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Hypergeometric Systems and Radon Transforms for Hermitian Symmetric Spaces

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Abstract.

Generalizing the Gelfand hypergeometric system on the Grassmann manifold we introduce certain differential equations on compact Hermitian symmetric spaces and investigate their properties. Especially, we give a criterion for their holonomicity, and show the existence of integral representations of their solutions. We also study properties of certain Radon transforms.

§0. Introduction

0.1 Let V be a finite dimensional vector space over the complex number field $\mathbb{C}$, and let X be the Grassmann manifold consisting of k-dimensional subspaces of V. We denote by L the rank one subbundle of the product bundle $X \times \wedge^k V$ on X whose fiber at $W \in X$ is $\wedge^k W$. Note that the group $G = SL(V)$ acts on X transitively and that L is a G-equivariant line bundle on X. Let $D_{X,L}$ be the sheaf of differential operators acting on the sections of L. By differentiating the action of G on L we get a Lie algebra homomorphism $\mathfrak{g} = \mathrm{Lie}(G) \to \Gamma(X, D_{X,L})$ $(a \mapsto \partial_a^L)$. Fix a maximal torus K of G and set $\mathfrak{k} = \mathrm{Lie}(K)$. The hypergeometric equation investigated in Gelfand [6] and Gelfand-Gelfand [7] is a differential equation whose unknown function is a section of L, i.e. a left $D_{X,L}$-module. It is of the form

$$M_\xi = D_{X,L}/(\mathcal{J} + \sum_{a \in \mathfrak{k}} D_{X,L}(\partial_a^L - \xi(a))), \tag{0.1}$$

where ξ is any fixed character of $\mathfrak{k}$, and $\mathcal{J}$ is a certain G-stable left ideal of $D_{X,L}$.

0.2 Our starting point is the fact that G-stable left ideals of $D_{X,L}$ correspond to submodules of a certain $\mathfrak{g}$-module called the generalized

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Verma module, and that the above $\mathcal{J}$ actually corresponds to the maximal proper submodule. By this observation we can formally generalize the above construction of M_ξ as follows.

Let G be a connected simply-connected semisimple algebraic group over $\mathbb{C}$. Let P be a parabolic subgroup of G and consider the generalized flag manifold $X = G/P$. Set $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{p} = \mathrm{Lie}(P)$. To each character λ of $\mathfrak{p}$ one can associate a sheaf of rings of twisted differential operators $D_{X,\lambda}$ on X, and a natural Lie algebra homomorphism $\mathfrak{g} \to \Gamma(X, D_{X,\lambda})$ $(a \mapsto \partial_a^\lambda)$ (see [2], [3], [14]). Then G-stable left ideals of $D_{X,\lambda}$ correspond to submodules of the right $\mathfrak{g}$-module $M(\lambda)$ which is induced from the one-dimensional $\mathfrak{p}$-module corresponding to λ. Let $\mathcal{J}_\lambda$ be the G-stable left ideal of $D_{X,\lambda}$ corresponding to the maximal proper submodule of $M(\lambda)$. Take a closed subgroup K of G with Lie algebra $\mathfrak{k}$ and a character ξ of $\mathfrak{k}$. Then the left $D_{X,\lambda}$-module

$$M_{\lambda,K,\xi} = D_{X,\lambda}/(\mathcal{J}_\lambda + \sum_{a\in\mathfrak{k}} D_{X,\lambda}(\partial_a^\lambda - \xi(a))) \tag{0.2}$$

is an obvious generalization of (0.1). However, we have to specify the choices of G, P, λ, K, and ξ in order that (0.2) is an interesting system. In fact we have $\mathcal{J}_\lambda = 0$ if λ is generic.

We restrict our attention to the case where the unipotent radical of P is commutative. In this case we have a finite set $\mathcal{A}$ of characters of $\mathfrak{p}$ such that $\mathcal{J}_\lambda$ for $\lambda \in \mathcal{A}$ is nontrivial and is explicitly described in a geometric manner, and we take λ from this set $\mathcal{A}$. The subgroup K may be chosen freely as long as the system $M_{\lambda,K,\xi}$ is holonomic. We shall give a criterion for $M_{\lambda,K,\xi}$ to be holonomic in §3.

For certain λ's in $\mathcal{A}$ one can associate another parabolic subgroup $Q = Q_\lambda$ of G and a character $\mu = \mu_\lambda$ of $\mathrm{Lie}(Q)$ satisfying the following properties. Let $D_{Y,\mu}$ be the sheaf of twisted differential operators on $Y = G/Q$ corresponding to μ. For each $D_{Y,\mu}$-module N one can define its Radon transform $R(N)$ as a complex of $D_{X,\lambda}$-modules (see §3 below). Set $N_{\mu,K,\xi} = D_{Y,\mu}/\sum_{a\in\mathfrak{k}} D_{Y,\mu}(\partial_a^\mu - \xi(a))$. Then one has a canonical morphism $M_{\lambda,K,\xi} \to R(N_{\mu,K,\xi})$. This gives integral representations of solutions to the differential equation corresponding to $M_{\lambda,K,\xi}$.

We remark that the D-module considered in Gelfand-Zelevinsky-Kapranov [9, §3.3] is a quotient of the restriction of $M_{\lambda,K,\xi}$ to an open subset of X, where λ is some particular element of $\mathcal{A}$ and K is a maximal torus.

0.3 We can define Radon transforms in a more general context. Let P and Q be parabolic subgroups of G and set $X = G/P$, $Y = G/Q$ as above. Let Z be a G-orbit on $X \times Y$, and let $p_1 : Z \to X$, $p_2 : Z \to Y$ be the canonical morphisms. Let λ be a character of $\mathrm{Lie}(P)$, and let μ

be a character of Lie(Q). If λ and μ satisfy a certain condition, then we can define, for a $D_{Y,\mu}$-module N, a complex $R(N)$ of $D_{X,\lambda}$-modules by

$$R(N) = \int_{p_1} (\Omega_{p_1}^{\otimes -1} \otimes p_2^* N). \tag{0.3}$$

We call this $R(N)$ the Radon transform of N. Several properties of this functor R are investigated recently by a number of people (see D'Agnolo-Schapira [4], Marastoni [17], Kashiwara-Tanisaki [15]). In this paper we deal with the case where P and Q are certain special maximal parabolic subgroups and Z is the closed orbit.

A conclusion of our analysis is an explicit description of $R(D_{Y,\mu})$ for the case $G = SL_n(\mathbb{C})$. It is a complex version of results in Kakehi [13], Oshima [21] and Sekiguchi [23].

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§1. Twisted differential operators on generalized flag manifolds

1.1 Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{h}^*$ the set of roots, and W its Weyl group. For $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We fix a set of simple roots $\{\alpha_i\}_{i \in I_0}$, and set $\Delta^{\pm} = \pm(\Delta \cap \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i)$. For $i \in I_0$ let $h_i \in \mathfrak{h}$ and $\varpi_i \in \mathfrak{h}^*$ be the simple coroot and the fundamental weight corresponding to α_i, respectively. For a subset I of I_0 set $\Delta_I = \sum_{i \in I} \mathbb{Z}\alpha_i \cap \Delta$, and let W_I be its Weyl group. Define subalgebras $\mathfrak{l}_I, \mathfrak{n}_I^+, \mathfrak{n}_I^-, \mathfrak{p}_I^+, \mathfrak{p}_I^-$ of $\mathfrak{g}$ by

$$\mathfrak{l}_I = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha), \qquad \mathfrak{n}_I^{\pm} = \oplus_{\alpha \in \Delta^{\pm} \setminus \Delta_I} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_I^{\pm} = \mathfrak{l}_I \oplus \mathfrak{n}_I^{\pm}. \tag{1.1}$$

Let G be a connected simply-connected simple algebraic group with Lie algebra $\mathfrak{g}$, and let $H, L_I, N_I^{\pm}, P_I^{\pm}$ be the connected closed subgroups of G corresponding to $\mathfrak{h}, \mathfrak{l}_I, \mathfrak{n}_I^{\pm}, \mathfrak{p}_I^{\pm}$, respectively. Let $\mathrm{Hom}(\mathfrak{p}_I^+, \mathbb{C})$ (resp. $\mathrm{Hom}(P_I^+, \mathbb{C}^\times)$) be the set of homomorphisms of Lie algebras (resp. algebraic groups) from $\mathfrak{p}_I^+$ (resp. P_I^+) to $\mathbb{C}$ (resp. $\mathbb{C}^\times$). Since the natural linear map $\mathfrak{h} \to \mathfrak{p}_I^+/[\mathfrak{p}_I^+, \mathfrak{p}_I^+]$ is surjective with kernel $\sum_{i \in I} \mathbb{C}h_i$, $\mathrm{Hom}(\mathfrak{p}_I^+, \mathbb{C})$ is naturally identified with $\sum_{i \in I_0 \setminus I} \mathbb{C}\varpi_i \subset \mathfrak{h}^*$. Moreover, $\lambda \in \mathrm{Hom}(\mathfrak{p}_I^+, \mathbb{C})$ corresponding to $\sum_{i \in I_0 \setminus I} a_i \varpi_i$ is integrated to a character of P_I^+ if and only if $a_i \in \mathbb{Z}$ for any $i \in I_0 \setminus I$, and hence $\mathrm{Hom}(P_I^+, \mathbb{C}^\times)$ is naturally identified with $\sum_{i \in I_0 \setminus I} \mathbb{Z}\varpi_i \subset \mathfrak{h}^*$.

1.2 For a smooth algebraic variety over $\mathbb{C}$, we denote by $\mathcal{O}_X, \Omega_X, D_X$ the structure sheaf, the canonical sheaf, and the sheaf of differential operators on X. A sheaf of rings on X is called a sheaf of twisted differential operators (a TDO-ring) if it is locally isomorphic to D_X (see Kashiwara [14] for more precise definition).

1.3 We shall consider TDO-rings on the generalized flag manifolds

$$X_I = G/P_I^+ \tag{1.2}$$

in the following. For $\lambda \in \mathrm{Hom}(P_I^+, \mathbb{C}^\times)$ let L_λ be the G-equivariant line bundle over X_I such that the action of P_I^+ on the fiber of L_λ at eP_I^+ is given by λ, and let $\mathcal{O}_{X_I}(\lambda)$ be the rank one locally free $\mathcal{O}_{X_I}$-module consisting of sections of L_λ. We denote by $D_{X_I,\lambda}$ the sheaf of rings on X_I consisting of differential operators acting on sections of L_λ. Then $D_{X_I,\lambda}$ is apparently a TDO-ring, and we have $D_{X_I,\lambda} \simeq \mathcal{O}_{X_I}(\lambda) \otimes_{\mathcal{O}_{X_I}} D_{X_I} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(-\lambda)$.

More generally, we can construct TDO-rings $D_{X_I,\lambda}$ for any $\lambda \in \mathrm{Hom}(\mathfrak{p}_I^+, \mathbb{C})$. We recall its construction following Beilinson-Bernstein [2]. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Define a $\mathbb{C}$-algebra homomorphism $U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(\mathcal{O}_{X_I}) \quad (u \mapsto \partial_u)$ by

$$\partial_a(f)(x) = \frac{d}{dt} f(\exp(-ta)x)|t = 0 \quad (a \in \mathfrak{g}, f \in \mathcal{O}_{X_I}, x \in X_I). \tag{1.3}$$

Set $U_{X_I}(\mathfrak{g}) = \mathcal{O}_{X_I} \otimes_{\mathbb{C}} U(\mathfrak{g})$. A $\mathbb{C}$-algebra structure on $U_{X_I}(\mathfrak{g})$ is uniquely determined by the following properties:

$$\mathcal{O}_{X_I} \to U_{X_I}(\mathfrak{g}) \quad (f \mapsto f \otimes 1) \text{ is an algebra homomorphism,} \tag{1.4}$$

$$U(\mathfrak{g}) \to U_{X_I}(\mathfrak{g}) \quad (u \mapsto 1 \otimes u) \text{ is an algebra homomorphism,} \tag{1.5}$$

$$(f \otimes 1)(1 \otimes u) = f \otimes u \qquad (f \in \mathcal{O}_{X_I}, u \in U(\mathfrak{g})), \tag{1.6}$$

$$[1 \otimes a, f \otimes 1] = \partial_a(f) \otimes 1 \qquad (f \in \mathcal{O}_{X_I}, a \in \mathfrak{g}). \tag{1.7}$$

We shall identify $\mathcal{O}_{X_I}$ and $U(\mathfrak{g})$ with subalgebras of $U_{X_I}(\mathfrak{g})$ via (1.4) and (1.5). For $x \in X_I$ let $U_{X_I}(\mathfrak{g})_x \to U(\mathfrak{g}) \quad (R \mapsto R(x))$ be the natural map given by $f \otimes u \mapsto f(x)u$. For an $\mathrm{ad}(\mathfrak{p}_I^+)$-stable subspace D of $U(\mathfrak{g})$ we denote by $\mathcal{L}^0_{X_I}(D)$ the subsheaf of $U_{X_I}(\mathfrak{g})$ consisting of $R \in U_{X_I}(\mathfrak{g})$ such that $R(gP_I^+) \in \mathrm{Ad}(g)D$ for any $gP_I^+ \in X_I$.

Lemma 1.1. *Let D be an $\mathrm{ad}(\mathfrak{p}_I^+)$-stable subspace of $U(\mathfrak{g})$.*

(i) $\mathcal{O}_{X_I}\mathcal{L}^0_{X_I}(D) = \mathcal{L}^0_{X_I}(D)$.

(ii) $[\mathfrak{g}, \mathcal{L}^0_{X_I}(D)] \subset \mathcal{L}^0_{X_I}(D)$.

(iii) $\mathcal{L}^0_{X_I}(DU(\mathfrak{g})) = U_{X_I}(\mathfrak{g})\mathcal{L}^0_{X_I}(D)$.

Proof. (i) This is clear by the definition.

(ii) Let $\pi : G \to X_I$ be the natural map. Let $Q \in \Gamma(U, U_{X_I}(\mathfrak{g}))$, and define a $U(\mathfrak{g})$-valued function v on $\pi^{-1}(U)$ by $v(g) = \mathrm{Ad}(g^{-1})(Q(gP_I^+))$. It is sufficient to show

$$[a, Q](gP_I^+) = \mathrm{Ad}(g)\left(\frac{d}{dt}v(\exp(-ta)g)|t = 0\right)$$

for any $a \in \mathfrak{g}$. Write $Q = \sum_i f_i u_i$ where $f_i \in \mathcal{O}_{X_I}$ and $u_i \in U(\mathfrak{g})$. Then we have

$$[a, Q] = \sum_i (\partial_a(f_i) + f_i a)u_i - \sum_i f_i u_i a = \sum_i \partial_a(f_i)u_i + \sum_i f_i[a, u_i],$$

and hence

$$[a, Q](gP_I^+) = \sum_i (\partial_a(f_i))(gP_I^+)u_i + \sum_i f_i(gP_I^+)[a, u_i].$$

On the other hand we have

$$\begin{aligned}
& \frac{d}{dt}v(\exp(-ta)g)|t = 0 \\
= \; & \frac{d}{dt}\left(\mathrm{Ad}(g^{-1}\exp ta)(\sum_i f_i(\exp(-ta)gP_I^+)u_i)\right)\Bigg| t = 0 \\
= \; & \mathrm{Ad}(g^{-1})\left(\mathrm{ad}(a)(\sum_i f_i(gP_I^+)u_i) + \sum_i (\partial_a(f_i))(gP_I^+)u_i\right) \\
= \; & \mathrm{Ad}(g^{-1})\left(\sum_i f_i(gP_I^+)[a, u_i] + \sum_i (\partial_a(f_i))(gP_I^+)u_i\right).
\end{aligned}$$

The statement (ii) is proved.

(iii) Set $D_1 = DU(\mathfrak{g})$. We have

$$\mathfrak{g}\mathcal{L}^0_{X_I}(D_1) \subset [\mathfrak{g}, \mathcal{L}^0_{X_I}(D_1)] + \mathcal{L}^0_{X_I}(D_1)\mathfrak{g} \subset \mathcal{L}^0_{X_I}(D_1) + \mathcal{L}^0_{X_I}(D_1)\mathfrak{g}$$

by (ii), and

$$\mathcal{L}^0_{X_I}(D_1)\mathfrak{g} \subset \mathcal{L}^0_{X_I}(D_1\mathfrak{g}) \subset \mathcal{L}^0_{X_I}(D_1)$$

by the definition of $\mathcal{L}^0_{X_I}$. Hence $\mathcal{L}^0_{X_I}(D_1)$ is a left ideal of $U_{X_I}(\mathfrak{g})$ by (i). Since $\mathcal{L}^0_{X_I}(D_1)$ is a left ideal containing $\mathcal{L}^0_{X_I}(D)$, we have

$$\mathcal{L}^0_{X_I}(DU(\mathfrak{g})) = \mathcal{L}^0_{X_I}(D_1) \supset U_{X_I}(\mathfrak{g})\mathcal{L}^0_{X_I}(D).$$

It remains to show $\mathcal{L}^0_{X_I}(D_1) \subset U_{X_I}(\mathfrak{g})\mathcal{L}^0_{X_I}(D)$. Since the question is local, it is sufficient to show it on the open subset $gN_I^- P_I^+/P_I^+ (\simeq N_I^-)$ for any $g \in G$. We can assume that $g = 1$ without loss of generality. Let $Q \in \mathcal{L}^0_{X_I}(D_1)$. For $x \in N_I^-$ we have $\mathrm{Ad}(x)^{-1}Q(xP_I^+) = \sum_i f_i(x)u_iv_i$ for $u_i \in D$, $v_i \in U(\mathfrak{g})$, and some locally defined functions f_i on N_I^-. Write

$$\mathrm{Ad}(x)u_i = \sum_k \varphi_i^k(x)u_i^k, \qquad \mathrm{Ad}(x)v_i = \sum_\ell \psi_i^\ell(x)v_i^\ell,$$

where $u_i^k, v_i^\ell \in U(\mathfrak{g})$ and φ_i^k, ψ_i^ℓ are functions on N_I^-. For a locally defined function f on N_I^- denote the corresponding locally defined function on $N_I^- P_I^+/P_I^+$ by F_f. We have

$$Q(xP_I^+) = \sum_{i,\ell} f_i(x)\psi_i^\ell(x)(\sum_k \varphi_i^k(x)u_i^k)v_i^\ell,$$

and hence

$$Q = \sum_{i,\ell} F_{f_i\psi_i^\ell}(\sum_k F_{\varphi_i^k}u_i^k)v_i^\ell.$$

Since

$$\mathrm{Ad}(x)^{-1}((\sum_k F_{\varphi_i^k}u_i^k)(xP_I^+)) = \mathrm{Ad}(x)^{-1}(\sum_k \varphi_i^k(x)u_i^k) = u_i \in D,$$

we have $Q \in \mathcal{O}_{X_I}\mathcal{L}^0_{X_I}(D)U(\mathfrak{g})$. By (i) and (ii) we see easily that $\mathcal{O}_{X_I}\mathcal{L}^0_{X_I}(D)U(\mathfrak{g}) \subset U_{X_I}(\mathfrak{g})\mathcal{L}^0_{X_I}(D)$. Hence (iii) is proved. Q.E.D.

1.4 For $\lambda \in \mathrm{Hom}(\mathfrak{p}_I^+, \mathbb{C})$ set

$$\mathfrak{p}_{I,\lambda}^+ = \{a - \lambda(a) \mid a \in \mathfrak{p}_I^+\}, \qquad \mathfrak{r}_{I,\lambda} = \mathfrak{p}_{I,\lambda}^+ U(\mathfrak{g}). \tag{1.8}$$

Lemma 1.2. *We have* $[\mathcal{O}_{X_I}, \mathcal{L}^0_{X_I}(\mathfrak{p}_{I,\lambda}^+)] = 0$.

Proof. Let $Q \in \mathcal{L}^0_{X_I}(\mathfrak{p}_{I,\lambda}^+)$ and $f \in \mathcal{O}_{X_I}$. We can write $Q = \sum_i f_ia_i - h$, where $f_i, h \in \mathcal{O}_{X_I}$ and $a_i \in \mathfrak{g}$ satisfy

$$a(g) := \mathrm{Ad}(g)^{-1}(\sum_i f_i(gP_I^+)a_i) \in \mathfrak{p}_I^+, \quad \lambda(a(g)) = h(gP_I^+).$$

Then we have

$$\begin{aligned}
[Q, f](gP_I^+) &= (\sum_i f_i \partial_{a_i}(f))(gP_I^+) \\
&= (\partial_{\sum_i f_i(gP_I^+)a_i}(f))(gP_I^+) \\
&= \frac{d}{dt} f(\exp(-t\sum_i f_i(gP_I^+)a_i)gP_I^+)|t=0 \\
&= \frac{d}{dt} f(g\exp(-ta(g))P_I^+)|t=0 \\
&= \frac{d}{dt} f(gP_I^+)|t=0 \\
&= 0
\end{aligned}$$

Q.E.D.

By Lemma 1.1 and Lemma 1.2 we see that $\mathcal{L}^0_{X_I}(\mathfrak{r}_{I,\lambda})$ is a two-sided ideal of $U_{X_I}(\mathfrak{g})$. Set

(1.9) $$D_{X_I,\lambda} = U_{X_I}(\mathfrak{g})/\mathcal{L}^0_{X_I}(\mathfrak{r}_{I,\lambda}).$$

Define a $\mathbb{C}$-algebra homomorphism $U(\mathfrak{g}) \to \Gamma(X_I, D_{X_I,\lambda})$ $(u \mapsto \partial_u^\lambda)$ by $\partial_u^\lambda = \overline{1 \otimes u}$. For the sake of completeness we give a proof of the following.

Lemma 1.3. *$D_{X_I,\lambda}$ is a TDO-ring on X_I.*

Proof. We shall show that $D_{X_I,\lambda}|(gN_I^- P_I^+/P_I^+)$ is isomorphic to the sheaf of ordinary differential operators $D_{gN_I^- P_I^+/P_I^+}$ for any $g \in G$. We may assume that $g = 1$ without loss of generality. We shall identify $N_I^- P_I^+/P_I^+$ with N_I^-. We have $D_{N_I^-} \simeq \mathcal{O}_{N_I^-} \otimes U(\mathfrak{n}_I^-)$ via the ring homomorphism $R: U(\mathfrak{n}_I^-) \to D_{N_I^-}$ given by

$$((R(a))(f))(x) = \frac{d}{dt} f(\exp(-ta)x)|t=0 \quad (a \in \mathfrak{n}_I^-, f \in \mathcal{O}_{N_I^-}, x \in N_I^-).$$

This gives an embedding of the ring

$$D_{N_I^-} \simeq \mathcal{O}_{N_I^-} \otimes U(\mathfrak{n}_I^-) \subset \mathcal{O}_{N_I^-} \otimes U(\mathfrak{g}) = U_{X_I}(\mathfrak{g})|N_I^-.$$

In other words $D_{N_I^-}$ is identified with the subring of $U_{X_I}(\mathfrak{g})|N_I^-$ consisting of $Q \in U_{X_I}(\mathfrak{g})|N_I^-$ satisfying $Q(xP_I^+) \in U(\mathfrak{n}_I^-)$ for any $x \in N_I^-$. By the definition $\mathcal{L}^0_{X_I}(\mathfrak{r}_{I,\lambda})|N_I^-$ consists of $Q \in U_{X_I}(\mathfrak{g})|N_I^-$ satisfying $Q(xP_I^+) \in \mathrm{Ad}(x)(\mathfrak{r}_{I,\lambda})$ for any $x \in N_I^-$. Since

$$U(\mathfrak{g}) = U(\mathfrak{n}_I^-) \oplus \mathfrak{r}_{I,\lambda}, \qquad \mathrm{Ad}(x)U(\mathfrak{n}_I^-) = U(\mathfrak{n}_I^-) \quad (x \in N_I^-),$$

we have $U_{X_I}(\mathfrak{g})|N_I^- = \mathcal{L}^0_{X_I}(\mathfrak{r}_{I,\lambda})|N_I^- \oplus D_{N_I^-}$, and hence $D_{X_I,\lambda}|N_I^- \simeq D_{N_I^-}$. Q.E.D.

We define an increasing filtration F of $D_{X_I,\lambda}$ by

$$(1.10)\quad F_p(D_{X_I,\lambda}) = \mathrm{Image}(\mathcal{O}_{X_I} \otimes F_p(U(\mathfrak{g})) \to D_{X_I,\lambda}) \qquad (p \in \mathbb{Z}_{\geq 0}),$$

where $F_p(U(\mathfrak{g}))$ denotes the subspace of $U(\mathfrak{g})$ consisting of elements with order $\leq p$.

Let $\delta : U_{X_I}(\mathfrak{g}) \to U_{X_I}(\mathfrak{g})$ be the anti-automorphism given by

$$(1.11)\qquad \delta(f) = f \quad (f \in \mathcal{O}_{X_I}), \qquad \delta(a) = -a \quad (a \in \mathfrak{g}).$$

Set

$$(1.12)\qquad \rho_I = \frac{1}{2} \sum_{\alpha \in \Delta^+ \setminus \Delta_I} \alpha \in \mathrm{Hom}(\mathfrak{p}_I^+, \mathbb{C}).$$

Then we see easily that $\delta(\mathcal{L}^0_{X_I}(\mathfrak{r}_{I,\lambda})) = \mathcal{L}^0_{X_I}(\mathfrak{r}_{I,-\lambda+2\rho_I})$, and hence we have

$$(1.13)\qquad D^{op}_{X_I,\lambda} \simeq D_{X_I,-\lambda+2\rho_I},$$

where $D^{op}_{X_I,\lambda}$ denotes the opposite ring of $D_{X_I,\lambda}$.

1.5 Define a right $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$(1.14)\qquad M_I(\lambda) = U(\mathfrak{g})/\mathfrak{r}_{I,\lambda}.$$

If we regard $M_I(\lambda)$ as a left $U(\mathfrak{g})$-module via the anti-automorphism of $U(\mathfrak{g})$ given by $a \mapsto -a$ for $a \in \mathfrak{g}$, it is a highest weight module with highest weight $-\lambda$ called a generalized Verma module. Especially it contains a unique maximal proper submodule J_λ. Let $V = \tilde{V}/\mathfrak{r}_{I,\lambda}$ be a $U(\mathfrak{g})$-submodule of $M_I(\lambda)$. For any $a \in \mathfrak{p}_I^+, b \in \tilde{V}$ we have

$$[a,b] = (a - \lambda(a))b - b(a - \lambda(a)) \subset \mathfrak{r}_{I,\lambda} + \tilde{V} \subset \tilde{V}$$

and hence $\tilde{V}$ is $\mathrm{ad}(\mathfrak{p}_I^+)$-stable. Therefore, we have obtained a G-stable left ideal

$$(1.15)\qquad \mathcal{L}_{X_I}(V) = \mathcal{L}^0_{X_I}(\tilde{V}) \bmod \mathcal{L}^0_{X_I}(\mathfrak{r}_{I,\lambda})$$

of $D_{X_I,\lambda}$ for a $U(\mathfrak{g})$-submodule V of $M_I(\lambda)$.

Theorem 1.4. *The map $\mathcal{L}_{X_I}$ gives a one-to-one correspondence between the set of $U(\mathfrak{g})$-submodules of $M_I(\lambda)$ and that of G-stable left ideals of $D_{X_I,\lambda}$.*

Proof. For an $\mathcal{O}_{X_I}$-submodule J of $D_{X_I,\lambda}$ and $gP_I^+ \in X_I$ we denote the image of $J_{gP_I^+}$ under $(D_{X_I,\lambda})_{gP_I^+} \to \mathbb{C} \otimes_{\mathcal{O}_{X_I,gP_I^+}} (D_{X_I,\lambda})_{gP_I^+}$ by $J(gP_I^+)$. We see from the proof of Lemma 1.3 that $\mathbb{C} \otimes_{\mathcal{O}_{X_I,eP_I^+}} (D_{X_I,\lambda})_{eP_I^+}$ is naturally identified with $M_I(\lambda)$. Then we see easily that $(\mathcal{L}_{X_I}(V))(eP_I^+) = V$ for any $U(\mathfrak{g})$-submodule V of $M_I(\lambda)$ under the above identification. It remains to show that $J = \mathcal{L}_{X_I}(J(eP_I^+))$ for any G-stable left ideal J of $D_{X_I,\lambda}$. Set $F_p(J) = J \cap F_p(D_{X_I,\lambda})$. Since $F_p(J)$ and $F_p(D_{X_I,\lambda})/F_p(J)$ are coherent $\mathcal{O}_{X_I}$-modules, they are locally free on an open subset of X_I (see [20, lecture 8]). Moreover, they are actually locally free on whole X_I by the G-equivariance. Set $J' = \mathcal{L}_{X_I}(J(eP_I^+))$. Then $F_p(J')$ and $F_p(D_{X_I,\lambda})/F_p(J')$ are also locally free by the same reason (or by the definition of $\mathcal{L}_{X_I}$). It follows that $F_p(J)$ and $F_p(J')$ coincide if $(F_p(J))(gP_I^+) = (F_p(J'))(gP_I^+)$ for any $gP_I^+ \in X_I$. By the definition we have $(F_p(J))(eP_I^+) = (F_p(J'))(eP_I^+)$, and by the G-equivariance we have $(F_p(J))(gP_I^+) = (F_p(J'))(gP_I^+)$ for any $gP_I^+ \in X_I$. Hence we have $F_p(J) = F_p(J')$. Since this holds for any p we have $J = J'$. Q.E.D.

§2. Highest weight modules associated to Hermitian symmetric spaces

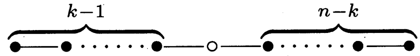
2.1 In this section we shall deal with the case:

(2.1) $$\mathfrak{n}_I^{\pm} \text{ is nonzero and commutative.}$$

Let $\theta = \sum_{i \in I_0} m_i \alpha_i$ be the highest root of $\mathfrak{g}$. It is well known that the condition (2.1) is equivalent to the following :

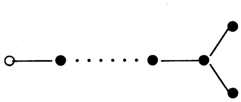
(2.2) $$I = I_0 \setminus \{i_0\} \text{ with } m_{i_0} = 1.$$

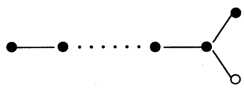
We have the following list of $(\mathfrak{g}, I)$ satisfying (2.1).

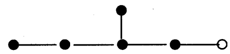
(I)  $(k-1 \leq n-k)$

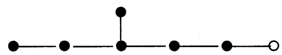
(II) 

(III) 

(IV) 

(V) 

(VI) 

(VII) 

Here, $I = I_0 \setminus \{i_0\}$, where i_0 corresponds to the white vertex of the Dynkin diagram.

2.2 Let us identify $\mathfrak{n}_I^+$ with an open subset U of G/P_I^- via the embedding $a \mapsto \exp(a)P_I^-$. Note that U is stable under the left multiplication by L_I and that the induced action of L_I on $\mathfrak{n}_I^+$ is the adjoint action. Let $\tilde{\mathcal{C}}$ denote the set of L_I-orbits on $\mathfrak{n}_I^+$.

Proposition 2.1 (see Richardson-Röhrle-Steinberg [22]). (i) *For any P_I^--orbit D on G/P_I^- the intersection $D \cap U$ consists of a single L_I-orbit.*

(ii) *The correspondence $D \mapsto D \cap U$ gives a bijection*

$$P_I^- \backslash G/P_I^- \simeq \tilde{\mathcal{C}}. \tag{2.3}$$

In particular, $\mathfrak{n}_I^+$ consists of only finitely many L_I-orbits.

By a well known result on the Bruhat decomposition we have the following natural one-to-one correspondence:

$$W_I \backslash W / W_I \simeq \tilde{\mathcal{C}}. \tag{2.4}$$

Proposition 2.2. *Let $C \in \tilde{\mathcal{C}}$, and set $O = \mathrm{Ad}(G)(C) \subset \mathfrak{g}$. Then we have $O \cap \mathfrak{n}_I^+ = C$, and $\dim C = \dim O/2$.*

Proof. Let $\mathcal{N}$ be the set of nilpotent orbits on $\mathfrak{g}$ which intersects with $\mathfrak{n}_I^+$. In order to show that $\mathrm{Ad}(G)(C) \cap \mathfrak{n}_I^+ = C$ for any $C \in \tilde{\mathcal{C}}$, it is sufficient to show that the map $\mathcal{C} \to \mathcal{N}$ given by $C \mapsto \mathrm{Ad}(G)(C)$ is injective. Since it is apparently surjective, it is sufficient to show $\sharp(\tilde{\mathcal{C}}) = \sharp(\mathcal{N})$. We can calculate $\sharp(\tilde{\mathcal{C}})$ in each individual case by using (2.4). Let $O_0 \in \mathcal{N}$ be the nilpotent orbit such that $O_0 \cap \mathfrak{n}_I^+$ is open dense in $\mathfrak{n}_I^+$. Since the moment map $T^*(G/P_I^+) \to \mathfrak{g}$ is a projective morphism, its image $\mathrm{Ad}(G)(\mathfrak{n}_I^+)$ is a closed subset of $\mathfrak{g}$. Hence $\mathcal{N}$ consists of nilpotent orbits O contained in $\overline{O}_0$. It follows that we can also determine $\sharp(\mathcal{N})$ using explicit description of the closure relations of the nilpotent orbits (see [12], [19]), and we conclude that $\sharp(\tilde{\mathcal{C}}) = \sharp(\mathcal{N})$. The latter half of our statement is also verified using case-by-case consideration. Details are omitted. Q.E.D.

Let $\mathcal{A}_0$ be the set of zeros of the b-function of the prehomogeneous vector space $(L_I, \mathfrak{n}_I^+)$, and set $\mathcal{A} = \{-a-1 \mid a \in \mathcal{A}_0\}$. The following explicit description of $\mathcal{A}$ is given in Gyoja [11].

(I)	$\mathcal{A} = \{0, 1, 2, \dots, k-1\}$	(II)	$\mathcal{A} = \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{n-1}{2}\}$
(III)	$\mathcal{A} = \{0, \frac{2n-3}{2}\}$	(IV)	$\mathcal{A} = \{0, n-2\}$
(V)	$\mathcal{A} = \{0, 2, 4, \dots, 2([\frac{n-2}{2}])\}$	(VI)	$\mathcal{A} = \{0, 3\}$
(VII)	$\mathcal{A} = \{0, 4, 8\}$		

Here, $n = \sharp(I_0)$.

Let $\mathcal{C}$ be the set of non-open L_I-orbits on $\mathfrak{n}_I^+$. By a case-by-case check we see that $\sharp(\mathcal{C}) = \sharp(\mathcal{A})$ and that $\mathcal{C}$ is a totally ordered set with respect to the closure relation. Hence there exists a unique bijection $\mathcal{A} \to \mathcal{C}\, (r \mapsto C_r)$ satisfying

$$C_r \subset \overline{C_s} \text{ if } r \leq s. \tag{2.5}$$

2.3 We shall give an explicit description of the maximal proper submodule $J_{r\varpi_{i_0}}$ of $M_I(r\varpi_{i_0})$ for $r \in \mathcal{A}$.

Let $\lambda \in \mathrm{Hom}(\mathfrak{p}_I^+, \mathbb{C})$. By the Poincaré-Birkhoff-Witt theorem the natural linear map $U(\mathfrak{n}_I^-) \to M_I(\lambda) = U(\mathfrak{g})/\mathfrak{r}_{I,\lambda}$ is an isomorphism. By the condition (2.1) $U(\mathfrak{n}_I^-)$ is isomorphic to the symmetric algebra $S(\mathfrak{n}_I^-)$. Via the Killing form of $\mathfrak{g}$ $S(\mathfrak{n}_I^-)$ is identified with the algebra $\mathbb{C}[\mathfrak{n}_I^+]$ consisting of polynomial functions on $\mathfrak{n}_I^+$. Hence we have a natural bijective linear map

$$F_\lambda : \mathbb{C}[\mathfrak{n}_I^+] \to M_I(\lambda). \tag{2.6}$$

For $C \in \mathcal{C}$ let $I(\overline{C})$ be the defining ideal of the closure $\overline{C}$ of C in $\mathfrak{n}_I^+$.

Proposition 2.3 (see [5], [26], and their references). *We have*

$$J_{r\varpi_{i_0}} = F_{r\varpi_{i_0}}(I(\overline{C}_r)) \tag{2.7}$$

for any $r \in \mathcal{A}$.

Remark 2.4. For $r_0 = \min(\mathcal{A} \setminus \{0\})$ the ideal $J_{r_0\varpi_{i_0}}$ is generated by polynomials with degree 2 by [10].

§3. Radon transforms

3.1 Let $f : X \to Y$ be a morphism of smooth algebraic varieties. For a TDO-ring A on Y one can associate a TDO-ring $f^\sharp A$ on X, a $(f^\sharp A, f^{-1}A)$-bimodule $A_{X\to Y}$, and a $(f^{-1}A, f^\sharp A)$-bimodule $A_{Y\leftarrow X}$ (see [3], [15]). If $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module and $A = \mathcal{L}\otimes_{\mathcal{O}_Y} D_Y \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes -1}$, then we have

$$\begin{aligned} f^\sharp A &= f^*\mathcal{L}\otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} f^*\mathcal{L}^{\otimes -1}, \\ A_{X\to Y} &= \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}A, \quad A_{Y\leftarrow X} = f^{-1}A \otimes_{f^{-1}\mathcal{O}_Y} \Omega_f. \end{aligned}$$

Here, $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{O}_Y)$, $f^*\mathcal{L} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}$, and $\Omega_f = \Omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}$.

For an A-module M set

$$\mathbb{L}f^*M = A_{X\to Y} \otimes^{\mathbb{L}}_{f^{-1}A} f^{-1}M. \tag{3.1}$$

It is a complex of $f^\sharp A$-modules. If f is smooth, we have $H^i(\mathbb{L}f^*M) = 0$ for $i \neq 0$, and in this case we simply write f^*M for $H^0(\mathbb{L}f^*M)$.

For an $f^\sharp A$-module M set

$$\int_f M = \mathbb{R}f_*(A_{Y\leftarrow X} \otimes^{\mathbb{L}}_{f^\sharp A} M). \tag{3.2}$$

It is a complex of A-modules.

3.2 Let I and J be subsets of I_0. Let

$$p_1 : X_{I\cap J} \to X_I, \qquad p_2 : X_{I\cap J} \to X_J \tag{3.3}$$

be the canonical projections. Set

$$\gamma_{IJ} = \sum_{\alpha\in(\Delta^+\setminus\Delta_J)\cap\Delta_I} \alpha \in \mathfrak{h}^*. \tag{3.4}$$

Let $\lambda \in \mathrm{Hom}(\mathfrak{p}_I, \mathbb{C}) \subset \mathfrak{h}^*$ and $\mu \in \mathrm{Hom}(\mathfrak{p}_J, \mathbb{C}) \subset \mathfrak{h}^*$ satisfying

$$\lambda = \mu - \gamma_{IJ}. \tag{3.5}$$

Then for a $D_{X_J,\mu}$-module N we can define a complex $R_{IJ}(N)$ of $D_{X_I,\lambda}$-module called the Radon transform of N by

$$R_{IJ}(N) = \int_{p_1} (\Omega_{p_1}^{\otimes -1} \otimes p_2^* N). \tag{3.6}$$

Indeed, $\Omega_{p_1}^{\otimes -1} \otimes p_2^* N$ is a $D_{X_{I\cap J},\lambda}$-module since $p_2^* N$ is a $D_{X_{I\cap J},\mu}$-module and we have $\Omega_{p_1} \simeq \mathcal{O}_{X_{I\cap J}}(\gamma_{IJ})$.

3.3 By the definition we have

$$R_{IJ}(N) = \mathbb{R}p_{1*}(A_{IJ} \otimes^{\mathbb{L}}_{p_2^{-1}D_{X_J,\mu}} p_2^{-1}N) \tag{3.7}$$

with

$$A_{IJ} = (p_1^{-1}D_{X_I,\lambda} \otimes_{p_1^{-1}\mathcal{O}_{X_I}} \mathcal{O}_{X_{I\cap J}}) \otimes^{\mathbb{L}}_{D_{X_{I\cap J},\mu}} (\mathcal{O}_{X_{I\cap J}} \otimes_{p_2^{-1}\mathcal{O}_{X_J}} p_2^{-1}D_{X_J,\mu}). \tag{3.8}$$

Lemma 3.1. *Set $i = (p_1, p_2) : X_{I\cap J} \to X_I \times X_J$. Then we have*

(3.9) $H^k(A_{IJ}) = 0$ *for* $k \neq 0$,

(3.10) $H^0(A_{IJ})$

$$= (i^{-1}(D_{X_I,\lambda} \boxtimes D^{op}_{X_J,\mu}) \otimes_{i^{-1}\mathcal{O}_{X_I \times X_J}} \mathcal{O}_{X_{I\cap J}}) \otimes_{D_{X_{I\cap J}}} \mathcal{O}_{X_{I\cap J}}.$$

This follows from the following general result.

Lemma 3.2. *Let $f : S \to X$ and $g : S \to Y$ be morphisms of smooth varieties, and let A_S, A_X, A_Y be TDO-rings on S, X, Y respectively such that $A_S = g^\sharp A_Y = (f^\sharp A_X^{op})^{op}$. Set $\varphi = (f, g) : S \to X \times Y$. Then we have*

$$\begin{aligned} &(f^{-1}A_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_S) \otimes^{\mathbb{L}}_{A_S} (\mathcal{O}_S \otimes_{g^{-1}\mathcal{O}_Y} g^{-1}A_Y) \\ \simeq\ &(\varphi^{-1}(A_X \boxtimes A_Y^{op}) \otimes_{\varphi^{-1}\mathcal{O}_{X\times Y}} \mathcal{O}_S) \otimes^{\mathbb{L}}_{D_S} \mathcal{O}_S. \end{aligned}$$

In particular, if φ is a closed embedding, then we have

$$H^k((f^{-1}A_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_S) \otimes^{\mathbb{L}}_{A_S} (\mathcal{O}_S \otimes_{g^{-1}\mathcal{O}_Y} g^{-1}A_Y)) = 0 \qquad (k \neq 0).$$

Proof. Let $\Delta : S \to S \times S$ be the diagonal embedding. In general, for a left A_S-module M and a right A_S-module N we have

$$\begin{aligned} N \otimes^{\mathbb{L}}_{A_S} M &= (N \otimes^{\mathbb{L}}_{\mathcal{O}_S} M) \otimes^{\mathbb{L}}_{D_S} \mathcal{O}_S \\ &= (\Delta^{-1}(N \boxtimes M) \otimes^{\mathbb{L}}_{\Delta^{-1}\mathcal{O}_{S\times S}} \mathcal{O}_S) \otimes^{\mathbb{L}}_{D_S} \mathcal{O}_S. \end{aligned}$$

Hence we have

$$\begin{aligned} &(f^{-1}A_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_S) \otimes^{\mathbb{L}}_{A_S} (\mathcal{O}_S \otimes_{g^{-1}\mathcal{O}_Y} g^{-1}A_Y) \\ =\ &(\Delta^{-1}((f^{-1}A_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_S) \boxtimes (g^{-1}A_Y^{op} \otimes_{g^{-1}\mathcal{O}_Y} \mathcal{O}_S)) \\ &\qquad \otimes^{\mathbb{L}}_{\Delta^{-1}\mathcal{O}_{S\times S}} \mathcal{O}_S) \otimes^{\mathbb{L}}_{D_S} \mathcal{O}_S \\ =\ &(\varphi^{-1}(A_X \boxtimes A_Y^{op}) \otimes_{\varphi^{-1}\mathcal{O}_{X\times Y}} \mathcal{O}_S) \otimes^{\mathbb{L}}_{D_S} \mathcal{O}_S. \end{aligned}$$

If φ is a closed immersion, then $\varphi^{-1}(A_X \boxtimes A_Y^{op}) \otimes_{\varphi^{-1}\mathcal{O}_{X\times Y}} \mathcal{O}_S$ is a locally free D_S-module. Hence $H^k((\varphi^{-1}(A_X \boxtimes A_Y^{op}) \otimes_{\varphi^{-1}\mathcal{O}_{X\times Y}} \mathcal{O}_S) \otimes^{\mathbb{L}}_{D_S} \mathcal{O}_S) = 0$ for $k \neq 0$. Q.E.D.

By the definition we see easily that

$$D_{X_{I\cap J},\mu} = U_{X_{I\cap J}}(\mathfrak{g})/\mathcal{I}, \tag{3.11}$$

$$p_1^{-1} D_{X_I,\lambda} \otimes_{p_1^{-1}\mathcal{O}_{X_I}} \mathcal{O}_{X_{I\cap J}} = U_{X_{I\cap J}}(\mathfrak{g})/\mathcal{J}, \tag{3.12}$$

$$\mathcal{O}_{X_{I\cap J}} \otimes_{p_2^{-1}\mathcal{O}_{X_J}} p_2^{-1} D_{X_J,\mu} = U_{X_{I\cap J}}(\mathfrak{g})/\mathcal{K}, \tag{3.13}$$

where

$$\mathcal{I} = \mathcal{L}^0_{X_{I\cap J}}(\mathfrak{r}_{I\cap J,\mu}), \tag{3.14}$$

$$\mathcal{J} = \delta(\mathcal{L}^0_{X_{I\cap J}}(\mathfrak{r}_{I,-\lambda+2\rho_I})) = p_1^{-1}\mathcal{L}^0_{X_I}(\mathfrak{r}_{I,\lambda}) U_{X_{I\cap J}}(\mathfrak{g}), \tag{3.15}$$

$$\mathcal{K} = \mathcal{L}^0_{X_{I\cap J}}(\mathfrak{r}_{J,\mu}) = U_{X_{I\cap J}}(\mathfrak{g}) p_2^{-1}\mathcal{L}^0_{X_J}(\mathfrak{r}_{J,\mu}). \tag{3.16}$$

Here, δ is as in (1.11). Hence we have

$$A_{IJ} = U_{X_{I\cap J}}(\mathfrak{g})/(\mathcal{J} + \mathcal{K}). \tag{3.17}$$

Let $n \in \Gamma(X_J, N)$. Since A_{IJ} has a canonical section $\overline{1}$, we have a morphism

$$\mathbb{C} \to A_{IJ} \otimes_{\mathbb{C}} p_2^{-1} N \to A_{IJ} \otimes^{\mathbb{L}}_{p_2^{-1} D_{X_J,\mu}} p_2^{-1} N$$

given by $1 \to \overline{1} \otimes n$. Hence the composition of

$$\mathbb{C} \to \mathbb{R}p_{1*}(p_1^{-1}\mathbb{C}) = \mathbb{R}p_{1*}(\mathbb{C}) \to \mathbb{R}p_{1*}(A_{IJ} \otimes^{\mathbb{L}}_{p_2^{-1} D_{X_J,\mu}} p_2^{-1} N) = R_{IJ}(N)$$

induces a section

$$R_{IJ}(n) \in \Gamma(X_I, H^0(R_{IJ}(N))). \tag{3.18}$$

3.4 In the rest of this section we fix a subset I of I_0 satisfying (2.1) and $r \in \mathcal{A}$. We set

$$\lambda = r\varpi_{i_0}. \tag{3.19}$$

Let J be a subset of I_0 and set $\mu = \lambda + \gamma_{IJ}$. We have $\mu \in \mathrm{Hom}(\mathfrak{p}_J^+, \mathbb{C})$ if and only if

$$\gamma_{IJ}(h_{i_0}) = -r, \tag{3.20}$$

where $I = I_0 \setminus \{i_0\}$. In this case we can define, for a $D_{X_J,\mu}$-module N, its Radon transform $R_{IJ}(N)$ as a complex of $D_{X_I,\lambda}$-modules.

Lemma 3.3. *Let J be a subset of I_0 satisfying (3.20), and set $\mu = \lambda + \gamma_{IJ}$. Let $\varphi : D_{X_I,\lambda} \to H^0(R_{IJ}(D_{X_J,\mu}))$ be the canonical morphism given by $1 \mapsto R_{IJ}(1)$ (see (3.18)). Assume that $\mathfrak{n}_I^+ \cap \mathfrak{n}_J^+ \subset \overline{C}_r$. Then we have $\mathcal{L}_{X_I}(J_\lambda) \subset \operatorname{Ker}\varphi$.*

Proof. By (3.17) we have

$$R_{IJ}(D_{X_J,\mu}) = \mathbb{R}p_{1*}(U_{X_{I\cap J}}(\mathfrak{g})/(\mathcal{J}+\mathcal{K})),$$

where $\mathcal{J}$ and $\mathcal{K}$ are as in (3.15) and (3.16). Let $\overline{1}$ be the canonical section of the left $p_1^{-1}D_{X_I,\lambda}$-module $U_{X_{I\cap J}}(\mathfrak{g})/(\mathcal{J}+\mathcal{K})$. Let m be the canonical generator of the right $\mathfrak{g}$-module $M_I(\lambda)$, and set $\tilde{J}_\lambda = \{u \in U(\mathfrak{g}) \mid um \in J_\lambda\}$. It is sufficient to show $p_1^{-1}(\mathcal{L}_{X_I}(J_\lambda)) \cdot \overline{1} = 0$, or equivalently,

$$p_1^{-1}(\mathcal{L}^0_{X_I}(\tilde{J}_\lambda)) \subset \mathcal{J}+\mathcal{K}. \tag{3.21}$$

Since the problem is local, we have only to show (3.21) on the open subsets $gN^-_{I\cap J}P^+_{I\cap J}/P^+_{I\cap J}$ of $X_{I\cap J}$. We may assume $g=1$ without loss of generality. We can identify $N^-_{I\cap J}P^+_{I\cap J}/P^+_{I\cap J}$ with $N^-_{I\cap J}$ via $x \mapsto xP^+_{I\cap J}$. Note that $N^-_{I\cap J} \simeq N^-_I \times (N^-_J \cap L_I)$ via the multiplication. Define $\varphi_1 : N^-_{I\cap J} \to N^-_I$ and $\varphi_2 : N^-_{I\cap J} \to N^-_J \cap L_I$ by $g = \varphi_1(g)\varphi_2(g)$ for $g \in N^-_{I\cap J}$. Let $R \in p_1^{-1}(\mathcal{L}^0_{X_I}(\tilde{J}_\lambda))$. Then there exists a (locally defined) $U(\mathfrak{g})$-valued function $\overline{R}$ on N^-_I such that

$$\overline{R}(x) \in \operatorname{Ad}(x)(\tilde{J}_\lambda) \quad (x \in N^-_I), \qquad R(gP^+_{I\cap J}) = \overline{R}(\varphi_1(g)) \quad (g \in N^-_{I\cap J}).$$

Set $V = \tilde{J}_\lambda \cap U(\mathfrak{n}^-_I)$. Then we have $\tilde{J}_\lambda = V \oplus \mathfrak{r}_{I,\lambda}$. Since $\mathfrak{n}^-_I$ is commutative, we have $\operatorname{Ad}(x)\tilde{J}_\lambda = V \oplus \operatorname{Ad}(x)\mathfrak{r}_{I,\lambda}$ for $x \in N^-_I$. Hence we can decompose $\overline{R}$ into the form $\overline{R} = \overline{R}_1 + \overline{R}_2$ with

$$\overline{R}_1(x) \in V, \quad \overline{R}_2(x) \in \operatorname{Ad}(x)\mathfrak{r}_{I,\lambda} \quad (x \in N^-_I).$$

Correspondingly, we have $R = R_1 + R_2$ with

$$R_1(gP^+_I) = \overline{R}_1(\varphi_1(g)) \in V \quad (g \in N^-_{I\cap J}), \quad R_2 \in p_1^{-1}(\mathcal{L}^0_{X_I}(\mathfrak{r}_{I,\lambda})) \subset \mathcal{J}.$$

Then we have only to show $R_1 \in \mathcal{K}$. This is equivalent to showing $V \subset \operatorname{Ad}(g)\mathfrak{r}_{J,\mu}$ for any $g \in N^-_{I\cap J}$. Let us show that V is $\operatorname{ad}(\mathfrak{n}^-_{I\cap J})$-stable. Since $\mathfrak{n}^-_{I\cap J} = \mathfrak{n}^-_I \oplus (\mathfrak{n}^-_J \cap \mathfrak{l}_I)$, we have only to show that V is stable under the adjoint actions of $\mathfrak{n}^-_I$ and $\mathfrak{n}^-_J \cap \mathfrak{l}_I$. Since $\mathfrak{n}^-_I$ is commutative, we have $[\mathfrak{n}^-_I, V] = 0$. Let $x \in \mathfrak{n}^-_J \cap \mathfrak{l}_I$. By the definition of $\tilde{J}_\lambda$ we see easily that $\tilde{J}_\lambda$ is $\operatorname{ad}(\mathfrak{l}_I)$-stable. In particular, we have $[x, V] \subset \tilde{J}_\lambda$. On the other hand, since $\mathfrak{n}^-_I$ is $\operatorname{ad}(\mathfrak{l}_I)$-stable we have $[x, V] \subset U(\mathfrak{n}^-_I)$. Hence, V is $\operatorname{ad}(\mathfrak{n}^-_{I\cap J})$-stable. Therefore, we have only to show $V \subset \mathfrak{r}_{J,\mu}$. Since

$\mathfrak{r}_{J,\mu} \supset [\mathfrak{p}_J^+, \mathfrak{p}_J^+]U(\mathfrak{g}) \supset (\mathfrak{p}_J^+ \cap \mathfrak{n}_I^-)U(\mathfrak{n}_I^-)$, it is sufficient to show $V \subset (\mathfrak{p}_J^+ \cap \mathfrak{n}_I^-)U(\mathfrak{n}_I^-)$. Since V corresponds to $I(\overline{C}_r)$ under $U(\mathfrak{n}_I^-) \simeq \mathbb{C}[\mathfrak{n}_I^+]$, this follows from our assumption $\mathfrak{n}_I^+ \cap \mathfrak{n}_J^+ \subset \overline{C}_r$. Q.E.D.

3.5 In the rest of this section we fix a subset J of I_0 satisfying the conditions (3.20) and

$$\overline{\mathrm{Ad}(L_I)(\mathfrak{n}_I^+ \cap \mathfrak{n}_J^+)} = \overline{C}_r. \tag{3.22}$$

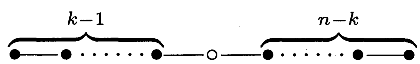
We set

$$\mu = \lambda + \gamma_{IJ}. \tag{3.23}$$

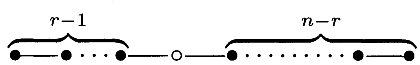
By Proposition 2.2 the condition (3.22) is equivalent to

$$\overline{\mathrm{Ad}(G)(\mathfrak{n}_I^+ \cap \mathfrak{n}_J^+)} = \overline{\mathrm{Ad}(G)(C_r)}. \tag{3.24}$$

We note that such J does not necessarily exist. If $r = 0$, then (3.20) and (3.22) obviously hold for $J = I_0$. In the case where $\mathfrak{g}$ is a classical Lie algebra, there exists such J if and only if $r \in \mathbb{Z}$. We list below possible choices for J when $\mathfrak{g}$ is a classical Lie algebra and $r \in \mathbb{Z}_{>0}$.

(I) $I \longleftrightarrow$ 

$r \in \{1, 2, \ldots, k-1\}, \quad k - 1 \le n - k$

$J \longleftrightarrow$ 

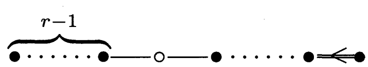
$\mu = k\varpi_{j_0}$ with $J = I_0 \setminus \{j_0\}$

$J \longleftrightarrow$ 

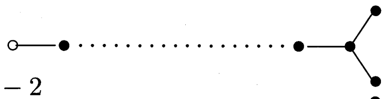
$\mu = (n + 1 - k)\varpi_{j_0}$ with $J = I_0 \setminus \{j_0\}$

(II) $I \longleftrightarrow$ 

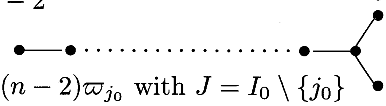
$r \in \{1, 2, \ldots, [\frac{n-1}{2}]\}$

$J \longleftrightarrow$ 

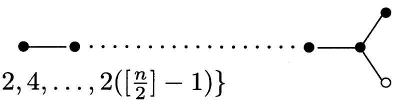
$\mu = n\varpi_{j_0}$ with $J = I_0 \setminus \{j_0\}$

(IV) I $\longleftrightarrow$ 

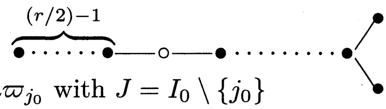
$r = n - 2$

J $\longleftrightarrow$ 

$\mu = 2(n-2)\varpi_{j_0}$ with $J = I_0 \setminus \{j_0\}$

(V) I $\longleftrightarrow$ 

$r \in \{2, 4, \ldots, 2([\frac{n}{2}] - 1)\}$

J $\longleftrightarrow$ 

$\mu = n\varpi_{j_0}$ with $J = I_0 \setminus \{j_0\}$

3.6 By Lemma 3.3 we have a canonical morphism

$$\varphi : D_{X_I,\lambda}/\mathcal{L}_{X_I}(J_\lambda) \to H^0(R_{IJ}(D_{X_J,\mu})). \tag{3.25}$$

Let

$$i_0 : V = P_I^+ P_J^+ / P_J^+ \to X_J \tag{3.26}$$

be the natural embedding. We see easily by (3.23) that $i_0^\sharp D_{X_J,\mu}^{op} \simeq \Omega_{i_0}^{\otimes -1} \otimes D_V \otimes \Omega_{i_0}$, and hence we can consider a $D_{X_J,\mu}^{op}$-module $\int_{i_0} \Omega_{i_0}^{\otimes -1}$ with a canonical section m_0. Let T^*X_J be the cotangent bundle of X_J, and let $T_V^*X_J$ be the conormal bundle of V. Let $\gamma : T^*X_J \to \mathfrak{g}$ be the moment map. Here we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form. By (3.22) we have $\gamma(T_V^*X_J) = \overline{C}_r$. Let

$$\overline{\gamma} : T_V^*X_J \to \overline{C}_r \tag{3.27}$$

be the induced morphism.

Theorem 3.4. (i) *We have*

$$H^k(R_{IJ}(D_{X_J,\mu})) = 0 \qquad (k \neq 0), \tag{3.28}$$

$$\varphi : D_{X_I,\lambda}/\mathcal{L}_{X_I}(J_\lambda) \simeq H^0(R_{IJ}(D_{X_J,\mu})) \tag{3.29}$$

if and only if

$$H^k(Y, \int_{i_0} \Omega_{i_0}^{\otimes -1}) = 0 \qquad (k \neq 0), \tag{3.30}$$

$$\Gamma(Y, \int_{i_0} \Omega_{i_0}^{\otimes -1}) = U(\mathfrak{n}_I^-) m_0. \tag{3.31}$$

(ii) *The conditions* (3.30) *and* (3.31) *are satisfied if*

$$H^k(T_V^*X_J, \mathcal{O}_{T_V^*X_J}) = 0 \qquad (k \neq 0), \tag{3.32}$$

$$\overline{\gamma} \text{ has connected fibers.} \tag{3.33}$$

Proof. (i) Set $X = X_I$, $Y = X_J$, $Z = X_{I\cap J}$. Let $p_1 : Z \to X$ and $p_2 : Z \to Y$ be the canonical morphisms, and set $i = (p_1, p_2) : Z \to X \times Y$. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections. Since i is a closed embedding, we have

$$R_{IJ}(D_{X_J,\mu}) = \mathbb{R}p_{1*}(A_{IJ}) = \mathbb{R}\pi_{1*}(i_*A_{IJ}).$$

By the definition of A_{IJ} we see easily that $i_*A_{IJ} \simeq \int_i \Omega_i^{\otimes -1}$ as a $D_{X,\lambda} \boxtimes D_{Y,\mu}^{op}$-module. Note that we have $i^\sharp(D_{X,\lambda} \boxtimes D_{Y,\mu}^{op}) \simeq \Omega_i^{\otimes -1} \otimes D_Z \otimes \Omega_i$ by (3.23). Hence we have

$$R_{IJ}(D_{X_J,\mu}) \simeq \mathbb{R}\pi_{1*}(M) \quad \text{with} \quad M = \int_i \Omega_i^{\otimes -1}.$$

We have a canonical section m of M, and we see easily that the morphism $\varphi : D_{X_I,\lambda}/\mathcal{L}_{X_I}(J_\lambda) \to \pi_{1*}(M)$ is given by $\overline{1} \mapsto \pi_{1*}(m)$. Since $\pi_{1*}(m) \neq 0$, we have $\varphi \neq 0$. Since φ is G-equivariant, and since $\mathcal{L}_{X_I}(J_\lambda)$ is the unique maximal G-stable left ideal, we see that φ is injective. Hence the conditions (3.28) and (3.29) hold if and only if

$$R^k\pi_{1*}(M) = 0 \qquad (k \neq 0), \tag{3.34}$$

$$\pi_{1*}(M) = D_{X,\lambda}\pi_{1*}(m). \tag{3.35}$$

Set $N = N_I^-$, and identify N with an open subset of X via the embedding $j : N \to X \quad (x \mapsto xP_I^+)$. Since M is G-equivariant, it is sufficient to consider (3.34) and (3.35) on N. Consider the following Cartesian diagrams.

$$\begin{array}{ccccc}
Z & \xrightarrow{\ i\ } & X \times Y & \xrightarrow{\ \pi_1\ } & X \\
\uparrow{\scriptstyle \overline{j}} & & \uparrow{\scriptstyle j\times 1} & & \uparrow{\scriptstyle j} \\
N \times V & \xrightarrow[\ \overline{i}\ ]{} & N \times Y & \xrightarrow[\ \overline{\pi}_1\ ]{} & N
\end{array}$$

Here $\overline{\pi}_1$ is the projection, and $\overline{i}(x, yP_J^+) = (x, xyP_J^+)$, $\overline{j}(x, yP_J^+) = xyP_{I\cap J}^+$ for $x \in N$, $y \in P_I^+$. Then we have

$$R^k\pi_{1*}(M)|N = R^k\overline{\pi}_{1*}((j \times 1)^* \int_i \Omega_i^{\otimes -1}) = R^k\overline{\pi}_{1*} \int_{\overline{i}} \Omega_{\overline{i}}^{\otimes -1}.$$

Let $f : N \times Y \to N \times Y$ be the isomorphism given by $f(x, yP_J^+) = (x, xyP_J^+)$ for $x \in N$ and $y \in G$. Then we have $\bar{i} = f \circ (1 \times i_0)$ and $\overline{\pi}_{1*} = \overline{\pi}_{1*} \circ f$. Hence we have

$$\begin{aligned} R^k\overline{\pi}_{1*} \int_{\bar{i}} \Omega_{\bar{i}}^{\otimes -1} &= R^k\overline{\pi}_{1*} \int_{1\times i_0} \Omega_{1\times i_0}^{\otimes -1} = R^k\overline{\pi}_{1*}(\mathcal{O}_N \boxtimes \int_{i_0} \Omega_{i_0}^{\otimes -1}) \\ &= \mathcal{O}_N \otimes_{\mathbb{C}} H^k(Y, \int_{i_0} \Omega_{i_0}^{\otimes -1}). \end{aligned}$$

Therefore we have obtained

$$\text{(3.36)} \qquad R^k\pi_{1*}(M)|N \simeq \mathcal{O}_N \otimes_{\mathbb{C}} H^k(Y, \overline{M}) \quad \text{with} \quad \overline{M} = \int_{i_0} \Omega_{i_0}^{\otimes -1}.$$

In order to consider the condition (3.35) let us examine the action of $D_{X,\lambda}|N \simeq D_N$ on $\pi_{1*}(M)|N \simeq \mathcal{O}_N \otimes_{\mathbb{C}} \Gamma(Y, \overline{M})$. Define a ring homomorphism $L : U(\mathfrak{n}_I^-) \to \Gamma(N, D_N)$ by

$$((L(a)(f))(x) = \frac{d}{dt} f(x \exp(ta))|t = 0 \qquad (a \in \mathfrak{n}_I^-, f \in \mathcal{O}_N, x \in N).$$

This induces an isomorphism $\mathcal{O}_N \otimes_{\mathbb{C}} U(\mathfrak{n}_I^-) \simeq D_N$. Let $\Phi : D_N \to D_N \otimes_{\mathbb{C}} U(\mathfrak{g})$ be the ring homomorphism given by

$$\Phi(f) = f \otimes 1 \quad (f \in \mathcal{O}_N), \qquad \Phi(L(a)) = L(a) \otimes 1 + 1 \otimes a \quad (a \in \mathfrak{n}_I^-).$$

Regard $\Gamma(Y, \overline{M})$ as a $U(\mathfrak{g})$-module via $U(\mathfrak{g}) \to D_{Y,\mu} \quad (u \mapsto \partial_u^\mu = \overline{1 \otimes u})$. Then we have

$$Pv = \Phi(P)v \qquad (P \in D_N, v \in \mathcal{O}_N \otimes_{\mathbb{C}} \Gamma(Y, \overline{M})).$$

Since $\pi_{1*}(m)|N = 1 \otimes m_0 \in \mathcal{O}_N \otimes_{\mathbb{C}} \Gamma(Y, \overline{M})$, we have $D_{X,\lambda}\pi_{1*}(m)|N = \mathcal{O}_N \otimes U(\mathfrak{n}_I^-)m_0$. The statement (i) is proved.

(ii) Define a good filtration of $\overline{M}$ by $F_p(\overline{M}) = F_p(D_{Y,\mu})\overline{m}$. Then we have

$$\overline{M} = \cup_p F_p(\overline{M}), \qquad \mathrm{gr}_F \overline{M} := \oplus_p F_p(\overline{M})/F_{p-1}(\overline{M}) \simeq \pi_* \mathcal{O}_{T_V^* Y},$$

where $\pi : T_V^* Y \to Y$ is the canonical map. Since π is an affine morphism, we have

$$H^k(Y, \pi_* \mathcal{O}_{T_V^* Y}) = H^k(T_V^* Y, \mathcal{O}_{T_V^* Y}) = 0 \qquad (k \neq 0)$$

by (3.32). Hence $H^k(Y, F_p(\overline{M})/F_{p-1}(\overline{M})) = 0$ for any $k \neq 0$ and any p. By the exact sequence

$$H^k(Y, F_{p-1}(\overline{M})) \longrightarrow H^k(Y, F_p(\overline{M})) \longrightarrow H^k(Y, F_p(\overline{M})/F_{p-1}(\overline{M})) = 0$$

we see by induction on p that $H^k(Y, F_p(\overline{M})) = 0$ for any $k \neq 0$ and any p. Hence $H^k(Y, \overline{M}) = 0$ for any $k \neq 0$, and (3.30) holds.

Define a filtration of $\Gamma(Y, \overline{M})$ by

$$F_p(\Gamma(Y, \overline{M})) = \Gamma(Y, F_p(\overline{M})) \subset \Gamma(Y, \overline{M}).$$

Then we have

$$F_p(U(\mathfrak{n}_I^-))F_q(\Gamma(Y, \overline{M})) \subset F_{p+q}(\Gamma(Y, \overline{M})),$$

where $F_p(U(\mathfrak{n}_I^-))$ denotes the subspace of $U(\mathfrak{n}_I^-)$ consisting of elements with order $\leq p$. Hence in order to show (3.31) it is sufficient to show that

$$\mathrm{gr}_F \Gamma(Y, \overline{M}) := \oplus_p F_p(\Gamma(Y, \overline{M}))/F_{p-1}(\Gamma(Y, \overline{M}))$$

is generated by the canonical element $[m_0] \in F_0(\Gamma(Y, \overline{M}))/F_{-1}(\Gamma(Y, \overline{M}))$ as a module over $\mathrm{gr}_F U(\mathfrak{n}_I^-) = S(\mathfrak{n}_I^-) = \mathbb{C}[\mathfrak{n}_I^+]$. On the other hand by the exact sequence

$$\begin{aligned} 0 \to \Gamma(Y, F_{p-1}(\overline{M})) &\to \Gamma(Y, F_p(\overline{M})) \\ &\to \Gamma(Y, F_p(\overline{M})/F_{p-1}(\overline{M})) \to H^1(Y, F_{p-1}(\overline{M})) = 0 \end{aligned}$$

we have

$$\begin{aligned} \mathrm{gr}_F \Gamma(Y, \overline{M}) &= \Gamma(Y, \mathrm{gr}_F \overline{M}) = \Gamma(Y, \pi_* \mathcal{O}_{T_V^* Y}) = \Gamma(T_V^* Y, \mathcal{O}_{T_V^* Y}) \\ &= \Gamma(\overline{C}_r, \gamma_*(\mathcal{O}_{T_V^* Y})). \end{aligned}$$

Let $\psi : \mathcal{O}_{\overline{C}_r} \to \overline{\gamma}_*(\mathcal{O}_{T_V^* Y})$ be the canonical morphism. Since $[m_0]$ corresponds to $\psi(1)$, it is sufficient to show that ψ is an isomorphism. Since $\overline{\gamma}$ is a proper morphism, we have the Stein factorization $\overline{\gamma} = \gamma_1 \circ \gamma_2$ of γ, where $\gamma_2 : T_V^* Y \to \mathrm{Spec}(\overline{\gamma}_*(\mathcal{O}_{T_V^* Y}))$ is a projective morphism, and $\gamma_1 : \mathrm{Spec}(\overline{\gamma}_*(\mathcal{O}_{T_V^* Y})) \to \overline{C}_r$ is a finite morphism. By (3.33) γ_1 is bijective. Moreover, by Lemma 2.1 and the normality of the Schubert varieties the variety $\overline{C}_r$ is normal. Since we are working in characteristic 0, we see that γ_1 is an isomorphism of algebraic varieties, and hence ψ is an isomorphism. Q.E.D.

Proposition 3.5. *The conditions* (3.32) *and* (3.33) *are satisfied for* $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

Proof. The condition (3.33) is easily checked by a direct calculation. Let us show (3.32). Since the projection $p : T_V^* X_J \to V$ is affine, it is sufficient to show $H^k(V, p_* \mathcal{O}_{T_V^* X_J}) = 0$ for $k \neq 0$. Note that $V = P_I^+ P_J^+ / P_J^+ = L_I / L_I \cap P_J^+$ is the generalized flag manifold of the smaller group L_I. For an $L_I \cap P_J^+$-module U let $\mathcal{L}_U$ denote the locally free $\mathcal{O}_V$-module consisting of sections of the L_I-equivariant vector bundle on $V = L_I / L_I \cap P_J^+$ corresponding to U. Then we have $p_* \mathcal{O}_{T_V^* X_J} = \mathcal{L}_{S((\mathfrak{n}_I^+ \cap \mathfrak{n}_J^+)^*)}$. Since $S((\mathfrak{n}_I^+ \cap \mathfrak{n}_J^+)^*)$ is a union of finite dimensional $L_I \cap P_J^+$-modules, $p_* \mathcal{O}_{T_V^* X_J}$ is a union of locally free $\mathcal{O}_V$-modules with finite rank. Hence it is sufficient to show that for any irreducible $L_I \cap P_J^+$-module U appearing in $S((\mathfrak{n}_I^+ \cap \mathfrak{n}_J^+)^*)$ we have $H^k(V, \mathcal{L}_U) = 0$ for $k \neq 0$. Let j_0 be the unique element of $I \setminus J$. By the theorem of Borel-Weil-Bott we have only to verify $\langle \gamma, h_{j_0} \rangle \geq 0$ for any highest weight γ appearing in the $L_I \cap L_J$-module $S(\mathfrak{n}_I^+ \cap \mathfrak{n}_J^+)$. This condition holds if $\langle \gamma, h_{j_0} \rangle \geq 0$ for any weight γ appearing in the $L_I \cap L_J$-module $\mathfrak{n}_I^+ \cap \mathfrak{n}_J^+$. We can easily check it directly. Q.E.D.

Corollary 3.6. *If* $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, *we have*

$$H^k(R_{IJ}(D_{X_J,\mu})) = 0 \qquad (k \neq 0),$$
$$\varphi : D_{X_I,\lambda} / \mathcal{L}_{X_I}(J_\lambda) \simeq H^0(R_{IJ}(D_{X_J,\mu})).$$

Real versions of this result are given in Kakehi [13], Oshima [21], and Sekiguchi [23].

§4. Hypergeometric systems

4.1 In this section we fix a subset I of I_0 satisfying (2.1).

For $r \in \mathcal{A}$, a closed subgroup K of G with Lie algebra $\mathfrak{k}$, and a character $\xi \in \mathrm{Hom}(\mathfrak{k}, \mathbb{C})$ we define a $D_{X_I, r\varpi_{i_0}}$-module $M_{r,K,\xi}$ by

$$(4.1) \quad M_{r,K,\xi} = D_{X_I, r\varpi_{i_0}} / (\mathcal{L}_{X_I}(J_{r\varpi_{i_0}}) + \sum_{a \in \mathfrak{k}} D_{X_I, r\varpi_{i_0}} (\partial_a^{r\varpi_{i_0}} - \xi(a)))$$

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. If $r = 1$ and K is a maximal torus, then the system $M_{r,K,\xi}$ is nothing but the hypergeometric system investigated in Gelfand [6] and Gelfand-Gelfand [7] (see also Aomoto [1]). The case $r = 1$ and K is the centralizer of a (not necessarily semisimple) regular element was also treated in Gelfand-Retakh-Serganova [8] and Kimura-Haraoka-Takano [16]. Moreover, Oshima [21] recently considered the case where $1 \leq r \leq k - 1$ and K is an appropriate subgroup of G. Our systems $M_{r,K,\xi}$ are natural generalization of the systems mentioned above, and we call them the hypergeometric systems on the Hermitian symmetric spaces X_I.

4.2 We identify the cotangent bundle T^*X_I of our flag manifold X_I with

$$\{(gP_I, a) \in X_I \times \mathfrak{g} \,|\, a \in \mathrm{Ad}(g)\mathfrak{n}_I^+\}$$

via the Killing form. For $s \in \mathcal{A}$ let O_s be the nilpotent conjugacy class of $\mathfrak{g}$ satisfying $\overline{\mathrm{Ad}(G)C_s} = \overline{O}_s$. For $s \in \mathcal{A}$ and a closed subgroup K of G we define a locally closed subset $\Lambda^0_{s,K}$ and a closed subset $\Lambda_{s,K}$ of T^*X_I by

$$\Lambda^0_{s,K} = \{(gP_I, a) \in T^*X_I \,|\, a \in O_s \cap \mathfrak{k}^\perp\}, \tag{4.2}$$

$$\Lambda_{s,K} = \{(gP_I, a) \in T^*X_I \,|\, a \in \overline{O}_s \cap \mathfrak{k}^\perp\}, \tag{4.3}$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{k}^\perp = \{a \in \mathfrak{g} \,|\, \langle a, \mathfrak{k}\rangle = 0\}$. Here $\langle\ ,\ \rangle$ is the Killing form of $\mathfrak{g}$. We have $\Lambda_{s,K} = \sqcup_{s' \in \mathcal{A}, s' \leq s} \Lambda^0_{s',K}$.

For a coherent module M over a TDO-ring on a smooth algebraic variety X one can define its characteristic variety $\mathrm{Ch}(M)$ as a closed subset of the cotangent bundle T^*X. Then M is called a holonomic module if $\dim \mathrm{Ch}(M) = \dim X$. By Proposition 2.2 we see easily the following.

Lemma 4.1. *We have* $\mathrm{Ch}(M_{r,K,\xi}) \subset \Lambda_{r,K}$.

Lemma 4.2. *We have*

$$\dim \Lambda^0_{r,K} = \dim X_I + \dim(O_r \cap \mathfrak{k}^\perp) - \frac{1}{2}\dim O_r.$$

Proof. Set

$$Z = \{(gP_I, a) \in X_I \times \mathfrak{g} \,|\, a \in \mathrm{Ad}(g)C_r\} = \{(gP_I, a) \in T^*X_I \,|\, a \in O_r\}.$$

The first projection $Z \to X_I$ is a G-equivariant fibering onto the homogeneous space X_I whose fiber at the origin is C_r, and hence we have $\dim Z = \dim X_I + \dim C_r = \dim X_I + (\dim O_r/2)$, The second projection $p : Z \to O_r$ is also a G-equivariant fibering onto the homogeneous space O_r, and hence we have $\dim p^{-1}(a) = \dim Z - \dim O_r = \dim X_I - (\dim O_r/2)$ for any $a \in O_r$. Since $\Lambda^0_{r,K} = p^{-1}(O_r \cap \mathfrak{k}^\perp)$, we have $\dim \Lambda^0_{r,K} = \dim(O_r \cap \mathfrak{k}^\perp) + \dim X_I - (\dim O_r/2)$. Q.E.D.

Hence the $D_{X_I, r\varpi_{i_0}}$-module $M_{r,K,\xi}$ is holonomic if $\dim(O_s \cap \mathfrak{k}^\perp) \leq \dim O_s/2$ for any $s \in \mathcal{A}$ such that $s \leq r$.

Remark 4.3. (i) The equality $\dim(O_s \cap \mathfrak{k}^\perp) = \dim O_s/2$ apparently holds true for $s = 0$. Hence, for $r = r_0 = \min(\mathcal{A} \setminus \{0\})$, the system $M_{r_0,K,\xi}$ is holonomic if $\dim(O_{r_0} \cap \mathfrak{k}^\perp) = \dim O_{r_0}/2$. One may suspect that a candidate for a closed subgroup K of G satisfying $\dim(O_{r_0} \cap \mathfrak{k}^\perp) = \dim O_{r_0}/2$ is the maximal torus H. However, we have $\dim(O_{r_0} \cap \mathfrak{h}^\perp) = \dim O_{r_0}/2$ if and only if $(\mathfrak{g}, I)$ is of type (I) or type (II). In other cases we have $\dim \Lambda_{r_0,H} > \dim X_I$.

(ii) In [9] a certain $D_{\mathfrak{n}_I^-}$-module is investigated as a special case of the so called A-hypergeometric systems. Identifying $\mathfrak{n}_I^-$ with an open subset of X_I via the embedding $\mathfrak{n}_I^- \to X_I \quad (a \mapsto \exp(a)P_I)$, this $D_{\mathfrak{n}_I^-}$-module coincides with the restriction of $M_{r_0,H,\xi}$ to $\mathfrak{n}_I^-$ when $(\mathfrak{g}, I)$ is of the type (I) or the type (II). In other cases it is some quotient of $M_{r_0,H,\xi}|\mathfrak{n}_I^-$.

4.3 We fix $r \in \mathcal{A}$ and a subset J of I_0 satisfying the conditions (3.20) and (3.22), and set

$$\lambda = r\varpi_{i_0}, \qquad \mu = \lambda + \gamma_{IJ}. \tag{4.4}$$

Then, for a $D_{X_J,\mu}$-module N, we can define its Radon transform $R_{IJ}(N)$ as a complex of $D_{X_I,\lambda}$-modules. Let K be a closed subgroup of G with Lie algebra $\mathfrak{k}$, and let $\xi \in \mathrm{Hom}(\mathfrak{k}, \mathbb{C})$. Set

$$N_{r,K,\xi} = D_{X_J,\mu} / \sum_{a \in \mathfrak{k}} D_{X_J,\mu}(\partial_a^\mu - \xi(a)). \tag{4.5}$$

By the argument in the proof of Lemma 3.3, we see easily the following.

Proposition 4.4. *There exists a canonical homomorphism*

$$M_{r,K,\xi} \to R_{IJ}(N_{r,K,\xi}).$$

Proposition 4.5. *Assume that there exist only finitely many K-orbits on X_J.*

(i) *We have* $\dim \Lambda^0_{s,K} \le \dim X_I + (\dim O_s/2) - \dim(O_s \cap \mathfrak{n}_J^+)$ *for any* $s \le r$.

(ii) *We have* $\dim \Lambda^0_{r,K} \le \dim X_I$. *Especially, if* $r = r_0 = \min(\mathcal{A} \setminus \{0\})$, *then the system* $M_{r,K,\xi}$ *is holonomic.*

(iii) *If* $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, *then we have* $\dim \Lambda^0_{s,K} \le \dim X_I$ *for any* $s \le r$. *Especially, the system* $M_{r,K,\xi}$ *is holonomic.*

Proof. (i) By Lemma 4.2 it is sufficient to show

$$\dim(O \cap \mathfrak{k}^\perp) \le \dim O - \dim(O \cap \mathfrak{n}_J^+) \tag{4.6}$$

for any nilpotent conjugacy class O in $\mathfrak{g}$ such that $O \cap \mathfrak{n}_J^+ \neq \emptyset$. Let $\gamma : T^*X_J \to \mathfrak{g}$ be the moment map. By the assumption O is contained in the image of γ. Set $Z = \gamma^{-1}(O)$. Since the projection $Z \to X_J$ is a G-equivariant fibering onto the homogeneous space X_J whose typical fiber is $O \cap \mathfrak{n}_J^+$, we have

$\dim Z = \dim X_J + \dim(O \cap \mathfrak{n}_J^+)$. Since the natural morphism $\overline{\gamma} : Z \to O$ is a G-equivariant fibering onto the homogeneous space O, we have $\dim \overline{\gamma}^{-1}(a) = \dim Z - \dim O = \dim X_J + \dim(O \cap \mathfrak{n}_J^+) - \dim O$ for any $a \in O$. Hence we have

(4.7)
$$\dim \overline{\gamma}^{-1}(O \cap \mathfrak{k}^\perp) = \dim X_J + \dim(O \cap \mathfrak{n}_J^+) - \dim O + \dim(O \cap \mathfrak{k}^\perp).$$

By the definition $\gamma^{-1}(\mathfrak{k}^\perp)$ is the union of the conormal bundles of the K-orbits on X_J. Hence by the assumption on K we have $\dim \gamma^{-1}(\mathfrak{k}^\perp) = \dim X_J$, and thus

$$\dim \overline{\gamma}^{-1}(O \cap \mathfrak{k}^\perp) \leq \dim \gamma^{-1}(\mathfrak{k}^\perp) = \dim X_J. \tag{4.8}$$

The assertion (4.6) follows from (4.7) and (4.8).

(ii) By (i) it is sufficient to show

$$\dim(O \cap \mathfrak{n}_I^+) = \dim(O \cap \mathfrak{n}_J^+) = \dim O/2 \tag{4.9}$$

for any nilpotent conjugacy class O in $\mathfrak{g}$ such that there exists some $w \in W$ satisfying $\overline{\mathrm{Ad}(G)(\mathfrak{n}_I^+ \cap w(\mathfrak{n}_J^+))} = \overline{O}$. Let Λ be the union of the conormal bundles of the G-orbits on $X_I \times X_J$. Since there exists only finitely many G-orbits on $X_I \times X_J$, Λ is a closed subvariety of $T^*(X_I \times X_J) \simeq T^*X_I \times T^*X_J$ with pure dimension $\dim X_I + \dim X_J$. We can identify Λ with

$$\{(g_1P_I^+, g_2P_J^+, a) \in X_I \times X_J \times \mathfrak{g} \mid a \in \mathrm{Ad}(g_1)\mathfrak{n}_I^+ \cap \mathrm{Ad}(g_2)\mathfrak{n}_J^+\}.$$

Let $\varphi : \Lambda \to \mathfrak{g}$ be the natural morphism given by $(g_1P_I^+, g_2P_J^+, a) \mapsto a$. Let $\Lambda_w \subset \Lambda$ be the conormal bundle of the G-orbit containing $(eP_I^+, wP_J^+) \in X_I \times X_J$. By the assumption on O we see that $\varphi^{-1}(O) \cap \Lambda_w$ is an open subset of Λ_w, and hence we have $\dim \varphi^{-1}(O) = \dim X_I + \dim X_J$. Fix $a \in O$ and set

$$X_I^a = \{gP_I^+ \mid a \in \mathrm{Ad}(g)(\mathfrak{n}_I^+)\}, \qquad X_J^a = \{gP_J^+ \mid a \in \mathrm{Ad}(g)(\mathfrak{n}_J^+)\}.$$

Since $\varphi^{-1}(O) \to O$ is a G-equivariant fibering onto O whose typical fiber is $X_I^a \times X_J^a$, we have

$$\dim O = (\dim X_I - \dim X_I^a) + (\dim X_J - \dim X_J^a) \tag{4.10}$$

Set $D = \{(gP_I^+, x) \in X_I \times O \mid a \in \mathrm{Ad}(g)(\mathfrak{n}_I^+)\}$. Considering the natural morphisms $D \to X_I$ and $D \to O$ we obtain $\dim X_I - \dim X_I^a = \dim O - \dim(O \cap \mathfrak{n}_I^+)$. Similarly we have $\dim X_J - \dim X_J^a = \dim O - \dim(O \cap \mathfrak{n}_J^+)$. Hence we have

$$\dim O = \dim(O \cap \mathfrak{n}_I^+) + \dim(O \cap \mathfrak{n}_J^+). \tag{4.11}$$

On the other hand, we have $\dim(O \cap \mathfrak{n}^+) = \dim O/2$, where $\mathfrak{n}^+ = \mathfrak{n}_\emptyset^+$ for $\emptyset \subset I_0$ (Spaltenstein [25]), and hence

$$\dim(O \cap \mathfrak{n}_I^+) \leq \dim O/2, \qquad \dim(O \cap \mathfrak{n}_J^+) \leq \dim O/2. \tag{4.12}$$

We obtain (4.9) from (4.11) and (4.12)

(iii) By Spaltenstein [24] we always have $\dim(O \cap \mathfrak{n}_{I_1}^+) = \dim O/2$ for any nilpotent conjugacy class O in $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and any $I_1 \subset I_0$ (unless $O \cap \mathfrak{n}_{I_1}^+ = \emptyset$). Hence the assertion follows from (i). Q.E.D.

4.4 We give an example in this subsection.

Let V be a $2n$-dimensional vector space over $\mathbb{C}$ with nondegenerate skew-symmetric bilinear form $\phi : V \times V \to \mathbb{C}$. Choose a basis $\langle e_1, \dots, e_n, f_1, \dots, f_n \rangle$ of V such that

$$\phi(e_i, e_j) = \phi(f_i, f_j) = 0, \qquad \phi(e_i, f_j) = \delta_{ij}. \tag{4.13}$$

Set

$$\begin{aligned} \mathfrak{g} &= \{f \in \mathrm{End}(V) \mid \phi(fv_1, v_2) + \phi(v_1, fv_2) = 0 \quad (v_1, v_2 \in V)\}, \\ G &= \{g \in GL(V) \mid \phi(gv_1, gv_2) = \phi(v_1, v_2) \quad (v_1, v_2 \in V)\}, \\ \mathfrak{h} &= \{f \in \mathfrak{g} \mid f(e_i) \in \mathbb{C}e_i, \quad f(f_i) \in \mathbb{C}f_i \quad (i = 1, \dots, n)\}. \end{aligned}$$

Define $\epsilon_i \in \mathfrak{h}^*$ by $f(e_i) = \epsilon_i(f)e_i$ for $f \in \mathfrak{h}$ and $i = 1, \dots, n$. Set

$$I_0 = \{1, 2, \dots, n\}, \quad I = \{1, 2, \dots, n-1\}, \tag{4.14}$$

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad (i \in I_0, i \neq n), \qquad \alpha_n = 2\epsilon_n. \tag{4.15}$$

Then $\mathfrak{g}$ is a simple Lie algebra of type (C_n), G is the corresponding algebraic group, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\{\alpha_i\}_{i \in I_0}$ is a set of simple roots, and $(\mathfrak{g}, I)$ is of type (II). The generalized flag manifold X_I is identified with the set of n-dimensional subspaces W of V satisfying $\phi(w_1, w_2) = 0$ for any $w_1, w_2 \in W$. We shall identify $\mathfrak{n}_I^-$ with

$$\mathcal{S}_n = \{z = (z_{ij}) \in M_n(\mathbb{C}) \mid {}^t z = z\} \tag{4.16}$$

via

$$\mathfrak{n}_J^- \ni f \leftrightarrow z = (z_{ij}) \in \mathcal{S}_n \qquad (f(e_j) = \sum_{i=1}^n z_{ij} f_i). \tag{4.17}$$

We further identify $\mathfrak{n}_I^-$ with an open subset of X_I via the embedding $\mathfrak{n}_I^- \ni a \mapsto \exp(a)P_I \in X_I$. Note that $\mathcal{S}_n \to X_I$ is given by $z \mapsto \sum_{i=1}^n \mathbb{C}(e_i + \sum_{j=1}^n z_{ij} f_j)$.

We consider the case $r = 1 \in \mathcal{A}$ in the following. Then we have

$$\lambda = r\varpi_{i_0} = \varpi_n = (\epsilon_1 + \cdots + \epsilon_n)/2. \tag{4.18}$$

In this case the subset

$$J = \{2, \dots, n\} \tag{4.19}$$

of I_0 satisfies the conditions (3.20), (3.22), and we have

$$\mu = \lambda + \gamma_{IJ} = n\varpi_1 = n\epsilon_1. \tag{4.20}$$

Fix $0 < m < n$. Set $W_m = \sum_{i=1}^m \mathbb{C}e_i$, $W'_m = \sum_{i=1}^m \mathbb{C}f_i$. Identify the group $K_1 = GL_m(\mathbb{C}) \times (\mathbb{C}^\times)^{n-m}$ with a subgroup of G via the following action of K_1 on V:

$$(g, a_{m+1}, \dots, a_n) \cdot v = gv \quad (v \in W_m), \tag{4.21}$$

$$(g, a_{m+1}, \dots, a_n) \cdot e_i = a_i e_i \quad (i = m+1, \dots, n), \tag{4.22}$$

$$(g, a_{m+1}, \dots, a_n) \cdot v = {}^t g^{-1} v \quad (v \in W'_m), \tag{4.23}$$

$$(g, a_{m+1}, \dots, a_n) \cdot f_i = a_i^{-1} f_i \quad (i = m+1, \dots, n), \tag{4.24}$$

where W_m and W'_m are identified with $\mathbb{C}^m$ through the bases $\langle e_i \,|\, i = 1, \dots m\rangle$ and $\langle f_i \,|\, i = 1, \dots m\rangle$, respectively. We define a subalgebra $\mathfrak{k}_2$ of $\mathfrak{n}_I^- \simeq \mathcal{S}_n$ by $\mathfrak{k}_2 = \{z = (z_{ij}) \in \mathcal{S}_n \,|\, z_{ij} = 0 \text{ unless } i, j \geq m+1\}$, and set $K_2 = \exp(\mathfrak{k}_2)$. We take K to be the semidirect product $K = K_1 K_2$.

Let us give an explicit description of $M_{1,K,\xi}|\mathcal{S}_n$. We use $(z_{ij})_{1 \leq i \leq j \leq n}$ as a coordinate of $\mathcal{S}_n$. Set $\partial_{ij} = \partial_{ji} = (1 + \delta_{ij})\partial/\partial z_{ij}$ for $i \leq j$. For $1 \leq i_1 < \cdots < i_N \leq n$ and $1 \leq j_1 < \cdots < j_N \leq n$ set $D^{i_1, \dots, i_N}_{j_1, \dots, j_N} = \det(\partial_{i_p j_q})_{1 \leq p,q \leq N}$. For $(\xi_1, \xi_{m+1}, \dots, \xi_n) \in \mathbb{C}^{n-m+1}$ define a character ξ of $\mathfrak{k} = \mathrm{Lie}(K_1) \oplus \mathrm{Lie}(K_2)$ by

$$\xi(a, a_{m+1}, \dots, a_n) = \xi_1 \operatorname{tr}(a) + \sum_{i=m}^n \xi_i a_i \qquad ((a, a_{m+1}, \dots, a_n) \in \mathrm{Lie}(K_1)), \tag{4.25}$$

$$\xi(b) = 0 \quad (b \in \mathrm{Lie}(K_2)), \tag{4.26}$$

where we identify $\mathrm{Lie}(K_1)$ with $\mathfrak{gl}_m(\mathbb{C}) \times \mathbb{C}^{n-m}$. Then $M_{1,K,\xi}|\mathcal{S}_n$ corresponds to the following system of differential equations for an unknown function f.

$$D^{i_1,i_2,i_3}_{j_1,j_2,j_3}(f) = 0 \qquad \text{for } 1 \leq i_1 < i_2 < i_3 \leq n, 1 \leq j_1 < j_2 < j_3 \leq n, \tag{4.27}$$

$$\partial_{ij}(f) = 0 \quad \text{for } m+1 \leq i, j \leq n, \tag{4.28}$$

$$\sum_{k=1}^{n} z_{ik}\partial_{jk}(f) = \delta_{ij}(\xi_1 - \frac{1}{2})f \quad \text{for } 1 \leq i, j \leq m, \tag{4.29}$$

$$\sum_{k=1}^{n} z_{ik}\partial_{ik}(f) = (\xi_i - \frac{1}{2})f \quad \text{for } m+1 \leq i \leq n. \tag{4.30}$$

The equation (4.28) allows us to rewrite the system (4.27), … ,(4.30) into a system of differential equations on $\mathcal{S}_m \times M_{m,n-m}(\mathbb{C})$ with the coordinate $(z_{ij})_{1\leq i\leq j\leq m} \times (z_{ip})_{1\leq i\leq m, m+1\leq p\leq n}$. If $n - m \geq 3$ and $\sharp\{p \mid m+1 \leq p \leq n, \xi_p = 1/2\} \leq 2$, then the rewritten system is equivalent to the following.

$$D^{i_1,i_2,i_3}_{j_1,j_2,j_3}(f) = 0 \quad \text{for } i_1 < i_2 < i_3 \leq m, \quad j_1 < j_2 < j_3 \leq m, \tag{4.31}$$

$$D^{i_1,i_2,p}_{j_1,j_2,j_3}(f) = 0 \quad \text{for } i_1 < i_2 \leq m < p, \quad j_1 < j_2 < j_3 \leq p, \tag{4.32}$$

$$D^{i_1,i_2,p}_{j_1,j_2,q}(f) = 0 \quad \text{for } i_1 < i_2 \leq m < p, \quad j_1 < j_2 \leq m < q, \tag{4.33}$$

$$D^{i_1,i_2}_{p_1,p_2}(f) = 0 \quad \text{for } i_1 < i_2 \leq m < p_1 < p_2, \tag{4.34}$$

$$\left(\sum_{k=1}^{m} z_{ik}\partial_{jk} + \sum_{p=m+1}^{n} z_{ip}\partial_{jp}\right)(f) = \delta_{ij}(\xi_1 - \frac{1}{2})f \quad \text{for } i, j \leq m, \tag{4.35}$$

$$\sum_{i=1}^{m} z_{ip}\partial_{ip}(f) = (\xi_p - \frac{1}{2})f \quad \text{for } p > m. \tag{4.36}$$

Here $\partial_{pq} = 0$ in (4.33).

Next we give integral representations of solutions of this system. The $D_{X_J,\mu}$-module $N_{1,K,\xi}$ is nonzero at the generic point of X_J if and only if $\xi_1 = 0$. Hence we restrict ourselves to this case. Define an $(n-1)$-form τ on $\mathbb{C}^n \setminus \{0\}$ by

$$\tau = \sum_{i=1}^{n} (-1)^{i+1} u_i du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_n, \tag{4.37}$$

where $(u_1, \ldots, u_n)$ is the coordinate of $\mathbb{C}^n \setminus \{0\} \subset \mathbb{C}^n$. Let

$$(4.38) \qquad z_1 = (z_{ij}) \in \mathcal{S}_m, \qquad z_2 = (z_{ip}) \in M_{m,n-m}(\mathbb{C}),$$

$$(4.39) \qquad \mathbf{u}_1 = {}^t(u_1, \ldots, u_m), \qquad \mathbf{u}_2 = {}^t(u_{m+1}, \ldots, u_n).$$

Then the $(n-1)$-form

(4.40)

$$\tilde{\omega}(u_1, \ldots, u_n) = ({}^t\mathbf{u}_1 z_1 \mathbf{u}_1 + {}^t\mathbf{u}_1 z_2 \mathbf{u}_2)^{(\xi_{m+1}+\cdots+\xi_n-n)/2} u_{m+1}^{-\xi_{m+1}} \cdots u_n^{-\xi_n} \tau$$

on $\mathbb{C}^n \setminus \{0\}$ induces an $(n-1)$-form ω on $\mathbb{P}^{n-1} = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^\times$. We see that the function

$$(4.41) \qquad f(z_1, z_2) = \int_\Gamma \omega$$

on $\mathcal{S}_m \times M_{m,n-m}(\mathbb{C})$ is a solution to the system (4.31), ... , (4.36) with $\xi_1 = 0$ for any twisted $(n-1)$-cycle Γ on $\mathbb{P}^{n-1}$.

This example is related to the system investigated in Matsumoto-Sasaki [18].

References

[1] K. Aomoto, On the structures of integrals of power products of linear functions, Sc. Papers Coll. Gen. Education, Univ. Tokyo, **27** (1977), 49–61.

[2] A. Beilinson, J. Bernstein, Localisation de **g**-modules, C. R. Acad. Sci. Paris, **292** (1981), 15–18.

[3] A. Beilinson, J. Bernstein, A generalization of Casselman's submodule theorem, in Representation Theory of Reductive Groups, Progr. Math., **40**, Birkhäuser, Boston, 1983, 35–52.

[4] A. D'Agnolo, P. Schapira, Radon-Penrose transform for $\mathcal{D}$-modules, J. Funct. Anal., **139** (1996), 349–382.

[5] T.J. Enright, A. Joseph, An intrinsic analysis of unitarizable highest weight modules, Math. Ann., **288** (1990), 571–594.

[6] I. M. Gelfand, General theory of hypergeometric functions, Soviet Math. Dokl., **33** (1986), 573–577.

[7] I. M. Gelfand, S. I. Gelfand, Generalized hypergeometric equations, Soviet Math. Dokl., **33** (1986), 643–646.

[8] I. M. Gelfand, V. S. Retakh, V. V. Serganova, Generalized Airy functions, Schubert cells, and Jordan groups, Soviet Math. Dokl., **37** (1988), 8–12.

[9] I. M. Gelfand, A.V. Zelevinsky, M. M. Kapranov, Hypergeometric functions and toric varieties, Functional Anal. Appl., **23** (1989), 94–106.

[10] A. B. Goncharov, Construction of the Weil representations of certain simple Lie algebras, Functional Anal. Appl., **16** (1982), 70–71.

[11] A. Gyoja, Highest weight modules and b-functions of semiinvariants, Publ. RIMS Kyoto Univ., **30** (1994), 353–400.
[12] W. Hesselink, Singularities in the nilpotent scheme of a classical group, Trans. Amer. Math. Soc., **222** (1976), 1–32.
[13] T. Kakehi, Integral geometry on Grassmann manifolds and calculus on invariant differential operators, preprint.
[14] M. Kashiwara, Representation theory and D-modules on flag varieties, Astérisque, **173-174** (1989), 55–109.
[15] M. Kashiwara, T. Tanisaki, Kazhdan-Lusztig conjecture for affine Lie algebras with negative level II non-integral case, Duke Math. J., **84** (1996), 771–813.
[16] H. Kimura, Y. Haraoka, K. Takano, The Generalized confluent hypergeometric functions, Proc. Japan Acad., **68** (1992), 290–295.
[17] C. Marastoni, La dualité de Grassmann pour les $\mathcal{D}$-modules, C. R. Acad. Sci. Paris, **322** (1996), 929–933.
[18] K. Matsumoto, T. Sasaki, On the system of differential equations associated with a quadric and hyperplanes, Kyushu J. Math., **50** (1996), 93–131.
[19] K. Mizuno, The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8, Tokyo J. Math. **3** (1980), 391–461.
[20] D. Mumford, Lectures on curves on an algebraic surface, Ann. of Math. Studies, **59**, Princeton Univ. Press, 1966.
[21] T. Oshima, Generalized Capelli identities and boundary value problems for $GL(n)$, Structure of Solutions of Differential Equations, Katata/Kyoto 1995, 307–335, World Scientific, 1996.
[22] R. Richardson, G. Röhrle, R. Steinberg, Parabolic subgroup with abelian unipotent radical, Invent. Math., **110** (1992), 649–671.
[23] H. Sekiguchi, The Penrose transform for certain non-compact homogeneous manifolds of $U(n,n)$, J. Math. Sci. Univ. Tokyo **3** (1996), 655–697.
[24] N. Spaltenstein, On the fixed point set of a unipotent transformation on the flag manifold, Proc. Kon. Ak. v. Wet., **79** (1976), 452–456.
[25] N. Spaltenstein, On the fixed point set of a unipotent element on the variety of Borel subgroups, Topology, **16** (1977), 203–204.
[26] T. Tanisaki, Highest weight modules associated to parabolic subgroups with commutative unipotent radicals, in:Algebraic groups and their representations (ed. R. W. Carter and J. Saxl). 73–90, 1998, Kluwer Academic Publishers.

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