Advanced Studies in Pure Mathematics 21, 1992
Zeta Functions in Geometry
pp. 281–325

On Hermitian Forms attached to Zeta Functions

Hiroyuki Yoshida

§0. Introduction

In this paper, we shall deal with some problems of analysis which arise naturally from explicit formulas. For $F \in C_c^\infty(\mathbf{R})$, set

$$\Phi(s) = \int_{-\infty}^{\infty} F(x) e^{(s-1/2)x}\, dx, \quad s \in \mathbf{C}, \qquad \hat{F}(t) = \Phi(\frac{1}{2} + it), \quad t \in \mathbf{R}.$$

Then the explicit formula for $\zeta(s)$ reads as

$$\begin{aligned}\sum_\rho \Phi(\rho) = &\int_{-\infty}^{\infty} F(x)(e^{x/2} + e^{-x/2}) dx - (\log \pi) F(0) \\ &- \sum_p \sum_{m=1}^{\infty} \frac{\log p}{p^{m/2}} (F(m \log p) + F(-m \log p)) \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(t)\, \mathrm{Re}\, (\psi(\frac{1}{4} + \frac{it}{2}))\, dt,\end{aligned}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ and ρ extends over all non-trivial zeros of $\zeta(s)$. The functional $T(F) = \sum_\rho \Phi(\rho)$ defines a distribution on $\mathbf{R}$. A well known observation of Weil states that T is positive definite i.e. $T(\alpha * \tilde{\alpha}) \geq 0$ for every $\alpha \in C_c^\infty(\mathbf{R})$ if and only if the Riemann hypothesis holds for $\zeta(s)$. We can define a hermitian form $\langle\ ,\ \rangle$ on $C_c^\infty(\mathbf{R})$ by

$$\langle \varphi_1, \varphi_2 \rangle = T(\varphi_1 * \tilde{\varphi}_2), \qquad \varphi_1, \varphi_2 \in C_c^\infty(\mathbf{R}).$$

For $a > 0$, we set

$$C(a) = \{\ \varphi \in C_c^\infty(\mathbf{R}) \mid \ \mathrm{supp}(\varphi) \subseteq [-a, a]\ \}.$$

Then R.H. is equivalent to the positive definiteness of $\langle\ ,\ \rangle | C(a)$ for every $a > 0$ (cf. Proposition 2). It can easily be verified that $\langle\ ,\ \rangle | C(a)$ is positive definite if a is sufficiently small. Now we can naturally ask:

Received September 3, 1990.

(I) When a is given, can one determine whether $\langle\ ,\ \rangle|C(a)$ is positive definite or not?
(II) Study the deformation of $\langle\ ,\ \rangle|C(a)$ when a changes. What shall happen at the point $a = a_0$ beyond which $\langle\ ,\ \rangle|C(a)$ is not positive definite?

The problem (II) arises from the author's study of unitarizability problem of group representations [9] and from the hope that the positive definiteness may become more tractable by cutting the support of functions, since the sum $\sum_p$ for $T(\varphi_1 * \tilde{\varphi}_2)$, φ_1, $\varphi_2 \in C(a)$ becomes a finite sum. The result obtained is a reduction of R.H. to the non-degeneracy of the hermitian form $\langle\ ,\ \rangle$ extended to a certain space. To explain this in more detail, let us introduce a space $K(a)$:

$$K(a) = \{\ \varphi \mid \varphi(x) = f(x) \text{ for } |x| \le a, \varphi(x) = 0 \text{ for } |x| > a \\ \text{with } f \in C^\infty(\mathbf{R}) \text{ which has period } 2a\ \}.$$

We can extend $\langle\ ,\ \rangle$ to $K(a)$. For a non-negative integer N, set

$$K_N(a) = \{\ \varphi \in K(a) \mid \int_{-a}^{a} \varphi(x)\exp(\frac{\pi i n x}{a})\, dx = 0 \\ \text{for all } n \in \mathbf{Z}, |n| \le N\ \}.$$

For a given a, $\langle\ ,\ \rangle|K_N(a)$ is positive definite if N is sufficiently large. More precisely, we can find $\mu > 0$ and N such that

$$\langle \varphi, \varphi \rangle \ge \mu \|\varphi\|_{L^2}^2 \quad \text{for every} \quad \varphi \in K_N(a).$$

Now decompose $K(a) = W \oplus K_N(a)$ with $W \subset K(a)$, $\dim W = 2N+1$. Let $\widehat{K_N(a)}$ be the completion of the pre-Hilbert space $K_N(a)$ with respect to $\langle\ ,\ \rangle$ and set $\widehat{K(a)} = W \oplus \widehat{K_N(a)}$. Then $\widehat{K(a)}$ can be canonically embedded as a subspace of $L^2([-a, a])$ and does not depend on the choice of W and N. Furthermore, $\langle\ ,\ \rangle$ extends to a hermitian form on $\widehat{K(a)}$ and $\widehat{K_N(a)}$ has the orthogonal complement $W(a)$ in $\widehat{K(a)}$:

$$\widehat{K(a)} = W(a) \oplus \widehat{K_N(a)} \qquad \text{(orthogonal direct sum)}.$$

We shall prove in §7 that the hermitian matrix obtained by $\langle\ ,\ \rangle$ on $W(a)$ varies continuously with respect to a when a basis of $W(a)$ is suitably chosen. This permits us to reduce the Riemann hypothesis to *the non degeneracy of* $\langle\ ,\ \rangle$ *on* $\widehat{K(a)}$ (Theorem 2). We can prove the non-degeneracy of $\langle\ ,\ \rangle$ on $C(a)$ and on $K(a)$ (Propositions 2 and 7), but to prove it on $\widehat{K(a)}$ certainly requires more ideas.

Concerning the problem (I), the existence of an algorithm to solve it is guaranteed by the estimate given by Lemma 9, §7. For an explicitly given a, the actual computation is not easy however. In §6, we shall give a detailed sample computation, though it does not follow the algorithm faithfully, when $a = \log 2/2$: We find $\langle\ ,\ \rangle | K(a)$ is positive definite for $a \leq \log 2/2$ (Theorem 1). The idea is to calculate the hermitian matrix on $W(a)$ with sufficient approximation for a suitably chosen N.

§1. Remarks on distributions arising from zeta functions

For a function α on $\mathbf{R}$, we set

$$\check{\alpha}(x) = \alpha(-x), \qquad \tilde{\alpha}(x) = \overline{\alpha(-x)}, \quad x \in \mathbf{R}.$$

If α is an integrable function with compact support, we define its Mellin transform $M(\alpha)$ by

$$M(\alpha)(s) = \int_{-\infty}^{\infty} \alpha(x) e^{(s-1/2)x}\, dx, \quad s \in \mathbf{C}.$$

We see that $M(\alpha)$ is an entire funcion. We set

$$\hat{\alpha}(t) = M(\alpha)(\frac{1}{2} + it) = \int_{-\infty}^{\infty} \alpha(x) e^{itx}\, dx, \quad t \in \mathbf{R},$$

which is the Fourier transform of α. It can be verified immediately that

$$M(\check{\alpha})(s) = M(\alpha)(1-s), \tag{1.1}$$

$$M(\tilde{\alpha})(s) = \overline{M(\alpha)(1-\bar{s})}, \tag{1.2}$$

$$M(\alpha * \beta)(s) = M(\alpha)(s) M(\beta)(s), \tag{1.3}$$

where β is also an integrable function with compact support. Let $\psi(s) = \Gamma'(s)/\Gamma(s)$ be the logarithmic derivative of the Gamma function. Let F be a continuous function with compact support which is continuously differentiable except for a finite number of points. For discontinuous points of F', we assume the existence of right and left limits of F'. In this paper, we call such F *admissible* . We set

$$V_1(F) = \lim_{T \to +\infty} \frac{1}{2\pi} \int_{-T}^{T} \hat{F}(t)\, \mathrm{Re}\,(\psi(\frac{1}{4} + \frac{it}{2}))\, dt,$$

$$V_2(F) = \lim_{T \to +\infty} \frac{1}{2\pi} \int_{-T}^{T} \hat{F}(t)\, \mathrm{Re}\,(\psi(\frac{1}{2} + it))\, dt,$$

$$\mathrm{Pf}\,(\frac{1}{|x|})(F) = \lim_{\epsilon \to +0} (\int_{|x| \geq \epsilon} \frac{F(x)}{|x|}\, dx + 2F(0) \log \epsilon).$$

$\mathrm{Pf}\,(\frac{1}{|x|})$ is a distribution (a pseudo function) defined in Schwartz [5]. We have

$$V_1 = -\frac{1}{2}\,\mathrm{Pf}\,(\frac{1}{|x|}) - (\gamma + \log 2)\delta_0 - (\frac{e^{|x|/2}}{|e^x - e^{-x}|} - \frac{1}{2|x|}), \tag{1.4}$$

$$V_2 = -\frac{1}{2}\,\mathrm{Pf}\,(\frac{1}{|x|}) - \gamma\delta_0 - \frac{1}{2}(\frac{1}{|e^{x/2} - e^{-x/2}|} - \frac{1}{|x|}), \tag{1.5}$$

where δ_0 denotes the Dirac distribution supported on 0 and γ is Euler's constant.

Let k be an algebraic number field of finite degree. Let r_1 and r_2 be the numbers of real and complex archimedean places of k respectively. Let D_k be the discriminant of k and put $A_k = \pi^{-r_1}(2\pi)^{-2r_2}|D_k|$. Let $\zeta_k(s)$ be the Dedekind zeta function of k. Then the explicit formula for $\zeta_k(s)$ can be written as follows (cf. Weil [7], Poitou [4]). Let F be admissible and set $\Phi = M(F)$.

$$\begin{aligned} &\lim_{T\to+\infty} \sum_{|\mathrm{Im}(\rho)|<T} \Phi(\rho) \\ &= \int_{-\infty}^{\infty} F(x)(e^{x/2} + e^{-x/2})dx + (\log A_k)F(0) \\ &\quad - \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{m/2}}(F(m\log N(\mathfrak{p})) + F(-m\log N(\mathfrak{p}))) \\ &\quad + r_1V_1(F) + 2r_2V_2(F), \end{aligned} \tag{1.6}$$

where ρ extends over all non-trivial zeros of $\zeta_k(s)$ and $\mathfrak{p}$ extends over all prime ideals of k.

Hereafter in this section, ρ denotes a non-trivial zero of $\zeta_k(s)$. For an admissible function F, set

$$T_k(F) = \lim_{T\to+\infty} \sum_{|\mathrm{Im}(\rho)|<T} \Phi(\rho), \quad \Phi = M(F).$$

It is well known and can easily be verified using the Riemann-von Mangoldt formula that $\sum_\rho \frac{1}{|\rho|^2}$ converges. It follows that if $F \in C_c^\infty(\mathbf{R})$, $\sum_\rho \Phi(\rho)$ converges absolutely and the functional T_k is continuous on $C_c^\infty(\mathbf{R})$, i.e. T_k defines a distribution in the sense of Schwartz.

Let T be a distribution on $\mathbf{R}$. Recall that T is called positive definite if $T(\alpha * \tilde{\alpha}) \geq 0$ for every $\alpha \in C_c^\infty(\mathbf{R})$. We call T evenly (resp. oddly) positive definite if $T(\alpha * \tilde{\alpha}) \geq 0$ for every even (resp. odd) function $\alpha \in C_c^\infty(\mathbf{R})$. Evenly positive definite distributions are studied in detail in Gel'fand-Vilenkin [2]. It is a well known observation of Weil that T_k is positive definite if and only if the Riemann hypothesis holds for $\zeta_k(s)$. We shall present a slight sharpening of Weil's result, since the part (1) of the next proposition, though easily proved, seems to be of some interest.

Proposition 1.

(1) *T_k is oddly positive definite if and only if R.H. holds for $\zeta_k(s)$.*
(2) *T_k is evenly positive definite if and only if R.H. holds for $\zeta_k(s)$ with possible exceptions of real zeros, i.e. every non-trivial zero of $\zeta_k(s)$ lies on the critical line if it is not real.*

First we shall prove a Lemma.

Lemma 1. *Suppose that a non-trivial zero ρ_0 of $\zeta_k(s)$ and any positive number ϵ are given. Then there exists an $\alpha \in C_c^\infty(\mathbf{R})$ such that*

$$M(\alpha)(\rho_0) = 1, \qquad |M(\alpha)(\rho)| \leq \epsilon/|\rho - \rho_0|^2 \quad \textit{for every} \quad \rho \neq \rho_0.$$

Proof. We may assume $\epsilon < 1$. First we take $\alpha_0 \in C_c^\infty(\mathbf{R})$ so that $M(\alpha_0)(\rho_0) = 1$. Since $M(\alpha_0''')(s) = -(s-1/2)^3 M(\alpha_0)(s)$, we have

$$M(\alpha_0) = O(|s-1/2|^{-3}) \quad \text{for} \quad 0 \leq \mathrm{Re}(s) \leq 1,\ |\mathrm{Im}(s)| \to +\infty.$$

Hence we can find $R > 1$ such that

$$(1.7)\ |M(\alpha_0)(\rho)| \leq \epsilon/|\rho-\rho_0|^2 \quad \text{for every} \quad \rho \quad \text{such that} \quad |\rho-\rho_0| \geq R.$$

Let $\rho_1, \rho_2, \cdots, \rho_M$ be all the non-trivial zeros which satisfy $|\rho-\rho_0| < R$. For each i, $1 \leq i \leq M$, we can find $\alpha_i \in C_c^\infty(\mathbf{R})$ such that $M(\alpha_i)(\rho_0) = 1$, $M(\alpha_i)(\rho_i) = 0$. Put

$$\alpha = \alpha_1 * \alpha_2 * \cdots * \alpha_M * \alpha_0 * \cdots * \alpha_0 \quad (\alpha_0 \text{ is convoluted } N\text{-times}),$$
$$\Phi = M(\alpha), \qquad \Phi_i = M(\alpha_i), \quad 0 \leq i \leq M.$$

By (1.3), we have

$$\Phi(s) = \Phi_0(s)^N \prod_{i=1}^{M} \Phi_i(s).$$

Hence we get $\Phi(\rho_0) = 1$, $\Phi(\rho) = 0$ if $\rho \neq \rho_0$, $|\rho - \rho_0| < R$. Since $\Phi_i(s)$ is bounded in the strip $0 \leq \mathrm{Re}(s) \leq 1$, we can find a constant C such that

$$|\prod_{i=1}^{M} \Phi_i(s)| \leq C, \quad 0 \leq \mathrm{Re}(s) \leq 1.$$

If ρ is a non-trivial zero such that $|\rho - \rho_0| \geq R$, then we have, by (1.7),

$$|\Phi(\rho)| \leq C \frac{\epsilon^{N-1}}{R^{2N-2}} \frac{\epsilon}{|\rho - \rho_0|^2}.$$

Taking N sufficiently large, α satisfies the required condition.

Proof of Proposition 1. Take $\alpha \in C_c^\infty(\mathbf{R})$ and put $\Phi_0 = M(\alpha)$, $\Phi = M(\alpha * \tilde{\alpha})$. By (1.2), we have $\Phi(s) = \Phi_0(s)\overline{\Phi_0(1-\bar{s})}$. Hence we get $T_k(\alpha * \tilde{\alpha}) = \sum_\rho \Phi(\rho) = \Phi_0(\rho)\overline{\Phi_0(1-\bar{\rho})}$. If R.H. holds for $\zeta_k(s)$, we have $1 - \bar{\rho} = \rho$. Hence $T_k(\alpha * \tilde{\alpha}) = \sum_\rho |\Phi_0(\rho)|^2 \geq 0$. If R.H. holds for $\zeta_k(s)$ except for real zeros and if α is even, then we have

$$T_k(\alpha * \tilde{\alpha}) = \sum_{\rho \notin \mathbf{R}} |\Phi_0(\rho)|^2 + \sum_{\rho \in \mathbf{R}} \Phi_0(\rho)\overline{\Phi_0(1-\rho)} = \sum_\rho |\Phi_0(\rho)|^2$$

by $\check{\alpha} = \alpha$ and (1.1). This proves if parts of (1) and (2).

Now assume R.H. does not hold for $\zeta_k(s)$ and let ρ_0 be a non-trivial zero such that $\mathrm{Re}(\rho_0) \neq 1/2$. For any $\epsilon > 0$, we can choose, by Lemma 1, α_1, $\alpha_2 \in C_c^\infty(\mathbf{R})$ so that

$$\begin{aligned} &M(\alpha_1)(\rho_0) = 1, \qquad |M(\alpha_1)(\rho)| \leq \epsilon/|\rho - \rho_0|^2, \quad \rho \neq \rho_0, \\ &M(\alpha_2)(\bar{\rho}_0) = 1, \qquad |M(\alpha_2)(\rho)| \leq \epsilon/|\rho - \bar{\rho}_0|^2, \quad \rho \neq \bar{\rho}_0. \end{aligned}$$

Put

$$\alpha = \alpha_1 + \alpha_2 - \check{\alpha}_1 - \check{\alpha}_2, \quad \Phi_0 = M(\alpha_1 + \alpha_2), \quad \Phi = M(\alpha * \tilde{\alpha}).$$

Then α is an odd function and we have

$$\Phi(s) = (\Phi_0(s) - \Phi_0(1-s))(\overline{\Phi_0(1-\bar{s}) - \Phi_0(\bar{s})}).$$

Hence we obtain

$$\Phi(1-s) = \Phi(s), \quad \Phi(\bar{s}) = \overline{\Phi(s)}, \quad \Phi(1-\bar{s}) = \overline{\Phi(s)}.$$

By our choice of α_1, α_2, we see easily that $\Phi(\rho_0)$, $\Phi(\bar{\rho}_0)$, $\Phi(1-\rho_0)$, $\Phi(1-\bar{\rho}_0)$ converges to -1 (resp. -4) if $\rho_0 \notin \mathbf{R}$ (resp. $\rho_0 \in \mathbf{R}$) for $\epsilon \to +0$. On the other hand, if ρ is a non-trivial zero different from ρ_0, $\bar{\rho}_0$, $1-\rho_0$, $1-\bar{\rho}_0$, then we have

$$|\Phi(\rho)| \leq \epsilon^2\Big(\frac{1}{|\rho - \rho_0|^2} + \frac{1}{|\rho - \bar{\rho}_0|^2} + \frac{1}{|1-\rho-\rho_0|^2} + \frac{1}{|1-\rho-\bar{\rho}_0|^2}\Big)^2.$$

Since $\sum_{\rho \neq \eta} 1/|\rho - \eta|^2$ converges for every $\eta \in \mathbf{C}$, we see that $T_k(\alpha * \tilde{\alpha})$ becomes negative when ϵ is sufficiently small. This proves only if part of (1). The only if part of (2) can be proved similarly. This completes the proof.

Let T be a distribution on $\mathbf{R}$. We define distributions $\check{T}$, $\bar{T}$ and $\tilde{T}$ by

$$\check{T}(\alpha) = T(\check{\alpha}), \qquad \bar{T}(\alpha) = \overline{T(\bar{\alpha})}, \qquad \tilde{T} = \bar{\check{T}}, \quad \alpha \in C_c^\infty(\mathbf{R}),$$

as in [5]. From the functional equation of $\zeta_k(s)$ and $\zeta_k(\bar{s}) = \overline{\zeta_k(s)}$, we easily obtain

$$(1.8) \qquad \check{T}_k = \bar{T}_k = \tilde{T}_k = T_k.$$

§2. Local positive definiteness

For $a > 0$, we set

$$\begin{aligned} C(a) &= \{\ \varphi \in C_c^\infty(\mathbf{R}) \mid \ \mathrm{supp}(\varphi) \subseteq [-a, a]\ \}, \\ K(a) &= \{\ \varphi \mid \ \text{there exists } f \in C^\infty(\mathbf{R}) \text{ whose period is } 2a \text{ such that} \\ &\qquad \varphi(x) = f(x) \text{ for } |x| \le a,\ \varphi(x) = 0 \text{ for } |x| > a\ \}. \end{aligned}$$

For φ_1, $\varphi_2 \in K(a)$, we set

$$\langle \varphi_1, \varphi_2 \rangle = T_k(\varphi_1 * \tilde{\varphi}_2).$$

By $\widetilde{T_k} = T_k$, we see easily that $\langle\ ,\ \rangle$ defines a hermitian form on $K(a)$. Let $K_{\mathrm{odd}}(a)$ (resp. $K_{\mathrm{even}}(a)$) denote the space of all odd (resp. even) functions in $K(a)$. We have $K(a) = K_{\mathrm{odd}}(a) \oplus K_{\mathrm{even}}(a)$. Since $\check{T}_k = T_k$, we see easily that $T_k(\alpha) = 0$ if α is odd and admissible. Hence $K_{\mathrm{odd}}(a)$ and $K_{\mathrm{even}}(a)$ are orthogonal with respect to $\langle\ ,\ \rangle$.

Let $\varphi \in K(a)$ and put $F = \varphi * \tilde{\varphi}$. We have

$$F(x) = \int_{-\infty}^{\infty} \varphi(y)\overline{\varphi(y-x)}\, dy, \qquad x \in \mathbf{R}.$$

Hence we get $F(0) = \|\varphi\|_{L^2}^2$ and $|F(x)| \le \|\varphi\|_{L^2}^2$, $x \in \mathbf{R}$ by the Schwarz inequality. By (1.2) and (1.3), we have $\hat{F}(t) = |\hat{\varphi}(t)|^2$. Put

$$\begin{aligned} C_1(a) &= \int_{-2a}^{2a} (e^{x/2} + e^{-x/2})\, dx, \\ C_2(a) &= |\{(\mathfrak{p}, m) \mid m \log N(\mathfrak{p}) \le 2a\}|. \end{aligned}$$

Since $\mathrm{supp}(F) \subseteq [-2a, 2a]$, we have

$$\begin{aligned}\langle\varphi,\varphi\rangle &= T_k(F)\\ &\geq (\log A_k - C_1(a) - 2C_2(a))\|\varphi\|^2_{L^2}\\ &\quad + r_1\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\varphi}(t)|^2 \,\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt \\ &\quad + 2r_2\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\varphi}(t)|^2 \,\mathrm{Re}\,(\psi(\frac{1}{2}+it))\,dt.\end{aligned} \tag{2.1}$$

Let $C > 0$. Since

$$\mathrm{Re}(\psi(\sigma+it)) \sim \log|t|, \qquad |t| \to +\infty$$

for a fixed σ, we can find $t_0 > 0$ so that

$$\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2})) \geq C \quad \text{and} \quad \mathrm{Re}\,(\psi(\frac{1}{2}+it)) \geq C \qquad \text{if} \quad |t| \geq t_0. \tag{2.2}$$

Put

$$C_0 = \max_{|t|\leq t_0} |\,\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))|, \qquad C_0' = \max_{|t|\leq t_0} |\,\mathrm{Re}\,(\psi(\frac{1}{2}+it))|. \tag{2.3}$$

Then we have

$$\begin{aligned}&\int_{-\infty}^{\infty} |\hat{\varphi}(t)|^2 \,\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt\\ \geq\, &C\int_{|t|\geq t_0} |\hat{\varphi}(t)|^2\,dt - C_0\int_{|t|\leq t_0} |\hat{\varphi}(t)|^2\,dt\\ =\, &C\int_{-\infty}^{\infty} |\hat{\varphi}(t)|^2\,dt - (C_0+C)\int_{|t|\leq t_0} |\hat{\varphi}(t)|^2\,dt\\ =\, &2\pi C\|\varphi\|^2_{L^2} - (C_0+C)\int_{|t|\leq t_0} |\hat{\varphi}(t)|^2\,dt\end{aligned}$$

by the Plancherel formula. Similarly we have

$$\int_{-\infty}^{\infty} |\hat{\varphi}(t)|^2 \,\mathrm{Re}\,(\psi(\frac{1}{2}+it))\,dt \geq 2\pi C\|\varphi\|^2_{L^2} - (C_0'+C)\int_{|t|\leq t_0} |\hat{\varphi}(t)|^2\,dt.$$

Therefore, by (2.1), we obtain

$$\begin{aligned}\langle\varphi,\varphi\rangle &\geq \{(r_1+2r_2)C + \log A_k - C_1(a) - 2C_2(a)\}\|\varphi\|^2_{L^2}\\ &\quad - \{r_1(C_0+C) + 2r_2(C_0'+C)\}\frac{1}{2\pi}\int_{|t|\leq t_0} |\hat{\varphi}(t)|^2\,dt.\end{aligned} \tag{2.4}$$

Lemma 2. *There exists $a_0 > 0$ such that*

$$\langle \varphi, \varphi \rangle > 0 \qquad \textit{for every} \quad \varphi \in K(a),\ \varphi \neq 0 \quad \textit{if} \quad a \leq a_0.$$

Proof. Let $a_0 > 0$, $a \leq a_0$ and $\varphi \in K(a)$. Then, by the Schwarz inequality, we have

$$|\hat{\varphi}(t)| \leq \sqrt{2a} \|\varphi\|_{L^2} \qquad t \in \mathbf{R}. \tag{2.5}$$

Choose $C > 0$ so that $(r_1 + 2r_2)C > C_1(a_0) + 2C_2(a_0) - \log A_k$. Define t_0 by (2.2) and C_0, C_0' by (2.3). Then, by (2.4) and (2.5), we obtain

$$\begin{aligned}\langle \varphi, \varphi \rangle \geq \{&(r_1 + 2r_2)C + (\log A_k - C_1(a_0) - 2C_2(a_0)) \\ &- \frac{2a_0}{\pi} r_1 (C_0 + C)t_0 - \frac{4a_0}{\pi} r_2 (C_0' + C)t_0\} \|\varphi\|_{L^2}^2.\end{aligned}$$

If a_0 is sufficiently small, we can find $\mu > 0$ such that

$$\langle \varphi, \varphi \rangle > \mu \|\varphi\|_{L^2}^2 \qquad \text{for every} \quad \varphi \in K(a),\ a \leq a_0.$$

This completes the proof.

§3. Positive definiteness for highly oscillating functions

For $a > 0$ and $n \in \mathbf{Z}$, we define $\chi_n \in K(a)$ by

$$\chi_n(x) = \begin{cases} \dfrac{1}{\sqrt{2a}} \exp(\dfrac{\pi i n x}{a}) & \text{if} \quad |x| \leq a, \\ 0 & \text{if} \quad |x| > a. \end{cases}$$

Then χ_n makes an orthonormal basis of $L^2([-a, a])$. Take any $\varphi \in K(a)$. Then we have the Fourier expansion

$$\varphi(x) = \sum_{n \in \mathbf{Z}} c_n \chi_n(x), \qquad |x| \leq a,$$
$$c_n = \frac{1}{\sqrt{2a}} \int_{-a}^{a} \varphi(x) \exp(\frac{-\pi i n x}{a})\, dx.$$

By partial integration, we see that

$$c_n = O(|n|^{-k}), \qquad |n| \to \infty \tag{3.1}$$

for any natural number k. For a non-negative integer N, we set

$$K_N(a) = \{ \varphi \in K(a) \mid \int_{-a}^{a} \varphi(x)\exp(\frac{\pi i n x}{a})\, dx = 0 \text{ for all } n \in \mathbf{Z}, |n| \leq N \},$$
$$C_N(a) = C(a) \cap K_N(a).$$

Lemma 3. *Suppose that $a_0 > 0$ and $\mu > 0$ are given. Then there exists a non-negative integer N such that*

$$\langle \varphi, \varphi \rangle \geq \mu \|\varphi\|_{L^2}^2 \qquad \textit{for every} \quad \varphi \in K_N(a),\ 0 < a \leq a_0.$$

Proof. Choose $C > 0$ so that

$$(r_1 + 2r_2)C + \log A_k - C_1(a_0) - 2C_2(a_0) > \mu.$$

Define t_0, C_0 and C_0' by (2.2) and (2.3). Then, by (2.4), we get

$$\langle \varphi, \varphi \rangle \geq \{(r_1 + 2r_2)C + \log A_k - C_1(a_0) - 2C_2(a_0)\}\|\varphi\|_{L^2}^2 - \{r_1(C_0 + C) + 2r_2(C_0' + C)\}\frac{1}{2\pi}\int_{|t|\leq t_0} |\hat{\varphi}(t)|^2\, dt$$

for every $\varphi \in K_N(a)$, $a \leq a_0$.

Now assume $\varphi \in K_N(a)$ and let $\varphi = \sum_{|n|>N} c_n \chi_n$ be its Fourier expansion. Then we have $\|\varphi\|_{L^2}^2 = \sum_{|n|>N} |c_n|^2$ and

$$\begin{aligned}\hat{\varphi}(t) &= \frac{1}{\sqrt{2a}}\int_{-a}^{a} \sum_{|n|>N} c_n \exp(\frac{\pi i n x}{a})\exp(itx)\, dx \\ &= \frac{1}{\sqrt{2a}}[\sum_{|n|>N} c_n \frac{a}{\pi i n}\exp(\frac{\pi i n x}{a})\exp(itx)]_{-a}^{a} \\ &\quad - \frac{1}{\sqrt{2a}}\int_{-a}^{a} \sum_{|n|>N} c_n \frac{a}{\pi i n}\exp(\frac{\pi i n x}{a}) it \exp(itx)\, dx\end{aligned}$$

by termwise partial integration which is legitimate by (3.1). Hence we obtain

$$|\hat{\varphi}(t)| \leq \sqrt{2a}(1 + a|t|)(\textstyle\sum_{|n|>N} |\frac{c_n}{\pi n}|), \qquad t \in \mathbf{R}.$$

By the Schwarz inequality, we get

$$|\hat{\varphi}(t)| \leq \sqrt{2a}(1 + a|t|)(\textstyle\sum_{|n|>N} (\frac{1}{\pi n})^2)^{1/2}\|\varphi\|_{L^2}, \qquad t \in \mathbf{R}.$$

Therefore we obtain

$$\langle\varphi,\varphi\rangle$$
$$\geq \{(r_1+2r_2)C+\log A_k - C_1(a_0) - 2C_2(a_0) - C_3 \sum_{|n|>N} (\frac{1}{\pi n})^2\}\|\varphi\|_{L^2}^2$$

where

$$C_3 = \frac{\{r_1(C_0+C)+2r_2(C_0'+C)\}}{2\pi} \int_{|t|\leq t_0} 2a_0(1+a_0|t|)^2\,dt.$$

Choosing N sufficiently large, we get $\langle\varphi,\varphi\rangle \geq \mu\|\varphi\|_{L^2}^2$. This completes the proof.

Proposition 2. *For any $a>0$, the restriction of the hermitian form $\langle\ ,\ \rangle$ to $C(a)$ is non-degenerate.*

Proof. Put $V_0 = \{\ \varphi \in C(a) \mid \langle\varphi,\psi\rangle = 0 \quad \text{for all} \quad \psi \in C(a)\ \}$. It suffices to show $V_0 = \{0\}$. By Lemma 3, we can take N so that $\langle\ ,\ \rangle|C_N(a)$ is positive definite. Then it is obvious that $V_0 \cap C_N(a) = \{0\}$. Hence V_0 can be mapped injectively into $C(a)/C_N(a)$. We see easily that the codimension of $C_N(a)$ in $C(a)$ is $2N+1$. Hence we obtain $\dim V_0 \leq 2N+1$. Now take any $\varphi \in V_0$. Then we have $\varphi' * \psi = \varphi * \psi'$ for every $\psi \in C(a)$. Hence $\varphi' \in V_0$. Therefore φ, φ', $\cdots$, $\varphi^{(2N+1)}$ are linearly dependent over $\mathbf{C}$. In other words, φ satisfies a differential equation

$$(3.2) \qquad (\frac{d}{dx})^n\varphi + c_1(\frac{d}{dx})^{n-1}\varphi + \cdots + c_{n-1}(\frac{d}{dx})\varphi + c_n\varphi = 0$$

with $n \leq 2N+1$, $c_i \in \mathbf{C}$. We see easily that a non-zero solution of (3.2) cannot belong to $C_c^\infty(\mathbf{R})$. This completes the proof.

Corollary. *The hermitian form $\langle\ ,\ \rangle$ considered on $C_c^\infty(\mathbf{R})$ is non-degenerate.*

§4. Existence of orthogonal complements

For a bounded function F on $\mathbf{R}$, we put $\|F\|_{L^\infty} = \sup_{x\in\mathbf{R}} |F(x)|$.

Lemma 4. *Let a, $\eta > 0$. Then there exists a positive constant c which depends only on a and η such that*

$$|T_k(F)| \leq c\|F\|_{L^\infty} + 2(r_1+2r_2)\eta\|F'\|_{L^\infty}^\eta$$

for every admissible function F *such that* $\operatorname{supp}(F) \subseteq [-a,a]$ *and that* $F'(x)$ *is continuous on* $[-\eta,\eta]$ *except at* $x = 0$. *Here* $\|F'\|^{\eta}_{L^\infty} = \sup_{0<|x|\le\eta} |F'(x)|$.

Proof. Obviously it suffices to show the existence of positive constants c_1 and c_2 which depends only on a and η such that

$$|V_1(F)| \le c_1\|F\|_{L^\infty} + 2\eta\|F'\|^{\eta}_{L^\infty}, \tag{4.1}$$

$$|V_2(F)| \le c_2\|F\|_{L^\infty} + 2\eta\|F'\|^{\eta}_{L^\infty}. \tag{4.2}$$

We have

$$\begin{aligned} V_1(F) &= -\frac{1}{2}\operatorname{Pf}(\frac{1}{|x|})F - (\gamma + \log 2)F(0) \\ &\quad - \int_{-\infty}^{\infty} (\frac{e^{|x|/2}}{|e^x - e^{-x}|} - \frac{1}{2|x|})F(x)\,dx. \end{aligned}$$

The sum of the second and the third terms can be estimated by $c_3\|F\|_{L^\infty}$ with $c_3 > 0$ which depends only on a. We have

$$\begin{aligned} \operatorname{Pf}(\frac{1}{|x|})(F) &= (\lim_{\epsilon\to+0} \int_{|x|\ge\epsilon} \frac{F(x)}{|x|}\,dx + 2F(0)\log\epsilon) \\ &= \int_{|x|\ge 1} \frac{F(x)}{|x|}\,dx + \int_{|x|\le 1} \frac{F(x)-F(0)}{|x|}\,dx. \end{aligned}$$

We may assume $\eta < 1$. Clearly we have

$$|\int_{|x|\ge 1} \frac{F(x)}{|x|}\,dx| \le 2\max(0,\log a)\|F\|_{L^\infty},$$

$$|\int_{\eta\le|x|\le 1} \frac{F(x)-F(0)}{|x|}\,dx| \le 4\log\frac{1}{\eta}\|F\|_{L^\infty}.$$

Since $F'(x)$ is continuous for $0 < |x| \le \eta$ and $\lim_{\epsilon\to\pm0} F'(\epsilon)$ exists, we have

$$|\int_{|x|\le\eta} \frac{F(x)-F(0)}{|x|}\,dx| \le 2\eta\|F'\|^{\eta}_{L^\infty}.$$

Hence (4.1) follows; (4.2) can be proved similarly.

Lemma 5. *Let* $a > 0$. *There exists a positive constant* c *which depends only on* a *such that*

$$|\langle\varphi_1,\varphi_2\rangle| \le c(\|\varphi_1\|_{L^2} + \|\varphi_1'\|_{L^2})\|\varphi_2\|_{L^2}$$

for every $\varphi_1 \in C(a)$, $\varphi_2 \in K(a)$.

Proof. Put $F = \varphi_1 * \tilde{\varphi}_2$. Then $F \in C(2a)$ and we have $F' = \varphi_1' * \tilde{\varphi}_2$. Hence we obtain $\|F\|_{L^\infty} \leq \|\varphi_1\|_{L^2}\|\varphi_2\|_{L^2}$, $\|F'\|_{L^\infty} \leq \|\varphi_1'\|_{L^2}\|\varphi_2\|_{L^2}$. Now the assertion follows from Lemma 4.

Let $a > 0$. By Lemma 3, we can choose N and $\mu > 0$ so that

$$(4.3) \qquad \langle \varphi, \varphi \rangle \geq \mu \|\varphi\|_{L^2}^2 \qquad \text{for every} \quad \varphi \in K_N(a).$$

We can choose a $2N+1$-dimensional subspace V (resp. W) of $C(a)$ (resp. $K(a)$) so that

$$C(a) = V \oplus C_N(a), \qquad K(a) = W \oplus K_N(a).$$

With the positive definite hermitian inner product $\langle\ ,\ \rangle$, $C_N(a)$ and $K_N(a)$ are pre-Hilbert spaces. Let $\widehat{C_N(a)}$ and $\widehat{K_N(a)}$ be the completions of $C_N(a)$ and $K_N(a)$ respectively. It is clear that $\langle\ ,\ \rangle$ extends on $\widehat{K_N(a)}$, and we denote this extended hermitian form by the same symbol $\langle\ ,\ \rangle$. Put $\|v\| = \sqrt{\langle v, v \rangle}$ for $v \in \widehat{K_N(a)}$. We set

$$\widehat{C(a)} = V \oplus \widehat{C_N(a)}, \qquad \widehat{K(a)} = W \oplus \widehat{K_N(a)}.$$

Lemma 6. *The hermitian form* $\langle\ ,\ \rangle$ *extends to a hermitian form on* $\widehat{K(a)}$ *and on* $\widehat{C(a)}$.

Proof. Let $w \in W$ and $u \in K_N(a)$. We can write $w = w_1 + w_2$ with $w_1 \in V$, $w_2 \in K_N(a)$. By Lemma 5 and (4.3), we have

$$(4.4) \qquad |\langle w_1, u \rangle| \leq c\mu^{-1/2}(\|w_1\|_{L^2} + \|w_1'\|_{L^2})\|u\|.$$

Since $\|\langle w_2, u \rangle\| \leq \|w_2\|\,\|u\|$, we obtain

$$(4.5) \qquad |\langle w, u \rangle| \leq \{c\mu^{-1/2}(\|w_1\|_{L^2} + \|w_1'\|_{L^2}) + \|w_2\|\}\|u\|.$$

Let $v \in \widehat{K_N(a)}$ and $\{v_i\}$ be a Cauchy sequence in $K_N(a)$ which represents v. By (4.5), we see that $\lim_{i\to\infty}\langle w, v_i \rangle$ exists and does not depend on the choice of $\{v_i\}$. We set $\langle w, v \rangle = \lim_{i\to\infty}\langle w, v_i \rangle$. Similarly we set $\langle v, w \rangle = \lim_{i\to\infty}\langle v_i, w \rangle$. Then we have $\langle w, v \rangle = \overline{\langle v, w \rangle}$, $w \in W$, $v \in \widehat{K_N(a)}$. For $w_i + v_i \in \widehat{K(a)}$, $w_i \in W$, $v_i \in \widehat{K_N(a)}$, $i = 1, 2$, we set

$$(4.6) \quad \langle w_1 + v_1, w_2 + v_2 \rangle = \langle w_1, w_2 \rangle + \langle w_1, v_2 \rangle + \langle v_1, w_2 \rangle + \langle v_1, v_2 \rangle.$$

Now it is immediate to see that a hermitian form on $\widehat{K(a)}$ is defined by (4.6). The latter assertion can be proved similarly. This completes the proof.

We can show the existence of the orthogonal complement of $\widehat{K_N(a)}$ in $\widehat{K(a)}$.

Proposition 3. *Let* $a > 0$ *and take* N *which satisfies* (4.3). *Put*

$$W(a) = \{\ w \in \widehat{K(a)} \mid \langle w, v \rangle = 0 \quad \textit{for all} \quad v \in \widehat{K_N(a)}\ \},$$
$$V(a) = \{\ w \in \widehat{C(a)} \mid \langle w, v \rangle = 0 \quad \textit{for all} \quad v \in \widehat{C_N(a)}\ \}.$$

Then we have

$$\widehat{K(a)} = W(a) \oplus \widehat{K_N(a)}, \qquad \widehat{C(a)} = V(a) \oplus \widehat{C_N(a)}.$$

Proof. Take any $w \in W$ and consider a linear functional

$$\widehat{K_N(a)} \ni v \longrightarrow \langle v, w \rangle \in \mathbf{C}.$$

We write $w = w_1 + w_2$ with $w_1 \in V$, $w_2 \in K_N(a)$. By (4.5), we easily obtain

$$|\langle w, v \rangle| \leq \{c\mu^{-1/2}(\|w_1\|_{L^2} + \|w_1'\|_{L^2}) + \|w_2\|\}\|v\|.$$

This shows that the above functional is bounded. Hence by Riesz' representation theorem, there exists $v_0 \in \widehat{K_N(a)}$ such that

$$\langle v, w \rangle = \langle v, v_0 \rangle \qquad \text{for every} \qquad v \in \widehat{K_N(a)}.$$

We have $w - v_0 \in W(a)$, $w \in W(a) + \widehat{K_N(a)}$. Since w is arbitrary, we get $W \subseteq W(a) + \widehat{K_N(a)}$. By definition, it is obvious that $W(a) \cap \widehat{K_N(a)} = \{0\}$. Therefore we obtain $\widehat{K(a)} = W(a) \oplus \widehat{K_N(a)}$. The latter assertion can be proved similarly. This completes the proof.

Let $\{\varphi_n\}$ be a Cauchy sequence in $K_N(a)$ with respect to $\|\ \|$. By (4.3), we see that $\{\varphi_n\}$ is also a Cauchy sequence with respect to $\|\ \|_{L^2}$. Hence we obtain a linear map $\widehat{K_N(a)} \longrightarrow L^2([-a, a])$. Since $\widehat{K(a)} = W \oplus \widehat{K_N(a)}$, $W \subset K(a)$, this map extends to a canonical linear map $\widehat{K(a)} \longrightarrow L^2([-a, a])$.

Proposition 4. *The canonical map $\widehat{K(a)} \longrightarrow L^2([-a,a])$ is injective.*

Proof. Suppose that $\eta + \varphi \in \widehat{K(a)}$, $\eta \in W$, $\varphi \in \widehat{K_N(a)}$ has zero image in $L^2([-a,a])$. Then we have

$$\int_{-a}^{a} (\eta+\varphi)(x)\exp(\frac{\pi i n x}{a})\, dx = \int_{-a}^{a} \eta(x)\exp(\frac{\pi i n x}{a})\, dx = 0$$

for all $n \in \mathbf{Z}$, $|n| \leq N$. By the choice of W, this clearly implies $\eta = 0$. Therefore it suffices to prove the injectivity of $\widehat{K_N(a)} \longrightarrow L^2([-a,a])$. For this purpose, let $\{\varphi_n\}$ be a Cauchy sequence in $K_N(a)$ with respect to $\|\ \|$ such that $\lim_{n\to\infty} \|\varphi_n\|_{L^2} = 0$. It is enough to prove $\lim_{n\to\infty} \|\varphi_n\| = 0$. Since

$$\|\varphi_n\|^2 = T_k(\varphi_n * \tilde{\varphi}_n), \qquad |(\varphi_n * \tilde{\varphi}_n)(x)| \leq \|\varphi_n\|_{L^2}^2, \quad x \in \mathbf{R},$$

it is sufficient to show

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\hat{\varphi}_n(t)|^2 \operatorname{Re}(\psi(\frac{1}{4}+\frac{it}{2}))\, dt = 0, \tag{4.7}$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\hat{\varphi}_n(t)|^2 \operatorname{Re}(\psi(\frac{1}{2}+it))\, dt = 0. \tag{4.8}$$

Choose $t_0 > 0$ so that $\operatorname{Re}(\psi(\frac{1}{4}+\frac{it}{2})) \geq 1$ if $|t| \geq t_0$ and put

$$C = \max_{|t|\leq t_0} |\operatorname{Re}(\psi(\frac{1}{4}+\frac{it}{2}))|, \qquad \Omega = \{t \in \mathbf{R} \mid |t| \geq t_0\}.$$

We have

$$\begin{aligned} |\int_{|t|\leq t_0} |\hat{\varphi}_n(t)|^2 \operatorname{Re}(\psi(\frac{1}{4}+\frac{it}{2}))\, dt| &\leq C\int_{|t|\leq t_0} |\hat{\varphi}_n(t)|^2\, dt \leq C\|\hat{\varphi}_n\|_{L^2}^2 \\ &= 2\pi C\|\varphi_n\|_{L^2}^2. \end{aligned}$$

Therefore (4.7) is equivalent to

$$\lim_{n\to\infty} \int_{\Omega} |\hat{\varphi}_n(t)|^2 \operatorname{Re}(\psi(\frac{1}{4}+\frac{it}{2}))\, dt = 0. \tag{4.9}$$

Since $\{\varphi_n\}$ is a Cauchy sequence with respect to $\|\ \|$, we see easily that for any $\epsilon > 0$, there exists M such that

$$\lim_{n\to\infty} \int_{\Omega} |\hat{\varphi}_n(t) - \hat{\varphi}_m(t)|^2 \operatorname{Re}(\psi(\frac{1}{4}+\frac{it}{2}))\, dt < \epsilon \tag{4.10}$$

if n, $m \geq M$. Hence $\{\hat{\varphi}_n(t)\sqrt{\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))}\,|\,\Omega\}$ is a Cauchy sequence in $L^2(\Omega)$. Let $\alpha(t)\sqrt{\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))}$ be its limit in $L^2(\Omega)$. Then (4.9) is equivalent to

$$\int_\Omega |\alpha(t)|^2\,\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt = 0. \tag{4.11}$$

We see $\alpha \in L^2(\Omega)$ and

$$\int_\Omega |\alpha(t)-\hat{\varphi}_n(t)|^2\,dt \leq \int_\Omega |\alpha(t)-\hat{\varphi}_n(t)|^2\,\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt.$$

Hence $\alpha(t)$ is the limit of $\hat{\varphi}_n|\Omega$ in $L^2(\Omega)$. Therefore we have $\int_\Omega |\alpha(t)|^2\,dt = 0$. From this, we can easily deduce (4.11). Thus we have proved (4.7); (4.8) can be proved similarly. This completes the proof.

Remark. Suppose that we have chosen (possibly) another N' and $W' \subset K(a)$ such that $K(a) = W' \oplus K_{N'}(a)$. We assume that (4.3) holds with μ', N' in the places of μ, N. Assume $N' \leq N$. Obviously $\widehat{K_N(a)}$ can be regarded canonically as a subspace of $\widehat{K_{N'}(a)}$. Combined with the linear map $W \to W' \oplus K_{N'}(a)$ obtained by the inclusion, we get a canonical linear map

$$\iota : W \oplus \widehat{K_N(a)} \longrightarrow W' \oplus \widehat{K_{N'}(a)}.$$

The composition of ι with $W' \oplus \widehat{K_{N'}(a)} \to L^2([-a,a])$ is the canonical map $W \oplus \widehat{K_N(a)} \to L^2([-a,a])$, which is injective by Proposition 4. Hence ι is injective.

Let us show that ι is surjective. It is obvious that $\mathrm{Image}(\iota) \supseteq W'$. Let $\varphi \in \widehat{K_{N'}(a)}$ and let $\{\varphi_i\}$ be a Cauchy sequence in $K_{N'}(a)$ which represents φ. Let $\varphi_i = \sum_{|n|>N'} c_{in}\chi_n$ be its Fourier expansion. Since $\{\varphi_i\}$ converges to φ in $L^2([-a,a])$, we have

$$c_n := \frac{1}{\sqrt{2a}}\int_{-a}^{a} \varphi(x)\exp(\frac{-\pi inx}{a})\,dx = \lim_{i\to\infty} c_{in}.$$

We can take a basis $\{\eta_n \mid |n| \leq N\}$ of W so that

$$\frac{1}{\sqrt{2a}}\int_{-a}^{a} \eta_n(x)\exp(\frac{-\pi imx}{a})\,dx = \delta_{nm}, \qquad |n|, |m| \leq N.$$

Put $\varphi_i^* = \varphi_i - \sum_{N'<|n|\leq N} c_{in}\eta_n$. Then we have $\varphi_i^* \in \widehat{K_N(a)}$ and

$$\|\varphi_i^* - \varphi_j^*\| \leq \|\varphi_i - \varphi_j\| + \sum_{N'<|n|\leq N} |c_{in} - c_{jn}|\|\eta_n\|.$$

Hence $\{\varphi_i^*\}$ is a Cauchy sequence in $K_N(a)$. Let $\varphi^* \in \widehat{K_N(a)}$ be its limit. Then we see easily that $\iota(\varphi^*) = \varphi - \sum_{N'<|n|\leq N} c_n\eta_n$. Therefore we have $\varphi \in \iota(W' + \widehat{K_N(a)})$. This proves that ι is surjective.

We have verified that $\widehat{K(a)}$ *does not depend on the choice of* N *and* W. We could have taken $W = V \subset C(a)$. This shows that $\widehat{C(a)}$ can be regarded canonically as a subspace of $\widehat{K(a)}$.

§5. Some matrix coefficients

The positive definiteness of $\langle\ ,\ \rangle|K(a)$ is, roughly speaking, equivalent to the positive definiteness of the infinite dimensional hermitian matrix $((\langle\chi_n, \chi_m\rangle))$. Thus we are interested in calculating $\langle\chi_n, \chi_m\rangle$ explicitly.

Fix n, $m \in \mathbf{Z}$ and put $F = \chi_n * \tilde{\chi}_m$. By direct computation, we have

$$\begin{aligned}(5.1)\qquad &\int_{-\infty}^{\infty} F(x)(e^{x/2} + e^{-x/2})\,dx \\ &= (-1)^{n+m}\frac{4}{a}(e^{a/2} - e^{-a/2})^2 \frac{1 - \frac{4\pi^2 nm}{a^2}}{\{1 + (\frac{2\pi n}{a})^2\}\{1 + (\frac{2\pi m}{a})^2\}},\end{aligned}$$

$$(5.2)\qquad F(x) = \begin{cases} \dfrac{1}{2a}(2a - x)\exp(\dfrac{\pi i n x}{a}), & 0 \leq x \leq 2a, \\ \dfrac{1}{2a}(2a + x)\exp(\dfrac{\pi i n x}{a}), & -2a \leq x \leq 0, \\ 0, & |x| > 2a, \end{cases} \quad \text{if } n = m.$$

$$(5.3)\quad F(x) = \begin{cases} \dfrac{(-1)^{n-m}}{2\pi i(n-m)}\{\exp(\dfrac{\pi i m x}{a}) - \exp(\dfrac{\pi i n x}{a})\}, & 0 \leq x \leq 2a, \\ \dfrac{(-1)^{n-m}}{2\pi i(n-m)}\{\exp(\dfrac{\pi i n x}{a}) - \exp(\dfrac{\pi i m x}{a})\}, & -2a \leq x \leq 0, \\ 0, & |x| > 2a, \end{cases}$$

if $n \neq m$. The calculations of $V_1(F)$ and $V_2(F)$ is not so trivial. Set

$$\Phi(s) = M(F), \qquad \Phi_n(s) = M(\chi_n). \tag{5.4}$$

Then we have

$$\Phi(s) = \Phi_n(s)\overline{\Phi_m(1-\bar{s})}, \tag{5.5}$$

$$\begin{aligned} \Phi_n(s) = (-1)^n \frac{1}{\sqrt{2a}} \frac{1}{s - \frac{1}{2} + \frac{\pi i n}{a}} \{\exp(a(s - \frac{1}{2})) \\ - \exp(-a(s-\frac{1}{2}))\}. \end{aligned} \tag{5.6}$$

$$\begin{aligned} V_1(F) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{F}(t) \operatorname{Re}(\psi(\frac{1}{4} + \frac{it}{2}))\, dt \\ &= \frac{1}{2} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} (\Phi(s) + \Phi(1-s))\psi(s/2)\, ds. \end{aligned}$$

Of course, $\Phi_n(s)$ is an entire function of s. Set

$$I_{n,m} = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Phi_n(s)\overline{\Phi_m(1-\bar{s})}\, \psi(s/2)\, ds. \tag{5.7}$$

Since $\Phi(1-s) = M(\chi_{-n} * \tilde{\chi}_{-m})$, we have

$$V_1(F) = (I_{n,m} + I_{-n,-m})/2. \tag{5.8}$$

By (5.6), we get

$$\begin{aligned} I_{n,m} &= (-1)^{n+m} \frac{1}{2a} \frac{1}{2\pi i} \\ &\times \int_{1/2-i\infty}^{1/2+i\infty} \frac{\{\exp(a(s-\frac{1}{2})) - \exp(-a(s-\frac{1}{2}))\}^2}{(s - \frac{1}{2} + \frac{\pi i n}{a})(s - \frac{1}{2} + \frac{\pi i m}{a})} \psi(s/2)\, ds. \end{aligned} \tag{5.9}$$

We have

$$\psi(s+1) = \psi(s) + \frac{1}{s}. \tag{5.10}$$

By a well known integration formula

$$\psi(s) = \log s - \frac{1}{2s} - 2\int_0^{\infty} \frac{u\, du}{(u^2 + s^2)(e^{2\pi u} - 1)}, \quad \operatorname{Re}(s) > 0,$$

we get

$$|\psi(s) - (\log s - \frac{1}{2s})| \leq \frac{1}{12t^2}, \quad t = \mathrm{Im}(s),\ \mathrm{Re}(s) > 0, \tag{5.11}$$

since

$$|\int_0^\infty \frac{u\,du}{(u^2+s^2)(e^{2\pi u}-1)}| \leq \frac{1}{t^2}\int_0^\infty \frac{u\,du}{(e^{2\pi u}-1)} = \frac{1}{24t^2},$$

where $\log s$ takes the principal value. By (5.10) and (5.11), we get

$$\begin{aligned} &|\psi(\sigma+it)| \sim \log|t| \quad \text{for} \quad |t| \to \infty \\ &\qquad\qquad \text{uniformly when } -\infty < \sigma_1 \leq \sigma \leq \sigma_2 < +\infty. \end{aligned} \tag{5.12}$$

Take any $\sigma > 1/2$. By (5.12), the line of integration in (5.9) may be shifted to $\int_{\sigma-i\infty}^{\sigma+i\infty}$. Set

$$g(s) = \frac{1}{(s-\frac{1}{2}+\frac{\pi in}{a})(s-\frac{1}{2}+\frac{\pi im}{a})},$$

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \exp(2a(s-\frac{1}{2}))g(s)\psi(s/2)\,ds, \\ I_2 &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} g(s)\psi(s/2)\,ds, \\ I_3 &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \exp(-2a(s-\frac{1}{2}))g(s)\psi(s/2)\,ds. \end{aligned}$$

Then we have

$$I_{n,m} = (-1)^{n+m}(I_1 - 2I_2 + I_3)/2a.$$

For I_i, $i = 1, 2, 3$, we may shift the line of integration to $\int_{\tau-i\infty}^{\tau+i\infty}$ for any $\tau > \sigma$. For I_2 and I_3, controlling the order of $\psi(s/2)$ by (5.10) and (5.11), we see easily that

$$\begin{aligned} &\lim_{\tau\to+\infty}\int_{\tau-i\infty}^{\tau+i\infty} g(s)\psi(s/2)\,ds = 0, \\ &\lim_{\tau\to+\infty}\int_{\tau-i\infty}^{\tau+i\infty} \exp(-2a(s-\frac{1}{2}))g(s)\psi(s/2)\,ds = 0. \end{aligned}$$

Thus we get $I_2 = I_3 = 0$. Let $\tau < 0$, $\tau \notin 2\mathbf{Z}$. By (5.12), we may shift the line of integration for I_1 to $\int_{\tau-i\infty}^{\tau+i\infty}$ picking up residues. We obtain

$$I_1 = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \exp(2a(s-\frac{1}{2}))g(s)\psi(s/2)\,ds$$

$$+\sum \text{Residues of } \exp(2a(s-\frac{1}{2}))g(s)\psi(s/2) \text{ for } \tau < \mathrm{Re}(s) < \sigma.$$

By (5.10) and (5.11), we find

$$\lim_{\tau\to-\infty}\int_{\tau-i\infty}^{\tau+i\infty} \exp(2a(s-\frac{1}{2}))g(s)\psi(s/2)\,ds = 0.$$

Therefore we get

$$I_1 = \sum \text{Residues of } \exp(2a(s-\frac{1}{2}))g(s)\psi(s/2) \text{ for } \mathrm{Re}(s) < \sigma.$$

These residues can be easily calculated, and since $I_{n,m} = (-1)^{n+m}I_1/2a$, we obtain

$$\begin{aligned} I_{n,n} = \frac{1}{2a}\{-2\sum_{k=0}^{\infty}\frac{1}{(2k+\frac{1}{2}-\frac{\pi i n}{a})^2}\exp(-2a(2k+\frac{1}{2})) \\ + 2a\psi(\frac{1}{4}-\frac{\pi i n}{2a}) + \frac{1}{2}\psi'(\frac{1}{4}-\frac{\pi i n}{2a})\}, \end{aligned} \tag{5.13}$$

$$\begin{aligned} I_{n,m} = \frac{(-1)^{n+m}}{2a}\{-2\sum_{k=0}^{\infty}\frac{\exp(-2a(2k+\frac{1}{2}))}{(2k+\frac{1}{2}-\frac{\pi i n}{a})(2k+\frac{1}{2}-\frac{\pi i m}{a})} \\ + \frac{a}{\pi i(n-m)}(\psi(\frac{1}{4}-\frac{\pi i m}{2a}) - \psi(\frac{1}{4}-\frac{\pi i n}{2a}))\}. \end{aligned} \tag{5.14}$$

By (5.8), we obtain a formula for $V_1(F)$; $V_2(F)$ can be calculated similarly. The final formula for $\langle \chi_n, \chi_m\rangle$ is as follows.

(5.15)

$$\begin{aligned}\langle\chi_n,\chi_n\rangle &= \frac{4}{a}(e^{a/2}-e^{-a/2})^2\frac{1-\frac{4\pi^2n^2}{a^2}}{\{1+(\frac{2\pi n}{a})^2\}^2}+\log A_k\\ &-\frac{1}{a}\sum_{\mathfrak{p},e,e\log N(\mathfrak{p})\le 2a}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{e/2}}(2a-e\log N(\mathfrak{p}))\cos(\frac{\pi ne\log N(\mathfrak{p})}{a})\\ &+\frac{r_1}{2a}\{-2\sum_{k=0}^{\infty}\frac{(2k+\frac12)^2-(\frac{\pi n}{a})^2}{\{(2k+\frac12)^2+(\frac{\pi n}{a})^2\}^2}\exp(-2a(2k+\frac12))\\ &+2a\mathrm{Re}(\psi(\frac14+\frac{\pi in}{2a}))+\frac12\mathrm{Re}(\psi'(\frac14+\frac{\pi in}{2a}))\}\\ &+\frac{r_2}{a}\{-\sum_{k=0}^{\infty}\frac{(k+\frac12)^2-(\frac{\pi n}{a})^2}{\{(k+\frac12)^2+(\frac{\pi n}{a})^2\}^2}\exp(-2a(k+\frac12))\\ &+2a\mathrm{Re}(\psi(\frac12+\frac{\pi in}{a}))+\frac12\mathrm{Re}(\psi'(\frac12+\frac{\pi in}{a}))\}.\end{aligned}$$

(5.16)

$$\begin{aligned}(-1)^{n+m}\langle\chi_n,\chi_m\rangle &= \frac{4}{a}(e^{a/2}-e^{-a/2})^2\frac{1-\frac{4\pi^2nm}{a^2}}{\{1+(\frac{2\pi n}{a})^2\}^2\}\{1+(\frac{2\pi m}{a})^2\}}\\ &-\frac{1}{\pi(n-m)}\times\\ &\sum_{\mathfrak{p},e,e\log N(\mathfrak{p})\le 2a}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{e/2}}\{\sin(\frac{\pi me\log N(\mathfrak{p})}{a})-\sin(\frac{\pi ne\log N(\mathfrak{p})}{a})\}\\ &+\frac{r_1}{2a}\{-2\sum_{k=0}^{\infty}\frac{\{(2k+\frac12)^2-(\frac{\pi^2nm}{a^2})\}\exp(-2a(2k+\frac12))}{\{(2k+\frac12)^2+(\frac{\pi n}{a})^2\}\{(2k+\frac12)^2+(\frac{\pi m}{a})^2\}}\\ &+\frac{a}{\pi(n-m)}(\mathrm{Im}(\psi(\frac14+\frac{\pi in}{2a})-\mathrm{Im}(\psi(\frac14+\frac{\pi im}{2a})))\}\\ &+\frac{r_2}{a}\{-\sum_{k=0}^{\infty}\frac{\{(k+\frac12)^2-(\frac{\pi^2nm}{a})\}\exp(-2a(2k+\frac12))}{\{(k+\frac12)^2+(\frac{\pi n}{a})^2)\}\{(k+\frac12)^2+(\frac{\pi m}{a})^2)\}}\\ &+\frac{a}{\pi(n-m)}(\mathrm{Im}(\psi(\frac12+\frac{\pi in}{a}))-\mathrm{Im}(\psi(\frac12+\frac{\pi im}{a})))\},\end{aligned}$$

if $n\neq m$.

From (5.15) and (5.16), we can easily deduce the following. There exists $\kappa>0$ which depends only on a such that

(5.17) $$|\langle\chi_n,\chi_m\rangle|\le\kappa/|n-m|\qquad\text{if}\quad n\neq m,$$

$$(5.18) \qquad \langle \chi_n, \chi_n \rangle \sim (r_1 + 2r_2) \log |n| \qquad \text{for} \quad |n| \to \infty.$$

The above formulas (5.15) and (5.16) are suitable for numerical computations, since the sums $\sum_{k=0}^{\infty}$ in them converge rapidly. If R.H. holds, the finite dimensional matrix $(\langle \chi_n, \chi_m \rangle \mid |n|, |m| \leq N)$ must be positive definite for every N and a. We have verified this for several instances of a and N when $k = \mathbf{Q}$.

§6. A numerical example

Let $k = \mathbf{Q}$ and $a = \log 2/2$. In this section, we shall prove

$$(6.1) \qquad \langle \varphi, \varphi \rangle \geq 0 \qquad \text{for every} \quad \varphi \in K(a).$$

For this purpose, it is necessary to determine the constant μ in Lemma 3 more precisely. We note that we may assume φ is odd or even function in (6.1) (cf. §2). By a direct computation, we have

$$(6.2) \qquad \int_{-\infty}^{\infty} (\varphi * \tilde{\varphi})(x)(e^{x/2} + e^{-x/2})\, dx = 2\epsilon \left| \int_{-\infty}^{\infty} \varphi(x) e^{x/2}\, dx \right|^2,$$

where $\epsilon = 1$ (resp. -1) if φ is even (resp. odd). Assume $\|\varphi\|_{L^2} = 1$. If φ is odd, we have

$$\left| \int_{-a}^{a} \varphi(x) e^{x/2}\, dx \right|^2 = \left| \int_{0}^{a} \varphi(x)(e^{x/2} - e^{-x/2})\, dx \right|^2 \leq (e^a - e^{-a} - 2a)/2,$$

by the Schwarz inequality. Hence we obtain

$$(6.3) \qquad \begin{aligned} \langle \varphi, \varphi \rangle \geq & -\log \pi - \left(\frac{1}{\sqrt{2}} - \log 2\right) \\ & + \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\varphi}(t)|^2 \operatorname{Re}\left(\psi\left(\frac{1}{4} + \frac{it}{2}\right)\right) dt, \\ & \varphi \in K(a),\ a = \log 2/2,\ \|\varphi\|_{L^2} = 1,\ \check{\varphi} = \pm\varphi. \end{aligned}$$

We have

$$\frac{d}{ds}\psi(s) = \sum_{n=0}^{\infty} \frac{1}{(n+s)^2}, \qquad \operatorname{Re}(s) > 0.$$

(cf. Whittaker-Watson [8], p. 250.) Let $\sigma > 0$. We get

$$\begin{aligned} \frac{d}{dt} \operatorname{Re}(\psi(\sigma + it)) &= -\operatorname{Im}\left(\frac{d}{ds}\psi(s)|_{s=\sigma+it}\right) \\ &= \sum_{n=0}^{\infty} \frac{2(\sigma+n)t}{\{(\sigma+n)^2 + t^2\}^2} > 0 \end{aligned}$$

for $t > 0$. Therefore $\mathrm{Re}(\psi(\sigma + it))$ is monotone increasing for $t \geq 0$.
Choose $C > 0$. Define $0 < t_0 < t_1$ so that

$$\mathrm{Re}\,(\psi(\frac{1}{4} + \frac{it}{2})) \geq 0 \quad \text{if} \quad |t| \geq t_0, \quad \text{equality for} \quad |t| = t_0,$$
$$\mathrm{Re}\,(\psi(\frac{1}{4} + \frac{it}{2})) \geq C \quad \text{if} \quad |t| \geq t_1, \quad \text{equality for} \quad |t| = t_1.$$

We have $t_0 = 2.0320\cdots$. Similarly as in §2, we get

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\varphi}(t)|^2\,\mathrm{Re}\,(\psi(\frac{1}{4} + \frac{it}{2}))\,dt \\ \geq{}& C - \frac{C}{2\pi}\int_{|t|\leq t_1} |\hat{\varphi}(t)|^2\,dt + \frac{1}{2\pi}\int_{|t|\leq t_0} |\hat{\varphi}(t)|^2\,\mathrm{Re}\,(\psi(\frac{1}{4} + \frac{it}{2}))\,dt. \end{aligned}$$

By (6.3), we have

$$\tag{6.4} \begin{aligned} \langle\varphi, \varphi\rangle \geq C - \log\,\pi - (\frac{1}{\sqrt{2}} - \log\,2) - \frac{C}{2\pi}\int_{|t|\leq t_1} |\hat{\varphi}(t)|^2\,dt \\ + \frac{1}{2\pi}\int_{|t|\leq t_0} |\hat{\varphi}(t)|^2\,\mathrm{Re}\,(\psi(\frac{1}{4} + \frac{it}{2}))\,dt. \end{aligned}$$

Now let $\varphi \in K_N(a)$ and $\varphi = \sum_{|n|>N} c_n\chi_n$ be its Fourier expansion. We have

$$\hat{\varphi}(t) = \frac{1}{\sqrt{2a}}\int_{-a}^{a} \sum_{|n|>N} c_n\exp(\frac{\pi inx}{a})\exp(itx)\,dx.$$

By termwise integration, we obtain

$$\tag{6.5} \hat{\varphi}(t) = \frac{\sqrt{2a}}{\pi}\sum_{|n|>N} (-1)^n c_n \frac{1}{n}\frac{\pi n}{at + \pi n}\sin\,at \quad \text{if} \quad |at| < \pi(N+1).$$

Since φ is odd or even, $\hat{\varphi}$ is also odd or even. Assume $at_1 < \pi(N+1)$. Then we have

$$\begin{aligned} &(\int_{|t|\leq t_1} |\hat{\varphi}(t)|^2\,dt)^{1/2} = \sqrt{2}(\textstyle\int_0^{t_1} |\hat{\varphi}(t)|^2\,dt)^{1/2} \\ \leq{}& \frac{\sqrt{4a}}{\pi}\sum_{|n|>N} |c_n\frac{1}{n}|(\int_0^{t_1} (\frac{\pi n}{at + \pi n})^2\sin^2 at\,dt)^{1/2}. \end{aligned}$$

For $0 \le t \le t_1$, we have

$$|\frac{\pi n}{at+\pi n}| \le 1 \quad \text{if } n>0, \quad |\frac{\pi n}{at+\pi n}| \le \frac{\pi|n|}{\pi|n|-at_1} \quad \text{if } n<0.$$

Since $c_n = \pm c_{-n}$, we obtain

$$(\int_{|t|\le t_1} |\hat{\varphi}(t)|^2\,dt)^{1/2} \le \frac{\sqrt{4a}}{\pi}\sum_{n=N+1}^{\infty}|c_n|\frac{1}{n}(1+\frac{\pi n}{\pi n - at_1}) \times (\int_0^{t_1}\sin^2 at\,dt)^{1/2}.$$

By $\sum_{n=N+1}^{\infty}|c_n|^2 = \|\varphi\|_{L^2}^2/2 = 1/2$ and the Schwarz inequality, we obtain

$$\begin{aligned}\frac{C}{2\pi}\int_{|t|\le t_1}|\hat{\varphi}(t)|^2\,dt &\le \frac{aC}{2\pi^3}(t_1 - \frac{\sin 2at_1}{2a}) \\ &\times \sum_{n=N+1}^{\infty}\frac{1}{n^2}(1+\frac{\pi n}{\pi n - at_1})^2 \quad \text{if} \quad t_1 < \frac{\pi(N+1)}{a}.\end{aligned} \tag{6.6}$$

Assume $at_0 < \pi(N+1)$. By (6.5), we immediately obtain

$$|\hat{\varphi}(t)| \le \frac{\sqrt{2a}}{\pi}\sum_{n=N+1}^{\infty}|c_n|\frac{1}{n}(1+\frac{\pi n}{\pi n - at_0}), \quad 0 \le t \le t_0.$$

Hence we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{|t|\le t_0}|\hat{\varphi}(t)|^2\,\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt \\ &= \frac{1}{\pi}\int_0^{t_0}|\hat{\varphi}(t)|^2\,\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt \\ &\ge \frac{2a}{\pi^3}(\sum_{n=N+1}^{\infty}|c_n|\frac{1}{n}(1+\frac{\pi n}{\pi n - at_0}))^2\int_0^{t_0}\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt \\ &\ge \frac{a}{\pi^3}\sum_{n=N+1}^{\infty}\frac{1}{n^2}(1+\frac{\pi n}{\pi n - at_0})^2\int_0^{t_0}\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt.\end{aligned}$$

We have

$$\begin{aligned}&\int_0^{t_0}\mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt \\ &= 2\int_0^{t_0/2}\mathrm{Re}(\psi(\frac{1}{4}+it))\,dt = 2\,\mathrm{Im}(\log\Gamma(\frac{1}{4}+\frac{it_0}{2})),\end{aligned}$$

since $\mathrm{Re}(\psi(\sigma+it)) = \dfrac{d}{dt}\mathrm{Im}(\log\Gamma(\sigma+it))$. By Stirling's formula, we can compute $\mathrm{Im}(\log\Gamma(\frac{1}{4}+\frac{it_0}{2}))$ easily and obtain

$$\int_0^{t_0} \mathrm{Re}\,(\psi(\frac{1}{4}+\frac{it}{2}))\,dt = -2.7626\cdots.$$

Hence by (6.4) and (6.6), we get

$$\begin{aligned}
&\langle\varphi,\varphi\rangle \geq C - \log\pi - (\frac{1}{\sqrt{2}} - \log 2)\\
&\qquad - \frac{aC}{2\pi^3}(t_1 - \frac{\sin 2at_1}{2a}) \sum_{n=N+1}^{\infty} \frac{1}{n^2}(1+\frac{\pi n}{\pi n - at_1})^2 \qquad (6.7)\\
&\qquad - 2.77\frac{a}{\pi^3} \sum_{n=N+1}^{\infty} \frac{1}{n^2}(1+\frac{\pi n}{\pi n - at_0})^2,\\
&\varphi \quad \text{is odd or even},\ \|\varphi\|_{L^2} = 1,\ at_1 < \pi(N+1).
\end{aligned}$$

Take $t_1 = 50$ and $N = 10$ in (6.7). Then $C = 3.2188\cdots$ and we have

$$\langle\varphi,\varphi\rangle \geq 1.52\|\varphi\|^2_{L^2}, \qquad \varphi\in K_{10}(a),\ \varphi \text{ is odd or even.} \tag{6.8}$$

Take $t_1 = 700$ and $N = 199$ in (6.7). Then $C = 5.9914\cdots$ and we have

$$\langle\varphi,\varphi\rangle \geq 4.08\|\varphi\|^2_{L^2}, \qquad \varphi\in K_{199}(a),\ \varphi \text{ is odd or even.} \tag{6.9}$$

Now we shall proceed to prove (6.1). For a non-negative integer N, we set

$$K_{N,\mathrm{odd}}(a) = K_N(a)\cap K_{\mathrm{odd}}(a), \qquad K_{N,\mathrm{even}}(a) = K_N(a)\cap K_{\mathrm{even}}(a).$$

If N satisfies (4.3), let $\widehat{K_{N,\mathrm{odd}}}(a)$ and $\widehat{K_{N,\mathrm{even}}}(a)$ denote the completions of the respective spaces with respect to $\langle\ ,\ \rangle$. First we shall prove (6.1) for $\varphi\in K_{\mathrm{odd}}(a)$. For a positive integer n, we set

$$\omega_n(x) = \begin{cases} \dfrac{1}{\sqrt{a}}\sin(\dfrac{\pi i n x}{a}) & \text{if } \ |x|\leq a,\\ 0 & \text{if } \ |x|>a.\end{cases}$$

Then we have $\omega_n\in K_{\mathrm{odd}}(a)$, $\omega_n = \frac{\sqrt{2}}{2i}(\chi_n - \chi_{-n})$, $\|\omega_n\|_{L^2} = 1$,

$$\langle\omega_n,\omega_m\rangle = \langle\chi_n,\chi_m\rangle - \langle\chi_n,\chi_{-m}\rangle. \tag{6.10}$$

For $1 \leq i \leq 10$, we consider a linear functional

$$\widehat{K_{10,\mathrm{odd}}(a)} \ni v \longrightarrow \langle v, \omega_i \rangle \in \mathbf{C}.$$

This functional is bounded as can be seen from the proof of Proposition 3. We can estimate its bound as follows. For $n \geq 1$, set $\eta_n = \omega_{n+10}$. Let $v \in \widehat{K_{10,\mathrm{odd}}(a)}$, $\|v\| = 1$. By Proposition 4, we may write $v = \sum_{k=1}^{\infty} a_k \eta_k$; then we have $\|v\|_{L^2}^2 = \sum_{k=1}^{\infty} |a_k|^2$. By (6.8), which may be applied to $\varphi \in \widehat{K_{10}(a)}$, we get $\sum_{k=1}^{\infty} |a_k|^2 \leq 1/1.52$. We have

$$|\langle v, \omega_i \rangle| = |\sum_{k=1}^{\infty} a_k \langle \eta_k, \omega_i \rangle| \leq (\sum_{k=1}^{\infty} |a_k|^2)^{1/2} (\sum_{k=1}^{\infty} |\langle \eta_k, \omega_i \rangle|^2)^{1/2}.$$

We note that $\langle \eta_k, \omega_i \rangle = O(k^{-1})$ by (5.17) and (6.10). Therefore we obtain

$$|\langle v, \omega_i \rangle|^2 \leq (\sum_{k=1}^{\infty} |\langle \eta_k, \omega_i \rangle|^2)/1.52, \quad v \in \widehat{K_{10,\mathrm{odd}}(a)}, \ \|v\| = 1. \tag{6.11}$$

By Riesz' representation theorem, there exists a $v_i \in \widehat{K_{10,\mathrm{odd}}(a)}$ such that

$$\langle v, \omega_i \rangle = \langle v, v_i \rangle \qquad \text{for every} \quad v \in \widehat{K_{10,\mathrm{odd}}(a)}. \tag{6.12}$$

Furthermore, by (6.11), we have

$$\|v_i\|^2 \leq (\sum_{k=1}^{\infty} |\langle \eta_k, \omega_i \rangle|^2)/1.52. \tag{6.13}$$

Put $\omega_i' = \omega_i - v_i$, $1 \leq i \leq 10$. Then it is clear that ω_i', $1 \leq i \leq 10$ span the orthogonal complement of $\widehat{K_{10,\mathrm{odd}}(a)}$ in $\widehat{K_{\mathrm{odd}}(a)}$, where $\widehat{K_{\mathrm{odd}}(a)} = \langle \omega_i; 1 \leq i \leq 10 \rangle_{\mathbf{C}} \oplus \widehat{K_{10,\mathrm{odd}}(a)}$. Therefore, to prove (6.1) for $\varphi \in K_{\mathrm{odd}}(a)$, it suffices to prove the positive definiteness of the 10×10 hermitian matrix $(\langle \omega_i', \omega_j' \rangle; 1 \leq i, j \leq 10)$. By (6.12), we have

$$\langle \omega_i', \omega_j' \rangle = \langle \omega_i, \omega_j \rangle - \langle v_i, v_j \rangle, \qquad 1 \leq i, j \leq 10. \tag{6.14}$$

Suppose that we have proved

$$\|v_i\|^2 \leq \epsilon, \qquad 1 \leq i \leq 10. \tag{6.15}$$

Then, by (6.14), we have

$$|\langle \omega_i', \omega_j' \rangle - \langle \omega_i, \omega_j \rangle| \leq \epsilon, \qquad 1 \leq i, j \leq 10.$$

For $(x_1, \cdots, x_{10}) \in \mathbf{C}^{10}$, we have

$$\sum_{1\leq i,j\leq 10} \langle \omega_i', \omega_j' \rangle x_i \bar{x}_j \geq \sum_{i=1}^{10} (\langle \omega_i, \omega_i \rangle - \epsilon)|x_i|^2 - \sum_{1\leq i,j\leq 10, i\neq j} (|\langle \omega_i, \omega_j \rangle| + \epsilon)|x_i x_j|.$$

Put

$$u_{ii} = \langle \omega_i, \omega_i \rangle - \epsilon, \qquad 1 \leq i \leq 10,$$
$$u_{ij} = -|\langle \omega_i, \omega_j \rangle| - \epsilon, \qquad 1 \leq i, j \leq 10, \ i \neq j.$$

Then the positive definiteness of $(\langle \omega_i', \omega_j' \rangle)$ follows from the positive definiteness of the symmetric matrix $U := (u_{ij}; 1 \leq i, j \leq 10)$. We find that U is positive definite if we can take $\epsilon = 1/40$ in (6.15). In fact, $\langle \omega_i, \omega_j \rangle$ can be easily calculated by (5.15), (5.16) and (6.10), and it suffices to show det $(u_{ij}; 1 \leq i, j \leq k) > 0$ for $1 \leq k \leq 10$. The verification can be done by a simple triangulation process applied to U. Thus, by (6.13), (6.1) for odd φ reduces to

$$(6.16) \qquad (\sum_{k=1}^{\infty} |\langle \omega_i, \omega_{k+10} \rangle|^2) \leq \frac{1.52}{40} = 0.038 \quad \text{for} \quad 1 \leq i \leq 10.$$

By (5.16) and (6.10), we have

$$\begin{aligned} &(-1)^{n+m} \langle \omega_n, \omega_m \rangle \\ &= \frac{4}{a}(e^{a/2} - e^{-a/2})^2 \frac{\frac{8\pi^2 nm}{a^2}}{\{1 + (\frac{2\pi n}{a})^2\}^2 \{1 + (\frac{2\pi m}{a})^2\}} \\ (6.17) \qquad &+ \frac{2}{a} \sum_{k=0}^{\infty} \frac{\frac{\pi^2 nm}{a^2} \exp(-2a(2k + \frac{1}{2}))}{\{(2k + \frac{1}{2})^2 + (\frac{\pi n}{a})^2\}\{(2k + \frac{1}{2})^2 + (\frac{\pi m}{a})^2\}} \\ &+ \frac{1}{2\pi(n - m)}(y_n - y_m) - \frac{1}{2\pi(n + m)}(y_n + y_m), \qquad n \neq m, \end{aligned}$$

for $a = \log 2/2$, where

$$y_n = \mathrm{Im}(\psi(\frac{1}{4} + \frac{\pi i n}{2a})), \qquad y_m = \mathrm{Im}(\psi(\frac{1}{4} + \frac{\pi i m}{2a})).$$

We have

$$\frac{4}{a}(e^{a/2}-e^{-a/2})^2\frac{\frac{8\pi^2 nm}{a^2}}{\{1+(\frac{2\pi n}{a})^2\}^2\}\{1+(\frac{2\pi m}{a})^2\}}\le\frac{2a}{\pi^2}(e^{a/2}-e^{-a/2})^2\frac{1}{nm},$$

$$\begin{aligned}&\frac{2}{a}\sum_{k=0}^{\infty}\frac{\frac{\pi^2 nm}{a^2}\exp(-2a(2k+\frac{1}{2}))}{\{(2k+\frac{1}{2})^2+(\frac{\pi n}{a})^2\}\{(2k+\frac{1}{2})^2+(\frac{\pi m}{a})^2\}}\\&\le\sum_{k=0}^{\infty}\frac{2a}{\pi^2 nm}\exp(-2a(2k+\frac{1}{2}))=\frac{2a}{\pi^2}\frac{\exp(-a)}{1-\exp(-4a)}\frac{1}{nm},\end{aligned}$$

$$\frac{1}{2\pi(n-m)}(y_n-y_m)-\frac{1}{2\pi(n+m)}(y_n+y_m)=\frac{1}{\pi(n^2-m^2)}(my_n-ny_m).$$

By (5.11), we easily get

$$|\mathrm{Im}(\psi(\frac{1}{4}+it))-\frac{\pi}{2}-\frac{1}{4t}|\le\frac{1}{10t^2}\qquad\text{for}\quad t\ge 3.$$

Assume $m>n$. Then we obtain

$$\begin{aligned}&-\frac{1}{m+n}\frac{\pi}{2}+\frac{a}{2\pi}\frac{1}{mn}+\frac{1}{10}(\frac{2a}{\pi})^2\frac{m^2-mn+n^2}{(m-n)m^2n^2}\\&\ge\frac{1}{n^2-m^2}(my_n-ny_m)\\&\ge-\frac{1}{m+n}\frac{\pi}{2}+\frac{a}{2\pi}\frac{1}{mn}-\frac{1}{10}(\frac{2a}{\pi})^2\frac{m^2-mn+n^2}{(m-n)m^2n^2}.\end{aligned}$$

Then we see easily that

$$|\frac{1}{2\pi(n-m)}(y_n-y_m)-\frac{1}{2\pi(n+m)}(y_n+y_m)|\le(\frac{1}{2}+\frac{a}{2\pi^2})\frac{1}{m}.$$

Thus we obtain

(6.18) $$|\langle\omega_n,\omega_m\rangle|\le C_1 m^{-1}\qquad\text{if}\quad n<m,$$

(6.19) $$C_1=\frac{2a}{\pi^2}(e^{a/2}-e^{-a/2})^2+\frac{2a}{\pi^2}\frac{\exp(-a)}{1-\exp(-4a)}+(\frac{1}{2}+\frac{a}{2\pi^2}).$$

We have $C_1^2=0.3508\cdots$. Hence we get

$$\sum_{k=1}^{\infty}|\langle\omega_i,\omega_{k+10}\rangle|^2\le C_1^2\sum_{m=11}^{\infty}m^{-2}\le C_1^2/10=0.03508\cdots$$

for $1 \le i \le 10$. This proves (6.16) and we complete the proof of (6.1) for odd φ.

Now we shall prove (6.1) for even φ. We can argue similarly as for the odd case, but the actual computation becomes more cumbersome. Set

$$\omega_0(x) = \begin{cases} \dfrac{1}{\sqrt{2a}}, & |x| \le a, \\ 0 \ , & |x| > a, \end{cases} \qquad \omega_n(x) = \begin{cases} \dfrac{1}{\sqrt{a}} \cos(\dfrac{\pi i n x}{a}), & |x| \le a, \\ 0 \ , & |x| > a, \end{cases}$$

for $n \ge 1$. Then we have

$$\omega_0 = \chi_0, \quad \omega_n = \frac{1}{\sqrt{2}}(\chi_n + \chi_{-n}), \ n \ge 1, \quad \|\omega_n\|_{L^2} = 1, \ n \ge 0.$$

For $0 \le i \le 199$, there exists a $v_i \in \widehat{K_{10,\text{even}}(a)}$ such that

$$\langle v, \omega_i \rangle = \langle v, v_i \rangle \qquad \text{for every} \quad v \in \widehat{K_{10,\text{even}}(a)}.$$

Set $\omega_i' = \omega_i - v_i$. Then it suffices to prove the positive definiteness of the 200×200 hermitian matrix $(\langle \omega_i', \omega_j' \rangle; 0 \le i,j \le 199)$. We have

$$\text{(6.20)} \qquad \langle \omega_i', \omega_j' \rangle = \langle \omega_i, \omega_j \rangle - \langle v_i, v_j \rangle, \qquad 0 \le i,j \le 199.$$

For $n \ge 1$, set $\eta_n = \omega_{n+199}$. By (6.9), we get

$$\text{(6.21)} \qquad \|v_i\|^2 \le (\sum_{k=1}^{\infty} |\langle \eta_k, \omega_i \rangle|^2)/4.08.$$

as in the odd case. Suppose for a moment that

$$\text{(6.22)} \qquad \|v_i\|^2 \le 1/2000, \qquad 0 \le i \le 199.$$

is proved. Then, by (6.20), we have

$$\text{(6.23)} \qquad |\langle \omega_i', \omega_j' \rangle - \langle \omega_i, \omega_j \rangle| \le 1/2000, \qquad 0 \le i,j \le 199.$$

Set $u_{ij} = \langle \omega_{i-1}', \omega_{j-1}' \rangle$, $1 \le i,j \le 200$, $U = (u_{ij})$. The first step of the reduction of U to a diagonal form is done by adding $-u_{i1}u_{1j}/u_{11}$ to u_{ij}, $2 \le i \le 200$, $1 \le j \le 200$. Repeated applications of this procedure succeed provided the diagonal entries are kept non-zero at every step. Using (6.23) and numerical values of $\langle \omega_i, \omega_j \rangle$, we can explicitly determine the lower and upper bounds which the matrix entries may take at every step. For our purpose, it suffices to observe that the lower bounds of every diagonal entries are positive at the final step. This fact can be verified rather easily on a computer.

Thus our task is to prove (6.22). By (6.21), it suffices to show

$$\sum_{k=1}^{\infty} |\langle \omega_i, \omega_{k+199}\rangle|^2 \le \frac{4.08}{2000} = 0.00204 \quad \text{for} \quad 0 \le i \le 199. \tag{6.24}$$

By (5.16), we have

$$\begin{aligned}
&(-1)^{n+m}\langle \omega_n, \omega_m\rangle \\
&= \frac{4}{a}(e^{a/2} - e^{-a/2})^2 \frac{2}{\{1+(\frac{2\pi n}{a})^2\}\{1+(\frac{2\pi m}{a})^2\}} \\
&- \frac{2}{a}\sum_{k=0}^{\infty} \frac{(2k+\frac{1}{2})^2 \exp(-2a(2k+\frac{1}{2}))}{\{(2k+\frac{1}{2})^2+(\frac{\pi n}{a})^2\}\{(2k+\frac{1}{2})^2+(\frac{\pi m}{a})^2\}} \\
&+ \frac{1}{2\pi(n-m)}(y_n - y_m) + \frac{1}{2\pi(n+m)}(y_n + y_m), \quad n \ne m,
\end{aligned} \tag{6.25}$$

for $a = \log 2/2$, where

$$y_n = \mathrm{Im}(\psi(\frac{1}{4} + \frac{\pi i n}{2a})), \qquad y_m = \mathrm{Im}(\psi(\frac{1}{4} + \frac{\pi i m}{2a})).$$

(This formula has to be multiplied by $1/\sqrt{2}$ if $nm = 0$.) By (6.25), similar calculations as in the odd case yield

$$|\langle \omega_n, \omega_m\rangle| \le C_2 m^{-1} \qquad \text{if} \quad 0 \le n < m, \tag{6.26}$$

$$C_2 = \frac{2a}{\pi^2}(e^{a/2} - e^{-a/2})^2 + \frac{2a}{\pi^2}\frac{\exp(-a)}{1-\exp(-4a)} + (\frac{1}{2} + \frac{(2a)^2}{10\pi^3}). \tag{6.27}$$

We have $C_2^2 = 0.3321\cdots$. Hence we get

$$\sum_{k=1}^{\infty} |\langle \omega_i, \omega_{k+199}\rangle|^2 = \sum_{k=200}^{\infty} |\langle \omega_i, \omega_k\rangle|^2 \le C_2^2/199 = 0.001668\cdots$$

for $0 \le i \le 199$. This proves (6.24) and we have proved (6.1) also for the even space. Summing up, we obtain

Theorem 1. *Let $a = \log\ 2/2$. We have*

$$\langle \varphi, \varphi\rangle = T_{\mathbf{Q}}(\varphi * \tilde{\varphi}) \ge 0 \qquad \textit{for every} \quad \varphi \in K(a),$$

where equality holds if and only if $\varphi = 0$.

§7. Continuity

For a function α on $\mathbf{R}$ and $t > 0$, we define a function α_t on $\mathbf{R}$ by

$$\alpha_t(x) = t^{-1}\alpha(t^{-1}x), \qquad x \in \mathbf{R}.$$

Then we have

$$(7.1) \qquad (\tilde{\alpha})_t = \widetilde{(\alpha_t)}, \qquad (\alpha_t)_u = \alpha_{tu}, \quad u > 0.$$

If α and β are integrable functions with compact support, we have

$$(7.2) \qquad \alpha_t * \beta_t = (\alpha * \beta)_t.$$

Let $a > 0$. By Lemma 3, we can find a positive integer N and $\mu > 0$ so that

$$(7.3) \qquad \langle \varphi, \varphi \rangle \geq \mu \|\varphi\|_{L^2}^2 \qquad \text{for every} \quad \varphi \in K_N(b),\ 0 < b \leq 2a.$$

Let $K(a) = W \oplus K_N(a)$ with $W \subset K(a)$ and let $\alpha_1, \alpha_2, \cdots, \alpha_{2N+1}$ be a basis of W. For $t > 0$, set $W_t = \langle (\alpha_1)_t, (\alpha_2)_t, \cdots, (\alpha_{2N+1})_t \rangle_{\mathbf{C}}$. Then we have $K(ta) = W_t \oplus K_N(ta)$. Assume $t \leq 2$. Then, by (7.3), we can consider the space $\widehat{K(ta)} = W_t \oplus \widehat{K_N(ta)}$. Let $W(ta)$ be the orthogonal complement of $\widehat{K_N(ta)}$ in $\widehat{K(ta)}$ whose existence is guaranteed by Proposition 3. For $1 \leq i \leq 2N+1$, we can find $v_i(t) \in \widehat{K_N(ta)}$ so that

$$(7.4) \qquad \langle v, (\alpha_i)_t \rangle = \langle v, v_i(t) \rangle \qquad \text{for every} \qquad v \in \widehat{K_N(ta)}.$$

Then $(\alpha_1)_t - v_1(t)$, $(\alpha_2)_t - v_2(t)$, $\cdots$, $(\alpha_{2N+1})_t - v_{2N+1}(t)$ make a basis of $W(ta)$. The purpose of this section is to prove the following result which shows the continuity of the hermitian form on $W(ta)$ induced by $\langle\ ,\ \rangle$ with respect to t.

Proposition 5. *Let the notation and the assumption be the same as above. Then the matrix coefficients $\langle (\alpha_i)_t - v_i(t), (\alpha_j)_t - v_j(t) \rangle$ are continuous functions of t for $0 < t \leq 2$, $1 \leq i, j \leq 2N+1$.*

It suffices to prove the continuity at $t = 1$. By (7.4), we obtain

$$\langle (\alpha_i)_t - v_i(t), (\alpha_j)_t - v_j(t) \rangle = \langle (\alpha_i)_t, (\alpha_j)_t \rangle - \langle v_i(t), v_j(t) \rangle.$$

By (7.2), we have $\langle (\alpha_i)_t, (\alpha_j)_t \rangle = T_k((\alpha_i * \tilde{\alpha}_j)_t)$. The continuity of this inner product follows from the next Lemma.

Lemma 7. *Let α be an admissible function. Then $T_k(\alpha_t)$ is a continuous function of t.*

Proof. It suffices to prove the continuity at $t=1$. Take $b>0$ so that $\mathrm{supp}(\alpha_t) \subseteq [-2b, 2b]$. Take $\eta > 0$ so that $\alpha'(x)$ is continuous for $|x| \le 2\eta$ except at $x=0$. By Lemma 4, it is enough to prove

$$\lim_{t\to 1} \|\alpha - \alpha_t\|_{L^\infty} = 0, \qquad \lim_{t\to 1} \|\alpha' - (\alpha)_t'\|_{L^\infty}^{\eta} = 0. \tag{7.5}$$

We have

$$\begin{aligned} &\|\alpha' - (\alpha)_t'\|_{L^\infty}^{\eta} = \sup_{0<|x|\le\eta} |\alpha'(x) - t^{-2}\alpha'(t^{-1}x)| \\ &\le \sup_{0<|x|\le\eta} |\alpha'(x) - \alpha'(t^{-1}x)| + |1-t^{-2}| \sup_{0<|x|\le\eta} |\alpha'(t^{-1}x)|. \end{aligned}$$

The second term obviously converges to 0 for $t \to 1$. By setting $\alpha'(0) = \lim_{\epsilon\to+0}\alpha'(\epsilon)$, α' is uniformly continuous on $[0, 2\eta]$. Hence we obtain $\lim_{t\to1}\sup_{0<x\le\eta}|\alpha'(x) - \alpha'(t^{-1}x)| = 0$. Similarly we get $\lim_{t\to1}$ $\sup_{-\eta\le x<0} |\alpha'(x) - \alpha'(t^{-1}x)| = 0$. This proves the latter part of (7.5). The first part of (7.5) can be proved similarly. This completes the proof.

By Lemma 7, Proposition 5 reduces to the continuity of $\langle v_i(t), v_j(t)\rangle$ at $t=1$. For $n \in \mathbf{Z}$, define $\chi_n(t) \in K(ta)$ by

$$\chi_n(t)(x) = \begin{cases} \dfrac{1}{\sqrt{2ta}} \exp(\dfrac{\pi i n x}{ta}) & \text{if} \quad |x| \le ta, \\ 0 & \text{if} \quad |x| > ta. \end{cases}$$

We put

$$\begin{aligned} \eta_1(t) = \chi_{N+1}(t), \eta_2(t) = \chi_{-(N+1)}(t), \cdots \\ \eta_{2n-1}(t) = \chi_{N+n}(t), \eta_{2n}(t) = \chi_{-(N+n)}(t), \cdots \end{aligned}$$

Let $\{\psi_k(t)\}$ be the orthonormal basis of $\widehat{K_N(ta)}$ obtained from $\{\eta_k(t)\}$ by the Schmidt orthogonalization process. We have

$$\begin{aligned} &\psi_1(t) = \eta_1(t)/\|\eta_1(t)\|, \cdots \\ &\psi_k(t) = \frac{\eta_k(t) - \sum_{n=1}^{k-1}\langle \eta_k(t), \psi_n(t)\rangle\psi_n(t)}{\|\eta_k(t) - \sum_{n=1}^{k-1}\langle \eta_k(t), \psi_n(t)\rangle\psi_n(t)\|}, \cdots \end{aligned} \tag{7.6}$$

By (7.4), we have

$$v_i(t) = \sum_{k=1}^{\infty} f_{ik}(t)\psi_k(t), \quad f_{ik}(t) = \langle v_i(t), \psi_k(t)\rangle = \langle (\alpha_i)_t, \psi_k(t)\rangle.$$

Lemma 8. *For $t < 2$, $f_{ik}(t)$ is a continuous function of t.*

Proof. Put $\alpha = \alpha_i$. We have $\chi_n(t) = (\chi_n(1))_t \times \sqrt{t}$. Hence we get

$$\langle \alpha_t, \chi_n(t)\rangle = \sqrt{t}\, T_k((\alpha * \widetilde{\chi_n(1)})_t).$$

By Lemma 7, we see that $\langle \alpha_t, \chi_n(t)\rangle$ is a continuous function for every $n \in \mathbf{Z}$. In particular, $\langle \alpha_t, \eta_k(t)\rangle$ is continuous for every $k \in \mathbf{N}$. By (7.6), we have $\psi_k(t) = \sum_{l=1}^{k} d_{kl}(t)\eta_l(t)$ with $d_{kl}(t) \in \mathbf{C}$. It suffices to show the continuity of $d_{kl}(t)$. By Lemma 7 or by (5.15), (5.16), we see that $\langle \eta_l(t), \eta_m(t)\rangle$ is continuous for every l, $m \in \mathbf{N}$. Now the continuity of $d_{kl}(t)$ can be shown by induction on k.

We have

$$\langle v_i(t), v_j(t)\rangle = \sum_{k=1}^{\infty} f_{ik}(t)\overline{f_{jk}(t)}$$

$$\sum_{k=M}^{\infty} |f_{ik}(t)\overline{f_{jk}(t)}| \le (\sum_{k=M}^{\infty} |f_{ik}(t)|^2)^{1/2} (\sum_{k=M}^{\infty} |f_{jk}(t)|^2)^{1/2}.$$

Therefore Proposition 5 reduces to the uniformity of convergence in a neighbourhood of t of $\sum_{k=1}^{\infty} |f_{ik}(t)|^2$ for every i, $1 \le i \le 2N+1$. This fact is by no means trivial but follows from the next Lemma.

Lemma 9. *Let H be an infinite dimensional vector space over $\mathbf{C}$. Let $(\ ,\ )_1$ and $(\ ,\ )_2$ be two positive definite hermitian forms on H. We set*

$$\|v\|_i = \sqrt{(v,v)_i}, \qquad v \in H,\ i = 1, 2.$$

We assume that H is a separable Hilbert space with respect to $\|\ \|_1$ and that H is embedded in the completion H^* of H with respect to $\|\ \|_2$. Let $\{\psi_n\}$ be an orthonormal basis of H with respect to $\|\ \|_1$ and let $\{\eta_n\}$, $\eta_n \in H$ be an orthonormal basis of H^*. We assume that $\{\psi_n\}$ is obtained from $\{\eta_n\}$ by the Schmidt orthogonalization process. Let T be a linear functional on H. We assume the following:

(I) *There exists $\kappa_1 > 0$ such that*

$$\|v\|_1 \ge \kappa_1 \|v\|_2 \qquad \text{for every} \quad v \in H.$$

(I$'$) *There exists a sequence of positive numbers $\mu(M)$, $M \in \mathbf{N}$ such that $\lim_{M\to\infty} \mu(M) = +\infty$ and that*

$$\|v\|_1 \ge \mu(M)\|v\|_2 \quad \text{if} \quad v \in H \text{ satisfies } (v, \eta_i)_2 = 0 \quad \text{for every } i \le M.$$

(II) *There exists* $\kappa_2 > 0$ *such that* $|(\eta_j, \eta_l)_1| \leq \kappa_2/|j-l|$ *if* $j \neq l$.
(III) *There exists* $\kappa_3 > 0$ *such that* $|T(\eta_n)| \leq \kappa_3 n^{-1}$, $n \geq 1$.
Then we have

$$\Big(\sum_{i>M} |T(\psi_i)|^2\Big)^{1/2} \leq \kappa_3(1+\kappa_1^{-2}\nu)(\mu(M)-\kappa_1^{-1}\nu)^{-1}\sqrt{\pi^2/6}$$

for $\mu(M) > \kappa_1^{-1}\nu$, *where* $\nu = \kappa_2^2(\frac{7}{6}\pi^2 + 5\pi)$.

Let us prove Proposition 5 taking Lemma 9 for granted. We set

$$H = \widehat{K_N(ta)}, \qquad (\ ,\ )_1 = \langle\ ,\ \rangle,$$
$$(\alpha,\beta)_2 = \int_{-ta}^{ta} \alpha(x)\overline{\beta(x)}\,dx, \quad \alpha,\beta \in \widehat{K_N(ta)} \subset L^2([-ta,ta]).$$

Take $\eta_k = \eta_k(t)$, $\psi_k = \psi_k(t)$, $k \in \mathbf{N}$. Now the assumption (I) is included in the assumption of Proposition 5 (cf. (7.3)) and (I′) follows from Lemma 3; (II) follows from (5.17). We take $(\alpha_i)_t \in K(ta)$ as before and set

$$T(v) = \langle v, (\alpha_i)_t\rangle, \qquad v \in \widehat{K_N(ta)}.$$

Let $\alpha_i = \sum_{k\in\mathbf{Z}} c_k\chi_k(1)$ be the Fourier expansion of α_i. Then we have

$$(\alpha_i)_t = \sqrt{t}^{-1}\textstyle\sum_{k\in\mathbf{Z}} c_k\chi_k(t),$$
$$\langle\chi_n(t), (\alpha_i)_t\rangle = \sqrt{t}^{-1}\textstyle\sum_{k\in\mathbf{Z}} c_k\langle\chi_n(t),\chi_k(t)\rangle.$$

By (3.1) and (5.17), we easily obtain

$$\sum_{|k-n|\leq|n|/2} |c_k\langle\chi_n(t),\chi_k(t)\rangle| = O(|n|^{-1}), \qquad \text{for} \quad |n|\to\infty,$$
$$\sum_{|k-n|>|n|/2} |c_k\langle\chi_n(t),\chi_k(t)\rangle| \leq \Big(\sum_{k\in\mathbf{Z}}|c_k|\Big)2\kappa|n|^{-1}.$$

Hence we obtain (III). We note that the constants κ_1, κ_2, κ_3 and $\mu(M)$ can be taken independently of t when $1/2 \leq t \leq 2$. Then the conclusion of Lemma 9 implies the uniform convergence of $\sum_{k=1}^{\infty}|f_{ik}(t)|^2$.

Proof of Lemma 9. Since $\{\psi_k\}$ is obtained from $\{\eta_k\}$ by the Schmidt orthogonalization process, we can write

$$\psi_i = \sum_{j=1}^{i} d_{ij}\eta_j, \qquad \eta_i = \sum_{j=1}^{i} c_{ij}\psi_j. \tag{7.7}$$

Set $C = (c_{ij})$, $D = (d_{ij})$. Then C and D are infinite dimensional lower triangular matrices (cf. (7.6)). For a positive integer M, we set

$$C = \begin{pmatrix} X_1 & 0 \\ X_3 & X_4 \end{pmatrix}, \qquad D = \begin{pmatrix} Y_1 & 0 \\ Y_3 & Y_4 \end{pmatrix},$$

where X_1 and Y_1 denote the first $M \times M$-blocks. Similarly dividing into blocks, we set

$$((\eta_i, \eta_j)_1) = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}.$$

Since

$$(\eta_i, \eta_j)_1 = (\sum_{k=1}^{i} c_{ik}\psi_k, \sum_{l=1}^{j} c_{jl}\psi_l) = \sum_{k=1}^{\min(i,j)} c_{ik}\bar{c}_{jk},$$

we have

$$C^t\bar{C} = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}. \tag{7.8}$$

Note that $C^t\bar{C}$ can be defined since C is lower triangular. For an infinite dimensional matrix A which maps ℓ^2 into ℓ^2, let $\|A\|$ denote the operator norm of A: $\|A\| = \sup_{\|x\|_{\ell^2}=1} \|xA\|_{\ell^2}$. We shall show

$$\|Z_3\| \le \nu, \tag{7.9}$$

where ν is the constant given in Lemma 9. Since Z_3 is a $\infty \times M$-matrix, ${Z_3}^t\bar{Z}_3$ is meaningful. Set $Z_3 = (u_{ij})$, ${Z_3}^t\bar{Z}_3 = (v_{ij})$. By the assumption (II), we have

$$|u_{ij}| \le \kappa_2/|M+i-j|, \qquad 1 \le i < \infty, 1 \le j \le M. \tag{7.10}$$

Hence xZ_3 is meaningful if $x \in \ell^2$. To prove (7.9), it is enough to show

$$(xZ_3)^t\overline{(xZ_3)} \le \nu x^t\bar{x} \qquad \text{for every} \quad x \in \ell^2. \tag{7.11}$$

From (7.10), we have

$$|v_{ik}| \le \kappa_2^2 \sum_{j=1}^{M} \frac{1}{M+i-j}\frac{1}{M+k-j} \tag{7.12}$$

In particular, we have

$$|v_{ii}| \le \kappa_2^2 \sum_{j=i}^{\infty} \frac{1}{j^2} \le \frac{\pi^2}{6}\kappa_2^2,$$

$$(7.13)\qquad |v_{ik}| \leq \kappa_2^2 \frac{1}{k-i}\Big(\sum_{j=i}^{M+i-1}\frac{1}{j} - \sum_{j=k}^{M+k-1}\frac{1}{j}\Big) \quad \text{if } i \neq k.$$

Set

$$v'_{ij} = \begin{cases} 0 & \text{if } i = j, \\ v_{ij} & \text{if } i \neq j, \end{cases}$$

and $V = (v'_{ij})$. Then, for $x = (x_1, x_2, \cdots) \in \ell^2$, we have

$$(xZ_3)^t\overline{(xZ_3)} = xV^t\bar{x} + \sum_{i=1}^{\infty} v_{ii}|x_i|^2 \leq |xV^t\bar{x}| + \frac{\pi^2}{6}\kappa_2^2\|x\|_{\ell^2}^2.$$

(This computation is justified if $\sum_{i,j}(\sum_k u_{ik}\bar{u}_{jk})x_i\bar{x}_j$ is absolutely convergent which shall be shown below through the proof of (7.14).) Thus (7.11) reduces to

$$(7.14)\qquad |xV^t\bar{x}| \leq \kappa_2^2(\pi^2 + 5\pi)\|x\|_{\ell^2}^2 \qquad \text{for every} \quad x \in \ell^2.$$

We have

$$(7.15)\qquad \Big|\sum_{j=i}^{l}\frac{1}{j} - \log\frac{l}{i}\Big| \leq \frac{1}{i} \qquad \text{for} \quad i, l \in \mathbf{Z},\ l \geq i \geq 1.$$

Hence we get

$$\begin{aligned} &\frac{1}{k-i}\Big(\sum_{j=i}^{M+i-1}\frac{1}{j} - \sum_{j=k}^{M+k-1}\frac{1}{j}\Big) \\ &\leq \frac{1}{k-i}\Big(\log\frac{M+i-1}{i} - \log\frac{M+k-1}{k}\Big) + \frac{1}{|k-i|}\Big(\frac{1}{i} + \frac{1}{k}\Big). \end{aligned}$$

We have

$$0 < \frac{1}{k-i}\Big(\log\frac{M+i-1}{i} - \log\frac{M+k-1}{k}\Big) < \frac{1}{k-i}\log\frac{k}{i},$$

$$\frac{1}{|k-i|}\Big(\frac{1}{i} + \frac{1}{k}\Big) \leq \frac{5}{i+k}.$$

Hence, by (7.13), we obtain

$$(7.16)\qquad |v_{ik}| \leq \kappa_2^2\Big(\frac{1}{k-i}\log\frac{k}{i} + \frac{5}{i+k}\Big), \quad i \neq k.$$

Set

$$v''_{ik} = \begin{cases} 0 & \text{if } i = k, \\ \frac{1}{k-i} \log \frac{k}{i}, & \text{if } i \neq k, \end{cases} \qquad v'''_{ik} = \begin{cases} 0 & \text{if } i = k, \\ \frac{5}{i+k}, & \text{if } i \neq k, \end{cases}$$

$V_1 = (v''_{ik})$, $V_2 = (v'''_{ik})$. Then (7.14) follows from

$$\text{(7.17)} \qquad |xV_1{}^t\bar{x}| \leq \pi^2 \|x\|^2_{\ell^2} \qquad \text{for} \quad x \in \ell^2,$$

$$\text{(7.18)} \qquad |xV_2{}^t\bar{x}| \leq 5\pi \|x\|^2_{\ell^2} \qquad \text{for} \quad x \in \ell^2.$$

(7.17) (resp. (7.18)) is given in Hardy-Littlewood-Pólya [3], p. 255(resp. p. 226). We have proved (7.9).

Now let

$$v = \sum_{i=1}^{M} x_i \eta_i \in H, \qquad x = (x_1, x_2, \cdots, x_M).$$

Then we have

$$\|v\|_1^2 = \sum_{j=1}^{M} \Big| \sum_{i=1}^{M} x_i c_{ij} \Big|^2 = \|xX_1\|^2_{\ell^2}, \quad \|v\|_2^2 = \sum_{i=1}^{M} |x_i|^2.$$

Hence, by (I), we get $\|xX_1\|_{\ell^2} \geq \kappa_1 \|x\|_{\ell^2}$. Inserting xX_1^{-1} into x, we get $\|x\|_{\ell^2} \geq \kappa_1 \|xX_1^{-1}\|_{\ell^2}$. Thus we obtain

$$\text{(7.19)} \qquad \|X_1^{-1}\| \leq \kappa_1^{-1}.$$

From (7.8), we get $X_3 = Z_3{}^t\bar{X}_1^{-1}$. Hence we have

$$\text{(7.20)} \qquad \|X_3\| \leq \kappa_1^{-1} \nu$$

by (7.9) and (7.20). Let

$$v = \sum_{i>M} x_i \eta_i \in H \subset H^*, \qquad x = (x_{M+1}, x_{M+2}, \cdots) \in \ell^2.$$

Then we have

$$v = \sum_{i>M} \sum_{j=1}^{i} x_i c_{ij} \psi_j, \quad \|v\|_1 = \|x(X_3 X_4)\|_{\ell^2}, \qquad \|v\|_2 = \|x\|_{\ell^2}.$$

From (I′), we obtain

$$\|x(X_3X_4)\|_{\ell^2} \geq \mu(M)\|x\|_{\ell^2}. \tag{7.21}$$

We note that (7.21) holds for every $x \in \ell^2$ which satisfies $x(X_3X_4) \in \ell^2$. By (7.20), we easily get

$$\|xX_4\|_{\ell^2} \geq (\mu(M) - \kappa_1^{-1}\nu)\|x\|_{\ell^2} \qquad \text{if} \quad x,\ xX_4 \in \ell^2.$$

Inserting x in xX_4, we get

$$\|x\|_{\ell^2} \geq (\mu(M) - \kappa_1^{-1}\nu)\|xX_4^{-1}\|_{\ell^2} \qquad \text{if} \quad x,\ xX_4^{-1} \in \ell^2.$$

Assume M is sufficiently large so that $\mu(M) - \kappa_1^{-1}\nu > 0$. Then we have

$$\|xX_4^{-1}\|_{\ell^2} \leq (\mu(M) - \kappa_1^{-1}\nu)^{-1}\|x\|_{\ell^2} \qquad \text{if} \quad x,\ xX_4^{-1} \in \ell^2. \tag{7.22}$$

Let $x = (y_1, y_2, \cdots, y_n, \cdots) \in \ell^2$ and set $x_i = (y_1, y_2, \cdots, y_i, 0, \cdots, 0, \cdots)$. Since X_4^{-1} is lower triangular, we have $x_iX_4^{-1} \in \ell^2$ and $\{x_iX_4^{-1}\}$ is a Cauchy sequence in ℓ^2 by (7.22). Therefore $x_iX_4^{-1}$ converges to some $z \in \ell^2$. We see easily that $\|z\|_{\ell^2} \leq (\mu(M) - \kappa_1^{-1}\nu)^{-1}\|x\|_{\ell^2}$. Thus we have $xX_4^{-1} \in \ell^2$ if $x \in \ell^2$ and obtain

$$\|X_4^{-1}\| \leq (\mu(M) - \kappa_1^{-1}\nu)^{-1}. \tag{7.23}$$

Now we are going to estimate $\sum_{i>M} |T(\psi_i)|^2$. Since $\sum_{i>M} |T(\psi_i)|^2 = \sum_{i>M} |\sum_{j=1}^{i} d_{ij}T(\eta_j)|^2$, we get

$$\sum_{i>M} |T(\psi_i)|^2 = \|\xi \begin{pmatrix} {}^tY_3 \\ {}^tY_4 \end{pmatrix} \|_{\ell^2}^2, \tag{7.24}$$

where $\xi = (T(\eta_1), T(\eta_2), \cdots, T(\eta_j), \cdots)$. By (III), we have

$$\|\xi\|_{\ell^2} \leq \kappa_3\sqrt{\pi^2/6}. \tag{7.25}$$

From $CD = 1$, we get $Y_4 = X_4^{-1}$, $Y_3 = -X_4^{-1}X_3X_1^{-1}$. The products of these matrices are meaningful since C and D are lower triangular. We obtain

$$\sum_{i>M} |T(\psi_i)|^2 = \|\xi \begin{pmatrix} -{}^tX_1^{-1}{}^tX_3 \\ 1 \end{pmatrix} {}^tX_4^{-1}\|_{\ell^2}^2. \tag{7.26}$$

From (7.19) and (7.20), we have

$$\|{}^tX_1^{-1}{}^tX_3\| \leq \kappa_1^{-2}\nu, \qquad \| \begin{pmatrix} -{}^tX_1^{-1}{}^tX_3 \\ 1 \end{pmatrix} \| \leq (1 + \kappa_1^{-2}\nu).$$

From (7.23), we have $\|{}^tX_4^{-1}\| \leq (\mu(M) - \kappa_1^{-1}\nu)^{-1}$, since the norm does not change when passing to the dual operator. Therefore we obtain

$$\|\xi \begin{pmatrix} -{}^tX_1^{-1}{}^tX_3 \\ 1 \end{pmatrix} {}^tX_4^{-1}\|_{\ell^2} \leq (1 + \kappa_1^{-2}\nu)(\mu(M) - \kappa_1^{-1}\nu)^{-1}\|\xi\|_{\ell^2}.$$

Taking account of (7.25) and (7.26), we complete the proof of Lemma 9.

§8. Reduction to non-degeneracy

Lemma 10. *Let α be an admissible function. For $\epsilon > 0$, let ρ_ϵ be a mollifier, that is $\rho_\epsilon \in C_c^\infty(\mathbf{R})$, $\mathrm{supp}(\rho_\epsilon) \subseteq [-\epsilon, \epsilon]$, $\rho_\epsilon(x) \geq 0$ for every $x \in \mathbf{R}$, $\int_{-\infty}^{\infty} \rho_\epsilon(x)\,dx = 1$. Then we have $\lim_{\epsilon \to +0} T_k(\alpha * \rho_\epsilon) = T_k(\alpha)$.*

Proof. Let $\mathrm{supp}(\alpha) \subseteq [-a, a]$ and take $-a = a_0 < a_1 < \cdots < a_n = a$ so that $\alpha'(x)$ is continuous except for $x = a_i$, $0 \leq i \leq n$. We have

$$\begin{aligned}\frac{d}{dx}(\alpha * \rho_\epsilon)(x) &= \int_{-a}^{a} \alpha(y)\rho_\epsilon'(x-y)\,dy = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} \alpha(y)\rho_\epsilon'(x-y)\,dy \\ &= (\alpha' * \rho_\epsilon)(x) + \alpha(-a)\rho_\epsilon(x+a) - \alpha(a)\rho_\epsilon(x-a),\end{aligned}$$

by partial integration. Take $0 < \eta < a/2$. Then we have

$$(8.1) \qquad \frac{d}{dx}(\alpha * \rho_\epsilon)(x) = (\alpha' * \rho_\epsilon)(x) \qquad \text{if} \quad \epsilon \leq \eta,\ |x| \leq \eta.$$

Since

$$\{(\alpha' * \rho_\epsilon) - \alpha'\}(x) = \int_{-\infty}^{\infty} (\alpha'(y) - \alpha'(x))\rho_\epsilon(x-y)\,dy,$$

there exists $A > 0$ which depends only on α such that

$$(8.2) \qquad |\{(\alpha' * \rho_\epsilon) - \alpha'\}(x)| \leq A \qquad \text{for all} \quad \epsilon > 0,\ x \in \mathbf{R}.$$

By (8.1) and (8.2), we have

$$\|\frac{d}{dx}\{(\alpha * \rho_\epsilon) - \alpha\}\|_{L^\infty}^{\eta} \leq A \qquad \text{if} \quad \epsilon \leq \eta.$$

By Lemma 4, we obtain

$$|T_k((\alpha * \rho_\epsilon) - \alpha)| \leq c\|(\alpha * \rho_\epsilon) - \alpha\|_{L^\infty} + 2(r_1 + 2r_2)A\eta \quad \text{if} \quad \epsilon \leq \eta.$$

It is easy to see that the first term converges to 0 for $\epsilon \longrightarrow +0$. Hence we have

$$\limsup_{\epsilon \longrightarrow +0} |T_k((\alpha * \rho_\epsilon) - \alpha)| \leq 2(r_1 + 2r_2)A\eta \quad .$$

Since $\eta < a/2$ is arbitrary, we obtain $\lim_{\epsilon \longrightarrow +0} T_k((\alpha * \rho_\epsilon) - \alpha) = 0$. This completes the proof.

Proposition 6. *Assume that the Riemann hypothesis does not hold for $\zeta_k(s)$. Then there exists $a_0 > 0$ which has the following properties.*

(1) *If $a \leq a_0$, $\langle\ ,\ \rangle | K(a)$ is positive semi-definite and $\langle\ ,\ \rangle | C(a)$ is positive definite.*
(2) *If $a > a_0$, both of $\langle\ ,\ \rangle | K(a)$ and $\langle\ ,\ \rangle | C(a)$ are not positive semi-definite.*

Proof. Set

$$I = \{\ a \in \mathbf{R}_+ \mid \langle\ ,\ \rangle | K(a) \text{ is positive semi-definite }\}, \quad J = \mathbf{R}_+ - I.$$

If $a \in J$, there exists $\alpha \in K(a)$ such that $\langle \alpha, \alpha \rangle < 0$. For $t > 0$, we have $\alpha_t \in K(ta)$,

$$\langle \alpha_t, \alpha_t \rangle = T_k((\alpha_t * \tilde{\alpha}_t)) = T_k((\alpha * \tilde{\alpha})_t).$$

Since $\alpha * \tilde{\alpha}$ is an admissible function, we obtain $\langle \alpha_t, \alpha_t \rangle < 0$ if t is sufficiently close to 1 by Lemma 7. Hence J is open and I is a closed subset of $\mathbf{R}_+$. Assume that the Riemann hypothesis does not hold for $\zeta_k(s)$. Then I is bounded. In fact, if I is not bounded, I contains an increasing sequence of numbers $\{a_i\}$ such that $\lim_{i \to \infty} a_i = +\infty$. From $C_c^\infty(\mathbf{R}) = \cup_{i=1}^\infty C(a_i)$ and $C(a_i) \subset K(a_i)$, we see that $\langle\ ,\ \rangle | C_c^\infty(\mathbf{R})$ is positive semi-definite, which is a contradiction. Let a_1 be the maximum of I. Set

$$I' = \{\ a \in \mathbf{R}_+ \mid \langle\ ,\ \rangle | C(a) \text{ is positive semi-definite }\}.$$

By the same argument as above, we see that there exists a maximum a_0 of I'. Since $C(u) \subseteq C(v)$ if $u \leq v$ and by Proposition 2, the assertions (1) and (2) for $C(a)$ is clear. From $C(a_1) \subseteq K(a_1)$, we have $a_1 \in I'$. Hence $a_1 \leq a_0$. Assume $a_1 < a_0$. Then we can find a and $\alpha \in K(a)$ so that $a_1 < a < a_0$, $\langle \alpha, \alpha \rangle < 0$. For $\epsilon > 0$, we have $\alpha * \rho_\epsilon \in C(a + \epsilon)$,

$$\langle \alpha * \rho_\epsilon, \alpha * \rho_\epsilon \rangle = T_k(\alpha * \tilde{\alpha} * \rho_\epsilon * \tilde{\rho}_\epsilon).$$

Since $\rho_\epsilon * \tilde{\rho}_\epsilon$ also satisfies the condition of a mollifier of Lemma 10 (with 2ϵ in the place of ϵ), we have $\langle \alpha * \rho_\epsilon, \alpha * \rho_\epsilon \rangle < 0$ when ϵ is sufficiently small.

This is a contradiction. Hence we obtain $a_0 = a_1$. Now the assertion (2) for $K(a)$ is obvious and (1) for $K(a)$ can be proved similarly using a mollifier. This completes the proof.

Theorem 2. *The Riemann hypothesis for $\zeta_k(s)$ holds if and only if the hermitian form $\langle\ ,\ \rangle$ on $\widehat{K(a)}$ is non-degenerate for every $a > 0$.*

Proof. First we assume that the Riemann hypothesis for $\zeta_k(s)$ does not hold. Take $a_0 > 0$ as in Proposition 6. Choose any $a_1 > a_0$, $\mu > 0$ and N so that

$$\langle \varphi, \varphi \rangle \geq \mu \|\varphi\|_{L^2}^2 \qquad \text{for every} \quad \varphi \in K_N(a),\ 0 < a \leq a_1.$$

As in §4, we can decompose

$$\widehat{K(a)} = W(a) \oplus \widehat{K_N(a)}, \qquad a \leq a_1$$

as the orthogonal direct sum. Obviously $\langle\ ,\ \rangle$ on $\widehat{K(a)}$ is positive semi-definite if and only if $\langle\ ,\ \rangle|W(a)$ is positive semi-definite. By Proposition 6, we see easily that $\langle\ ,\ \rangle|W(a)$ is positive semi-definite for $a \leq a_0$ and is not positive semi-definite for $a_0 < a \leq a_1$. Proposition 5 states that $\langle\ ,\ \rangle|W(a)$ is represented by a hermitian matrix whose matrix coefficients are continuous functions of a when we choose a basis of $W(a)$ suitably. These facts immediately imply that $\langle\ ,\ \rangle|W(a)$ degenerates at $a = a_0$.

Conversely we assume the Riemann hypothesis for $\zeta_k(s)$. Then the hermitian form $\langle\ ,\ \rangle$ on $\widehat{K(a)}$ is positive semi-definite for every $a > 0$. Fix $a > 0$ and assume $\varphi \in \widehat{K(a)}$ satisfies $\langle \varphi, \varphi \rangle = 0$. It suffices to show $\varphi = 0$. For this purpose, let

$$\widehat{K(a)} = W \oplus \widehat{K_N(a)}, \qquad W \subset K(a)$$

as in §4 and let $\varphi = \alpha + \psi$, $\alpha \in W$, $\psi \in \widehat{K_N(a)}$. Let $\{\psi_n\}$ be a Cauchy sequence in $K_N(a)$ which represents ψ and put $\varphi_n = \alpha + \psi_n$. By Proposition 4, we may assume $\varphi \in L^2([-a,a])$ and $\varphi_n \in K(a)$ converges to φ in $L^2([-a,a])$. Set $\Phi = M(\varphi)$, $\Phi_n = M(\varphi_n)$. Then for any fixed $s \in \mathbf{C}$, $\Phi_n(s)$ converges to $\Phi(s)$. We have $\langle \varphi_n, \varphi_n \rangle = \sum_\rho |\Phi_n(\rho)|^2$ where ρ extends over all non-trivial zeros, and $\langle \varphi_n, \varphi_n \rangle$ converges to $\langle \varphi, \varphi \rangle$. Hence we immediately obtain $\Phi(\rho) = 0$ for every non-trivial zero ρ of $\zeta_k(s)$. On the other hand, we have

$$\begin{aligned} |\Phi(s)| &\leq \int_{-a}^{a} |\varphi(x)||e^{(s-1/2)x}|\, dx \\ &\leq \int_{-a}^{a} |\varphi(x)|\, dx\, e^{|\sigma-1/2|a} \leq \sqrt{2a}\|\varphi\|_{L^2}\, e^{|\sigma-1/2|a}, \quad \sigma = \mathrm{Re}(s). \end{aligned}$$

This shows that $\Phi(s)$ is an entire function of order ≤ 1, exponential type a (cf. Boas [1], p. 8). Let $n(r)$ be the number of zeros of $\Phi(s)$ in the disk $|s| \leq r$ counted with multiplicity. Then we have $n(r) = O(r)$ if $\Phi \neq 0$ (cf. [1], p. 16). Let $N(r)$ be the number of distinct zeros of $\zeta_k(s)$ in $|s| \leq r$. It is known that $N(r) \neq O(r)$ (cf. Siegel [6], Satz 2). This is a contradiction if $\Phi \neq 0$. Hence we have $\Phi = 0$ which implies $\varphi = 0$. This completes the proof.

We shall prove the following result supplementing Proposition 2.

Proposition 7. *If $k = \mathbf{Q}$, the hermitian form $\langle\ ,\ \rangle$ on $K(a)$ is non-degenerate for every $a > 0$.*

Proof. By Theorem 1, we may assume $a > \log 2/2$. Set

$$V_0 = \{\ \varphi \in K(a) \mid \langle \varphi, \psi \rangle = 0 \quad \text{for all} \quad \psi \in C(a)\ \}.$$

We shall show $V_0 = \{0\}$. Take $\check{\varphi} \in V_0$. We have $\langle \check{\varphi}, \psi \rangle = T_{\mathbf{Q}}(\check{\varphi} * \tilde{\psi}) = (T_{\mathbf{Q}} * \varphi)(\psi)$. Note that $T_{\mathbf{Q}} * \varphi$ is well defined as a distribution since φ is compactly supported. From the assumption, we have $\mathrm{supp}(T_{\mathbf{Q}} * \varphi) \subseteq \mathbf{R} - (-a, a)$. For $u \in \mathbf{R}$, let δ_u denote the Dirac distribution supported on $\{u\}$. We have

$$\begin{aligned} T_{\mathbf{Q}} &= e^{x/2} + e^{-x/2} - (\log \pi)\delta_0 \\ &\quad - \sum_p \sum_{e=1}^{\infty} \frac{\log p}{p^{e/2}} (\delta_{e\log p} + \delta_{-e\log p}) + V_1. \end{aligned} \tag{8.3}$$

It is immediate to see

$$\{(e^{x/2} + e^{-x/2}) * \varphi\}(y) = \Phi(0)e^{y/2} + \Phi(1)e^{-y/2}, \quad \Phi = M(\varphi),$$

$$(\delta_u * \varphi)(y) = \varphi(y - u) \quad \text{for every} \quad u \in \mathbf{R}.$$

We shall show that $V_1 * \varphi$ is a continuous function except at $x = \pm a$. For this purpose, it suffices to prove the same assertion for $\mathrm{Pf}\,(\frac{1}{|x|}) * \varphi$.

If $\alpha \in C(a-\epsilon)$ for some $0<\epsilon<a$, we have

$$
\begin{aligned}
&(\mathrm{Pf}\,(\frac{1}{|x|}) * \varphi)(\alpha) = \mathrm{Pf}\,(\frac{1}{|x|})(\check{\varphi} * \alpha) \\
&= \mathrm{Pf}\,(\frac{1}{|x|}) \int_{-a+\epsilon}^{a-\epsilon} \alpha(y)\varphi(y-x)\,dy \\
&= \int_{|x|\geq 1} \int_{-a+\epsilon}^{a-\epsilon} \frac{\alpha(y)\varphi(y-x)}{|x|}\,dy\,dx \\
&\qquad + \int_{|x|\leq 1} \int_{-a+\epsilon}^{a-\epsilon} \frac{\alpha(y)\varphi(y-x)-\alpha(y)\varphi(y)}{|x|}\,dy\,dx \\
&= \int_{-a+\epsilon}^{a-\epsilon} (\int_{|x|\geq 1} \frac{\varphi(y-x)}{|x|}\,dx)\alpha(y)\,dy \\
&\qquad + \int_{-a+\epsilon}^{a-\epsilon} (\int_{|x|\leq 1} \frac{\varphi(y-x)-\varphi(y)}{|x|}\,dx)\alpha(y)\,dy.
\end{aligned}
$$

This shows that

$$
\begin{aligned}
(\mathrm{Pf}\,(\frac{1}{|x|}) * \varphi)(y) &= \int_{|x|\geq 1} \frac{\varphi(y-x)}{|x|}\,dx \\
&\quad + \int_{|x|\leq 1} \frac{\varphi(y-x)-\varphi(y)}{|x|}\,dx \quad \text{for } |y|<a.
\end{aligned} \tag{8.4}
$$

We see easily that the right hand side of (8.4) is a continuous function of y for $|y|<a$. Similarly (8.4) holds for $|y|>a$ and we see that $V_1 * \varphi$ is a continuous function on $\mathbf{R}-\{-a,a\}$. By (8.3), we have

$$
\begin{aligned}
(T_{\mathbf{Q}} * \varphi)(x) &= \Phi(0)e^{x/2} + \Phi(1)e^{-x/2} - (\log \pi)\varphi(x) + (V_1 * \varphi)(x) \\
&\quad - \sum_p \sum_{e=1}^{\infty} \frac{\log p}{p^{e/2}}(\varphi(x+e\log p)+\varphi(x-e\log p)),
\end{aligned}
$$

which vanishes identically for $|x|<a$. Thus we obtain

$$
\begin{aligned}
&\Phi(0)e^{x/2} + \Phi(1)e^{-x/2} - (\log \pi)\varphi(x) + (V_1 * \varphi)(x) \\
&= \sum_{p,e,e\log\, p<2a} \frac{\log p}{p^{e/2}}(\varphi(x+e\log p)+\varphi(x-e\log p)), \quad |x|<a.
\end{aligned} \tag{8.5}
$$

Now assume $\varphi(a) \neq 0$. Since $a > \log 2/2$, there exists a prime p_1 and $e_1 \in \mathbf{N}$ such that $e_1 \log p_1 < 2a$. The left hand side of (8.5) is continuous for $|x|<a$ as shown above, but $\varphi(x+e_1\log p_1)$ is discontinuous at

$x = a - e_1 \log p_1$. Hence there must exist a prime p_2 and $e_2 \in \mathbf{Z}$ such that $e_2 \neq 0$, $|e_2| \log p_2 < 2a$, $(p_1, e_1) \neq (p_2, e_2)$, $a - e_1 \log p_1 = \pm a - e_2 \log p_2$. If $a - e_1 \log p_1 = a - e_2 \log p_2$, we get $p_1 = p_2$, $e_1 = e_2$, contradiction. Therefore we have

$$e_1 \log p_1 - e_2 \log p_2 = 2a. \tag{8.6}$$

Assume $a > (\log 5)/2$. Then we can find a prime p_3 which is different from p_1, p_2, and $e_3 \in \mathbf{N}$ such that $e_3 \log p_3 < 2a$. Applying the same argument as above, we see that there exist a prime p_4 and $e_4 \in \mathbf{Z}$ such that $e_4 \neq 0$, $|e_4| \log p_4 < 2a$, $(p_3, e_3) \neq (p_4, e_4)$,

$$e_3 \log p_3 - e_4 \log p_4 = 2a. \tag{8.7}$$

From (8.6) and (8.7), we get $p_1^{e_1} p_2^{-e_2} = p_3^{e_3} p_4^{-e_4}$. Since $p_3 \neq p_1, p_2$, this implies $p_3 = p_4$, $e_3 = e_4$, contradiction. Assume $a \leq (\log 5)/2$. Take $p_1 = 2$, $e_1 = 1$ in (8.6). Clearly we have $e_2 < 0$. If $p_2 > 2$ or $|e_2| \geq 2$, we get $\log 5 \geq 2a \geq \log 2 + \log 3$, contradiction. Hence we have $p_2 = 2$, $e_2 = -1$. This shows $a = \log 2$. Since $2a = \log 4 > \log 3$, we may take $p_1 = 3$, $e_1 = 1$ in (8.6). Then we must have $3 p_2^{-e_2} = 4$ with some prime p_2 and $e_2 \in \mathbf{Z}$, contradiction.

Thus we have proved $\varphi(a) = 0$. Applying this result to $(T_{\mathbf{Q}} * \varphi)^{\vee} = T_{\mathbf{Q}} * \check{\varphi}$ whose support is contained in $\mathbf{R} - (-a, a)$, we obtain $\varphi(-a) = 0$. From $\varphi(a) = \varphi(-a) = 0$, we have $\check{\varphi} * \psi' = (\check{\varphi})' * \psi$ for every $\psi \in C(a)$. Hence $(\check{\varphi})' \in V_0$. Then we have $\varphi'(a) = \varphi'(-a) = 0$ from the above result. Since this process can be repeated indefinitely, we obtain $\varphi \in C(a)$. Now $V_0 = \{0\}$ follows from Proposition 2. This completes the proof.

References

[1] B.P.Boas, "Entire functions", Academic Press, 1954.

[2] I.M.Gel'fand and N.Ya.Vilenkin, "Generalized functions, vol. 4", Academic Press, 1964.

[3] G.H.Hardy, J.E.Littlewood and G.Pólya,, "Inequalities, second edition", Cambridge University Press, 1952.

[4] G.Poitou, Sur les petits discriminants, Séminaires Delange-Pisot-Poitou, **no.6** (1976/1977).

[5] L.Schwartz, "Théorie des distribution, Nouvelle édition", Hermann, 1966.

[6] C.L.Siegel, Wurzeln Heckescher Zetafunktionen, Göttingen Nachrichten (1972), 11–20.

[7] A.Weil, Sur les " formules explicites" de la théorie des nombres premiers, Comm. Lund. (1952), 252–267.
[8] E.T.Whittaker and G.N.Watson, "A course of Modern analysis, fourth edition", Cambridge University Press, 1927.
[9] H.Yoshida, On the unitarizability of principal series representations of $\mathfrak{p}$-adic Chevalley groups, preprint, 1989 (revised 1991).
[10] K.Yosida, "Functional analysis, sixth edition", Springer Verlag, 1980.

Department of Mathematics
Kyoto University
Kyoto 606, Japan