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From Chern classes to Milnor classes A history of characteristic classes for singular varieties

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Abstract.

In this paper, we give a survey and recent developments about the definitions of characteristic classes for possibly singular complex analytic (or algebraic) varieties. We recall the classical construction of characteristic classes in the case of manifolds, by obstruction theory and using Schubert cycles. Then, we present various generalizations of characteristic classes to singular varieties, due to M.H. Schwartz, W.T. Wu, J. Mather, R. MacPherson, W. Fulton and K. Johnson and we discuss relations among these definitions. More recent results concern the definition and properties of so-called Milnor classes, as developped by P. Aluffi, J.P. Brasselet-D. Lehmann-J. Seade-T. Suwa, A. Parusiński-P. Pragacz and S. Yokura.

§0. Introduction

The Euler-Poincaré characteristic was the first characteristic class (or number) to be introduced. For a triangulable (possibly singular) compact variety X without boundary, it can be defined, as

$$\chi(X) = \sum (-1)^i n_i \,,$$

where n_i is the number of i-dimensional simplices. It is also equal to $\sum(-1)^i b_i$ where $b_i = \mathrm{rk} H_i(X)$. The Poincaré-Hopf theorem says that, if X is a manifold and v a (continuous) vector field with a finite number of isolated singularities a_k of indices $I(v, a_k)$, then

$$\chi(X) = \sum I(v, a_k).$$

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This means that the Euler-Poincaré characteristic measures the obstruction to the existence of a non-zero vector field tangent to X.

On another hand, characteristic classes of projective varieties have been defined by Severi, Todd and others using polar varieties. Then Chern defined such characteristic clases for hermitian manifolds, in several ways, in particular as measuring the obstruction to the construction of complex r-frames tangent to the manifold, and using Schubert varieties (related to polar varieties). During some time, the attractiveness of the axiomatic properties of Chern classes caused the viewpoint of polar varieties to be somewhat forgotten.

For singular varieties, it appears that Wu and Mather classes can be defined in terms of polar varieties, with a formula similar to the non-singular case. On the other hand, the obstruction theory Chern's point of view has been generalized by M.H. Schwartz, and the axiomatic point of view by R. MacPherson. The Schwartz and MacPherson classes coincide, via Alexander duality.

The Fulton and Fulton-Johnson classes use Segre classes definition, without reference to the original definitions of Chern classes of varieties (for complete intersections they correspond to the Chern classes of the virtual bundle, generalization of the tangent bundle).

A natural question was to compare the Schwartz-MacPherson and the Fulton-Johnson classes. A result of Suwa shows that in the case of isolated singularities, the difference is given by the Milnor numbers in the singular points. It was natural to call Milnor classes the difference arising in the general case. This difference has been described by several authors by different means.

In this paper, cohomology classes will be constructed in the context of cell decompositions in order to keep things consistent with Poincaré duality. We will denote by M a complex manifold, by (K) a triangulation of M, (K') a barycentric subdivision of (K) and (D) the associated dual cell decomposition. The dual cell of a simplex $\sigma \in K$ will be denoted by $d(\sigma)$ or simply d if there is no possible confusion. The barycenter $\hat{\sigma}$ is the intersection point $\hat{\sigma} = \hat{d}(\sigma) = d(\sigma) \cap \sigma$. The (D)-cochain whose value is 1 at $d(\sigma)$ and 0 at other cells of (D) will be denoted by $\widetilde{d}(\sigma)$.

In the sequel, all homology and cohomology groups will be understood with integer coefficients. Recall that if M is a compact complex m-dimensional manifold, the Poincaré duality isomorphism

$$H^{2m-i}(M) \longrightarrow H_i(M),$$

the cap-product with the fundamental class $[M] \in H_{2m}(M)$, is represented at the chain level as the homomorphism $C^{2m-i}_{(D)}(M) \to C^{(K)}_i(M)$

sending the elementary (D)-cochain $\widetilde{d}(\sigma)$ to the elementary (K)-chain σ.

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§1. Chern classes in the non-singular case.

In his original paper [Ch], Chern gave several constructions of characteristic classes for Hermitian manifolds: by obstruction theory, using the decomposition of the Grassmann manifold in Schubert cycles, using differential forms and by transgression cocycles. We will briefly recall the first two definitions, which extend to singular varieties. The paper [Ch] is highly recommended for the study of Chern classes.

1.1. Chern classes by obstruction theory.

Let us recall the idea of constructing Chern classes by obstruction theory (see [Ch]), following Steenrod [Ste], part III.

We denote by TM the complex tangent bundle to the complex m-dimensional manifold M and by T_rM the bundle of complex r-frames tangent to M. The fiber of T_rM over a point $x \in M$ is the Stiefel manifold $W_{r,m}$ of complex r-frames in $\mathbf{C}^m$. Let $d = d(\sigma)$ be a k-cell in a trivialization domain U of T_rM, i.e. $T_rM|_U \cong U \times W_{r,m}$. Let us suppose that we are given an r-frame $v^{(r)} = (v_1, \ldots, v_r)$ on the boundary ∂d of d. This defines a section of T_rM over ∂d and, by composition, a map

$$S^{k-1} \cong \partial d \xrightarrow{v^{(r)}} T_rM|_U \xrightarrow{\mathrm{pr}_2} W_{r,m}$$

where pr_2 denotes the projection to the second factor. We thus obtain an element

$$[v^{(r)}; \partial d] \in \pi_{k-1}(W_{r,m})$$

which vanishes if and only if the r-frame $v^{(r)}$ can be extended without singularity to all of d. We remark that if this element is non-zero, then we can extend the r-frame to the relative interior of d by homothety centered at the barycenter $\hat{d} = \hat{d}(\sigma)$, thus obtaining an isolated singularity of index $[v^{(r)}; \partial d]$.

Let us recall ([Ste], §25.7) that

$$\pi_i(W_{r,m}) = \begin{cases} 0 & \text{for } i < 2m - 2r + 1, \\ \mathbf{Z} & \text{for } i = 2m - 2r + 1. \end{cases}$$

This result implies that we can construct an r-frame $v^{(r)}$, i.e. a section of T_rM, by induction on the dimension of cells of the given cell decomposition of M without singularity up to the $(2m-2r+1)$-skeleton and with isolated singularities on the $2p = 2(m-r+1)$-skeleton. For each $2p$-cell $d(\sigma)$, the index of the complex r-frame $v^{(r)}$ at its only singular point $\hat{d} = d(\sigma) \cap \sigma$ in d is $I(v^{(r)}, \hat{d}) = [v^{(r)}; \partial d] \in \mathbf{Z}$. Associating to each p-cell $d(\sigma)$ the integer $I(v^{(r)}, \hat{d})$ defines a $2p$-cochain that actually is a cocycle, called the obstruction cocycle.

Definition ([Ch]). The p-th (cohomology) *Chern class* of M, $c^p(M) \in H^{2p}(M; \mathbf{Z})$ is the class of the obstruction cocycle.

By the Poincaré duality isomorphism, the image of $c^p(M)$ in $H_{2(r-1)}(M)$ is the $(r-1)$-st homology Chern class of M represented by the cycle

$$\sum_{\dim \sigma = 2(r-1)} I(v^{(r)}, \hat{d}(\sigma))\ \sigma. \tag{1}$$

In particular, the evaluation of $c^m(M)$ on the fundamental class $[M]$ of M yields the Euler-Poincaré characteristic.

1.2. Chern classes using Schubert cycles and polar varieties.

The construction of Chern classes using Schubert cycles was already present in Chern's original paper. This construction was emphasized by Gamkrelidze in [Ga1] and [Ga2]. A historical introduction and complete bibliography can be found in the Teissier's paper [T2].

The Schubert cell decomposition of the Grassmann manifold $\mathcal{G} = \mathcal{G}(n,m)$ of n-planes in $\mathbf{C}^m$ has been described by Ehresmann [Eh] and it was used by Chern to give an alternative definition of his characteristic classes. Let

$$\{0\} = D_m \subset D_{m-1} \subset \cdots \subset D_1 \subset D_0 = \mathbf{C}^m \tag{$\mathcal{D}$}$$

be a flag in $\mathbf{C}^m$, with $\mathrm{codim}_{\mathbf{C}} D_j = j$.

For each integer k, with $0 \leq k \leq n$, the k-th Schubert variety associated to $\mathcal{D}$, defined by

$$M_k(\mathcal{D}) = \{T \in \mathcal{G}(n,m) : \dim(T \cap D_{n-k+1}) \geq k\}$$

is an algebraic subvariety of $\mathcal{G}(n,m)$ of pure codimension k. The inequality condition is equivalent to saying that T and D_{n-k+1} do not span $\mathbf{C}^m$.

Let θ^n be the universal (sub)bundle over $\mathcal{G}(n,m)$. The cycle

$(-1)^k M_k(\mathcal{D})$ represents the image, under the Poincaré duality isomorphism, of the Chern class $c^k(\theta^n) \in H^{2k}(\mathcal{G}(n,m))$. If V is an n-dimensional complex analytic manifold and $f : V \to \mathcal{G}(n,m)$ is the classifying map for TV, i.e. such that $TV \cong f^*(\theta^n)$, then the cohomological Chern classes of V are $c^k(V) = c^k(TV) = f^*(c^k(\theta^n))$ (see [MS]).

Let us consider the projective situation. We denote by $G(n,m)$ the Grassmann manifold of n-dimensional linear subspaces in $\mathbf{P}^m$. We fix a flag of projective linear subspaces

$$(\mathcal{D}) \qquad L_m \subset L_{m-1} \subset \cdots \subset L_1 \subset L_0 = \mathbf{P}^m$$

where $\operatorname{codim}_{\mathbf{C}} L_j = j$. The k-th Schubert variety associated to $\mathcal{D}$ is defined by

$$M_k(\mathcal{D}) = \{\widetilde{T} \in G(n,m) : \dim(\widetilde{T} \cap L_{n-k+2}) \geq k-1\}$$

Let us remark that we always have $\dim(\widetilde{T} \cap L_{n-k+2}) \geq k-2$. The Schubert variety $M_k(\mathcal{D})$ has codimension k in $G(n,m)$.

Let us denote $N = nm = \dim_{\mathbf{C}} G(n,m)$ and fix $0 \leq s \leq m$. The Schubert variety

$$(2) \qquad \begin{aligned} M_k^{N-s} &= \{(x,T) : x \in L_{s-k},\ x \in T,\ \dim(T \cap L_{n-k+2}) \geq k-2\} \\ &= L_{s-k} \cap M_k(\mathcal{D}) \end{aligned}$$

is the intersection of $M_k(\mathcal{D})$ with a general $(s-k)$-codimensional plane and it has codimension s in $G(n,m)$. The (homological) Chern classes of $G(n,m)$ are

$$(3) \qquad c_{N-s}(G(n,m)) = \sum_{k=0}^{s} (-1)^k \binom{n-s+1}{n-k+1} M_k^{N-s}.$$

Let us now consider the case of an n-projective manifold $V \subset \mathbf{P}^m$. The k-th polar variety is defined by

$$P_k = \{x \in V : \dim(T_x(V) \cap L_{n-k+2}) \geq k-1\},$$

where $T_x(V)$ is the projective tangent space to V at x. For L_{n-k+2} sufficiently general, the codimension of P_k in V is equal to k. Also, the class $[P_k]$ of P_k modulo rational equivalence in the Chow group $A_{n-k}(V)$ does not depend on L_{n-k+2} for L_{n-k+2} sufficiently general. This class is called the k-th polar class of V.

Let $\gamma : V \to G(n,m)$ be the Gauss map, i.e. the map defined by

$$\gamma(x) = T_x(V) \subset \mathbf{P}^m.$$

Then

$$P_k = \gamma^{-1}(M^k(\mathcal{D})).$$

The relation between Chern classes and polar classes has been described by Gamkrelidze and Todd.

If $\mathcal{L} = \mathcal{O}_{\mathbf{P}^m}(1)|_V$, then we obtain the Todd formula (compare with (3)):

$$(4) \qquad c_{n-s}(V) = \sum_{k=0}^{s} (-1)^k \binom{n-s+1}{n-k+1} c^1(\mathcal{L})^{s-k} \cap [P_k]$$

where the cap-product with $c^1(\mathcal{L})^{s-k}$ is equivalent to the intersection with a general $(s-k)$-codimensional plane.

§2. Chern classes in the singular case.

In the singular case, there are different possible definitions of Chern classes, generalizing the ones in the non-singular case.

The Wu and Mather classes generalize the definitions by Schubert cycles and polar varieties. J. Zhou proved that Wu and Mather classes coincide.

The Schwartz classes use obstruction theory, and the MacPherson classes, defined in an algebraic geometry way, satisfy good functorial properties. J.P. Brasselet and M.H. Schwartz proved that Schwartz and MacPherson classes coincide, via Alexander duality.

The Fulton and Fulton-Johnson definitions of Chern classes use Segre classes and correspond to the class of the virtual tangent bundle in the case of local complete intersections (for example).

The relation between Wu-Mather classes and Schwartz-MacPherson classes appears in MacPherson's definition itself. The MacPherson construction uses Wu-Mather classes, taking into account the local complexity of the singular locus along Whitney strata. This is the role of the local Euler obstruction.

The difference between Schwartz-MacPherson and Fulton-Johnson classes is expressed, in the case of isolated singularities, in terms of the Milnor numbers (at the singularities) (Seade-Suwa). In the general case, this difference is called Milnor class and has been studied by several authors: P. Aluffi, J.P. Brasselet-D. Lehmann-J. Seade-T. Suwa, A. Parusiński-P. Pragacz and S. Yokura.

2.1. The Wu classes (1965).

In the singular case, Wu [Wu2] generalized Chern's and Gamkrelidze's constructions in the following way: Let $X^n \subset \mathbf{P}^m$ be a complex projective algebraic variety and let X' be a subvariety of X containing the singular part X_{sing}. Denoting by $A_*(X)$ the Chow group of classes of algebraic cycles of X and with $A_*(X, X')$ the subgroup of classes that have no component in X', there is a natural inclusion

$$J : A_*(X, X') \to A_*(X).$$

Wu defines a notion of transform of X, which coincides with the Nash transform (see [Z1]). We recall the definition of Nash transform, the original definition of Wu being slightly different.

Let us denote by $\nu : G \to \mathbf{P}^m$ the Grassmann bundle over $\mathbf{P}^m$ whose fibre over x is the Grassmann manifold $G(n, m)$ of n-linear subspaces in $T_x\mathbf{P}^m$. The Gauss map $\gamma : X_{\text{reg}} \to G$ is defined on the regular part $X_{\text{reg}} = X \setminus X_{\text{sing}}$ of X by

$$\gamma(x) = T_x(X_{\text{reg}}) \subset T_x\mathbf{P}^m$$

The *Wu* (or *Nash*) *transform* $\widetilde{X}$ is defined as the closure of the image of γ in G. In general $\widetilde{X}$ is singular; nevertheless, if X is an analytic variety, then $\widetilde{X}$ is also analytic, and the restriction $\nu : \widetilde{X} \to X$ of the projection $\nu : G \to \mathbf{P}^m$ is analytic. It induces a map

$$\nu_* : A_*(\widetilde{X}, \widetilde{X}') \to A_*(X, X')$$

where $\widetilde{X}' = \nu^{-1}(X')$.

The (transverse) intersection of cycles with $\widetilde{X}$ defines a map

$$A_{d-s}(G) \xrightarrow{I} A_{n-s}(\widetilde{X}, \widetilde{X}'),$$

with $\dim_{\mathbf{C}} G = d$.

Finally, let $D : A_s(G) \to A_{d-s}(G)$ be the duality map in G. The composition $W = J \circ \nu_* \circ I \circ D$ is a map

$$W_s : A_s(G) \to A_{n-s}(X)$$

In analogy to the formula (3), we have:

Definition ([Wu1]). The Wu classes are defined by

$$c^W_{n-s}(X) = \sum_{k=0}^{s} (-1)^k \binom{n-s+1}{n-k+1} W_s(M^s_k)$$

2.2. The Mather classes (1974).

R. MacPherson named Mather classes the classes that Mather described to him on a blackboard (see [M2]). Let us recall their definition.

Let X be an n-dimensional analytic complex subvariety X of an m-dimensional manifold M. We consider the Nash transform $\widetilde{X}$ and denote by E the tautological bundle over the Grassmann bundle G. The fiber of E over $P \in G$ is

$$E_P = \{\, v(x) \in T_x(M) : \; v(x) \in P, \; x = \nu(P)\,\}.$$

Let us denote by $\widetilde{E}$ the restriction of E to $\widetilde{X}$. We have a commutative diagram:

$$\begin{array}{ccc} \widetilde{E} & \hookrightarrow & E \\ \downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & G \\ \downarrow & & \downarrow \\ X & \hookrightarrow & M \end{array}$$

Definition ([M2]). The Mather class of X is defined by

$$c^M(X) = \nu_*(c^*(\widetilde{E}) \cap [\widetilde{X}]),$$

where $c^*(\widetilde{E})$ denotes the usual (total) Chern class of the bundle $\widetilde{E}$ in $H^*(\widetilde{X})$ and the cap-product with $[\widetilde{X}]$ is the Poincaré duality homomorphism (in general not an isomorphism).

The Mather class can be defined by using polar varieties in the following way: First of all, let us consider the local situation. For a general flag $\mathcal{D}$ and an affine variety $X^n \subset \mathbf{C}^m$, we define

$$\begin{array}{ccccccc} & & \widetilde{X} & \hookrightarrow & \mathcal{G}(n,m)\times \mathbf{C}^m & \xrightarrow{\pi_1} & \mathcal{G}(n,m) \\ & \sigma \nearrow & \downarrow \nu & & \downarrow \pi_2 & & \\ X_{\mathrm{reg}} & \hookrightarrow & X & \hookrightarrow & \mathbf{C}^m & & \end{array}$$

and we denote by $\widetilde{\gamma} = \pi_1|_{\widetilde{X}} : \widetilde{X} \to \mathcal{G}(n,m)$ the Gauss map.

Let us define the following analytic subspace of X [LT]:

$$N_k(\mathcal{D}) = \nu \circ \widetilde{\gamma}^{-1}(M_k(\mathcal{D})) = \overline{\nu(\widetilde{\gamma}^{-1}(M_k(\mathcal{D})) \cap \sigma(X_{\mathrm{reg}}))}$$

If the flag $\mathcal{D}$ is good (sufficiently general), i.e. $\widetilde{\gamma}$ is transverse to the strata

$$M_{k,i}(\mathcal{D}) = \{W \in \mathcal{G}(n,m) : \mathrm{codim}(W + D_{n-k+i-1}) = k+1\}$$

of $M_k(\mathcal{D})$, then the cycle $N_k(\mathcal{D})$ is well defined and independent of the choice of the (good) flag. In that case, it is called the polar variety (Lê-Teissier).

If the flag $\mathcal{D}$ is good, and still in the local situation, let $\pi : X \to \mathbf{C}^{n-k+1}$ be the restriction to X of a linear projection with kernel D_{n-k+1}, then $N_k(\mathcal{D})$ is the closure (in X) of the critical locus of the restriction of π to X_{reg} [LT].

In the projective case, the polar variety is the closure of

$$\{x \in X_{\text{reg}} : \dim(T_x(X_{\text{reg}}) \cap L_{n-k+2}) \geq k-1\} \tag{5}$$

where $\operatorname{codim}_{\mathbf{C}^m} L_{n-k+2} = n-k+2$.

Now, if $X^n \subset \mathbf{P}^m$ is a projective variety, then (see (4) and [Pi2])

$$c^M_{n-s}(X) = \sum_{k=0}^{s} (-1)^k \binom{n-s+1}{n-k+1} c^1(\mathcal{L})^{s-k} \cap [N_k(\mathcal{D})]$$

where $\mathcal{L} = \mathcal{O}_{\mathbf{P}^m}(1)|_X$.

Theorem ([Z1]). *Let X be a projective variety. Then the Mather and Wu classes of X coincide.*

The Mather classes can be also expressed in terms of conormal space, notion which is strongly related to the one of polar variety (see [T1] and [S]). The conormal space is the subvariety of the cotangent bundle T^*M of M defined as the closure of

$$T^*_X M = \{(x,\xi) \in T^*M : x \in X_{\text{reg}}, \xi|_{T_x(X_{\text{reg}})} \equiv 0\}.$$

We denote by $C(X,M) \subset \mathbf{P}T^*M$ the projectivization of the conormal space and by τ the projection $\tau : C(X,M) \to X$, restriction of the projection $\mathbf{P}T^*M \to M$ to $C(X,M)$. By [S] (see also [PP4] and [Ke1]), we have

$$c^M_*(X) = (-1)^{m-n-1} c(TM|_X) \cap \tau_* \left(c(\mathcal{L})^{-1} \cap [C(X,M)]\right).$$

The Mather classes do not verify the Deligne-Grothendieck axioms that we recall below. That is the MacPherson's motivation for introducing the so-called Schwartz-MacPherson classes.

2.3. The Schwartz classes (1965).

The first definition of Chern class for singular varieties was given in 1965 by M.H. Schwartz in two "Notes aux CRAS" [Sc1]. We briefly recall her construction. Let $X \subset M$ be a singular n-dimensional complex variety embedded in a complex m-dimensional manifold. Let us consider a Whitney stratification $\{V_\alpha\}$ of M [Wh] such that X is a union of strata and denote by (K) a triangulation of M compatible with the stratification, i.e. each open simplex is contained in a stratum.

As before, we denote by (K') a barycentric subdivision of (K) and (D) the associated dual cell decomposition. Each cell of (D) is transverse to the strata. This implies that if d is a cell of dimension $2p = 2(m-r+1)$ and V_α is a stratum of dimension $2s$, then $d \cap V_\alpha$ is a cell such that

$$\dim(d \cap V_\alpha) = 2(s - r + 1)$$

This means that if d is a cell whose dimension is the dimension of obstruction to the construction of an r-frame tangent to M, then $d \cap V_\alpha$ is a cell whose dimension is exactly the dimension of obstruction to the construction of an r-frame tangent to the stratum V_α.

This fact leads M.H. Schwartz to the very nice construction of a stratified radial r-frame in the following way:

An r-frame $v^{(r)}$, defined on a part $A \subset M$, is called a *stratified r-frame* if at each point $x \in A$, $v^{(r)}(x)$ is tangent to the stratum V_α containing x. In the following we write $v^{(r)}$ as $(v^{(r-1)}, v_r)$, the last vector being individualized.

Proposition ([Sc1] [Sc2]). *One can construct, on the $2p$-skeleton $(D)^{2p}$, a stratified r-frame $v^{(r)}$, called radial frame, whose singularities satisfy the following properties:*

(i) *$v^{(r)}$ has only isolated singular points, which are zeroes of the last vector v_r. On $(D)^{2p-1}$, the r-frame $v^{(r)}$ has no singular point and on $(D)^{2p}$ the $(r-1)$-frame $v^{(r-1)}$ has no singular point.*

(ii) *Let $a \in V_\alpha \cap (D)^{2p}$ be a singular point of $v^{(r)}$ in the $2s$-dimensional stratum V_α. If $s > r - 1$, the index of $v^{(r)}$ at a, denoted by $I(v^{(r)}, a)$, is the same as the index of the restriction of $v^{(r)}$ to $V_\alpha \cap (D)^{2p}$ considered as an r-frame tangent to V_α. If $s = r - 1$, then $I(v^{(r)}, a) = +1$.*

(iii) *Inside a $2p$-cell d which meets several strata, the only singularities of $v^{(r)}$ are inside the lowest dimensional one (in fact located in the barycenter of d).*

(iv) *The r-frame $v^{(r)}$ is "pointing outward" a (particular) regular neighborhood U of X in M. It has no singularity on ∂U.*

The procedure of the construction of radial frames is made by induction on the dimension of the strata, using the properties of Whitney stratifications for proving the existence of frames "pointing outward" regular neighborhoods and satisfying property (ii). An r-frame already known on a neighborhood of the boundary of a stratum is extended with isolated singularities inside (a suitable skeleton) of the stratum and then extended with property (ii) to a regular neighborhood of this stratum.

Let us denote by $\mathcal{T}$ the tubular neighborhood of X in M consisting of the (D)-cells which meet X. Let us recall that $\widetilde{d}$ is the elementary (D)-cochain whose value is 1 at d and 0 at all other cells. We can define a $2p$-dimensional (D)-cochain in $C^{2p}(\mathcal{T}, \partial\mathcal{T})$ by:

$$\sum_{d \in \mathcal{T}} I(v^{(r)}, \hat{d})\, \widetilde{d}.$$

This cochain is a cocycle whose class lies in

$$H^{2p}(\mathcal{T}, \partial\mathcal{T}) \cong H^{2p}(\mathcal{T}, \mathcal{T} \setminus X) \cong H^{2p}(M, M \setminus X),$$

where the first isomorphism is given by retraction and the second by excision.

Definition ([Sc1] [Sc2]). The p-th Schwartz class $c^p(X)$ is the class obtained in $H^{2p}(M, M \setminus X)$.

2.4. The MacPherson classes (1974).

Let us recall firstly some basic definitions.

A *constructible set* in a variety X is a subset obtained by finitely many unions, intersections and complements of subvarieties. A *constructible function* $\alpha : X \to \mathbf{Z}$ is a function such that $\alpha^{-1}(n)$ is a constructible set for all n. The constructible functions on X form a group denoted by $\mathbf{F}(X)$. If $A \subset X$ is a subvariety, we denote by $\mathbf{1}_A$ the characteristic function whose value is 1 over A and 0 elsewhere.

If X is triangulable, α is a constructible function if and only if there is a triangulation (K) of X such that α is constant on the interior of each simplex of (K). Such a triangulation of X is called α-adapted.

The correspondence $\mathbf{F} : X \to \mathbf{F}(X)$ defines a contravariant functor when considering the usual pull-back $f^* : \mathbf{F}(Y) \to \mathbf{F}(X)$ for a morphism $f : X \to Y$. One interesting fact is that it can be made a covariant functor when considering the pushforward defined on characteristic functions by:

$$f_*(\mathbf{1}_A)(y) = \chi(f^{-1}(y) \cap A),$$

for all $y \in Y$, and linearly extended to elements of $\mathbf{F}(X)$.

The following result was conjectured by Deligne and Grothendieck in 1969 and proved by R. MacPherson [M2] in 1974.

Theorem ([M2]). *Let* $\mathbf{F}$ *be the covariant functor of constructible functions and let* $H_*(\ ;\mathbf{Z})$ *be the usual covariant* $\mathbf{Z}$*-homology functor.*

Then there exists a unique natural transformation

$$c_* : \mathbf{F} \to H_*(\ ; \mathbf{Z})$$

satisfying that $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ *if* X *is a manifold.*

The MacPherson's construction uses Mather classes and local Euler obstruction that we briefly recall.

The notion of *local Euler obstruction* was defined originally by R. MacPherson [M2] in 1974. It has been shown in [BDK] that the local invariant of singularities which appear in the Kashiwara formula for the index of holonomic modules [Ka] is equal to the local Euler obstruction. Definitions equivalent to MacPherson's have been given by several authors. We recall the one in [BS]: Let v be a radial vector field with an isolated singularity at $x \in V_\alpha$. Let B be a ball centered at x, small enough to be transversal to every stratum V_β with $V_\alpha \subset \overline{V_\beta}$, and such that x is the unique zero of v inside B. Using the Whitney conditions, it is possible to prove that there is a canonical lifting $\widetilde{v}$ of $v|_{\partial B \cap X}$ as a section of $\widetilde{E}|_{\nu^{-1}(\partial B \cap X)}$ (see [BS], Proposition 9.1). The obstruction to the extension of $\widetilde{v}$, on $\nu^{-1}(B \cap X)$, as a non-zero section of $\widetilde{E}$, evaluated on the corresponding fundamental class, is an integer denoted by $\mathrm{Eu}_x(X)$.

The local Euler obstruction is a constructible function Eu_X, constant on each stratum of the Whitney stratification. The relation between the local Euler obstruction and the polar varieties is given by Lê and Teissier [LT]:

Theorem ([LT]). *For a sufficiently general flag* $\mathcal{D}$ *in* $\mathbf{C}^m$, *the local Euler obstruction is expressed as*

$$\mathrm{Eu}_x(X) = \sum_{i=0}^{n-1} (-1)^{n-1-i} m_x(N_{n-1-i}(\mathcal{D}))$$

where $m_x(C)$ *denotes the multiplicity of* C *at* x.

For a Whitney stratification, we have the following lemma:

Lemma ([M1]). *There are integers* n_α *such that, for every point* $x \in X$, *we have:*

$$\sum_\alpha n_\alpha \mathrm{Eu}_x(\overline{V_\alpha}) = 1.$$

Definition ([M1]). The MacPherson class of X is defined by

$$c_*(X) = c_*(\mathbf{1}_X) = \sum_\alpha n_\alpha \, i_* c_M(\overline{V_\alpha})$$

where i denotes the inclusion $\overline{V_\alpha} \hookrightarrow X$.

Note that we have the following relation: $c^M(X) = c_*(\mathrm{Eu}_X)$.

In [BS] was proved the following result:

Theorem ([BS]). *The MacPherson class is the image of the Schwartz class by the Alexander duality isomorphism*

$$H^{2(m-r+1)}(M, M \setminus X) \xrightarrow{\cong} H_{2(r-1)}(X).$$

One of the consequences of this result is that the $(r-1)$-st MacPherson class $c_{r-1}(X)$ is represented by the cycle

$$\sum_{\sigma \in X} I(v^{(r)}, \hat{d}(\sigma))\ \sigma$$

where $\dim \sigma = 2(r-1)$ (see (1)).

The following theorem gives an expression of the MacPherson class in terms of Segre classes (see 2.5).

Theorem ([A3]). *If X is a hypersurface in a nonsingular variety M and Y is its singular scheme, then*

$$c_*(X) = c(TM) \cap s(X \setminus Y, M)$$

Following Sabbah [S] (see also [PP4]), we obtain a formula giving the Schwartz-MacPherson classes in terms of characteristic cycles. Denoting by $\mathbf{PCh}(\mathbf{1}_X) \subset T^*M$ the characteristic cycle associated to the constructible function $\mathbf{1}_X$ on M, we have (see the analogous formula for the Mather classes):

$$c_*(X) = (-1)^{n-1} c(TM|_X) \cap \tau_* \left(c(\mathcal{L})^{-1} \cap [\mathbf{PCh}(\mathbf{1}_X)] \right).$$

2.5. The Fulton classes (1984) ([Fu] exemple 4.2.6 (a)).

If X is a proper subvariety of a variety M, the Segre class $s(X, M)$ of X in M is the class in $A_*(X)$ defined as follows (see [F], §4): the normal cone to the closed subscheme X in the scheme M is defined as

$$C = C_X M = \mathrm{Spec} \left(\sum_{i=0}^{\infty} \mathcal{I}^i / \mathcal{I}^{i+1} \right)$$

where $\mathcal{I}$ is the ideal sheaf defining X in M. We denote by $P(C)$ the projectivized normal cone and p the projection $p : P(C) \to X$. Then

$$s(X, M) = \sum_{i \geq 0} p_*(c^1(\mathcal{O}(1))^i \cap [P(C)]).$$

When X is regularly imbedded in M, $C = N_X M$ is the normal vector bundle, and

$$s(X, M) = c(N_X M)^{-1} \cap [X].$$

The following Plücker formula, due to R. Piene, gives the relation between polar varieties (hence Mather classes) and Segre classes:

Theorem ([Pi1]). *Let X be an hypersurface of degree d in $\mathbf{P}^m$ and let $\mathcal{L} = \mathcal{O}_{\mathbf{P}^m}(1)|_X$. Then the polar variety N_k is given by*

$$[N_k] = (d-1)^k c^1(\mathcal{L})^k \cap [X] + \sum_{i=0}^{k-1} \binom{k}{i} (d-1)^i c^1(\mathcal{L})^i \cap s_{k-i}(X_{\text{sing}}, X).$$

The Fulton classes are defined by:

Definition ([Fu]). Let X be an algebraic scheme which can be imbedded as a closed subscheme of a non-singular variety M. We define the Fulton class of X in $A_*(X)$ by the formula

$$c^F(X) = c(TM|_X) \cap s(X, M),$$

where $c(TM|_X)$ is the total Chern class of the tangent bundle of M restricted to X and $s(X, M)$ is the Segre class of X in M.

This definition is independent of the choice of the embedding.

If X is a *local complete intersection*, then the normal bundle of X_{reg} in M extends canonically to X as a vector bundle $N_X M$ and

$$c^F(X) = c(TM|_X) c(N_X M)^{-1} \cap [X] = c(\tau_X) \cap [X]. \tag{6}$$

Here $\tau_X = TM|_X - N_X M$ denotes the virtual tangent bundle on X, defined in the Grothendieck group of vector bundles on X.

Let M be a non-singular compact complex analytic variety of pure dimension $n+1$ and let L be a holomorphic line bundle on M. Take $f \in H^0(M, L)$, a holomorphic section of L, such that the variety X of zeroes of f is a (nowhere dense) hypersurface in M. Then, the Fulton class of X is

$$c^F(X) = c(TM|_X - L|_X) \cap [X].$$

In [A1] P. Aluffi defines a notion of "thickening" of the scheme X along its singular subscheme Y: if $\mathcal{I}_Y$ denotes the ideal of Y and $\mathcal{I}$ the locally principal ideal of X, we denote by X^k the subscheme of M defined by the ideal $\mathcal{I}.\mathcal{I}_Y^k$. Then the Schwartz-MacPherson class and the Fulton class satisfy:

$$c_*(X) = c_F(X^{-1}) \qquad c^F(X) = c^F(X^0)$$

2.6. The Fulton-Johnson classes (1980) ([FJ], [Fu] exemple 4.2.6 (c)).

The definition (6) can also be generalized to arbitrary singular varieties in another way : for any coherent sheaf $\mathcal{F}$ on an algebraic scheme, one defines the Segre class $s(\mathcal{F})$ in the group $A_*(X)$ of cycles modulo rational equivalence as follows: Let $P(\mathcal{F}) = \operatorname{Proj}(\operatorname{Sym}(\mathcal{F}))$, with projection $p : P(\mathcal{F}) \to X$. Let us denote by $\mathcal{O}_{\mathcal{F}}(1)$ the canonical invertible sheaf which is the universal quotient of $p^*(\mathcal{F})$. If the support of $\mathcal{F}$ is X, define its Segre class $s(\mathcal{F})$ in $A_*(X)$ by the formula

$$\begin{aligned} s(\mathcal{F}) &= p_* \left(\sum_{i \geq 0} c^1(\mathcal{O}_{\mathcal{F}}(1))^i \cap [P(\mathcal{F})] \right) \\ &= p_* \left(c(\mathcal{O}_{\mathcal{F}}(-1))^{-1} \cap [P(\mathcal{F})] \right) \end{aligned}$$

For an arbitrary coherent sheaf $\mathcal{F}$ on X, define $s(\mathcal{F})$ to be $s(\mathcal{F} \oplus \mathcal{E}^1)$, where $\mathcal{E}^1$ is the trivial locally free sheaf of rank one on X.

Definition ([FJ]). If X is an algebraic scheme which may be imbedded in a non-singular scheme M, we define the Fulton-Johnson class of X in $A_*(X)$ by the formula

$$c^{FJ}(X) = c(TM|_X) \cap s(\mathcal{N}),$$

where $c(TM|_X)$ is the total Chern class of the tangent bundle of M restricted to X and $s(\mathcal{N})$ is the Segre class of the conormal sheaf of the embedding of X in M.

Remark. In the case of local complete intersection, the Fulton and Fulton-Johnson classes coincide and are equal to

$$c(TM|_X - N_X M) \cap [X].$$

§3. The Milnor classes.

The comparison between the Schwartz-MacPherson classes and the Fulton-Johnson classes can be viewed in two ways, which coincide in some classical situations. We observe that, in the case of isolated singularities, the difference is given by the Milnor numbers at the singular points. On the other hand, for a radial vector field tangent to the singular locus and with isolated singularity at a singular point, the difference between the "Schwartz" (classical) index and the "virtual" (GSV)-index is the Milnor number at this point. This observation motivates definition 2 below.

3.1. Definition and main properties of Milnor classes.

The following general definition is given by the corresponding authors in particular cases.

Definition 1 ([A3], [BLSS1], [PP4], [Y2]). The difference class

$$\mu_*(X) = (-1)^n(c^F(X) - c_*(X))$$

is called the Milnor class of X.

Let us consider the following situation ($\mathcal{H}$): X is an n-subvariety in the m-manifold M defined by a regular section, i.e. a holomorphic section generically transverse to the zero section, of a holomorphic vector bundle E (of rank $k = m - n$) over M [Su2]. We set $N = E|_X$. The virtual tangent bundle of X is denoted by

$$\tau_X = TM|_X \setminus N$$

Let us consider a compact connected subset $S \subset X$ (in particular a component of $X_{\rm sing}$) and a neighborhood U of S in M such that $U \cap X - S \subset X_{\rm reg}$. For each r-frame $v^{(r)}$ tangent to $X_{\rm reg}$ on $\partial U \cap X \cap D^{(2p)}$ with $2p = 2(m - r + 1)$ (see §2.3), we can define:

a) the localized Schwartz (usual) class $\mathrm{Sch}(v^{(r)}, S) \in H_{2(r-1)}(S)$ which computes the obstruction to the extension of $v^{(r)}$ as a stratified r-frame inside $U \cap X \cap D^{(2p)}$. It is the contribution of S to $c_{r-1}(X) \in H_{2(r-1)}(X)$ ([BLSS1], Theorem 2.13),

b) the localized virtual class $\mathrm{Vir}(v^{(r)}, S) \in H_{2(r-1)}(S)$ which computes the "obstruction to the extension of $v^{(r)}$ as linearly independent sections of τ_X", i.e. which is the contribution of S to $c_{r-1}(\tau_X) \in H_{2(r-1)}(X)$ ([BLSS1], Theorem 5.9).

Definition 2 ([BLSS1]). The $(r-1)$-st localized Milnor class of X at a compact component S of $X_{\rm sing}$ is defined by

$$\mu_{r-1}(X, S) = (-1)^{n-1}(\mathrm{Sch}(v^{(r)}, S) - \mathrm{Vir}(v^{(r)}, S)) \qquad \text{in } H_{2(r-1)}(S)$$

The total Milnor class is the sum over the components of $X_{\rm sing}$:

$$\mu_{(r-1)}(X) = \sum_{S_\alpha \subset X_{\rm sing}} (i_\alpha)_* \mu_{(r-1)}(X, S_\alpha) \in H_{2(r-1)}(X)$$

where i_α denotes the inclusion $S_\alpha \hookrightarrow X$.

The Milnor class $\mu_*(X)$ is supported on the singular locus of X. When $k = 1$ and $r = 1$, $\mu_0(X, S)$ is the Parusiński generalized Milnor number [Pa]. Also, if S is a point p and X a complete intersection near p, then $\mu_0(X, S)$ is the usual Milnor number.

The two definitions coincide in the case of local complete intersections, in particular in the case of hypersurfaces.

In the case $r = 1$, i.e. $v^{(1)} = v$, and for an isolated singularity p, the Schwartz index is the usual index and the virtual index coincides with the GSV-index (see [GSV], [LSS], [SS]). The difference of these indices is the Milnor number of X at p:

$$\mathrm{Sch}(v, p) - \mathrm{Vir}(v, p) = (-1)^{n+1}\mu(X, p).$$

Theorem ([SS]). *In the situation $\mathcal{H}$, suppose that X is compact and the singularities of X are isolated points $\{x_i\}$ where X is a local complete intersection. Then*

$$\mu_0(X) = (-1)^{n+1} \sum_{i=1}^{q} \mu(X, x_i)[x_i] \in H_0(X)$$

Theorem ([Su]). *In the previous situation, $\mu_i(X) = 0$ for $i > 0$.*

This result was also proved by [Pa] and [PP] for hypersurfaces with arbitrary singularities. It is generalized in the following way:

Theorem ([BLSS1] [BLSS2]). *Let X be a subvariety of a complex manifold in the situation $\mathcal{H}$, if X is compact, then we have, for each $r = 0, \dots, n-1$:*

$$c_r(X) = c_r(TM|_X - N) + (-1)^{n+1}\mu_r(X) \qquad \textit{in } H_{2r}(X).$$

In other words, the difference between the total Schwartz-MacPherson class $c_*(X)$ of X and the total virtual class $c_*(TM|_X - N)$, regarded in homology, is the sum over the connected components of $\mathrm{Sing}(X)$ of the "total" localized Milnor classes $\mu_*(X, S) = \oplus_{i=0}^{n-1}\mu_i(X, S)$.

A similar formula for hypersurfaces is given by Aluffi (see [A1] for the notations):

Theorem ([A3]). *Let $X \subset M$ be a hypersurface with its singular subscheme Y and $\mathcal{L} = \mathcal{O}(X)$. Then we have*

$$c_*(X) = c^F(X) + c(\mathcal{L})^{\dim X} \cap (\mu_{\mathcal{L}}(Y)^\vee \otimes_M \mathcal{L}),$$

where $\mu_{\mathcal{L}}(Y) = c(T^*(M) \otimes \mathcal{L}) \cap s(Y, M)$.

We have the following Lefschetz-type formulae for the Milnor class:

Theorem ([BLSS1] [BLSS2]). *Let us denote by ℓ the complex dimension of S and let H be a complex $(m-\ell)$-dimensional plane transverse to S in M.*

a) *If X is a hypersurface in M, defined by a holomorphic section of a holomorphic line bundle E, and S a compact component of X_{sing}, then*

$$\mu_{r-1}(X,S) = (-1)^\ell \mu(X \cap H, p) \cdot [c(S)c(E)^{-1}]^{\ell-r+1} \cap [S]$$

b) *If $r = \ell+1$ and k is arbitrary, then*

$$\mu_{r-1}(X,S) = (-1)^\ell \mu(X \cap H, p) \cdot [S]$$

In the case where $\mu(X \cap H, p) = 1$, the formula (a) is proved in [A3].

3.2. Description in terms of constructible functions [PP4].

Consider the function $\chi : X \to \mathbf{Z}$ defined by $\chi(x) := \chi(F_x)$, where F_x denotes the Milnor fibre at x and $\chi(F_x)$ its Euler characteristic. Define also the function $\mu : X \to \mathbf{Z}$ by $\mu = (-1)^{n-1}(\chi - \mathbf{1}_X)$.

Fix any stratification $\mathcal{S}$ of X such that μ is constant on the strata of $\mathcal{S}$, for instance any Whitney stratification of X. The topological type of the Milnor fibre is constant along the strata of any Whitney stratification of Z. Let us denote the value of μ on the stratum S by μ_S.

Let

$$\alpha(S) = \mu_S - \sum_{S' \neq S, S \subset \overline{S'}} \alpha(S')$$

be the numbers defined inductively on descending dimensions of S.

Theorem ([PP4]). *We have*

$$\mu_*(X) = \sum_{S \in \mathcal{S}} \alpha(S) c(L|_X)^{-1} \cap (i_{\overline{S},X})_* c_*(\overline{S}) = c(L|_X)^{-1} \cap c_*(\mu),$$

where $i_{\overline{S},X} : \overline{S} \to X$ denotes the natural inclusion.

The formula was conjectured in [Y2] when X is projective. Under this last assumption, [PP2] proved earlier that

$$\int_X \mu_*(X) = \sum_{S \in \mathcal{S}} \alpha(S) \int_{\overline{S}} c(L|_{\overline{S}})^{-1} \cap c_*(\overline{S})$$

3.3. Description in terms of divisors [A3].

Let $B = Bl_Y M \to M$ be the blow-up of M along the singular subscheme Y of X. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the total transform of X and the exceptional divisor in B, respectively.

Theorem ([A2]). *Let $\pi : \mathcal{X} \to X$ be the restriction of the blow-up to X. Then*

$$c_*(X) = c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}] - [\mathcal{Y}]}{1 + \mathcal{X} - \mathcal{Y}} \right),$$

where, on the right hand side, $\mathcal{X}$ and $\mathcal{Y}$ mean the first Chern classes of the line bundles associated with $\mathcal{X}$ and $\mathcal{Y}$, i.e. those of $\pi^*(L|_X)$ and $\mathcal{O}_B(-1)$, the latter being the canonical line bundle on B.

Let us denote by $\mathcal{X}'$ the proper transform of X, the following formulae are also due to Aluffi [A3]

$$c^M(X) = c_*(\mathrm{Eu}_X) = c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}']}{1 + \mathcal{X} - \mathcal{Y}} \right)$$

$$c^F(X) = c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}]}{1 + \mathcal{X}} \right)$$

and we deduce [PP4]:

$$\mu_*(X) = (-1)^{n-1} c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{Y}]}{(1 + \mathcal{X})(1 + \mathcal{X} - \mathcal{Y})} \right).$$

3.4. Specialization (the hypersurface case) [PP4].

Suppose that $X = f^{-1}(0)$ where f is a section of the line bundle L over M. Suppose that there exists a section $g \in H^0(M, L)$ such that $g^{-1}(0)$ is non-singular and transverse to the strata of a (fixed) Whitney stratification of X. For $t \in \mathbf{C}$ denote $f_t = f - tg$ and set $X_t = f_t^{-1}(0)$. We denote by $\mathbf{X}$ the following correspondence in $M \times \mathbf{C}$:

$$\mathbf{X} = \{(x, t) \in M \times \mathbf{C} | x \in X_t\}.$$

Denoting by $p : \mathbf{X} \to \mathbf{C}$ the restriction to $\mathbf{X}$ of the projection onto the second factor, then $X_t = p^{-1}(t)$ for $t \in \mathbf{C}$ and $X = X_0$. Denote by

$$\sigma_F : \mathbf{F}(\mathbf{X}) \to \mathbf{F}(X)$$

the specialization map on constructible functions and

$$\sigma_H : H_*(X_t) \to H_*(X)$$

the specialization map of homology classes (see [Ve]). For $\varphi \in \mathbf{F(X)}$ and t sufficiently small, one has $\sigma_H c_*(\varphi|_{X_t}) = c_*(\sigma_F \varphi)$.

The Fulton class $c^F(X)$ is given, in terms of MacPherson class as:

$$c^F(X) = c_*(\sigma_F(\mathbf{1_X}))$$

and the Milnor class as:

$$\mu_*(X) = c_*(\sigma_F(\mathbf{1_X}) - \mathbf{1}_X).$$

References

[A1] P. Aluffi, *MacPherson's and Fulton's Chern classes of hypersurfaces*, Int. Math. Res. Notices, **11** (1994), 455–465.

[A2] P. Aluffi, *Singular schemes of hypersurfaces*, Duke Math. J., **80** (1995), no. 2, 325–351.

[A3] P. Aluffi, *Chern classes for singular hypersurfaces*, Trans. Amer. Math. Soc. **351** (1999), 3989–4026.

[A4] P. Aluffi, *Weighted Chern-Mather classes and Milnor classes for hypersurfaces*, Advanced Studies in Pure Math. (to appear).

[BS] J.-P. Brasselet et M.H. Schwartz, *Sur les classes de Chern d'un ensemble analytique complexe*, Astérisque **82-83** (1981), 93–146.

[BLSS1] J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, *Milnor classes of local complete intersections*, Hokkaido University preprints series, **413**, (1998).

[BLSS2] J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, *Milnor numbers and classes of local complete intersections*, Proc. Japan Acad., **75**, Ser. A (1999), 179–183.

[BDK] J.L. Brylinski, A. Dubson and M. Kashiwara, *Formule de l'indice pour les Modules Holonomes et obstruction d'Euler locale*, C.R. Acad. Sc. Paris, t. 293, 1981, 573–576.

[Ch] S.S. Chern, *Characteristic classes of Hermitian manifolds*, Annals of Math., **47**, No. 1 (1946), 85–121.

[Eh] C. Ehresmann, *Sur la topologie de certains espaces homogènes*, Annals of Math., **35**, No. 2 (1934).

[Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, (1984).

[FJ] W. Fulton and K. Johnson, *Canonical classes of singular varieties.* Manuscripta Math. **32**, no. 3-4 (1980), 381–389.

[Ga1] P.B. Gamkrelidze, *Computation of the Chern cycles of algebraic manifolds*, (in Russian) Doklady Akad. Nauk., **90**, No. 5 (1953), 719–722.

[Ga2] P.B. Gamkrelidze, *Chern's cycles of complex algebraic manifolds*, (in Russian) Izv. Akad. Nauk. SSSR, Math. Ser. **20** (1956), 685–706.

[GSV] X. Gómez-Mont, J. Seade and A. Verjovsky, *The index of a holomorphic flow with an isolated singularity*, Math. Ann. **291** (1991), 737–751.

[Iv] B. Iversen, *Local Chern classes*, Ann. Sci. Ecole Norm. Sup. **9** (1976), 155–169.

[Ka] M. Kashiwara, *Index theorem for a maximally overdetermined system of linear differential equations*, Proc. Japan Acad. **49** (1973), 803–804.

[Ke1] G. Kennedy, *MacPherson's Chern classes for singular algebraic varieties*, Comm. in Alg. **18** (1990), 2821–2839.

[Ke2] G. Kennedy, *Specialization of MacPherson's Chern classes*, Math. Scand. **66** (1990), 12–16.

[LSS] D. Lehmann, M. Soares and T. Suwa, *On the index of a holomorphic vector field tangent to a singular variety*, Bol. Soc. Bras. Mat. **26** (1995), 183–199.

[LT] Lê Dũng Tráng et B. Teissier, *Variétés polaires et classes de Chern des variétés singulières*, Ann. of Math. **114** (1981), 457–491.

[M1] R. MacPherson, *Characteristic classes for singular varieties*, Proc. 9th Brazilian Math. Coll. Poços de Caldas, (1973), Vol. II, 321–327.

[M2] R. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. **100**, no. 2 (1974), 423–432.

[MS] J. Milnor and J. Stasheff, *Characteristic Classes*, Princeton University Press (1974).

[Pa] A. Parusiński, *A generalization of the Milnor number*, Math. Ann., **281** (1988), 247–254.

[PP1] A. Parusiński and P. Pragacz, *Characteristic numbers of degeneracy loci*, Contemp. Math., **123** (1991), 189–198.

[PP2] A. Parusiński and P. Pragacz, *A formula for the Euler characteristic of singular hypersurfaces*, J. Alg. Geom., **4** (1995), 337–351.

[PP3] A. Parusiński and P. Pragacz, *Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors*, J. Amer. Math. Soc. **8** (1995), no. 4, 793–817.

[PP4] A. Parusiński and P. Pragacz, *Characteristic classes of hypersurfaces and characteristic cycles*, J. Alg. Geom., (to appear).

[Pi1] R. Piene, *Polar classes of singular varieties*, Ann. Sc. E.N.S. **11** (1978), 247–276.

[Pi2] R. Piene, *Cycles polaires et classes de Chern pour les variétés projectives singulières*, Séminaire Ecole Polytechnique, Paris, 1977-78 and Travaux en cours **37**, Hermann Paris (1988), 7–34.

[S] C. Sabbah, *Quelques remarques sur la géométrie des espaces conormaux*, Astérisque, **130** (1985), 161–192.

[Sc1] M.H. Schwartz, *Classes caractéristiques définies par une stratification d'une variété analytique complexe*, CRAS, **260**, (1965), 3262–3264 et 3535–3537.

[Sc2] M.-H. Schwartz, *Classes obstructrices des ensembles analytiques*, to appear in Actualités mathématiques, Hermann, Paris.

[SS] J. Seade and T. Suwa, *An adjunction formula for local complete intersections*, Int. J. Math. **9** (1998), 759–768.

[Se] B. Segre, *Nuovi metodi e risultati nella geometria sukke varietà algebriche*, Annali di Mat. **4**, 35, (1953), 1–128.

[Ste] N. Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press (1951).

[Su1] T. Suwa, *Classes de Chern des intersections complètes locales*, C.R. Acad. Sci. Paris, **324** (1996), 67–70.

[Su2] T. Suwa, *Dual class of a subvariety*, Tokyo Journal of Math. (to appear).

[T1] B. Teissier, *Variétés polaires. II. Multiplicités polaires, sections planes et conditions de Whitney*, Algebraic geometry (La Rábida 1981), LNM 961, Springer (1982), 314–491.

[T2] B. Teissier, *Quelques points de l'histoire des variétés polaires, de Poncelet à nos jours.*

[Ve] J.-L. Verdier, *Spécialisation des classes de Chern*, Astérisque, **82-83** (1981), 149–159.

[Wh] H. Whitney, *Tangents to an analytic variety*, Ann. of Math. **81** (1965), 496–549.

[Wu0] Wu. Wen-tsun, *Characteristic Chern classes of algebraic varieties*, in chinese, Shuxue Jinzhan, **8** no. 4 (1965), 395–401.

[Wu1] Wu. Wen-tsun, *Chern classes on algebraic varieties with arbitrary singularities*, Several complex variables (Hangzhou, 1981), Birkhäuser Boston, Boston, Mass., (1984), 247–249.

[Wu2] Wu. Wen-tsun, *On Chern numbers of algebraic varieties with arbitrary singularities*, Acta Math. Sinica (N.S.), **3** (1987), 227–236.

[Y1] S. Yokura, *On a Verdier-type Riemann-Roch for Chern-Schwartz-MacPherson class*, Topology and its applications, **94** (1999), 315–327.

[Y2] S. Yokura, *On a Milnor class*, Preprint (1997).

[Y3] S. Yokura, *On characteristic classes of complete intersections*, "Algebraic geometry-Hirzebruch 70", Contemporary Math. AMS **241**, 1999.

[Z1] J. Zhou, *Classes de Chern pour les variétés singulières, classes de Chern en théorie bivariante*, Thèse Marseille (1995).

[Z2] J. Zhou, *Classes de Wu et classes de Mather*, C. R. Acad. Sci. Paris, Sér. I, **319** (1994), no. 2, 171–174.

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