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Schur's Q-functions and Twisted Affine Lie Algebras

Tatsuhiro Nakajima and Hiro-Fumi Yamada

To the memory of our friend and former colleague,
Nobuo Sasakura

Abstract.

Weight vectors of the basic representations of $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ are studied. They are expressed in terms of Schur's Q-functions. The up and down motion along the string of the fundamental imaginary root is described as a combinatorial game.

§0. Introduction

The article deals with an explicit expression of the weight vectors of the twisted affine Lie algebras of type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$.

In 1981 Date et al. introduced a KP like hierarchy of nonlinear differential equations, which has the infinite dimensional Lie algebra of type B_∞ as the infinitesimal transformation group of solutions, and named it the KP hierarchy of type B or the BKP hierarchy for short. In [DJKM] they investigated the reductions of the BKP hierarchy and related them with the basic representations of the twisted affine Lie algebras of type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$. Using the principal realization of the basic representation on an algebra of polynomials with infinitely many variables, they expressed the polynomial solutions to the reduced BKP hierarchies by means of the Schur functions.

In this context the Schur function indexed by the partition λ of n is defined by

$$S_\lambda(x) = \sum_\rho \chi_\rho^\lambda \frac{x_1^{m_1} x_2^{m_2} \cdots}{m_1! m_2! \cdots}, \tag{0.1}$$

where the summation runs over the partitions $\rho = (1^{m_1} 2^{m_2} \cdots)$ of n and χ_ρ^λ denotes the irreducible character value of the symmetric group S_n. The original Schur function, as a symmetric function, is obtained

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from $S_\lambda(x)$ by putting $x_j = p_j/j$ $(j \geqslant 1)$, where p_j is the j-th power sum symmetric function. A determinant expression is also known as the Jacobi-Trudi formula [Mac]. Utilizing this determinant formula, the solutions to the KP hierarchy or its reductions are beautifully expressed by means of the Schur functions.

However, theory of the BKP hierarchy is that of Pfaffians by nature and does not fit the Schur functions well. Instead one expects that the BKP hierarchy is related to Schur's Q-functions defined by, for a strict partition λ of n,

$$Q_\lambda(t) = \sum_\rho 2^{\frac{\ell(\lambda)-\ell(\rho)+\epsilon}{2}} \zeta_\rho^\lambda \frac{t_1^{m_1} t_3^{m_3} \cdots}{m_1! m_3! \cdots}, \tag{0.2}$$

where the summation runs over the partitions $\rho = (1^{m_1} 3^{m_3} \cdots)$ of n consisting of odd parts, ζ_ρ^λ denotes the irreducible spin character value of S_n, and $\epsilon = 0$ or 1 according to that $n - \ell(\lambda)$ is even or odd. The original Q-function, as a symmetric function, is recovered from $Q_\lambda(t)$ by putting $t_j = 2p_j/j$ $(j \geqslant 1$, odd$)$.

You [Y] showed that $Q_\lambda(t)$ solves the BKP hierarchy for any strict partition λ, and later the present authors [NY1] investigated a relation between the reduced BKP hierarchies and the (reduced) Q-functions.

In the theory of the KP hierarchy, the r-reduction means the elimination of the time variables x_{rj} $(j \geqslant 1)$. For example, the KdV hierarchy is the 2-reduction of the KP hierarchy and has the time variables x_j $(j \geqslant 1$, odd$)$. As for the BKP hierarchy it has only the odd numbered time variables t_j $(j \geqslant 1$, odd$)$ from the beginning. When the reduction number r is odd $(\geqslant 3)$, the reduction procedure is achieved by eliminating the variables t_{rj} $(j \geqslant 1$, odd$)$ in the BKP hierarchy. As a result the infinitesimal transformation group becomes $A_{2\ell}^{(2)}$ if $r = 2\ell + 1$. On the contrary, if $r = 2\ell + 2$, the representation space $V = \mathbb{C}[t_j; j \geqslant 1, \text{odd}]$ remains unchanged, but the infinitesimal transformation group reduced to $D_{\ell+1}^{(2)}$.

The weighted homogeneous polynomial solutions to the r-reduced BKP hierarchy appear as the maximal weight vectors of the basic representation of $A_{2\ell}^{(2)}$ $(r = 2\ell + 1)$ or $D_{\ell+1}^{(2)}$ $(r = 2\ell + 2)$ when one realizes the representation on V. They are expressed in terms of the (reduced) Q-functions [NY2].

In this article we will make a close investigation of the weight vectors of the basic representations of $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$. The up and down motion of the weights along the string of the fundamental imaginary root is described as a combinatorial game on an abacus representing the strict partitions.

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§1. Neutral free fermions and Schur's Q-functions

We first review some ingredients of operator formalism for Schur's Q-functions. Let $\mathbb{B}$ be the $\mathbb{C}$-algebra generated by ϕ_n ($n \in \mathbb{Z}$) with respect to the relations

$$\phi_n\phi_m + \phi_m\phi_n = (-1)^n\delta_{n+m,0} \quad (n, m \in \mathbb{Z}). \tag{1.1}$$

In the literature (*e.g.* [DJKM]) the generators ϕ_n are referred to as the neutral free fermions. Define the degree on $\mathbb{B}$ by $\deg \phi_n = 1$ ($n \in \mathbb{Z}$). If we let $\mathbb{B}_0$ (resp. $\mathbb{B}_1$) be the subspace consisting of the elements of even (resp. odd) degree, then $\mathbb{B} = \mathbb{B}_0 \oplus \mathbb{B}_1$ has a structure of a superalgebra. Let $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 = \mathbb{B}_0|0\rangle \oplus \mathbb{B}_1|0\rangle$ (resp. $\mathcal{F}^* = \langle 0|\mathbb{B}_0 \oplus \langle 0|\mathbb{B}_1$) be the Fock space (resp. the dual Fock space), where the vacuum $|0\rangle$ (resp. $\langle 0|$) is defined by

$$\phi_n|0\rangle = 0 \quad (n < 0) \qquad (\text{resp. } \langle 0|\phi_n = 0 \quad (n > 0)). \tag{1.2}$$

The vacuum expectation value $\langle 0|a|0\rangle$ ($a \in \mathbb{B}$) is uniquely determined by setting $\langle 0|1|0\rangle = 1$, $\langle 0|\phi_0|0\rangle = 0$.

We construct a realization of $\mathcal{F}_0$. The normal ordering for the neutral free fermions is defined by

$$\,^\circ_\circ\phi_n\phi_m{}^\circ_\circ = \phi_n\phi_m - \langle 0|\phi_n\phi_m|0\rangle. \tag{1.3}$$

Define the Hamiltonian by

$$H(t) = \frac{1}{2}\sum_{\substack{j\geqslant 1\\ j:\text{odd}}}\sum_{n\in\mathbb{Z}}(-1)^{n+1}t_j\phi_n\phi_{-j-n}. \tag{1.4}$$

Let $V = \mathbb{C}[t_j;\ j \geqslant 1, \text{odd}]$ be the algebra of polynomials with infinitely many variables. There is an isomorphism from $\mathcal{F}_0$ to V defined by

$$a|0\rangle \longmapsto \langle 0|e^{H(t)}a|0\rangle \quad (a \in \mathbb{B}_0), \tag{1.5}$$

which is often called the boson-fermion correspondence. The subspace

$$B_\infty = \left\{ \sum_{n,m\in\mathbb{Z}} c_{nm}\,^\circ_\circ\phi_n\phi_m{}^\circ_\circ;\, c_{nm} = 0 \text{ if } |n+m| \gg 0 \right\} \tag{1.6}$$

of a completion $\overline{\mathbb{B}}_0$ admits a structure of a Lie algebra isomorphic to the one dimensional central extension of $\mathfrak{o}(\infty)$. The Cartan subalgebra of B_∞ consists of the (infinite) linear combinations of ${}^\circ_\circ\phi_n\phi_{-n}{}^\circ_\circ$ $(n \in \mathbb{Z})$. Although certain infinite sums are allowed as elements of B_∞, the space V affords an action of B_∞. This Fock representation of B_∞ on V is best described by the vertex operator

$$(1.7) \qquad Z_B(p,q) = \frac{p-q}{2(p+q)}\left(e^{\xi(t,p)+\xi(t,q)}e^{-2\xi(\tilde{\partial},p^{-1})+2\xi(\tilde{\partial},q^{-1})} - 1\right)$$

which corresponds to the action of ${}^\circ_\circ\phi(p)\phi(q){}^\circ_\circ$ on $\mathcal{F}_0$, where $\phi(p) = \sum_{n\in\mathbb{Z}}\phi_n p^n$. Here we have set $\xi(t,p) = \sum_{\substack{j\geqslant 1\\ j:\text{odd}}} t_j p^j$ and $\xi(\tilde{\partial},p^{-1}) = \sum_{\substack{j\geqslant 1\\ j:\text{odd}}} \frac{1}{j}\frac{\partial}{\partial t_j}p^{-j}$.

A strict partition is a sequence $\lambda = (\lambda_1,\dots,\lambda_{2d})$ of non-negative integers with $\lambda_1 > \cdots > \lambda_{2d} \geqslant 0$. The number of positive parts is called the length of λ and denoted by $\ell(\lambda)$. When fermions are associated with the strict partition λ, we always append 0 in the tail of λ if $\ell(\lambda)$ is odd. For a strict partition $\lambda = (\lambda_1,\dots,\lambda_{2d})$, put $v_\lambda = \phi_{\lambda_1}\cdots\phi_{\lambda_{2d}}|0\rangle \in \mathcal{F}_0$. It is shown in [DJKM] that $\{v_\lambda\ ;\ \lambda \text{ is a strict partition}\}$ forms a weight basis for the Fock representation of B_∞.

In order to express these weight vectors as elements in V via the boson-fermion correspondence, we recall Schur's Q-functions. Define polynomials $q_n(t) \in V$ by

$$e^{\xi(t,p)} = \sum_{n=0}^{\infty} q_n(t)p^n.$$

For positive integers m, n $(m > n \geqslant 0)$ put

$$Q_{(m,n)}(t) = q_m(t)q_n(t) + 2\sum_{k=1}^{n}(-1)^k q_{m+k}(t)q_{n-k}(t),$$
$$Q_{(n,m)}(t) = -Q_{(m,n)}(t).$$

And finally, for a strict partitions $\lambda = (\lambda_1,\dots,\lambda_{2d})$, set

$$Q_\lambda(t) = \mathrm{Pf}(Q_{(\lambda_i,\lambda_j)}(t)),$$

where Pf stands for the Pfaffian of a skew-symmetric matrix. We refer to this polynomial $Q_\lambda(t)$ as the Q-function associated with the strict partition λ. The original Q-function [*e.g.* Mac], as a symmetric function, is obtained from $Q_\lambda(t)$ by putting $t_j = 2p_j/j$ $(j \geqslant 1, \text{odd})$, where p_j is the j-th power sum symmetric function.

Using anti-commutation relations of the neutral free fermions, it is easy to see that

$$[H(t), \phi(p)] = \xi(t,p)\phi(p),$$

from which one deduces

$$e^{H(t)}\phi(p)e^{-H(t)} = e^{\mathrm{ad}H(t)}\phi(p) = e^{\xi(t,p)}\phi(p).$$

Looking at the coefficient of p^n, one has

$$e^{H(t)}\phi_n e^{-H(t)} = \sum_{k=0}^{\infty} q_k(t)\phi_{n-k}$$

for any $n \in \mathbb{Z}$. Consequently we have

$$\begin{aligned}\langle 0|e^{H(t)}\phi_m\phi_n|0\rangle &= \frac{1}{2}q_m(t)q_n(t) + \sum_{k=1}^{n}(-1)^k q_{m+k}(t)q_{n-k}(t)\\ &= \frac{1}{2}Q_{(m,n)}(t)\end{aligned}$$

for $m > n \geqslant 0$. A standard fermion calculus shows that

$$\langle 0|e^{H(t)}\phi_{\lambda_1}\cdots\phi_{\lambda_{2d}}|0\rangle = \frac{1}{2^d}\mathrm{Pf}(Q_{(\lambda_i,\lambda_j)}(t)) = \frac{1}{2^d}Q_\lambda(t)$$

for a strict partition $\lambda = (\lambda_1, \ldots, \lambda_{2d})$.

§2. Basic representation of $A_{2\ell}^{(2)}$

In this section we fix $\ell \geqslant 1$ and put $r = 2\ell + 1$. The Lie subalgebra $A_{2\ell}^{(2)}$ of B_∞ is realized by the following Chevalley generators:

(2.1a) $$e_0 = \sqrt{2}\sum_{k\in\mathbb{Z}}(-1)^{k+1} {}^\circ_\circ\phi_{kr}\phi_{-kr-1}{}^\circ_\circ,$$

(2.1b) $$e_i = \sum_{k\in\mathbb{Z}}(-1)^{k+i} {}^\circ_\circ\phi_{kr-i-1}\phi_{-kr+i}{}^\circ_\circ,$$

(2.1c) $$e_\ell = \sum_{k\in\mathbb{Z}}(-1)^{k+\ell+1} {}^\circ_\circ\phi_{kr+\ell}\phi_{-kr-\ell-1}{}^\circ_\circ,$$

(2.1d) $$f_0 = \sqrt{2}\sum_{k\in\mathbb{Z}}(-1)^{k+1} {}^\circ_\circ\phi_{kr}\phi_{-kr+1}{}^\circ_\circ,$$

(2.1e) $$f_i = \sum_{k\in\mathbb{Z}} (-1)^{k+i+1} {}^{\circ}_{\circ}\phi_{kr-i}\phi_{-kr+i+1}{}^{\circ}_{\circ},$$

(2.1f) $$f_\ell = \sum_{k\in\mathbb{Z}} (-1)^{k+\ell} {}^{\circ}_{\circ}\phi_{kr+\ell+1}\phi_{-kr-\ell}{}^{\circ}_{\circ},$$

(2.1g) $$\alpha_0^\vee = 2\sum_{k\in\mathbb{Z}} (-1)^{k+1} {}^{\circ}_{\circ}\phi_{kr-1}\phi_{-kr+1}{}^{\circ}_{\circ} + 1,$$

(2.1h) $$\alpha_i^\vee = \sum_{k\in\mathbb{Z}} (-1)^{k+i} \left({}^{\circ}_{\circ}\phi_{kr+i}\phi_{-kr+i}{}^{\circ}_{\circ} - {}^{\circ}_{\circ}\phi_{kr-i-1}\phi_{-kr+i+1}{}^{\circ}_{\circ}\right),$$

(2.1i) $$\alpha_\ell^\vee = \sum_{k\in\mathbb{Z}} (-1)^{k+\ell} {}^{\circ}_{\circ}\phi_{kr+\ell}\phi_{-kr-\ell}{}^{\circ}_{\circ},$$

for $1 \leqslant i \leqslant \ell - 1$. The fundamental imaginary root reads

$$\delta = 2\sum_{i=0}^{\ell-1} \alpha_i + \alpha_\ell.$$

As we have seen in the previous section, the polynomial ring $V = \mathbb{C}[t_j\ ;\ j \geqslant 1, \text{odd}]$ can be viewed as a B_∞-module via the boson-fermion correspondence. Moreover any Q-function $Q_\lambda(t)$ is a weight vector. The restriction to the Lie subalgebra $A_{2\ell}^{(2)}$ corresponds to the specialization $q^r = -p^r$ of the parameters in the vertex operator (1.7). The Lie algebra $A_{2\ell}^{(2)}$ stabilizes the ideal $I^{(r)}$ of V generated by t_{jr} ($j \geqslant 1$, odd), because $p^{jr} + q^{jr} = 0$ ($j \in \mathbb{Z}$). Hence $A_{2\ell}^{(2)}$ acts on the quotient algebra $V^{(r)} = V/I^{(r)}$, which will always be identified with the subalgebra $V = \mathbb{C}[t_j\ ;\ j \geqslant 1, \text{odd and } j \not\equiv 0 \pmod r]$. This action is shown to be irreducible and in fact isomorphic to the basic representation of $A_{2\ell}^{(2)}$ with the highest weight vector $1 \in V^{(r)}$.

For a polynomial $F(t) \in V$, denote the r-reduced polynomial of $F(t)$ by

$$F^{(r)}(t) = F(t)|_{t_r = t_{3r} = \cdots = 0} \in V^{(r)}$$

Since the Cartan subalgebra of $A_{2\ell}^{(2)}$ is contained in that of B_∞, all r-reduced Q-functions $Q_\lambda^{(r)}(t)$ are weight vectors under the action of the former.

To describe the weight of a given r-reduced Q-function, we need the following r-abacus:

$$
\begin{array}{ccccccc}
 & 1 & 2 & 3 & \cdots & r-2 & r-1 \\
r & r+1 & r+2 & r+3 & \cdots & 2r-2 & 2r-1 \\
2r & 2r+1 & 2r+2 & 2r+3 & \cdots & 3r-2 & 3r-1 \\
3r & 3r+1 & 3r+2 & 3r+3 & \cdots & 4r-2 & 4r-1
\end{array}
$$

$$\cdots\cdots\cdots\cdots$$

For a strict partition we put a set of beads on the assigned positions. The above figure is the r-abacus representing the strict partition $(2r+1, 2r, r+2, r-1, 1)$.

Theorem 2.1. *Let $Q_\lambda^{(r)}(t)$ be an r-reduced Q-function of weight $wt(\lambda)$. Then a strict partition corresponding to weight $wt(\lambda)+\delta$ is obtained by one of the following:*

(1) *Move a bead one position up along a runner.*
(2) *Remove the bead at the position r.*
(3) *Remove the two beads at positions s and $r-s$, simultaneously, for $1 \leqslant s \leqslant \ell$.*

Proof. Since each α_i $(0 \leqslant i \leqslant \ell)$ has degree 1 in this realization (the principal realization), $\delta = 2\sum_{i=0}^{\ell-1} \alpha_i + \alpha_\ell$ has degree r. Each move described above decreases the degree of the corresponding r-reduced Q-function by r. Accordingly it suffices to show that the fermion operators corresponding to the moves (1)–(3) commute with $\alpha_i^\vee$ $(0 \leqslant i \leqslant \ell)$. Let $v = \phi_{\lambda_1} \cdots \phi_{\lambda_{2d}}|0\rangle$ be the given weight vector in $\mathcal{F}_0$. Up to sign, the move (1) is achieved by multiplying v by ${}^\circ_\circ \phi_{k-r}\phi_{-k} {}^\circ_\circ$ $(k \geqslant r+1)$ from the left. The commutation relations

$$[\alpha_i^\vee, {}^\circ_\circ \phi_{k-r}\phi_{-k} {}^\circ_\circ] = 0$$

are verified by using the general formula:

(2.2)
$$
\begin{aligned}
[{}^\circ_\circ \phi_a\phi_b {}^\circ_\circ, {}^\circ_\circ \phi_c\phi_d {}^\circ_\circ] = {} & (-1)^b \delta_{b+c,0} {}^\circ_\circ \phi_a\phi_d {}^\circ_\circ - (-1)^b \delta_{b+d,0} {}^\circ_\circ \phi_a\phi_c {}^\circ_\circ \\
& - (-1)^a \delta_{a+c,0} {}^\circ_\circ \phi_b\phi_d {}^\circ_\circ + (-1)^a \delta_{a+d,0} {}^\circ_\circ \phi_b\phi_c {}^\circ_\circ \\
& + (-1)^{a+b} (\delta_{a+c,0}\delta_{b+c,0} - \delta_{a+d,0}\delta_{b+d,0}) \left(Y(-b) - Y(a)\right),
\end{aligned}
$$

where

$$Y(a) = \begin{cases} 1 & a > 0 \\ \frac{1}{2} & a = 0 \\ 0 & a < 0. \end{cases}$$

Likewise the moves (2) and (3) are achieved by multiplying ${}^{\circ}_{\circ}\phi_{-r}\phi_0{}^{\circ}_{\circ}$ and ${}^{\circ}_{\circ}\phi_{-s}\phi_{s-r}{}^{\circ}_{\circ}$ $(1 \leqslant s \leqslant \ell)$, respectively, from the left. By a direct computation these operators are shown to commute with $\alpha_i^\vee$ $(0 \leqslant i \leqslant \ell)$. ∎

We should remark that each move in the above theorem corresponds to the removal of a r-bar from a (shifted) Young diagram introduced by Morris [Mo2].

A weight Λ is said to be maximal if $\Lambda + \delta$ is no longer a weight. It is known [K, Lemma 12.6] that Λ is a maximal weight if and only if Λ lies on the Weyl group orbit through the highest weight Λ_0. The maximal weight vectors are the r-reduced Q-functions $Q_\lambda^{(r)}(t)$ with λ obtained as the "stalemates" of the game described in Theorem 2.1. These r-reduced Q-functions $Q_\lambda^{(r)}(t)$ coincide with the (full) Q-functions $Q_\lambda(t)$ and solve the r-reduced BKP hierarchy [DJKM]. For the case $r = 3$, the Q-functions associated with the strict partitions

$$\{\emptyset, (3n+1, 3n-2, \ldots, 4, 1), (3n+2, 3n-1, \ldots, 5, 2) \quad (n \geqslant 0)\}$$

cover the maximal weight vectors for $A_2^{(2)}$.

The r-reduced Q-functions $\left\{Q_\lambda^{(r)}(t); \lambda \text{ is a strict partition}\right\}$ are linearly dependent. The linear relations satisfied by those functions are fully investigated in [NY2]. Here we only restate Proposition 2.1 in [NY2].

Proposition 2.2. *The set*

$$\left\{Q_\lambda^{(r)}(t); \lambda \textit{ is a strict partition with no part divisible by } r\right\}$$

forms a weight basis for $V^{(r)}$.

Another combinatorial way to compute the weight of a given strict partition λ is known as follows. Draw the Young diagram λ and fill each cell with a non-negative integer in such a way that, in each row the sequence $(0, 1, 2, \ldots, \ell-1, \ell, \ell-1, \ldots, 2, 1, 0)$ repeats from the left as long as possible. Let k_i $(0 \leqslant i \leqslant \ell)$ be the number of i's written in the

Young diagram. Then we have

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{i=0}^{\ell} k_i \alpha_i.$$

For example, let $\lambda = (9,5,4,2,1)$. Then the $A_2^{(2)}$-weight of $Q_\lambda(t)$ is

$$\mathrm{wt}(\lambda) = \Lambda_0 - 7\alpha_0 - 6\alpha_1 - 5\alpha_2 - 3\alpha_3$$

since we have the Young diagram

0	1	2	3	2	1	0	0	1
0	1	2	3	2				
0	1	2	3					
0	1							
0								

§3. Basic representation of $D_{\ell+1}^{(2)}$

In this section we fix $\ell \geqslant 2$ and put $r = 2\ell + 2$. We discuss the basic representation of $D_{\ell+1}^{(2)}$. For the most part the arguments are parallel to those in Section 2.

Let $\omega = \exp(2\pi i/r)$. Consider the reduced vertex operators

$$Z_B(p, -\omega^j p) = \sum_{i\in\mathbf{Z}} Z_i^{(j)} p^i \qquad (1 \leq j \leq \ell).$$

Then the homogeneous components $Z_i^{(j)}$, together with the Heisenberg Lie algebra, constitute a Lie algebra acting on $V = \mathbf{C}[t_j;\ j \geqslant 1, \text{odd}\ ]$, which is isomorphic to the basic representation of $D_{\ell+1}^{(2)}$.

The Chevalley generators and coroots of $D_{\ell+1}^{(2)}$ are described in terms of the fermion operators as follows:

$$\text{(3.1a)} \qquad e_0 = \sqrt{2} \sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ} \phi_{kr-1} \phi_{-kr} {}^{\circ}_{\circ},$$

$$\text{(3.1b)} \qquad e_i = (-1)^i \sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ} \phi_{kr+i} \phi_{-kr-i-1} {}^{\circ}_{\circ},$$

$$e_\ell = (-1)^{\ell+1}\sqrt{2}\sum_{k\in\mathbb{Z}} {}^\circ_\circ \phi_{kr+\ell}\phi_{-kr-\ell-1} {}^\circ_\circ, \tag{3.1c}$$

$$f_0 = -\sqrt{2}\sum_{k\in\mathbb{Z}} {}^\circ_\circ \phi_{kr}\phi_{-kr+1} {}^\circ_\circ, \tag{3.1d}$$

$$f_i = (-1)^i\sum_{k\in\mathbb{Z}} {}^\circ_\circ \phi_{kr+i+1}\phi_{-kr-i} {}^\circ_\circ, \tag{3.1e}$$

$$f_\ell = (-1)^\ell\sqrt{2}\sum_{k\in\mathbb{Z}} {}^\circ_\circ \phi_{kr+\ell+1}\phi_{-kr-\ell} {}^\circ_\circ, \tag{3.1f}$$

$$\alpha_0^\vee = -2\sum_{k\in\mathbb{Z}} {}^\circ_\circ \phi_{kr-1}\phi_{-kr+1} {}^\circ_\circ + 1, \tag{3.1g}$$

$$\alpha_i^\vee = (-1)^i\sum_{k\in\mathbb{Z}}\left({}^\circ_\circ \phi_{kr+i}\phi_{-kr-i} {}^\circ_\circ - {}^\circ_\circ \phi_{kr-i-1}\phi_{-kr+i+1} {}^\circ_\circ \right), \tag{3.1h}$$

$$\alpha_\ell^\vee = (-1)^\ell 2\sum_{k\in\mathbb{Z}} {}^\circ_\circ \phi_{kr+\ell}\phi_{-kr-\ell} {}^\circ_\circ, \tag{3.1i}$$

for $1 \leqslant i \leqslant \ell - 1$. The fundamental imaginary root reads

$$\delta = \sum_{i=0}^{\ell} \alpha_i.$$

Note that the Cartan subalgebra of $D_{\ell+1}^{(2)}$ is the intersection of that of B_∞ and $D_{\ell+1}^{(2)}$. Hence any Q-function $Q_\lambda(t)$ for a strict partition is a $D_{\ell+1}^{(2)}$-weight vector. In contrast with the case of $A_{2\ell}^{(2)}$, we do not have to eliminate any variables of the Q-functions. Therefore these weight vectors are automatically linearly independent.

We will describe the weight of a given Q-function. To this end we need the following r-abacus:

	1	2	$\cdots$	ℓ	$\ell+2$	$\ell+3$	$\cdots$	$2\ell+1$
$\ell+1$								
$2\ell+2$	$2\ell+3$	$2\ell+4$	$\cdots$	$3\ell+2$	$3\ell+4$	$3\ell+5$	$\cdots$	$4\ell+3$
$3\ell+3$								
$4\ell+4$	$4\ell+5$	$4\ell+6$	$\cdots$	$5\ell+4$	$5\ell+6$	$5\ell+7$	$\cdots$	$6\ell+5$
$\cdots\cdots\cdots$								

For a strict partition we put a set of beads on the assigned positions. The above figure is the r-abacus representing the strict partition $(5\ell+7, 4\ell+3, 3\ell+5, 3\ell+3, 2\ell+3, \ell+1, 2)$.

Theorem 3.1. *Let $Q_\lambda(t)$ be a Q-function of weight $wt(\lambda)$. Then a strict partition corresponding to weight $wt(\lambda)+\delta$ is obtained by one of the following:*

(1) *Move a bead one position up along the leftmost runner.*
(2) *Remove the bead at the position $\ell+1$.*

A strict partition corresponding to weight $wt(\lambda)+2\delta$ is obtained by one of the following:

(3) *Iterate the procedures obtaining $wt(\lambda)+\delta$.*
(4) *Move a bead one position up along a runner except for the leftmost one.*
(5) *Remove the two beads at positions s and $r-s$, simultaneously, for $1 \leqslant s \leqslant \ell$.*

Proof. The idea is the same as in Theorem 2.1. In this case the degree of $\delta = \sum_{i=0}^{\ell} \alpha_i$ equals to $r/2 = \ell+1$. Since the moves (1) and (2) (*resp.* (3)–(5)) decreases the degree of corresponding Q-function by $r/2$ (*resp.* r), we only have to check that fermion operators corresponding to those moves commute with $\alpha_i^\vee$ $(0 \leqslant i \leqslant \ell)$. Let $v = \phi_{i_t} \cdots \phi_{i_1}|0\rangle$ be a given weight vector. Up to sign, the move (1) is achieved by multiplying ${}^{\circ}_{\circ}\phi_{k(\ell+1)}\phi_{-(k+1)(\ell+1)}{}^{\circ}_{\circ}$ from the left of v. Therefore it suffices to check that

$$[\alpha_i^\vee, {}^{\circ}_{\circ}\phi_{k(\ell+1)}\phi_{-(k+1)(\ell+1)}{}^{\circ}_{\circ}] = 0 \quad (0 \leqslant i \leqslant \ell),$$

which is easily verified by (2.2) and the the fermion expression (3.1) of $\alpha_i^\vee$. Likewise the moves (2), (4) and (5) are achieved by multiplying ${}^{\circ}_{\circ}\phi_{-(\ell+1)}\phi_0{}^{\circ}_{\circ}$, ${}^{\circ}_{\circ}\phi_{k(\ell+1)+s}\phi_{-(k+2)(\ell+1)-s}{}^{\circ}_{\circ}$ $(1 \leqslant s \leqslant \ell)$, and ${}^{\circ}_{\circ}\phi_s\phi_{r-s}{}^{\circ}_{\circ}$ $(1 \leqslant s \leqslant \ell)$, respectively, from the left. It is easily verified that these operators commute with $\alpha_i^\vee$ $(0 \leqslant i \leqslant \ell)$. ∎

The maximal weight vectors are the Q-functions $Q_\lambda(t)$ with λ corresponding to the "stalemates". These Q-functions solve the r-reduced BKP hierarchy [DJKM]. For example, the strict partitions whose Q-functions are maximal weight vectors for $D_4^{(2)}$ of degree up to 12 are

$$\{\emptyset, (1), (2), (3), (2,1), (3,1), (5), (3,2), (6), (5,1), (3,2,1),$$
$$(7), (6,1), (5,2), (5,2,1), (7,2), (6,3), (9,1), (7,3), (6,3,1),$$
$$(6,5), (10,2), (7,5), (9,2,1), (6,5,1), (7,3,2)\}$$

Another combinatorial way to compute the weight of a given strict partition λ is known as follows. Draw the Young diagram λ and fill each cell with a non-negative integer in such a way that, in each row the sequence $(0,1,2,\ldots,\ell-1,\ell,\ell,\ell-1,\ldots,2,1,0)$ repeats from the left as long as possible. Let k_i $(0 \leqslant i \leqslant \ell)$ be the number of i's written in the Young diagram. In other words,

$$k_i = \sum_{j \geqslant 0} (\mu_{rj+i+1} + \mu_{rj+r-i}),$$

where $\lambda' = (\mu_1, \mu_2, \ldots)$ is the transpose of λ. Then we have

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{i=0}^{\ell} k_i \alpha_i.$$

For example, let $\lambda = (9,5,4,2,1)$. Then the $D_4^{(2)}$-weight of $Q_\lambda(t)$ is

$$\mathrm{wt}(\lambda) = \Lambda_0 - 7\alpha_0 - 5\alpha_1 - 4\alpha_2 - 5\alpha_3$$

since we have the Young diagram

0	1	2	3	3	2	1	0	0
0	1	2	3	3				
0	1	2	3					
0	1							
0								

Finally we make a remark on the weight multiplicity for the basic representation of $D_{\ell+1}^{(2)}$. A converse move of (1), (2) (resp. (3), (4), (5)) makes a weight vector of weight $\mathrm{wt}(\lambda) - \delta$ (resp. $\mathrm{wt}(\lambda) - 2\delta$) from the given strict partition λ. Starting from the highest weight Λ_0, which corresponds to the empty partition $\emptyset$, the number of bead configurations for the strict partitions of $n(\ell+1)$ is equal to $b^{(\ell)}(n)$, defined by

$$\sum_{n=0}^{\infty} b^{(\ell)}(n) q^n = \frac{\phi(q^2)}{\phi(q)} \cdot \frac{1}{\phi(q^2)^\ell} = \frac{1}{\phi(q)\phi(q^2)^{\ell-1}}.$$

This enumeration is an easy exercise of combinatorics. Since the maximal weights are on the Weyl group orbit through Λ_0 and they are all of multiplicity one, we have

$$\mathrm{mult}(\Lambda - n\delta) = b^{(\ell)}(n)$$

for any maximal weight Λ [K].

§4. Application to the case of $A_1^{(1)}$

We now apply the results of the preceding section to the basic representation of $A_1^{(1)}$. To this end, we first recall the following theorem which is due to M. Wakimoto.

Theorem 4.1. *Let e_i and f_i $(i = 0, 1, 2, 3)$ be the Chevalley generators and $\alpha_i^\vee$ $(i = 0, 1, 2, 3)$ be the simple coroots of $D_4^{(2)}$. Put*

$$\begin{aligned} \tilde{e}_0 &= e_0 + e_3, & \tilde{e}_1 &= \sqrt{2}(e_1 + e_2), \\ \tilde{f}_0 &= f_0 + f_3, & \tilde{f}_1 &= \sqrt{2}(f_1 + f_2), \\ \tilde{\alpha}_0^\vee &= \alpha_0^\vee + \alpha_3^\vee, & \tilde{\alpha}_1^\vee &= 2(\alpha_1^\vee + \alpha_2^\vee). \end{aligned} \tag{4.1}$$

Then $\tilde{e}_i$ and $\tilde{f}_i$ $(i = 0, 1)$ generate a Lie subalgebra isomorphic to $A_1^{(1)}$. Moreover, the restriction of the basic representation of $D_4^{(2)}$ to this subalgebra remains irreducible and turns out to be the basic representation of $A_1^{(1)}$.

Proof. It is straightforward to check that $\tilde{e}_i$, $\tilde{f}_i$ and $\tilde{\alpha}_i^\vee$ $(i = 0, 1)$ satisfy the defining relations of $A_1^{(1)}$.

The latter can be proved by looking at the formal characters. The formal character of the basic representation of $D_4^{(2)}$ is given by the following infinite product:

$$\begin{aligned} \mathrm{ch}L(\Lambda_0; D_4^{(2)}) &= e^{\Lambda_0} \prod_j^{(-)} (1 + e^{j\delta}) \\ &\times \prod_{j:\mathrm{odd}}^{(-)} (1 + e^{j\delta \pm \alpha_3})(1 + e^{j\delta \pm (\alpha_2 + \alpha_3)})(1 + e^{j\delta \pm (\alpha_1 + \alpha_2 + \alpha_3)}), \end{aligned}$$

where $\delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Here $\prod_j^{(-)} (1 + e^{j\delta + \alpha})$ denotes the product where j runs over the integers such that $j\delta + \alpha$ is a negative root. Under the restriction we have $\alpha_0 = \alpha_3$ and $\alpha_1 = \alpha_2$. Hence, putting $\delta = \alpha_0 + \alpha_1$, the "restricted character" reads $e^{\Lambda_0} \prod_j^{(-)} (1 + e^{j\delta})(1 + e^{2j\delta \pm \alpha_0})$, which is nothing but the formal character $\mathrm{ch}L(\Lambda_0; A_1^{(1)})$. ∎

In a similar fashion, one can embed $D_{2^n}^{(2)}$ into $D_{2^n+1}^{(2)}$ so that the basic representation is preserved under the restriction. Since the above theorem is thought of as the special case $n = 1$, we denote by $D_2^{(2)}$ the subalgebra isomorphic to $A_1^{(1)}$.

As we have seen in Section 3, the basic representation of $D_4^{(2)}$ is realized on $V = \mathbf{C}[t_j; j \geqslant 1, \text{odd}]$. Therefore V is viewed also as the space of the basic representation of $D_2^{(2)} \cong A_1^{(1)}$. The same argument as in Section 3 shows that the $D_2^{(2)}$-weight vectors are Schur's Q-functions for the strict partitions. In this section we need the 4-abacus in order to determine the weight of a given Q-function. Here we give the 4-abacus representing the strict partition $\lambda = (9, 5, 3, 2)$.

$$\begin{array}{ccc}
 & 1 & 3 \\
2 & & \\
4 & 5 & 7 \\
6 & & \\
8 & 9 & 11 \\
\cdots & \cdots & \cdots
\end{array}$$

Suppose we are given a Q-function $Q_\lambda(t)$ of weight $\mathrm{wt}(\lambda)$. A strict partition corresponding to $\mathrm{wt}(\lambda) + \delta$ is obtained by one of the following:

(1) Move a bead one position up along the leftmost runner.
(2) Remove the bead at the position 2.

A strict partition corresponding to $\mathrm{wt}(\lambda) + 2\delta$ is obtained by one of the following:

(3) Iterate the procedure obtaining $\mathrm{wt}(\lambda) + \delta$.
(4) Move a bead one position up along the first or third runner.
(5) Remove the two beads at the positions 1 and 3, simultaneously.

The strict partitions corresponding to the maximal weights are thus characterized by the "stalemates", which constitute the following set:

(4.2)
$$HC_4 := \{\emptyset, (4n+1, 4n-3, \ldots, 5, 1), (4n+3, 4n-1, \ldots, 7, 3) \quad (n \geqslant 0)\}.$$

Again as in Section 3, there is another way to compute the weight of a given Q-function. Let $\lambda' = (\mu_1, \mu_2, \ldots)$ be the transpose of the strict partition λ. Then the weight of $Q_\lambda(t)$ is equal to

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{j\geqslant 0}(\mu_{4j+1} + \mu_{4j+4})\alpha_0 - \sum_{j\geqslant 0}(\mu_{4j+2} + \mu_{4j+3})\alpha_1.$$

We remark here the equivalence of the 4-reduced BKP hierarchy and the 2-reduced KP hierarchy, i.e., the KdV hierarchy [DJKM]. The vertex operator of the former looks

$$Z_B(p, -ip) = \frac{1+i}{2(1-i)}\left\{\exp\Big(\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}}(1-i^j)t_j p^j\Big)\exp\Big(-2\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}}\frac{(1+i^j)}{j}\frac{\partial}{\partial t_j}p^{-j}\Big)-1\right\}.$$

Hence, by changing the variables

$$x_j = t_j \cos\left(\frac{j\pi}{4}\right) \qquad (j \geqslant 1, \mathrm{odd}) \tag{4.3}$$

and $k = (1-i)p/\sqrt{2}$, it reads

$$\frac{1+i}{2(1-i)}\left\{\exp\Big(2\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}} x_j k^j\Big)\exp\Big(-2\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}}\frac{1}{j}\frac{\partial}{\partial x_j}k^{-j}\Big) - 1\right\},$$

which equals, up to constant, the vertex operator for the KdV hierarchy.

The set of all the weighted homogeneous polynomial solutions coincides with the set of the maximal weight vectors. As for the 4-reduced BKP hierarchy this set consists of $Q_\lambda(t)$ with λ in HC_4.

Let DP_4 denote the set of partitions $\lambda = (1^{m_1}2^{m_2}\cdots)$ for which $m_i \leqslant 1$ when i is odd [LT]. Let DPR_4 be the subset of DP_4 consisting of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $0 < \lambda_i - \lambda_{i+1} \leqslant 4$ for odd i and $0 \leqslant \lambda_i - \lambda_{i+1} < 4$ for even i, where $\lambda_{\ell+1} = 0$.

The "doubling" $d(\lambda)$ of $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_\ell) \in DP_4$ is defined by

$$d(\lambda) = \left(\left[\frac{\lambda_1+1}{2}\right], \left[\frac{\lambda_1}{2}\right], \left[\frac{\lambda_2+1}{2}\right], \left[\frac{\lambda_2}{2}\right], \ldots \left[\frac{\lambda_\ell+1}{2}\right], \left[\frac{\lambda_\ell}{2}\right]\right)$$

(cf. [BO]). Let λ° denote the transpose of $d(\lambda)$ for $\lambda \in DP_4$. If $\lambda' = (\mu_1, \mu_2, \ldots)$ is the transpose of λ, then one sees that $\lambda^\circ = (\mu_1 + \mu_2, \mu_3 + \mu_4, \ldots)$. It can be verified that $\lambda^\circ \in DP_4$ and the operation $\circ$ is an involution in DP_4. Moreover the set of strict partitions is mapped onto DPR_4 by $\circ$. Put $\Delta_r = (r, r-1, \ldots, 2, 1)$ for $r \geqslant 1$ and $\Delta_0 = \emptyset$. These are called the staircase partitions. The involution $\circ$ gives a one-to-one correspondence between the set HC_4 and the set of the staircase partitions.

Let denote by ϖ the change of variables (4.3). More precisely, define an algebra isomorphism

$$\varpi : \mathbb{C}[t_j; j \geqslant 1, \mathrm{odd}] \longrightarrow \mathbb{C}[x_j; j \geqslant 1, \mathrm{odd}]$$

by $\varpi(t_j) = x_j / \cos(j\pi/4)$ for $j \geqslant 1$, odd.

Theorem 4.2. *For $\lambda \in HC_4$, we have*

$$\varpi(Q_\lambda) = 2^{\ell(\lambda)/2} S_{\lambda^\circ}.$$

In order to prove this theorem, we define the polynomial $B_\mu(x)$ for a strict partition μ by

$$B_\mu(x) = \sum_\rho b_\rho^\lambda \frac{x_1^{m_1} x_3^{m_3} \cdots}{m_1! m_3! \cdots},$$

where the summation runs over the partitions $\rho = (1^{m_1} 3^{m_3} \cdots)$ consisting of odd parts, and b_ρ^λ denotes the irreducible Brauer character value for the symmetric group S_n of characteristic 2 [JK]. One knows that $\{B_\mu(x); \mu \text{ is a strict partition}\}$ gives a weight basis for $\mathbb{C}[x_j; j \geqslant 1, \mathrm{odd}]$, the space of the basic representation of $A_1^{(1)}$ in the KdV picture. It is worth noting that

$$S_{\Delta_r}(x) = B_{\Delta_r}(x) \qquad (r \geqslant 0).$$

Let $\tilde{S}_n$ be the double cover of S_n, which is generated by $\tau_1, \ldots \tau_{n-1}$ and z with respect to the relations:

$$\begin{aligned} z^2 = 1, \quad z\tau_i = \tau_i z, \quad \tau_i^2 &= z \quad (1 \leqslant i \leqslant n-1), \\ (\tau_i \tau_{i+1})^3 &= z \quad (1 \leqslant i \leqslant n-2), \\ \tau_i \tau_j &= z \tau_j \tau_i \quad (|i-j| \geqslant 2). \end{aligned}$$

Let $\theta : \tilde{S}_n \longrightarrow S_n$ be the canonical epimorphism. A partition $\rho = (1^{m_1} 2^{m_2} \cdots)$ of n determines a conjugacy class C_ρ of S_n consisting of the elements of cycle type ρ. It is a classical result due to Schur that $\theta^{-1}(C_\rho)$ splits into two $\tilde{S}_n$-conjugacy classes $\tilde{C}_\rho^+$ and $z\tilde{C}_\rho^+$ if and only if either (1) the parts of ρ are all odd, or (2) ρ is a strict partition such that $n - \ell(\rho)$ is odd. An irreducible representation of $\tilde{S}_n$ is said to be "negative" if the central element z is mapped to -1. The character ζ of a negative representation satisfies $\zeta(z\tilde{C}_\rho^+) = -\zeta(\tilde{C}_\rho^+)$. For a partition $\rho = (\rho_1, \rho_2, \ldots, \rho_\ell)$ of n consisting of odd parts, define $\tau_\rho = \pi_1 \pi_2 \cdots \pi_\ell$, where $\pi_j = \tau_{a+1} \tau_{a+2} \cdots \tau_{a+\rho_j-1}$ $(a = \rho_1 + \rho_2 + \cdots + \rho_{j-1})$

for $1 \leqslant j \leqslant \ell$. For example, if $\rho = (5,3,1)$, then $\tau_\rho = \tau_1\tau_2\tau_3\tau_4\tau_6\tau_7$. A direct computation shows that $\mathrm{ord}(\tau_\rho) \equiv f(\rho) + 1 \pmod 2$, where $f(\rho) = \sum_{j\equiv 3,5 \pmod 8} m_j$ for $\rho = (1^{m_1}3^{m_3}\cdots)$. The irreducible negative representations of $\tilde{S}_n$ are parametrized by the strict partitions of n. Let ζ_ρ^λ denote the character value of the irreducible negative representation $\langle\lambda\rangle$ corresponding to the strict partition λ, evaluated on the element τ_ρ. One finds the character tables (ζ_ρ^λ) for $n \leqslant 14$ in [HH]. The irreducible negative representation $\langle\lambda\rangle$ has a composition series under the reduction modulo 2. The irreducible 2-modular representations of $\tilde{S}_n$ are nothing but those of S_n, since the central element z is always mapped to 1 when the characteristic equals 2. Therefore they are also parametrized by the strict partitions of n. Denote by $\tilde{d}_{\lambda\mu}$ the number of occurrence of the irreducible 2-modular representation indexed by the strict partition μ in the composition series of $\langle\lambda\rangle$ ([B]).

Let $\tilde{Z}_n = \left((-1)^{f(\rho)}\zeta_\rho^\lambda\right)_{\lambda\rho}$, $\tilde{D}_n = (\tilde{d}_{\lambda\mu})_{\lambda\mu}$ and $B_n = (b_\rho^\mu)_{\mu\rho}$, where λ and μ are strict partitions of n and ρ is the partition of n consisting of odd parts. By a general theory of modular representations, one sees that

$$\tilde{Z}_n = \tilde{D}_n B_n.$$

The Brauer characters are defined on the elements of odd order. Hence b_ρ^λ is evaluated on $z\tau_\rho$ if τ_ρ is of even order. This is the reason why the signs in $\tilde{Z}_n$ appear.

Theorem 4.3. *For a strict partition λ of n, we have*

$$\varpi(Q_\lambda) = 2^{(\ell(\lambda)+\epsilon)/2} \sum_\mu \tilde{d}_{\lambda\mu} B_\mu,$$

where the summation runs over the strict partitions μ of n, and $\epsilon = 0$ or 1 according to that $n - \ell(\lambda)$ is even or odd.

Proof. Recall that the Q-functions are related with the power sum symmetric functions by

$$Q_\lambda(t) = \sum_\rho 2^{(\ell(\lambda)-\ell(\rho)+\epsilon)/2} \zeta_\rho^\lambda \frac{t_1^{m_1} t_3^{m_3}\cdots}{m_1! m_3! \cdots},$$

where the summation runs over $\rho = (1^{m_1}3^{m_3}\cdots)$. Here we have put $t_j = 2p_j/j$ as before. Therefore we have

$$\varpi(Q_\lambda) = 2^{(\ell(\lambda)+\epsilon)/2} \sum_\rho (-1)^{f(\rho)} \zeta_\rho^\lambda \frac{x_1^{m_1} x_3^{m_3}\cdots}{m_1! m_3! \cdots},$$

which implies the required formula. ▮

For $\lambda \in HC_4$, the decomposition number $\tilde{d}_{\lambda\mu}$ is equal to 1 if $\mu = \lambda^\circ$ and 0 otherwise [B]. This proves Theorem 4.2, since $n - \ell(\lambda)$ is even for $\lambda \in HC_4$.

Looking at the square matrices $\tilde{D}_n$, given in [B] for $n \leqslant 15$, one observes that $\det \tilde{D}_n$ is a power of 2. A proof of this fact will be given in a separate paper [TY].

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Tatsuhiro Nakajima
Faculty of Economics
Meikai University
Urayasu 279-8550, Japan
nakajima@phys.metro-u.ac.jp

Hiro-Fumi Yamada
Department of Mathematics
Faculty of Science
Okayama University
Okayama, 700-0082, Japan
yamada@math.okayama-u.ac.jp