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Asymptotics for the Number of Negative Eigenvalues of Three–Body Schrödinger Operators with Efimov Effect

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Introduction

The Efimov effect is one of the most interesting results in the spectral analysis for three–body Schrödinger operators. Roughly speaking, it can be explained as follows: If all three two–body subsystems have no negative eigenvalues and if at least two of these subsystems have a resonance state at zero energy, then the three–body system under consideration has an infinite number of negative eigenvalues accumulating at zero. This remarkable spectral property was first discovered by Efimov [1] and the mathematically rigorous proof has been given by the works [4, 8, 10]. In the present note, we study the asymptotic distribution of these negative eigenvalues below the bottom zero of essential spectrum which is a three–cluster threshold energy. Let $N(E)$, $E > 0$, be the number of negative eigenvalues less than $-E$ with repetition according to their multiplicities. Then the result obtained here is, somewhat loosely stating, that $N(E)$ behaves like $|\log E|$ as $E \to 0$.

We first formulate precisely the main theorem and then make a brief comment on the recent related result obtained by Sobolev [7]. We consider a system of three particles with masses $m_j > 0$, $1 \leq j \leq 3$, which move in the three–dimensional space R^3 and interact with each other through a pair potential $V_{jk}(r_j - r_k)$, $1 \leq j < k \leq 3$, where $r_j \in R^3$ denotes the position vector of the j–th particle. For such a system, the energy Hamiltonian H (three–body Schrödinger operator) takes the form

$$H = H_0 + V, \qquad V = \sum_{1 \leq j < k \leq 3} V_{jk}(r_j - r_k), \tag{0.1}$$

in the center–of–mass frame, where H_0 denotes the free Hamiltonian. Both the operators H_0 and H act on the space $L^2(R^6)$ and are repre-

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sented in various forms according to the choice of the Jacobi coordinates. The pair potential V_{jk} is assumed to satisfy the following condition:

$(V)_\rho$ $V_{jk}(x)$, $x \in R^3$, is real–valued and has the decay property

$$|V_{jk}(x)| \leq C\,(1+|x|)^{-\rho} \quad \text{for some } \rho > 2.$$

By this assumption, the Hamiltonian H formally defined above admits a unique self–adjoint realization in $L^2(R^6)$. We denote by the same notation H this self–adjoint realization.

We use the letters α, β and γ to denote one of three pairs (j,k) with $1 \leq j < k \leq 3$. For a pair $\alpha = (j,k)$, we define the reduced mass m_α through the relation $1/m_\alpha = 1/m_j + 1/m_k$ and the two–body subsystem Hamiltonian H^α as

$$H^\alpha = -\Delta/2m_\alpha + V_\alpha, \quad V_\alpha(x) = V_{jk}(x), \quad \text{on } L^2(R^3_x).$$

We further assume that these subsystem Hamiltonians H^α have the following spectral properties:

$(H.1)$ H^α has no negative bound state energies for all pairs α.

$(H.2)$ H^α has a resonance state at zero energy for all pairs α.

Roughly speaking, the second assumption $(H.2)$ means that the equation $H^\alpha\varphi = 0$ has a solution $\varphi(x)$, $x \in R^3$, behaving like $\varphi(x) \sim |x|^{-1}$ at infinity. Such a solution is called a resonance state at zero energy. It should be noted that φ is not an eigenstate of H^α at zero energy. By the HVZ theorem ([5]), it follows from $(H.1)$ that H has essential spectrum beginning at zero and negative discrete spectrum. If, in addition, $(H.2)$ is satisfied, then H has an infinite number of negative eigenvalues accumulating at zero. In assumption $(H.1)$, we have assumed that any pair of two particles does not have bound states at negative energies. Nevertheless the three–body system has an infinite number of bound states at negative energies. As stated above, this spectral property is called the Efimov effect.

With the above notations and assumptions, we are now in a position to formulate the first theorem.

Theorem 1. *Assume that $(V)_\rho$, $(H.1)$ and $(H.2)$ are fulfilled. Let $N(E)$, $E > 0$, be the number of negative eigenvalues less than $-E$ of H with repetition according to their multiplicities. Then $N(E)$ obeys the following asymptotic formula:*

$$N(E) = C_0\,|\log E|\,(1+o(1)), \qquad E \to 0,$$

for some $C_0 > 0$.

Remark 1. We should make some comments on the leading coefficient C_0 in the asymptotic formula. This constant C_0 does not depend on the pair potentials V_{jk} and is given as a positive function of only the ratios m_j/m_k between the masses. The constant is determined from an eigenvalue asymptotics for a certain compact integral operator and is in general difficult to write down in an explicit form. In the special case with identical masses, C_0 is determined as $C_0 = s/2\pi$ with the unique positive root $s > 0$ of the equation

$$s = 2^3 \cdot 3^{-1/2}(\sinh s\,\pi/6)/(\cosh s\,\pi/2).$$

Remark 2. (1) The following result can be also obtained in the course of proof: If at most one of two–body subsystem Hamiltonians H^α has a resonance state at zero energy, then H has only a finite number of negative eigenvalues; $N(E) = O(1)$, $E \to 0$. This result asserts the finiteness of discrete spectrum below the bottom of essential spectrum, even if the bottom coincides with a three–cluster threshold energy. (2) As previously stated, H has in general an infinite number of negative eigenvalues accumulating at zero except for a certain special case, if only two subsystem Hamiltonians have a resonance state at zero energy. Even in such a case, a similar asymptotic formula with another leading coefficient $C_0 > 0$ can be obtained.

The asymptotic formula for $N(E)$ has been first established by Sobolev [7] under the main assumption that pair potentials are non–positive $V_{jk} \leq 0$ and have the decay property $(V)_\rho$ with $\rho > 3$. The above properties of the leading coefficient C_0 has been also investigated in detail there. Theorem 1 is only a supplement of the interesting result obtained by Sobolev [7] and the proof is also based on the idea developed there. But the arguments undergo slight changes in many aspects, if the non–positivity assumption of pair potentials is not necessarily assumed.

The method here applies also to the problem on the eigenvalue asymptotics in the coupling limit. We consider the three–body Hamiltonian

$$(0.2) \qquad H(\lambda) = H - \lambda V = H_0 + (1-\lambda)V \qquad \text{on } L^2(R^6)$$

with a coupling constant λ, $0 < \lambda \ll 1$, small enough, where H is defined by (0.1) and is assumed to satisfy all the assumptions in Theorem 1. Let $N_0(\lambda)$ be the number of negative eigenvalues of $H(\lambda)$. For $\lambda > 0$, $H(\lambda)$

has only a finite number of negative eigenvalues but $N_0(\lambda) \to \infty$ as λ tends to the critical value 0. The theorem below gives the asymptotic formula as $\lambda \to 0$ for $N_0(\lambda)$.

Theorem 2. *Let the notations be as above. Suppose that the three-body Hamiltonian $H = H(0)$ fulfills the assumptions $(V)_\rho$, $(H.1)$ and $(H.2)$. Then $N_0(\lambda)$ behaves like*

$$N_0(\lambda) = 2\,C_0\,|\log\lambda|\,(1 + o(1)), \qquad \lambda \to 0,$$

with the same positive constant C_0 as in Theorem 1.

§1. Low energy analysis for two-body resolvents

The proof of the theorems above is based on the behavior at low energies of two-body resolvents with resonance at zero energy. We here make a brief review on this result. For details, see [2, 3].

Throughout the section, we work in the space $L^2 = L^2(R^3_x)$ and denote by $\langle\ ,\ \rangle$ the L^2 scalar product. We begin by defining precisely the resonance state at zero energy. Let $T = -\Delta + V_0$ be the two-body Schrödinger operator acting on L^2. We assume that the potential $V_0(x)$ has the decay property $(V)_\rho$ and that the operator T has the spectral properties $(H.1)$ and $(H.2)$. We now consider the equation $T\varphi = 0$. This equation can be put into the integral equation

$$\varphi(x) = -(1/4\pi)\int |x-y|^{-1}V_0(y)\varphi(y)\,dy, \tag{1.1}$$

where the integration with no domain attached is taken over the whole space. Equation (1.1) is considered in the weighted L^2 space $L^2_{-s} = L^2(R^3_x; \langle x\rangle^{-2s}\,dx)$ with weight $\langle x\rangle^{-s} = (1 + |x|^2)^{-s/2}$, $s > 1/2$ being taken close enough to 1/2. If $\varphi \in L^2_{-s}$ solves the equation (1.1), then it is easily seen that φ behaves like

$$\varphi(x) = -(1/4\pi)\langle V_0, \varphi\rangle |x|^{-1} + O(|x|^{-\rho+1}),$$
$$(\partial/\partial|x|)\varphi(x) = (1/4\pi)\langle V_0, \varphi\rangle |x|^{-2} + O(|x|^{-\rho})$$

as $|x| \to \infty$. We say that φ is a resonance state of T at zero energy, if $\langle V_0, \varphi\rangle \neq 0$ is satisfied. Thus the resonance state φ behaves like $\varphi(x) \sim |x|^{-1}$ as $|x| \to \infty$ and hence $\varphi \notin L^2$ is not a bound state at zero energy. On the other hand, if $\langle V_0, \varphi\rangle = 0$ is satisfied, then we obtain from (1.1) that $\varphi(x) = O(|x|^{-2})$, so that φ belongs to L^2 and becomes a bound state of T at zero energy. Conversely, if φ is a bound state at zero

energy, then we can easily see that φ satisfies the relation $\langle V_0, \varphi \rangle = 0$. This implies that a resonance state at zero energy is non–degenerate. Under assumptions $(V)_\rho$, $(H.1)$ and $(H.2)$, it also follows from Theorem A.3.1 of [6] that T cannot have a bound state at zero energy (bottom of its spectrum) and hence T has only a resonance state.

Assumption $(V)_\rho$ enables us to choose a non–negative potential $U_0 \geq 0$ satisfying $(V)_\rho$ so that

$$W_0(x) = U_0(x) - V_0(x) \geq U_0(x)/2 \geq 0. \tag{1.2}$$

We define the Schrödinger operator S with potential U_0 by

$$S = -\Delta + U_0 \qquad \text{on } L^2(R^3_x) \tag{1.3}$$

and denote the resolvent of S as $R(d^2; S) = (S + d^2)^{-1}$ for $d > 0$. Since U_0 is non–negative, $R(0; S)$ can be also defined as a bounded operator from L^2_s into L^2_{-s} for any $s > 1$ and the generalized eigenfunction $\theta_0(x)$ of S at zero energy is obtained as a unique solution to the Lippmann–Schwinger equation. Let $A(d) : L^2 \to L^2$ be the operator defined by

$$A(d) = \mathrm{Id} - W_0^{1/2} R(d^2; S) W_0^{1/2}, \quad d \geq 0, \tag{1.4}$$

Id being the identity operator. It should be noted that this operator can be defined even for $d = 0$. Let Σ_1 be the kernel of $A(0)$. The kernel Σ_1 can be shown to be a one–dimensional space. Denote by $\psi_1 \in L^2$ the normalized function spanning Σ_1. Then we can show that $\psi_1(x)$ falls off with order $O(|x|^{-1-\rho/2})$ and satisfies $\langle \theta_0, W_0^{1/2} \psi_1 \rangle \neq 0$. We decompose the space $L^2 = L^2(R^3_x)$ into the orthogonal sum $L^2 = \Sigma_1 \oplus \Sigma_2$ and we denote by P_j, $1 \leq j \leq 2$, the orthogonal projections onto Σ_j.

We study the behavior as $d \to 0$ of $A(d)$ defined above. To do this, we here introduce new notations. A bounded operator $T(d)$, $0 < d \ll 1$, acting on L^2 is said to be of class $Op(d^\nu)$, if its operator norm obeys the bound $\|T(d)\| = O(d^\nu)$ as $d \to 0$. When the difference $T_1(d) - T_2(d)$ is of class $Op(d^\nu)$, we denote this relation as $T_1(d) = T_2(d) + Op(d^\nu)$.

Lemma 1.1. *Let the notations be as above. Suppose that T fulfills $(V)_\rho$, $(H.1)$ and $(H.2)$. Then the operator $A(d)$ has the following properties.*

(i) *Let ϵ, $0 < \epsilon \ll 1$, be fixed arbitrarily. Then there exist positive constants c_ϵ and c'_ϵ such that*

$$c_\epsilon \, \mathrm{Id} \leq A(d) \leq c'_\epsilon \, \mathrm{Id}, \qquad d \geq \epsilon,$$

in the form sense.

(ii) *Define* $A_{jk}(d)$, $1 \leq j,\ k \leq 2$, *as* $A_{jk}(d) = P_j A(d) P_k$. *Then:*

(1) $A_{22}(d) \in Op(d^0)$ *and* $A_{22}(d) \geq c_2 P_2$ *for some* $c_2 > 0$.
(2) $A_{12}(d) \in Op(d^\nu)$ *for some* $\nu > 1/2$.
(3) $A_{11}(d) = \sigma_1 d\, P_1 + Op(d^\nu)$ *for some* $\nu > 1$, *where*

$$\sigma_1 = |\langle \theta_0, W_0^{1/2} \psi_1 \rangle|^2 / 4\pi > 0.$$

Remark 1.2. A similar argument applies to the Schrödinger operator $T = -\Delta/2m + V_0$ with reduced mass m. For such an operator, the constant σ_1 in the lemma is given as

$$\sigma_1 = 2^{-1/2} \pi^{-1} m^{3/2} |\langle \theta_0, W_0^{1/2} \psi_1 \rangle|^2,$$

where θ_0 is the generalized eigenfunction at zero energy of $S = -\Delta/2m + U_0$, U_0 being chosen to satisfy (1.2), and $\psi_1 \in L^2$ is the normalized function constructed for the operator S.

§2. Sketch of proof of Theorem 1

We here give a sketch for the proof of Theorem 1 only. See [9] for the detailed proof, including the proof of Theorem 2.

(0) We begin by introducing several basic notations used in the spectral analysis for three–body Schrödinger operators.

Let $\alpha = (j, k)$ be given pair and let l, $l \neq j,\ k$, be the index by which the third particle is labelled. Then the Jacobi coordinates associated with α are defined as

$$x_\alpha = r_j - r_k, \quad y_\alpha = r_l - (m_j r_j + m_k r_k)/(m_j + m_k). \tag{2.1}$$

We denote by $(p_\alpha, q_\alpha) \in R^{3\times 2}$ the coordinates dual to (x_α, y_α). In this coordinate system, the symbol $H_0(p_\alpha, q_\alpha)$ of the three–body free Hamiltonian H_0 is described as

$$H_0(p_\alpha, q_\alpha) = |p_\alpha|^2/2m_\alpha + |q_\alpha|^2/2n_\alpha,$$

where m_α again denotes the reduced mass associated with α and n_α is defined through the relation $1/n_\alpha = 1/m_l + 1/(m_j + m_k)$. Let $\beta \neq \alpha$ be another pair. Then a simple calculation yields

$$p_\alpha = \kappa^{\alpha\alpha} q_\alpha + \kappa^{\alpha\beta} q_\beta, \quad p_\beta = \kappa^{\beta\alpha} q_\alpha + \kappa^{\beta\beta} q_\beta, \tag{2.2}$$

where the coefficients $\kappa^{\alpha\alpha}$, $\kappa^{\beta\alpha}$, $\kappa^{\alpha\beta}$ and $\kappa^{\beta\beta}$ are explicitly expresssed in terms of the masses m_j, $1 \le j \le 3$, and, in particular, $\kappa^{\beta\alpha}$ and $\kappa^{\alpha\beta}$ satisfy $|\kappa^{\beta\alpha}| = |\kappa^{\alpha\beta}| = 1$. We also denote by $H_0(q_\alpha, q_\beta)$ the symbol representation for H_0 in the coordinate system (q_α, q_β). We further define the cluster Hamiltonian H_α as

$$H_\alpha = H_0 + V_\alpha, \quad V_\alpha = V_{jk}, \quad \text{on } L^2(R^6).$$

The base space $L^2(R^6)$ is decomposed as the tensor product

$$L^2(R^6) = L^2(R^3; dx_\alpha) \otimes L^2(R^3; dy_\alpha)$$

and hence the Hamiltonian H_α is represented as

$$H_\alpha = H^\alpha \otimes \mathrm{Id} + \mathrm{Id} \otimes T_\alpha \quad \text{on } L^2(R^3; dx_\alpha) \otimes L^2(R^3; dy_\alpha),$$

where H^α again denotes the two–body subsystem Hamiltonian associated with α and T_α is given as

$$T_\alpha = -\Delta/2n_\alpha \qquad \text{on } L^2(R^3; dy_\alpha). \tag{2.3}$$

We now choose a non–negative potential $U_\alpha = U_\alpha(x_\alpha) \ge 0$ to satisfy the property (1.2)

$$W_\alpha(x_\alpha) = U_\alpha(x_\alpha) - V_\alpha(x_\alpha) \ge U_\alpha(x_\alpha)/2 \ge 0$$

and define the Hamiltonians K^α and K_α as

$$\begin{aligned} K^\alpha &= -\Delta/2m_\alpha + U_\alpha \quad \text{on } L^2(R^3; dx_\alpha), \\ K_\alpha &= K^\alpha \otimes \mathrm{Id} + \mathrm{Id} \otimes T_\alpha \quad \text{on } L^2(R^3; dx_\alpha) \otimes L^2(R^3; dy_\alpha). \end{aligned} \tag{2.4}$$

We also define $A(d; K^\alpha) : L^2(R^3; dx_\alpha) \to L^2(R^3; dx_\alpha)$ as

$$A(d; K^\alpha) = \mathrm{Id} - W_\alpha^{1/2}(K^\alpha + d^2)^{-1} W_\alpha^{1/2}, \quad d \ge 0, \tag{2.5}$$

in a way similar to (1.4) and denote by P_j^α, $1 \le j \le 2$, the orthogonal projections associated with $A(0; K^\alpha)$, which are constructed in the same way as P_j in section 1. We further denote by $\theta_0^\alpha = \theta_0^\alpha(x_\alpha)$ the generalized eigenfunction of K^α at zero energy and by $\psi_1^\alpha \in L^2(R^3; dx_\alpha)$ the normalized function spanning the range of P_1^α, the range being a one–dimensional space. The operator $A(d; K^\alpha)$ defined above preserves the same properties as in Lemma 1.1 (see also Remark 1.2) and, in particular, we have

$$P_1^\alpha A(d; K^\alpha) P_1^\alpha = \sigma_\alpha \, d \, P_1^\alpha + Op(d^\nu), \quad d \to 0,$$

for some $\nu > 1$, where $\sigma_\alpha > 0$ is given as

$$\sigma_\alpha = 2^{-1/2}\pi^{-1} m_\alpha^{3/2} |\langle \theta_0^\alpha, W_\alpha^{1/2}\psi_1^\alpha\rangle|^2. \tag{2.6}$$

(1) We consider only E, $0 < E \ll 1$, small enough. For given self-adjoint operator A, we denote by $n(\mu; A)$ the number of eigenvalues greater than μ of A. Let $U = \sum_\alpha U_\alpha$ and $W = \sum_\alpha W_\alpha$, where the summation $\sum_\alpha$ is taken over all three pairs α. Define the Hamiltonian K by $K = H_0 + U = H + W$ and the bounded operator $M(E) : L^2(R^6) \to L^2(R^6)$ by

$$M(E) = (K+E)^{-1/2} W (K+E)^{-1/2} = \sum_\alpha M_\alpha(E)^* M_\alpha(E)$$

with $M_\alpha(E) = W_\alpha^{1/2}(K+E)^{-1/2}$. Then the quantity $N(E)$ in question coincides with $n(1; M(E))$ by the Birman–Schwinger principle. The next lemma is due to Sobolev [7].

Lemma 2.1. *Let $\mathcal{L}^2 = \sum \oplus L^2(R^6)$, three summands. Define the operator $\mathcal{M}(E) : \mathcal{L}^2 \to \mathcal{L}^2$ as*

$$\mathcal{M}(E) = \begin{pmatrix} M_\alpha(E)M_\alpha(E)^* & M_\alpha(E)M_\beta(E)^* & M_\alpha(E)M_\gamma(E)^* \\ M_\beta(E)M_\alpha(E)^* & M_\beta(E)M_\beta(E)^* & M_\beta(E)M_\gamma(E)^* \\ M_\gamma(E)M_\alpha(E)^* & M_\gamma(E)M_\beta(E)^* & M_\gamma(E)M_\gamma(E)^* \end{pmatrix},$$

where α, β and γ denote different three pairs. Then one has

$$N(E) = n(1; \mathcal{M}(E)).$$

(2) We denote by $\mathrm{Dia}\{B_\alpha, B_\beta, B_\gamma\}$ the 3×3 diagonal matrix with operators B_α, B_β and B_γ as diagonal entries. Let $\mathcal{M}(E)$ be as in Lemma 2.1. The off-diagonal entries of $\mathcal{M}(E)$ are all compact operators on $L^2(R^6)$ but the diagonal ones are not necessarily compact operators. Thus we look more carefully at the operator

$$M_\alpha(E)M_\alpha(E)^* = W_\alpha^{1/2}(K+E)^{-1}W_\alpha^{1/2}$$

in the diagonal entries of $\mathcal{M}(E)$.

Let K_α be defined by (2.4). We decompose the above operator into the sum $M_\alpha(E)M_\alpha(E)^* = M_{0\alpha}(E) + L_\alpha(E)$, where $M_{0\alpha}(E) = W_\alpha^{1/2}(K_\alpha + E)^{-1}W_\alpha^{1/2}$ and

$$L_\alpha(E) = W_\alpha^{1/2}((K+E)^{-1} - (K_\alpha+E)^{-1})W_\alpha^{1/2},$$

so that $\mathcal{M}(E)$ is represented as $\mathcal{M}(E) = \mathcal{M}_0(E) + \mathcal{M}_1(E)$ with

$$\mathcal{M}_0(E) = \mathrm{Dia}\{M_{0\alpha}(E), M_{0\beta}(E), M_{0\gamma}(E)\}.$$

We note that $\mathcal{M}_1(E) : \mathcal{L}^2 \to \mathcal{L}^2$ is a compact operator.

We now introduce a positive smooth function $\omega(s)$, $s > 0$, such that

$$\omega(s) = s \quad \text{for } 0 < s < 1, \qquad \omega(s) = 2 \quad \text{for } s > 2.$$

Let T_α be defined by (2.3) as an operator on $L^2(R^3; dy_\alpha)$. We define

$$\omega_\alpha(E) = \omega((T_\alpha + E)^{1/2}),$$

which is considered as an operator acting on $L^2(R^6)$ as well as on $L^2(R^3; dy_\alpha)$. We further define $A_\alpha(E) : L^2(R^6) \to L^2(R^6)$ as

$$A_\alpha(E) = \mathrm{Id} - M_{0\alpha}(E) = \mathrm{Id} - W_\alpha^{1/2}(K_\alpha + E)^{-1}W_\alpha^{1/2}.$$

By Lemma 1.1 (see also Remark 1.2), we can find strictly positive smooth bounded functions $f^\pm(s)$, $0 < c \le f^+(s) \le f^-(s)$, behaving like

$$f^\pm(s) = 1 + o(s^\nu), \qquad s \to 0,$$

for some $\nu > 0$ such that

$$A_\alpha(E) \ge f_\alpha^+(E)\omega_\alpha(E)P_1^\alpha + c_+P_2^\alpha, \tag{2.7}$$

$$A_\alpha(E) \le f_\alpha^-(E)\omega_\alpha(E)P_1^\alpha + c_-P_2^\alpha \tag{2.8}$$

for some positive constants $c_\pm$, $0 < c_+ < c_-$, where

$$f_\alpha^\pm(E) = \sigma_\alpha f^\pm((T_\alpha + E)^{1/2})$$

with $\sigma_\alpha > 0$ given by (2.6), and the inequality relations are understood in the form sense. Denote by $F_\alpha^+(E)$ and $F_\alpha^-(E)$ the operators on the right side of (2.7) and (2.8), respectively, and define

$$\mathcal{F}_0^\pm(E) = \mathrm{Dia}\{F_\alpha^\pm(E), F_\beta^\pm(E), F_\gamma^\pm(E)\}.$$

Then it follows from (2.7) and (2.8) that

$$\mathcal{F}_0^+(E) \le \mathrm{Id} - \mathcal{M}_0(E) \le \mathcal{F}_0^-(E)$$

and hence we obtain from Lemma 2.1 that

$$n(1; \mathcal{Q}^-(E)) \le N(E) \le n(1; \mathcal{Q}^+(E)), \tag{2.9}$$

where

$$\mathcal{Q}^{\pm}(E) = \mathcal{F}_0^{\pm}(E)^{-1/2}\mathcal{M}_1(E)\mathcal{F}_0^{\pm}(E)^{-1/2}.$$

(3) We study the behavior as $E \to 0$ of Hilbert–Schmidt norm of the entry operators $Q_{\alpha\beta}^{\pm}(E)$ in $\mathcal{Q}^{\pm}(E)$. To do this, we here introduce new notations. Let $B(E)$, $0 < E \ll 1$, be a compact operator on $L^2(R^6)$. We say that $B(E)$ is of class $(HS)_\epsilon$, if for any $\epsilon > 0$ small enough, $B(E)$ has a decomposition $B(E) = B_1(E;\epsilon) + B_2(E;\epsilon)$ such that: (i) the Hilbert–Schmidt norm of $B_1(E;\epsilon)$ obeys the bound $\|B_1(E;\epsilon)\|_{HS} \le C_\epsilon$ for some C_ϵ independent of E; (ii) the operator norm of $B_2(E;\epsilon)$ obeys the bound $\|B_2(E;\epsilon)\| \le \epsilon$. If the difference between two operators $B_1(E)$ and $B_2(E)$ is of class $(HS)_\epsilon$, we denote this relation as $B_1(E) \sim B_2(E)$.

Lemma 2.2. $Q_{\alpha\alpha}^{\pm}(E) \sim 0$.

We analyse the operators $Q_{\alpha\beta}^{\pm}(E)$, $\alpha \neq \beta$, in the off–diagonal entries of $\mathcal{Q}^{\pm}(E)$. Recall that $\psi_1^\alpha \in L^2(R^3; dx_\alpha)$ is the normalized function spanning the range of P_1^α (one–dimensional space). Let $\chi(x)$, $x \in R^3$, be the characteristic function of the unit ball B_1 in R^3. We set

$$\zeta_\alpha(q_\alpha; E) = \chi(q_\alpha)\,(|q_\alpha|^2/2n_\alpha + E)^{-1/4}$$

and denote by $\Pi_{\alpha\beta}(E) : L^2(R^6; dx_\beta\, dq_\beta) \to L^2(R^6; dx_\alpha\, dq_\alpha)$, $\alpha \neq \beta$, the integral operator with the kernel $\psi_1^\alpha(x_\alpha) J_{\alpha\beta}(q_\alpha, q_\beta; E)\psi_1^\beta(x_\beta)$, where $J_{\alpha\beta}(q_\alpha, q_\beta; E)$ is defined by

$$J_{\alpha\beta}(q_\alpha, q_\beta; E) = \tau_{\alpha\beta}\,\zeta_\alpha(q_\alpha; E)(H_0(q_\alpha, q_\beta) + E)^{-1}\zeta_\beta(q_\beta; E) \tag{2.10}$$

with

$$\tau_{\alpha\beta} = 2^{-5/2}\,\pi^{-2}\,(m_\alpha m_\beta)^{-3/4}.$$

Let $\Psi_\alpha : L^2(R^3; dy_\alpha) \to L^2(R^3; dq_\alpha)$ be the Fourier transformation in y_α. We further define $S_{\alpha\beta}(E) : L^2(R^6) \to L^2(R^6)$ by $S_{\alpha\beta}(E) = \Psi_\alpha^* \Pi_{\alpha\beta}(E)\Psi_\beta$, $\alpha \neq \beta$.

Lemma 2.3. $Q_{\alpha\beta}^{\pm}(E) \sim S_{\alpha\beta}(E), \quad \alpha \neq \beta.$

Let $\mathcal{S}(E) : \mathcal{L}^2 \to \mathcal{L}^2$, $\mathcal{L}^2$ being as in Lemma 2.1, be the self–adjoint compact operator defined by

$$\mathcal{S}(E) = \begin{pmatrix} 0 & S_{\alpha\beta}(E) & S_{\alpha\gamma}(E) \\ S_{\beta\alpha}(E) & 0 & S_{\beta\gamma}(E) \\ S_{\gamma\alpha}(E) & S_{\gamma\beta}(E) & 0 \end{pmatrix}, \quad S_{\beta\alpha}(E) = S_{\alpha\beta}(E)^*.$$

Then Lemmas 2.2 and 2.3, together with (2.9), yield that

$$n((1+\epsilon);\mathcal{S}(E)) - C_\epsilon \le N(E) \le n((1-\epsilon);\mathcal{S}(E)) + C_\epsilon$$

for any $\epsilon > 0$ small enough, where $C_\epsilon > 0$ is independent of E. This relation can be easily obtained by use of the Weyl inequality

$$n(\lambda_1+\lambda_2; A_1+A_2) \le n(\lambda_1; A_1) + n(\lambda_2; A_2)$$

for the sum of compact operators A_1 and A_2.

(4) The proof of the theorem is completed in this step. Let

$$\mathcal{L}^2(B_1) = \sum_\alpha \oplus L^2(B_1; dq_\alpha), \quad \text{three summands.}$$

We denote by $J_{\alpha\beta}(E) : L^2(B_1; dq_\beta) \to L^2(B_1; dq_\alpha)$ the integral operator with the kernel $J_{\alpha\beta}(q_\alpha, q_\beta; E)$ defined by (2.10), and define the operator $\mathcal{J}_0(E) : \mathcal{L}^2(B_1) \to \mathcal{L}^2(B_1)$ as

$$\mathcal{J}_0(E) = \begin{pmatrix} 0 & J_{\alpha\beta}(E) & J_{\alpha\gamma}(E) \\ J_{\beta\alpha}(E) & 0 & J_{\beta\gamma}(E) \\ J_{\gamma\alpha}(E) & J_{\gamma\beta}(E) & 0 \end{pmatrix}.$$

Then it is easily seen that $n(\mu; \mathcal{S}(E)) = n(\mu; \mathcal{J}_0(E))$ for $\mathcal{S}(E)$ defined above and hence we have

$$n((1+\epsilon);\mathcal{J}_0(E)) - C_\epsilon \le N(E) \le n((1-\epsilon);\mathcal{J}_0(E)) + C_\epsilon. \tag{2.11}$$

The eigenvalue asymptotics for the integral operator $\mathcal{J}_0(E)$ has been in detail studied in Sobolev [7] by employing an argument used in the calculation of the canonical distribution of Toeplitz operators. We here summarize the results obtained there.

Lemma 2.4. *Let $n(\mu; \mathcal{J}_0(E))$ be as above. Then:*

(1) *There exists a limit*

$$\Theta_0(\mu) = \lim_{E\to 0} n(\mu; \mathcal{J}_0(E))/|\log E|$$

as a continuous function of $\mu > 0$.

(2) *The constant $C_0 = \Theta_0(1)$ depends only on the ratios between the masses of three particles under consideration and obeys the lower bound*

$$C_0 > \log 2/2\pi^2 > 0.$$

This lemma, together with relation (2.11), completes the proof of the theorem.

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