Advanced Studies in Pure Mathematics 22, 1993
Progress in Differential Geometry
pp. 117–122

A Uniqueness Result for Minimal Surfaces in S^3

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§1. Introduction

In the study of minimal surfaces, the uniqueness for minimal surfaces bounded by a given contour is an important problem which is not yet solved completely.

The first uniqueness result was proved by Radó [4] for minimal surfaces in $\mathbf{R}^3$. He proved that if a Jordan curve Γ has a one-to-one parallel or central projection onto a convex plane Jordan curve, then Γ bounds a unique minimal disk. The second result is due to Nitsche [3] and states that if the total curvature of an analytic Jordan curve Γ does not exceed 4π, then Γ bounds a unique minimal disk. The third result is due to Tromba [6] and states that if a C^2-Jordan curve Γ is sufficiently closed to a C^2-plane Jordan curve in the C^2-topology, then Γ bounds a unique minimal disk.

For minimal surfaces in other Riemannian manifolds, uniqueness theorems in the three dimensional hemisphere of S^3 were proved by Sakaki [5] and Koiso [2]. Sakaki's result is an analogy of Tromba's uniqueness theorem, and Koiso's is an analogy of Radó's theorem.

In this paper we restrict ourselves to minimal surfaces in S^3 which are "graphs" in some sense (Definition 1.1).

Set $S^3 = \{\mathbf{x} \in \mathbf{R}^4; |\mathbf{x}| = 1\}$. Let Σ be a 2-plane in $\mathbf{R}^4$ containing the origin of $\mathbf{R}^4$. We denote by B the two dimensional unit open disk in Σ which is bounded by $\Sigma \cap S^3$.

Definition 1.1. Let D be a subset of the closed disk $\overline{B}$. A subset M of S^3 is called a "graph" over D if M intersects with each 2-plane containing a point of D which is orthogonal to Σ in $\mathbf{R}^4$ at precisely one point.

Definition 1.2. (1) A minimal surface M in S^3 is a continuous mapping Φ of a two dimensional compact C^∞-manifold R with boundary

Received April 5, 1991.
Revised May 27, 1991.

∂R into S^3 which is of class C^2 in the interior of R and which is a critical point of the area functional for every variation preserving the boundary values $\Phi|_{\partial R}$.

(2) We sometimes call the image $\Phi(R)$ of a minimal surface $\Phi : R \longrightarrow S^3$ to be a minimal surface. On such an occasion we call $\Phi(\partial R)$ to be the boundary of the minimal surface $\Phi(R)$, and denote $\Phi(\partial R)$ by $\partial\Phi(R)$.

(3) When we mention the uniqueness for minimal surfaces, we mean the uniqueness for the images of minimal surfaces.

Now we can state our uniqueness result:

Theorem 1.3. *Let D be a simply-connected domain whose closure $\overline{D}$ is contained in B. If M is a minimal surface which is a "C^2-graph" over $\overline{D}$, then M is the unique minimal surface bounded by ∂M which is a "C^2-graph" over $\overline{D}$.*

For the proof, we represent each "graph" over $\overline{D}$ in terms of a single real-valued function φ defined on $\overline{D}$. We prove that the considered "graph" is a minimal surface if and only if the function φ satisfies a certain quasilinear elliptic partial differential equation (Lemma 2.4). A uniqueness theorem for the Dirichlet problem for quasilinear elliptic operators assures the uniqueness of our minimal surface.

We conjecture that under the assumption of Theorem 1.3, the uniqueness of the area-minimizing surface bounded by ∂M is valid.

§2. Proof of Theorem 1.3

Throughout this section, we assume that D is a simply-connected domain whose closure is contained in B.

We introduce the orthogonal coordinates (x, y, z, w) in $\mathbf{R}^4$. Without loss of generality, we set Σ the (x, y)-plane. For simplicity we denote a point $(x, y, 0, 0)$ in Σ by (x, y). If f is a differentiable function of x and y, we denote $\partial f/\partial x, \partial f/\partial y, \partial^2 f/\partial x^2$ by f_x, f_y, f_{xx}, etc.

A "graph" over $\overline{D}$ is represented as follows:

$$\left(x, y, \sqrt{1-x^2-y^2}\cos\varphi(x,y), \sqrt{1-x^2-y^2}\sin\varphi(x,y)\right), \tag{2-1}$$

$$(x, y) \in \overline{D},$$

where $\varphi(x, y)$ is a real-valued function defined on $\overline{D}$.

Definition 2.1. A "graph" over $\overline{D}$ represented by (2-1) is called a "C^n-graph" over $\overline{D}$ if φ can be chosen to be of class C^n on $\overline{D}$.

Remark 2.2. If M is a "C^n-graph" represented by (2-1), then

$$\partial M = \left\{\left(x, y, \sqrt{1-x^2-y^2}\cos\varphi, \sqrt{1-x^2-y^2}\sin\varphi\right); (x,y) \in \partial D\right\}$$

is a Jordan curve of class C^n.

Remark 2.3. Since $\overline{D}$ is contained in B, $x^2+y^2<1$ for any point (x,y) in $\overline{D}$.

Lemma 2.4. *Let φ be of class $C^2(\overline{D}, \mathbf{R})$.*

$$M = \{\left(x, y, \sqrt{1-x^2-y^2}\cos\varphi(x,y), \sqrt{1-x^2-y^2}\sin\varphi(x,y)\right); \quad (x,y) \in \overline{D}\}$$

is a minimal surface if and only if

$$L\varphi = 0 \qquad inD,$$

where L is a quasilinear elliptic operator of the form

(2-2)
$$\begin{aligned} L\varphi = &\left\{1-x^2+(1-x^2-y^2)^2{\varphi_y}^2\right\}\varphi_{xx} \\ &-2\left\{xy+(1-x^2-y^2)^2\varphi_x\varphi_y\right\}\varphi_{xy} \\ &+\left\{1-y^2+(1-x^2-y^2)^2{\varphi_x}^2\right\}\varphi_{yy} \\ &-4x\varphi_x-4y\varphi_y \\ &+2(1-x^2-y^2)(-x+x^3){\varphi_x}^3 \\ &+(1-x^2-y^2)(-2y+6x^2y){\varphi_x}^2\varphi_y \\ &+(1-x^2-y^2)(-2x+6xy^2)\varphi_x{\varphi_y}^2 \\ &+2(1-x^2-y^2)(-y+y^3){\varphi_y}^3, \qquad (x,y)\in D. \end{aligned}$$

Proof. Set

$$\Phi(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\cos\varphi(x,y), \sqrt{1-x^2-y^2}\sin\varphi(x,y)\right),$$

$$(x,y) \in \overline{D}.$$

Then $\Phi \in C^2(\overline{D}, S^3)$. The area A of M is represented as

$$A = \iint_D \left\{ |\Phi_x|^2 |\Phi_y|^2 - (\Phi_x, \Phi_y)^2 \right\}^{1/2} dx\, dy,$$

where (Φ_x, Φ_y) is the usual inner product in $\mathbf{R}^4$ and $|\Phi_x|^2 = (\Phi_x, \Phi_x)$, $|\Phi_y|^2 = (\Phi_y, \Phi_y)$. By easy calculations we get

$$A = \iint_D \left\{ \left(1 - x^2 - y^2\right)^{-1} + \left(1 - x^2 - y^2\right) \left({\varphi_x}^2 + {\varphi_y}^2\right) \right.$$
$$\left. + \left(x\varphi_y - y\varphi_x\right)^2 \right\}^{1/2} dx\, dy.$$

Let $f = f(x, y)$ be a real-valued C^2-function on $\overline{D}$ which vanishes on the boundary ∂D. Then we get 1-parameter family of surfaces M_t represented as follows.

$$\left(x, y, \sqrt{1 - x^2 - y^2} \cos(\varphi + t\, f), \sqrt{1 - x^2 - y^2} \sin(\varphi + t\, f)\right),$$

$$(x, y) \in \overline{D}, \qquad t \in \mathbf{R}.$$

Denote the area of M_t by $A(t)$. Then $M = M_0$ is a minimal surface if and only if

$$\left.\frac{d}{dt} A(t)\right|_{t=0} = 0$$

for any f.

We observe that

$$\left.\frac{d}{dt} A(t)\right|_{t=0}$$
$$= \iint_D \frac{\left\{\left(1 - x^2\right) \varphi_x - xy\varphi_y\right\} f_x + \left\{\left(1 - y^2\right) \varphi_y - xy\varphi_x\right\} f_y}{Q} \, dx\, dy,$$

where

$$Q = \left\{ \frac{1}{1 - x^2 - y^2} + \left(1 - x^2 - y^2\right) \left({\varphi_x}^2 + {\varphi_y}^2\right) + \left(x\varphi_y - y\varphi_x\right)^2 \right\}^{\frac{1}{2}}.$$

By virtue of the Stokes' formula and the assumption $f|_{\partial D} = 0$, we see that

$$\left.\frac{d}{dt}A(t)\right|_{t=0} =$$

$$-\iint_D f\left[\left\{\frac{\left(1-x^2\right)\varphi_x - xy\varphi_y}{Q}\right\}_x + \left\{\frac{\left(1-y^2\right)\varphi_y - xy\varphi_x}{Q}\right\}_y\right] dxdy.$$

By lengthy but easy calculations we get

$$\left.\frac{d}{dt}A(t)\right|_{t=0} = -\iint_D f\left(1-x^2-y^2\right)^{-1} Q^{-3} L\varphi\, dx\, dy,$$

where $L\varphi$ is given by the equality (2-2) in the statement of Lemma 2.4. If $(d/dt)A(t)|_{t=0} = 0$ for any $f \in C^2(\overline{D}, \mathbf{R})$ with $f|_{\partial D} = 0$, then $L\varphi$ must vanish in D, and vice versa.

To see the ellipticity of L, we regard $L\varphi$ as a function of x, y, φ, φ_x, φ_y, φ_{xx}, φ_{xy}, φ_{yy}, and we set $p = \varphi_x$, $q = \varphi_y$, $r = \varphi_{xx}$, $s = \varphi_{xy}$, and $t = \varphi_{yy}$. Then

$$\begin{aligned} L_\varphi &= 0, \\ L_r L_t - (L_s/2)^2 & \\ &= 1 - x^2 - y^2 \\ &\qquad + \left(1-x^2-y^2\right)^2 \left\{\left(1-x^2-y^2\right)\left(p^2+q^2\right) + (yp - xq)^2\right\} \\ &> 0 \end{aligned}$$

for any point $(x, y) \in \overline{D}$, which implies that L is elliptic Q.E.D.

Proof of Theorem 1.3. If two functions $\varphi \in C^2(\overline{D}, \mathbf{R})$ and $\psi \in C^2(\overline{D}, \mathbf{R})$ define minimal surfaces

$$\Phi(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\cos\varphi(x,y), \sqrt{1-x^2-y^2}\sin\varphi(x,y)\right),$$

$$(x,y) \in \overline{D},$$

and

$$\Psi(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\cos\psi(x,y), \sqrt{1-x^2-y^2}\sin\psi(x,y)\right),$$

$$(x,y) \in \overline{D},$$

and if these two minimal surfaces have the same boundary, then we can assume that $\varphi = \psi$ on ∂D. Moreover, by Lemma 2.4, we see that $L\varphi = 0$

and $L\psi = 0$ in D. Therefore by virtue of the uniqueness theorem for the Dirichlet problem for quasilinear elliptic operators ([1, p.208, Theorem 9.3]), φ and ψ must coincide in D. Q.E.D.

§3. The final remark

Remark 3.1. The assumption that $\overline{D}$ is contained in B is essential in the following sense. Set

$$D = B = \left\{(x, y, 0, 0) \in \mathbf{R}^4; x^2 + y^2 < 1\right\}.$$

Then the uniqueness result does not hold. In fact,

$$\Phi(x,y) = \left(x, y, \frac{a\sqrt{1-x^2-y^2}}{\sqrt{a^2+b^2}}, \frac{b\sqrt{1-x^2-y^2}}{\sqrt{a^2+b^2}}\right),$$

$$(a,b) \in \mathbf{R}^2 - (0,0), \qquad (x,y) \in \overline{D}$$

is a half of a geodesic 2-sphere bounded by the geodesic circle ∂D, hence Φ is a minimal surface bounded by ∂D. Therefore we obtain 2-parameter family of minimal surfaces bounded by the same contour ∂D which are "C^∞-graphs" over $\overline{D}$ and all of which are area-minimizing.

References

[1] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[2] M. Koiso, The uniqueness for minimal surfaces in S^3, manuscripta math., **63** (1989), 193–207.

[3] J. C. C. Nitsche, A new uniqueness theorem for minimal surfaces, Arch. Rat. Mech. Anal., **52** (1973), 319–329.

[4] T. Radó, Contributions to the theory of minimal surfaces, Acta. Litt. Sci. Szeged, **6** (1932), 1–20.

[5] M. Sakaki, A uniqueness theorem for minimal surfaces in S^3, Kodai Math. J., **10** (1987), 39–41.

[6] A. Tromba, On the Number of Simply Connected Minimal Surfaces Spanning a Curve, Mem. Amer. Math. Soc., **194** (1977).

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