $G/Z(G) \cong L_n(q)$, $n \geq 5$, let $V = V(\lambda_2)$ and A be an offender as F-module. Then $|A| = q^{n-1} = |V : C_V(A)|$.*

§2. $2F$–modules for Lie type groups in even characteristic

Throughout this chapter we will assume that G is a quasisimple group with $G/Z(G)$ a group of Lie type over a field with $q = 2^n$ elements, including $G(2)'$ and A_6. We additionally assume that $G' \not\cong 3 \cdot A_6$, as this will be handled in the last chapter together with the alternating groups. As G will act faithfully on a $GF(2)$–module, G will always be a factor of the universal group.

Further V is an $2F$-module and $A \in \mathcal{P}_2(G, V)$ is a quadratic offending subgroup. If G_i is a maximal parabolic in G we set $Q_i = O_2(G_i)$.

If V is an irreducible $GF(2)$–module for G. Then $V \otimes GF(q)$ is a direct sum of algebraic conjugates of some irreducible $GF(q)$–module M. If $V \otimes GF(q)$ is an algebraic conjugate of a fundamental module for a weight λ we also write $V = V(\lambda)$. In fact as we usually will have strong quadratic modules, which then by [Str] are defined over $GF(q)$, we usually just have to handle $GF(q)$–modules.

Lemma 2.1. *Let $G/Z(G) = G(q)$ be classical or $F_4(q)$ and $V = V(\lambda_2)$ be a $2F$-module with quadratic offender A. Then V is an F–module too.*

Lemma 2.2. *Let $G/Z(G) = G(q)$ and V be an irreducible faithful $2F$–module with quadratic offender. If $C_V(Q_i) = C_V(Z(S))$ for some i, then $V = V(\lambda_i)$ or $G/Z(G) \cong L_2(q)$ and $V = X \otimes X^\sigma$ for some field automorphism σ, X the natural module.*

Lemma 2.3. *Let $G/Z(G) = G(q) \not\cong L_n(q)$ be classical and V be an irreducible faithful $2F$–module with quadratic offender $A \leq Q_1$. Then the pair (G, V) is one of the theorem.*

Lemma 2.4. *Let $G/Z(G) = G(q)$ be of rank at most two and V be an irreducible faithful $2F$–module with quadratic offender A. Then $G/Z(G) \cong L_2(q)$, $L_3(q)$, $Sp(4,q)$, $U_4(q)$, $U_5(q)$, $\Omega^-(6,q)$ or $G_2(q)$ and V is the natural module, or $G \cong L_2(q)$ and V is a tensor product.*

Lemma 2.5. *Let $G/Z(G) = G(q) \not\cong L_n(q)$ be classical of rank at least three and $V = V(\lambda_n)$ be a $2F$–module with quadratic offender A. Then (G, V) is one of the theorem.*

Lemma 2.6. *Let $G \cong \Omega^-(2n,q)$, $n > 3$, and V be an irreducible faithful $2F$–module with quadratic offender A. Then $V \cong V(\lambda_1)$, or $n = 4, 5$ and $V \cong V(\lambda_n)$.*

Lemma 2.7. *Let $G \cong Sp(2n,q)$, $n \leq 6$, and $V \cong V(\lambda_1 + \lambda_n)$. Then V is not a $2F$–module with quadratic offender.*

Proposition 2.8. *Let $G/Z(G) = G(q)$ be classical and V be an irreducible faithful $2F$–module with quadratic offender A. Then one of the following holds*

(i) *$G(q) \cong L_n(q)$, $V \cong V(\lambda_1)$, $V(\lambda_2)$, $V(\lambda_{n-2})$, $V(\lambda_{n-1})$.*
(ii) *$G(q) \cong Sp(2n,q)$, $\Omega^{\pm}(2n,q)$, or $U_n(q)$ and $V \cong V(\lambda_1)$.*
(iii) *$G(q) \cong L_6(q)$, or $U_6(q)$ and $V \cong V(\lambda_3)$.*
(iv) *$G(q) \cong Sp(2n,q)$, $n = 3, 4, 5$, and V is the spin module.*
(v) *$G(q) \cong \Omega^{\pm}(2n,q)$, $n = 4, 5$, or $\Omega^+(12,q)$ and V is the half spin module.*
(vi) *$G(q) \cong L_2(q^2)$ and $V \cong V_1^\sigma \otimes V_1$, where V_1 is the natural module and σ the field automorphism of order two.*

Proposition 2.9. *Let $G/Z(G) = E_n(q)$, $n = 6, 7, 8$, and V be an irreducible faithful $2F$–module with quadratic offender A. Then $n = 6$ and $V \cong V(\lambda_1)$ or $V(\lambda_6)$, or $n = 7$ and $V \cong V(\lambda_7)$.*

Lemma 2.10. *If V is an irreducible faithful $2F$–module with quadratic offender A for the group $G \cong F_4(q)$, then $V \cong V(\lambda_1)$ or $V(\lambda_4)$.*

Lemma 2.11. *The group $G/Z(G) \cong {}^2E_6(q)$ does not possess a $2F$–module with quadratic offender.*

So we have shown

Proposition 2.12. *Let $G/Z(G) = G(q)$ be of Lie type, V be an irreducible faithful $2F$–module over $GF(2)$ with quadratic offending group. Then one of the following holds*

(i) $G(q) \cong L_n(q)$, $V \cong V(\lambda_1)$, $V(\lambda_2)$, $V(\lambda_{n-2})$, $V(\lambda_{n-1})$.
(ii) $G(q) \cong Sp(2n, q)$, $\Omega^{\pm}(2n, q)$ *or* $U_n(q)$ *and* $V \cong V(\lambda_1)$.
(iii) $G(q) \cong L_6(q)$ *or* $U_6(q)$ *and* $V \cong V(\lambda_3)$.
(iv) $G(q) \cong Sp(2n, q)$, $n = 3, 4, 5$, *and* V *is the spin module.*
(v) $G(q) \cong \Omega^{\pm}(2n, q)$, $n = 4, 5$, *or* $\Omega^{+}(12, q)$ *and* V *is the half spin module.*
(vi) $G(q) \cong L_2(q^2)$ *and* $V \cong V_1^{\sigma} \otimes V_1$, *where* V_1 *is the natural module and* σ *the field automorphism of order two.*
(vii) $G(q) \cong E_6(q)$ *and* $V \cong V(\lambda_1)$ *or* $V(\lambda_6)$.
(viii) $G(q) \cong E_7(q)$ *and* $V \cong V(\lambda_7)$.
(ix) $G(q) \cong F_4(q)$ *and* $V \cong V(\lambda_1)$ *or* $V(\lambda_4)$.

§3. $2F$-modules for alternating, sporadic and Lie type groups in odd characteristic

Throughout this chapter we will assume that G is a perfect central extension of an alternating group, a sporadic group or a group of Lie type over a field of odd characteristic, which is not a group of Lie type over a field of characteristic 2 too. Further V is a $2F$-module over $GF(2)$ and $A \in \mathcal{P}_2(G, V)$ an offending subgroup which acts quadratically.

Lemma 3.1. *Let* $G = A_n$ *or* $G = 3A_m$, $m = 6, 7$, *and* V *be an irreducible faithful* $2F$*–module over* $GF(2)$ *with quadratic offender* A, *then either* V *is the permutation module or*

(i) $G \cong A_8 \cong L_4(2)$ *and* V *is the natural* $L_4(2)$*-module.*
(ii) $G \cong A_7$ *and* V *is as in* (i).
(iii) $G \cong 3A_6$ *and* $|V| = 2^6$.
(iv) $G \cong A_5 \cong L_2(4)$ *and* V *is the natural* $L_2(4)$*-module.*
(v) $G \cong A_9$ *and* V *is the eight dimensional spin module.*

Lemma 3.2. *Let* $G/Z(G)$ *be sporadic and* V *be an irreducible faithful* $2F$*-module with quadratic offender* A. *Then* $G \cong 3M_{22}$ *and* V *is the* 12*-dimensional module coming from the embedding into* $SU_6(2)$.

Lemma 3.3. *Let* V *be an irreducible faithful* $2F$*-module with quadratic offender* A. *If* G *is some covering group of a group of Lie type in odd characteristic then it is a group of Lie type in even characteristic too, or* $G \cong 3 \cdot U_4(3)$ *and* V *is the* 12*-dimensional module, coming from the embedding into* $SU_6(2)$.

References

[Asch] M. Aschbacher, $GF(2)$-representations of finite groups, Amer. J. of Math., **104** (1982), 683–771.

[CM] B. Cooperstein and G. Mason, Some questions concerning the representations of Chevalley groups in characteristic two, preprint 1977.

[Coop] B. Cooperstein, An enemies list for factorization theorems, Comm. Alg., **6** (1978), 1239–1288.

[CurRei] C.W. Curtis and I. Reiner, Representation theory of finite groups and association algebras, Interscience, New York, 1962.

[Go] D. Gorenstein, Finite groups, Harper and Row, New York 1968.

[JLPW] C. Jansen, R. Lux, R. Parker and R. Wilson, An atlas of Brauer characters, Clarendon Press, Oxford 1995.

[MeiStr1] U. Meierfrankenfeld and G. Stroth, Quadratic $GF(2)$ - modules for sporadic simple groups and alternating groups, Comm. Algebra, **18** (1990), 2099–2139.

[MeiStr2] U. Meierfrankenfeld and G. Stroth, On quadratic $GF(2)$-modules for Chevalley groups over fields of odd order, Arch. Math., **55** (1990), 105–110.

[MeiStr3] U. Meierfrankenfeld and G. Stroth, F-modules for finite simple groups, preprint.

[Sm] S. Smith, Irreducible modules and parabolic subgroups, J. Algebra, **75** (1982), 286–289.

[Stei] R. Steinberg, Lectures on Chevalley groups, Notes by J. Faulkner and R. Wilson, Mimeographed notes, Yale University, Mathematics Department, 1968.

[Str] G. Stroth, Strong quadratic modules, Israel J. Math., **79** (1992), 257–279.

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