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Self-dual Einstein Hermitian Surfaces

Takashi Koda and Kouei Sekigawa

§1. Introduction

N. Hitchin [4] has proved that a 4-dimensional compact half conformally flat Einstein space of positive scalar curvature is isometric to a 4-dimensional sphere or a complex projective surface with the respective standard metric.

A 4-dimensional almost Hermitian manifold $M = (M, J, g)$ with integrable almost complex structure J is called a Hermitian surface. In the present paper, concerning the above result by Hitchin, we shall prove the following

Theorem A. *Let $M = (M, J, g)$ be a compact self-dual Einstein Hermitian surface. Then M is a Kähler surface of constant holomorphic sectional curvature, i.e., M is one of the following*
(1) *flat,*
(2) *$P^2(\mathbf{C})$ with its standard Fubini-Study metric and*
(3) *a compact quotient of unit disk D^2 with the Bergman metric.*

Remark. C.P. Boyer [2] has asserted the above result without detailed proof. In the present paper, we shall give another explicit proof.

In the sequel, unless otherwise stated, we assume the manifold under consideration to be connected.

§2. Preliminaries

Let $M = (M, J, g)$ be a Hermitian surface and Ω the Kähler form of M given by $\Omega(X, Y) = g(X, JY)$, $X, Y \in \mathfrak{X}(M)$. ($\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M). We assume that M is oriented by the volume form $dM = \frac{1}{2}\Omega^2$. We have

$$d\Omega = \omega \wedge \Omega, \quad \omega = \delta\Omega \circ J. \tag{2.1}$$

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The 1-form $\omega = (\omega_i)$ is called the Lee form of M. We denote by $\nabla, R = (R_{ijk}{}^l), \rho = (\rho_{ij})$ and τ the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. The Ricci $*$-tensor $\rho^* = (\rho^*{}_{ij})$ and the $*$-scalar curvature τ^* are defined respectively by

$$\rho^*{}_{ij} = \frac{1}{2} J_j{}^s R_{isa}{}^b J_b{}^a, \tag{2.2}$$

$$\tau^* = g^{ij} \rho^*{}_{ij}. \tag{2.3}$$

The generalized Chern form $\gamma = (\gamma_{ij})$ is given by

$$8\pi\gamma_{ij} = -4J_j{}^k \rho^*{}_{ik} - J^{kl}(\nabla_j J_k{}^h)\nabla_i J_{lh}. \tag{2.4}$$

It is well-known that the 2-form γ represents the first Chern class of M in the de Rham cohomology group. The Lee form $\omega = (\omega_j)$ satisfies the following:

$$J^{ij}\nabla_i \omega_j = 0, \tag{2.5}$$

$$\begin{aligned} 2\nabla_i J_j{}^k =& \omega_a J_j{}^a \delta_i^k - \omega_a J^{ka} g_{ij} \\ & - \omega_j J_i{}^k + \omega^k J_{ij}, \end{aligned} \tag{2.6}$$

$$\tau - \tau^* = 2\delta\omega + \|\omega\|^2, \tag{2.7}$$

(cf. [7], [9], [10]).

We denote by $\chi(M), c_1(M), c_2(M)$ and $p_1(M)$ the Euler class, the first Chern class, the second Chern class and the first Pontrjagin class of M, respectively. We note that $c_2(M)$ is equal to $\chi(M)$ when M is compact. Now, we assume that $M = (M, J, g)$ is of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then we have ([7])

$$\begin{aligned} R_{ijkl} =& \frac{1}{4}\|\omega\|^2 C_{ijkl} + \left(\frac{c}{4} - \frac{\|\omega\|^2}{16}\right) H_{ijkl} \\ & + \frac{1}{96}\{g_{ik}A_{jl} - g_{il}A_{jk} + g_{jl}A_{ik} - g_{jk}A_{il} \\ & + J_{ik}B_{jl} - J_{il}B_{jk} + J_{jl}B_{ik} - J_{jk}B_{il} \\ & + 2J_{ij}B_{kl} + 2J_{kl}B_{ij}\}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} C_{ijkl} =& g_{il}g_{jk} - g_{ik}g_{jl}, \\ H_{ijkl} =& g_{il}g_{jk} - g_{ik}g_{jl} \\ & + J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl}, \end{aligned}$$

$$\begin{aligned}A_{ij} =&21(\nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j)\\ &- 3J_i{}^a J_j{}^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b),\\ B_{ij} =&7(J_j{}^a\nabla_i\omega_a - J_i{}^a\nabla_j\omega_a)\\ &- (J_j{}^a\nabla_a\omega_i - J_i{}^a\nabla_a\omega_j)\\ &+ 3(J_j{}^a\omega_i\omega_a - J_i{}^a\omega_j\omega_a).\end{aligned}$$

By (2.7) and (2.8), we have

$$\rho_{ij} =\{\frac{3}{2}c + \frac{3}{16}(\tau - \tau^*)\}g_{ij} - \frac{1}{4}T_{ij}, \tag{2.9}$$

$$\rho^*{}_{ij} =\{\frac{3}{2}c - \frac{1}{16}(\tau - \tau^*)\}g_{ij} + \frac{1}{4}T^*{}_{ij}, \tag{2.10}$$

where

$$\begin{aligned}T_{ij} =&\nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j\\ &- J_i{}^a J_j{}^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b),\end{aligned} \tag{2.11}$$

$$T^*{}_{ij} =\nabla_i\omega_j - \nabla_j\omega_i - J_i{}^a J_j{}^b(\nabla_a\omega_b - \nabla_b\omega_a). \tag{2.12}$$

By (2.9), we get

$$\tau + 3\tau^* = 24c. \tag{2.13}$$

By (2.13), (2.9) and (2.10) are rewritten by

$$\rho_{ij} =\frac{\tau}{4}g_{ij} - \frac{1}{4}T_{ij}, \tag{2.9$'$}$$

$$\rho^*{}_{ij} =\frac{\tau^*}{4}g_{ij} + \frac{1}{4}T^*{}_{ij}. \tag{2.10$'$}$$

We assume that the manifold M under consideration is compact. We shall recall several integral formulas which will be needed in the proof of Theorem A.

$$\begin{aligned}&\int_M \omega^i\omega^j J_i{}^a J_j{}^b\nabla_a\omega_b dM\\ &=\int_M\{\tau\delta\omega + \frac{1}{4}\|\omega\|^4 - \frac{1}{2}(\tau - \tau^*)^2\\ &\quad + 6c\|\omega\|^2 - \|d\omega\|^2\}dM,\end{aligned} \tag{2.14}$$

$$\chi(M) =\frac{1}{32\pi^2}\int_M\{12c^2 - \frac{1}{16}(\tau - \tau^*)^2 + \frac{1}{2}\tau^*\|\omega\|^2\}dM, \tag{2.15}$$

$$p_1(M) = \frac{1}{32\pi^2}\int_M \{\frac{1}{12}(\tau - 3\tau^*)^2 + \|d\omega\|^2\}dM, \tag{2.16}$$

$$c_1(M)^2 = \frac{1}{32\pi^2}\int_M \{(\tau^*)^2 + \tau^*\|\omega\|^2 + \|d\omega\|^2\}dM, \tag{2.17}$$

(see [7]). We define a tensor field $S = (S_{ij})$ of type (0,2) by

$$\begin{aligned} S_{ij} =& \nabla_i\omega_j - J_i{}^a J_j{}^b \nabla_a\omega_b \\ &+ \frac{1}{2}(\omega_i\omega_j - J_i{}^a J_j{}^b \omega_a\omega_b). \end{aligned} \tag{2.18}$$

Then we have

$$\int_M \|S\|^2 dM = \int_M \{\frac{1}{2}(\tau - \tau^*)^2 - \tau^*\|\omega\|^2\}dM. \tag{2.19}$$

We assume furthermore that the manifold M under consideration is Einsteinian. Then, by $(2.9)'$, we get $T_{ij} = 0$. Thus, taking account of (2.7), (2.13), (2.14) and (2.18), we get

$$\begin{aligned} 0 =& \int_M T_{ij}\omega^i\omega^j dM \\ =& \int_M \{\|\omega\|^2\delta\omega + \|\omega\|^4 - 2\omega^i\omega^j J_i{}^a J_j{}^b \nabla_a\omega_b\}dM \\ =& \int_M \{\|\omega\|^2\delta\omega + \frac{1}{2}\|\omega\|^4 + (\tau - \tau^*)^2 \\ &- \frac{\tau + 3\tau^*}{2}\|\omega\|^2 + 2\|d\omega\|^2\}dM \\ =& \int_M \{\frac{1}{2}\|\omega\|^2(\tau - \tau^*) + (\tau - \tau^*)^2 \\ &- \frac{\tau + 3\tau^*}{2}\|\omega\|^2 + 2\|d\omega\|^2\}dM \\ =& \int_M \{(\tau - \tau^*)^2 - 2\tau^*\|\omega\|^2\}dM \\ &+ 2\int_M \|d\omega\|^2 dM \\ =& 2\int_M \{\|S\|^2 + \|d\omega\|^2\}dM. \end{aligned}$$

Thus, we have

Proposition 2.1. *Let $M = (M, J, g)$ be a compact Einstein Hermitian surface of pointwise constant holomorphic sectional curvature.*

Then M is a locally conformal Kähler surface and the tensor field S vanishes.

By (2.6), (2.18) and Proposition 2.1, we get

$$
\begin{aligned}
(2.20)\qquad 0 =& 2\nabla^i\nabla_i\omega_j - 2(\nabla^i J_i{}^a)J_j{}^b\nabla_a\omega_b \\
&- 2J_i{}^a(\nabla^i J_j{}^b)\nabla_a\omega_b - 2J_i{}^a J_j{}^b\nabla^i\nabla_a\omega_b \\
&+ \{(\nabla^i\omega_i)\omega_j + \omega^i\nabla_i\omega_j \\
&- (\nabla^i J_i{}^a)J_j{}^b\omega_a\omega_b - J_i{}^a(\nabla^i J_j{}^b)\omega_a\omega_b \\
&- J_i{}^a J_j{}^b\omega_a\nabla^i\omega_b\} \\
=& 2\nabla^i\nabla_j\omega_i - 2(\omega^i J_i{}^a)J_j{}^b\nabla_a\omega_b \\
&- J^{ia}(\omega_c J_j{}^c\delta_i^b - \omega_c J^{bc}g_{ij} - \omega_j J_i{}^b + \omega^b J_{ij})\nabla_a\omega_b \\
&+ J^{ia}J_j{}^b R_{iab}{}^c\omega_c \\
&+ \{-(\delta\omega)\omega_j + \frac{1}{2}\nabla_j\|\omega\|^2 - J^{ia}J_j{}^b\omega_a\nabla_i\omega_b\} \\
=& -2\nabla_j\delta\omega + \frac{\tau}{2}\omega_j - 2(\omega^i J_i{}^a)J_j{}^b\nabla_a\omega_b \\
&- \omega_j\delta\omega - \frac{1}{2}\nabla_j\|\omega\|^2 + 2\rho^*{}_{ij}\omega^i - \omega_j\delta\omega + \frac{1}{2}\nabla_j\|\omega\|^2 \\
=& -2\nabla_j\delta\omega - 2\omega_j\delta\omega + 2\rho_{ji}\omega^i + 2\rho^*{}_{ji}\omega^i \\
&- 2(\omega^i J_i{}^a)J_j{}^b\nabla_a\omega_b.
\end{aligned}
$$

Taking account of $S = 0$, we get

$$
\begin{aligned}
(2.21)\qquad &(\omega^c J_c{}^a)J_j{}^b\nabla_a\omega_b \\
&= \omega^c\{\nabla_c\omega_j + \frac{1}{2}(\omega_c\omega_j - J_c{}^b J_j{}^d\omega_b\omega_d)\} \\
&= \frac{1}{2}\nabla_j\|\omega\|^2 + \frac{1}{2}\omega_j\|\omega\|^2.
\end{aligned}
$$

Thus, by (2.7), (2.19) and (2.20), we have

$$
-2\nabla_j\delta\omega + \frac{\tau+\tau^*}{2}\omega_j - \nabla_j\|\omega\|^2 - \omega_j\|\omega\|^2 - 2\omega_j\delta\omega = 0,
$$

and hence

$$
(2.22)\qquad \nabla_j(\tau - 3\tau^*) + \frac{3}{2}(\tau - 3\tau^*)\omega_j = 0.
$$

§3. Proof of Theorem A

First, we shall recall the following results by the first author of the present paper.

Proposition 3.1 ([5]). *Let $M = (M, J, g)$ be a self-dual Einstein almost Hermitian 4-manifold. Then M is of pointwise constant holomorphic sectional curvature.*

Remark. Conversely, we may see that if M is an almost Hermitian 4-manifold of pointwise constant holomorphic sectional curvature, then M is self-dual.

Proposition 3.2 ([5]). *Let $M = (M, J, g)$ be a compact Hermitian surface. Then M is anti-self-dual if and only if M is a locally conformal Kähler surface with $\tau = 3\tau^*$.*

On one hand, the second author has proved the following.

Proposition 3.3 ([9]). *Let $M = (M, J, g)$ be a compact Einstein Hermitian surface. If $\tau^* < 0$ on M, then M is a Kähler surface.*

Let $M = (M, J, g)$ be a compact self-dual Einstein Hermitian surface. Then by Proposition 3.1, M is of pointwise constant holomorphic sectional curvature, say, c. Hence, by Proposition 2.1, M is also a locally conformal Kähler surface.

We suppose that $\tau = 3\tau^*$ at some point of M. Then, taking account of (2.22), we may observe that $\tau = 3\tau^*$ holds everywhere on M. Thus, by Proposition 3.2, M is anti-self-dual and hence conformally flat. Since M is Einsteinian, M is thus a compact Hermitian surface of non-positive constant sectional curvature. If M is of negative constant sectional curvature c, then $\tau = 12c, \tau^* = 4c$, and hence from Proposition 3.3, it follows that M is a Kähler surface of negative constant curvature. But this is impossible. If M is locally flat, then $\tau = \tau^* = 0$ and hence from (2.7), it follows immediately that M is a locally flat Kähler surface.

Next, we assume that $\tau - 3\tau^* \neq 0$ at every point of M. Let $\widetilde{M} = (\widetilde{M}, \widetilde{J}, \widetilde{g})$ be the universal Hermitian covering of M and $\pi : \widetilde{M} \longrightarrow M$ be the covering projection. Then, by Proposition 2.1, $\widetilde{M}$ is a globally conformal Kähler surface with $\widetilde{\omega} = d\widetilde{f}$, for some differentiable function $\widetilde{f}$ on $\widetilde{M}$, where $\widetilde{\omega}$ is the Lee form of $\widetilde{M}$. We denote by $\widetilde{\tau}, \widetilde{\tau}^*$ the scalar curvature, the $*$-scalar curvature of $\widetilde{M}$, respectively. Then $\widetilde{\tau} = \tau \circ \pi, \widetilde{\tau}^* = \tau^* \circ \pi, \widetilde{\omega} = \pi^*\omega$. Solving the system of partial differential equations corresponding to (2.22), we have

$$\widetilde{\tau} - 3\widetilde{\tau}^* = \widetilde{c}e^{-\frac{3}{2}\widetilde{f}}, \tag{3.1}$$

$\widetilde{c}$ is a non-zero constant. By (3.1), we see that the function $\widetilde{f}$ is projectable, i.e., there exists a differentiable function f on M such that $\widetilde{f} = f \circ \pi$. Thus, $\widetilde{\omega} = d\widetilde{f} = \pi^* df$ and have $\pi^*(\omega - df) = 0$. Therefore, M is a globally conformal Kähler surface with $\omega = df$. Taking account of (2.7), we have

$$\tau - \tau^* = -2\Delta f + \|df\|^2, \tag{3.2}$$

where $\Delta = -\delta d$ is the Laplace-Beltrami operator acting differentiable functions on M.

First, we suppose that $\tau > 3\tau^*$ on M. Let $f(p_0) = \min_{p \in M} f(p)$. Then we have $\Delta f(p_0) \geq 0$. Thus, $\tau - \tau^* \leq 0$ at p_0 and hence $2\tau^* < \tau \leq \tau^*$ at p_0. Thus, $\tau^* < 0$ at p_0 (and hence $\tau < 0$). Since $\tau > 3\tau^*$, we see therefore that $\tau^* < 0$ on M. Thus, by Proposition 3.3, M is a Kähler surface of negative constant holomorphic sectional curvature.

Next, we assume that $\tau < 3\tau^*$ on M. Let $f(p_0) = \max_{p \in M} f(p)$. Then $\tau - \tau^* \geq 0$ at p_0. Thus, $\tau^* \leq \tau < 3\tau^*$ at p_0 and hence $\tau^* > 0$ at p_0. Thus, in this case, we see that $\tau > 0$ (and hence $\tau^* > 0$ on M). By (2.17), $c_1(M)^2 > 0$ and hence M is algebraic. Since $\tau > 0$ and $\tau^* > 0$ on M, taking account of the arguments in [8] and [12], we may see that the plurigenera of M all vanish, that is, the Kodaira dimension of M is equel to -1. Thus, the Noether's formula ([6]) is of the form

$$c_1(M)^2 + c_2(M) = 12(1 - q), \tag{3.3}$$

where $q = q(M)$ is the irregularity of M. Since $c_1(M)^2 > 0, c_2(M) = \chi(M) > 0$, from (3.3), we have $q = 0$. This reduces to

$$c_1(M)^2 + c_2(M) = 12. \tag{3.4}$$

Referring to the well-known classification of compact complex surfaces (see, e.g., [1] p.415), we may see that M is rational, equivalently, obtained by successive blowing up's from a complex projective plane $P^2(\mathbf{C})$ or a (geometrically) ruled surface over a complex projective line $P^1(\mathbf{C})$. Since $c_2(M) = \chi(M) > 0$, Miyaoka's inequality is of the form

$$c_1(M)^2 \leq 3c_2(M). \tag{3.5}$$

By (3.4) and (3.5), we have

$$c_2(M) \geq 3. \tag{3.6}$$

Furthermore, by Wu's theorem and $p_1(M) \geq 0$,

$$c_1(M)^2 \geq 2c_2(M). \tag{3.7}$$

By (3.4), (3.7) and (3.6), we have

$$c_2(M) = 3 \text{ or } 4. \tag{3.8}$$

We assume $c_2(M) = 4$. Then by (3.4), we have $c_1(M)^2 = 8$. Hence, by Wu's theorem, $p_1(M) = 0$. Thus, by (2.17) and Proposition 2.1, we have $\tau = 3\tau^*$. But, this is a contradiction. So, we see that $c_2(M) = 3$. Then, we have $c_1(M)^2 = 9$ and $p_1(M) = 3$. Thus, we may conclude that M is biholomorphically equivalent to a complex projective plane $P^2(\mathbf{C})$.

The new metric $\bar{g} = e^{-f}g$ on M is a self-dual Kähler metric. By the classification of self-dual Kähler surfaces [3], we see that $\bar{g}$ is the Fubini-Study metric on $P^2(\mathbf{C})$. Taking account of (3.2), the scalar curvature $\bar{\tau}$ of $\bar{g}$ is given by

$$\begin{aligned}\bar{\tau} &= e^f(\tau + 3\Delta f - \frac{3}{2}\|\text{grad } f\|^2)\\ &= e^f\{\tau - \frac{3}{2}(-2\Delta f + \|\text{grad } f\|^2)\}\\ &= e^f\{\tau - \frac{3}{2}(\tau - \tau^*)\}\\ &= e^f\left(\frac{-\tau + 3\tau^*}{2}\right).\end{aligned}$$

Here, by (2.22), we have

$$e^f = \left(\frac{\tau - 3\tau^*}{C}\right)^{-\frac{2}{3}}, \tag{3.9}$$

where C is a negative constant. Hence

$$\bar{\tau} = -\frac{(\tau - 3\tau^*)^{\frac{1}{3}}}{2C^{-\frac{2}{3}}}.$$

Since $\bar{\tau}$ is constant, so is $\tau - 3\tau^*$. Hence, by (3.9), we see that f is constant. Therefore, g is homothetic to the Fubini-Study metric. This completes the proof of Theorem A.

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Takashi Koda
Department of Mathematics
Toyama University
Toyama 930
Japan

Kouei Sekigawa
Department of Mathematics
Faculty of Science
Niigata University
Niigata 950–21
Japan