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Zeta Functions of Loop Groups

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Abstract.

We will make a preparation for defining the Selberg zeta function of $PSL(2, \mathbf{Z}[T])$, which is a discrete subgroup of the loop group G of $PSL(2, \mathbf{C})$. Conjugacy classes of $PSL(2, \mathbf{Z}[T])$ will be classified and the definition of the norm of hyperbolic classes will be proposed.

§1. Introduction

Selberg zeta function was introduced by Selberg [6] in 1956. The definition is an analogue of the Riemann zeta function as an Euler product. Instead of prime numbers, the product is taken over all the primitive hyperbolic conjugacy classes of the given discrete subgroup Γ of some Lie group G. In this paper we study an example of Selberg zeta functions of infinite dimensional groups. Throughout this paper we fix Γ to be $PSL(2, \mathbf{Z}[T_1, \cdots, T_n])$ $(n = 0, 1, 2, \ldots)$, which is a discrete subgroup of the n-ple loop group of $PSL(2, \mathbf{C})$. In the next section we will define hyperbolic, elliptic, and parabolic conjugacy classes of Γ. Primitive hyperbolic classes will be corresponded to "real" quadratic extensions of $\mathbf{Q}(T_1, \ldots, T_n)$. Some ideas in the next section come from the paper of Akagawa [1], who treated the case of $PSL(2, \mathbf{F}[T])$ with finite field $\mathbf{F}$. The last section is a proposal of the definition of the norm of hyperbolic classes. We will define the norm via the regulator map in algebraic K-theory. The definition is a natural generalization of the classical well-known case $(n = 0)$. Partial solution of the convergence of the zeta functions will be given.

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§2. Conjugacy classes

First we introduce three sequences of rings or fields;

$$\begin{aligned} R_0 &:= \mathbf{Z}, & R_n &:= R_{n-1}[T_n] \\ K_0 &:= \mathbf{Q}, & K_n &:= K_{n-1}(T_n) \\ F_0 &:= \mathbf{R}, & F_n &:= F_{n-1}((T_n^{-1})). \end{aligned}$$

The field F_n is the completion of K_n with respect to the place T_n^{-1}. An element of F_n has the form $\sum_{k=-\infty}^{m} a_k T_n^k$ ($a_k \in F_{n-1}$, $a_m \neq 0$). We define the homomorphism

$$\deg_n : F_n^* \ni \sum_{k=-\infty}^{m} a_k T_n^k \longmapsto m \in \mathbf{Z}.$$

Let Γ be the group $PSL(2, R_n)$, which acts on the algebraic closure $\overline{K_n}$ of K_n by the linear fractional transformation. The matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ fixes $\omega \in \overline{K_n}$ when $c\omega^2 + (d-a)\omega - b = 0$. Then ω is at most quadratic over K_n. When ω is quadratic, we put the minimal polynomial of ω by $C\omega^2 - B\omega + A = 0$, where $A, B, C \in R_n$, $(A, B, C) = 1$, and $B^2 - 4AC \notin R_n^2$. Comparing the above two quadratic equations, there exists $u \in R_n$ such that $Cu = c$, $Bu = a - d$, $Au = -b$. Putting $t := a + d$, we have the representation as

$$\gamma = \begin{pmatrix} \dfrac{t + Bu}{2} & -Au \\ Cu & \dfrac{t - Bu}{2} \end{pmatrix}.$$

In what follows in this paper we put D to be the discriminant $B^2 - 4AC$. As $\det \gamma = 1$, we have the Pell's type of equation $t^2 - Du^2 = 4$. In the purpose of defining hyperbolic and so on, we introduce the notion of real and imaginary. We call $x \in \overline{K_n}$ to be *n-real* when $x \in F_n$. We call $x \in \overline{K_n}$ to be *n-imaginary* when $x \notin F_n$. By definition, 0-real and 0-imaginary agree with the ordinary real and imaginary. For two integers $i < j$, i-real implies j-real. The solution $\omega = \dfrac{B \pm \sqrt{D}}{2C}$ of the minimal equation is n-real if and only if the degree of D is even and the coefficient of the highest degree term in D is $(n-1)$-real. Now we define hyperbolic and so on.

Definition. Let γ be an element of Γ which fixes ω. Then we call γ *hyperbolic* when ω is n-real quadratic. We call γ *elliptic* when ω is n-imaginary quadratic. We call γ *parabolic* when ω is not quadratic.

For given three polynomials $A, B, C \in R_n$ we consider the following two sets;

$$S_{ABC} := \left\{ \frac{t+\sqrt{D}}{2} \ ; \ t^2 - Du^2 = 4, D = B^2 - 4AC, \ t, u \in R_n \right\} / \{\pm 1\}$$

$$\Gamma_{ABC} := \{\gamma \in \Gamma \ ; \ \gamma\omega = \omega, \ C\omega^2 - B\omega + A = 0\}.$$

Proposition 1. *The set S_{ABC} is a multiplicative group and isomorphic to Γ_{ABC}. The isomorphism is given by*

$$S_{ABC} \ni \frac{t+\sqrt{D}}{2} \longmapsto \begin{pmatrix} \dfrac{t+Bu}{2} & -Au \\ Cu & \dfrac{t-Bu}{2} \end{pmatrix} \in \Gamma_{ABC}.$$

The above proposition is a generalization of the classical case ($n = 0$). The proof is a complete analogue.

Proposition 2. *Let $\gamma \in \Gamma$ be elliptic. Then the order of γ is either 2 or 3.*

Proof. As γ is elliptic, $\sqrt{\mathrm{tr}(\gamma)^2 - 4}$ is n-imaginary. Therefore $\mathrm{tr}(\gamma)$ is an negative integer. Then the Pell's equation tells us that t and u are negative integers. The proof is reduced to the classical case, where the desired result is well-known. Q.E.D.

The above propositions are generalizations of the well-known fact in the case of $n = 0$. But not all properties are analogous to the classical case. For example, there exist infinitely many elliptic conjugacy classes when $n > 0$. In what follows, we will treat hyperbolic element and assume $\sqrt{D} \in F_n$.

Proposition 3. *The group S_{ABC} is generated by a single element.*

Proof. First we prove in the case $D \notin R_{n-1}$ that $\deg_n |_{S_{ABC}}$ is an injective homomorphism into $\mathbf{Z}$. Take any $\epsilon = \dfrac{t+\sqrt{D}u}{2} \in \mathrm{Ker}(\deg_n)$. Then $\deg_n(\epsilon^{-1}) = -\deg_n(\epsilon) = 0$, and $\deg_n(t) = \deg_n(\epsilon + \epsilon^{-1}) \leq 0$. As t is in R_n, t is an integer. The Pell's equation shows that $u = 0$ by the assumption $D \notin R_{n-1}$. Hence $t = \pm 2$, and $\epsilon = \pm 1$. Next, when

$D \in R_{n-1}$, similarly $\sqrt{D} \in F_{n-1}$ and we deduce that either $\deg_{n-1}$ is injective or $\sqrt{D} \in F_{n-2}$. By repeating this way, we have that either $\deg_i$ is injective for some i $(i = 1, 2, \cdots, n)$ or $\sqrt{D} \in F_0 = \mathbf{R}$. In the latter case, the Pell's equation says that t and u are integers and the proof is reduced to the classical case. Q.E.D.

The above proposition assures the existence of the fundamental solution of the Pell's equation. We denote the generator of S_{ABC} by $\epsilon_0 = \dfrac{t_0 + \sqrt{D}u_0}{2}$, which corresponds to the generator $\gamma_\omega = \begin{pmatrix} \frac{t+Bu}{2} & -Au \\ Cu & \frac{t-Bu}{2} \end{pmatrix}$ of Γ_{ABC} for $\omega = \dfrac{B \pm \sqrt{D}}{2}$. The conjugate of a quadratic element $\omega = X + Y\sqrt{D}$ is denoted by $\bar{\omega} = X - Y\sqrt{D}$. These notations leads the following immediately.

Proposition 4. (1) $\gamma_{\bar{\omega}} = \gamma_\omega^{-1}$. (2) *For any* $\sigma \in \Gamma$, $\gamma_{\sigma\omega} = \sigma\gamma_\omega\sigma^{-1}$. (3) *There is a bijection between the set of* Γ*-equivalence classes of* n*-real quadratic elements over* K_n *and primitive hyperbolic conjugacy classes of* Γ. *The class of* ω *corresponds to the class of* γ_ω.

The above proposition lets us grasp the set of all the primitive hyperbolic conjugacy classes of Γ, which will be denoted by $\mathrm{Prim}(\Gamma)$. For investigating the trace formula in the future, the following will be useful.

Proposition 5. *Let* $\gamma \in \Gamma$ *be hyperbolic. Then the centralizer* Γ_γ *is generated by a single element.*

Proof. Thanks to the Propositions 1 and 3, it suffices to say that $\Gamma_\gamma \subset \Gamma_\omega$ with $\gamma\omega = \omega$. Choose any $\gamma_1 \in \Gamma_\gamma$ and put $\omega_1 = \gamma_1\omega$. Then $\gamma\omega_1 = \gamma\gamma_1\omega = \gamma_1\gamma\omega = \gamma_1\omega = \omega_1$. Hence $\omega_1 = \omega$ and $\gamma_1 \in \Gamma_\omega$. Q.E.D.

§3. Norm

We introduce an equivalence relation in $\mathrm{Prim}(\Gamma)$. We call $\gamma_\omega, \gamma_{\omega'} \in \mathrm{Prim}(\Gamma)$ *equivalent* if and only if the corresponding quadratic field $K_n(\omega)$ and $K_n(\omega')$ are isomorphic as fields. Then Proposition 4(3) induces a surjection from the set of equivalence classes of $\mathrm{Prim}(\Gamma)$ onto the isomorphism classes of the set of n-real quadratic fields over K_n. We denote the image of an equivalence class γ by K_γ, which is actually $K_n(\sqrt{D(T_1, \cdots, T_n)})$. The field K_γ is the function field of the algebraic variety

$$V_\gamma : X_{n+1}^2 = D(X_1, \cdot, X_n).$$

The number h_γ of classes in $\mathrm{Prim}(\Gamma)/_\sim$ which maps to an isomorphism class K_γ is called the class number of quadratic forms over $\mathbf{Q}(T)$. It is a generalization of the classical class number over $\mathbf{Q}$. Another generalization of the class number is known as the order H_γ of the Chow group of V_γ, which should appear in special values of L-functions as in Assumption 3 below. It is known that H_γ is finite ([5]). The relation of the two values H_γ and h_γ seems to be difficult to investigate and is not known at all. We will make the following assumption in Proposition 6.

Assumption 1. The value h_γ is finite and equal to H_γ.

Putting the rank of the $(n+1)$-th K-group of V_γ to be r, we have the regulator map

$$\mathrm{Reg} : K_{n+1}(V_\gamma) \longrightarrow \mathbf{R}^r.$$

Definition. The norm of the hyperbolic conjugacy class represented by γ is defined to be

$$N(\gamma) := e^{2R_\gamma},$$

where $R_\gamma > 0$ is the higher regulator of Bloch-Beilinson ([2], [3, Lec. 8,9]) (the volume of the cokernel of the regulator map).

When $n = 0$, the above definition coincides with the original one by Selberg, which is equal to the square of the fundamental unit of the corresponding quadratic field over $\mathbf{Q}$. In that case, $K_1(V_\gamma)$ is the unit group of the integer ring of K_γ, and the regulator map is the logarithm.

Definition. We put formally the Selberg zeta function of Γ by

$$\zeta_\Gamma(s) := \prod_{\gamma \in \mathrm{Prim}(\Gamma)/_\sim} (1 - N(\gamma)^{-s})^{-1}.$$

Next we consider the convergence of the zeta function in the case of $n = 1$. In this case V_γ is a hyperelliptic curve

$$C_\gamma : Y^2 = D(X).$$

Its genus is given by $g = \frac{1}{2}(\deg(D) - 2)$ as $\deg(D)$ is even. We can decompose the zeta function to the infinite product over g;

$$\zeta_\Gamma(s) = \prod_{g=1}^{\infty} \zeta_\Gamma^{(g)}(s),$$

where $\zeta_\Gamma^{(g)}(s)$ is defined as an Euler product over all C_γ with genus g. Here $\zeta_\Gamma^{(0)}(s)$ does not contribute to the product because the regulator map is trivial in the case of $g = 0$. When $g = 1$, the zeta function $\zeta_\Gamma^{(1)}(s)$ is defined as an Euler product over elliptic curves. The following conjectures are widely believed to be true.

Assumption 2 (a part of Taniyama-Weil conjecture). For any elliptic curve E over $\mathbf{Q}$ with conductor N, there exists a cusp form f of weight 2 for $\Gamma_0(N)$ such that $L(s, E) = L(s, f)$.

Assumption 3 (a part of Beilinson-Bloch conjecture). Let E_γ be an elliptic curve which contributes to $\zeta_\Gamma^{(1)}(s)$. Let N_γ be its conductor. Then up to some elementary factors,

$$L(2, E_\gamma) = \frac{H_\gamma R_\gamma}{N_\gamma^4}.$$

The above assumption can be regarded as a special case of the conjecture of Kato [4, Conjecture 7.5]

Lemma. *The special value $L(2, E_\gamma)$ is estimated by constants not depending on E_γ, namely,*

$$\left(\frac{\zeta(3)}{\zeta(\frac{3}{2})}\right)^2 \le L(2, E_\gamma) \le \frac{1-2^{-1-\epsilon}}{1-2^{-1/2}} \frac{1-3^{-1-\epsilon}}{1-2\cdot 3^{-3/2}} \zeta(1+\epsilon),$$

where ϵ is any real number satisfying

$$0 < \epsilon < \frac{1}{2} - \frac{\log 2}{\log 5}.$$

Proof. By the Euler product expression of the L-function

$$L(s, E_\gamma) = \prod_{p:\text{good}} (1 - a(p)p^{-s} + p^{1-2s})^{-1} \prod_{p:\text{bad}} (1 - a(p)p^{-s})^{-1},$$

with $a(p) := 1 + p - \sharp E_\gamma(\mathbf{F}_p)$, we have

$$L(2, E_\gamma) \ge \prod_{p:\text{prime}} (1 - a(p)p^{-2} + p^{-3})^{-1}.$$

It is known that the absolute value of $a(p)$ is less than $2\sqrt{p}$. Therefore

$$\begin{aligned} L(2, E_\gamma) &\geq \prod_p (1 + 2\sqrt{p}\, p^{-2} + p^{-3})^{-1} \\ &= \prod_p (1 + p^{-3/2})^{-2} \\ &= \left(\frac{\zeta(3)}{\zeta(\frac{3}{2})} \right)^2 . \end{aligned}$$

On the other hand,

$$\begin{aligned} L(2, E_\gamma) &\leq \prod_p (1 - 2p^{-3/2})^{-1} \\ &= (1 - 2^{-1/2})^{-1} (1 - 2 \cdot 3^{-3/2})^{-1} \prod_{p \geq 5} (1 - 2p^{-3/2})^{-1} \\ &\leq \frac{1 - 2^{-1-\epsilon}}{1 - 2^{-1/2}} \frac{1 - 3^{-1-\epsilon}}{1 - 2 \cdot 3^{-3/2}} \zeta(1 + \epsilon), \end{aligned}$$

because $2 < p^{\frac{1}{2} - \epsilon}$ for any prime $p \geq 5$. Q.E.D.

Assumption 4. The number R_γ tends to infinity when $N_\gamma \to \infty$.

Proposition 7. *Under the above Assumptions* 1 *to* 4, *the first part of the Selberg zeta function* $\zeta_\Gamma^{(1)}(s)$ *converges for* $\Re(s) \gg 0$.

Proof. By the above lemma and Assumption 4, we have $H_\gamma \sim N^\alpha$ for some $0 < \alpha < 4$ as $N_\gamma \to \infty$. The absolutely convergence of $\zeta_\Gamma^{(1)}(s)$ is equivalent to that of the following sum;

$$\begin{aligned} \sum_\gamma N(\gamma)^{-\Re s} h_\gamma &= \sum_\gamma \exp(-2R_\gamma \Re s) h_\gamma \\ &< \sum_\gamma (2R_\gamma)^{-\Re s} h_\gamma \\ &= \sum_\gamma \left(\frac{H_\gamma}{2N_\gamma^4 L(2, E_\gamma)} \right)^{\Re s} h_\gamma, \end{aligned}$$

where Assumption 3 is used. We regard it to be the sum over $N = N_\gamma$. By Assumption 2, the number of the elliptic curves of conductor N is equal to

$$\dim S_2(\Gamma_0(N)) = O(N^{3/2})$$

as N tends to infinity. Using Assumption 1, the proof is reduced to the convergence of

$$\sum_{N=1}^{\infty} N^{\frac{3}{2}-4\Re s+\alpha(s-1)}.$$

It converges when $\Re s > \dfrac{5+2\alpha}{2(4-\alpha)}$. Q.E.D.

By generalizing the classical definition of the norm of hyperbolic classes, there is another possibility of the definition. In the classical definition, for a metric space X with a distance d_X and its isometry γ, the norm of γ is defined as $\exp(\inf_{x\in X} d_X(x,\gamma x))$. Now we apply it to the case of loop groups. Let X be the n-ple loop space of the real three dimensional hyperbolic space H_3. The distance of X is given for any two elements u and v in X by

$$d_X(u,v) := \int_{(S^1)^n} d_{H_3}(u(T),v(T))dT,$$

which equals

$$\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} d_{H_3}(u(e^{i\theta}),v(e^{i\theta}))d\theta_1\cdots d\theta_n$$

where $e^{i\theta} = (e^{i\theta_1},\cdots,e^{i\theta_n})$. A hyperbolic element γ of Γ can be diagonalized to $\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$ in $PSL(2,\overline{K_n})$, which acts on X isometrically. Fix an element $T\in(S^1)^n$, and put $v(T) = z(T)+y(T)j \in X$ with $z(T)\in\mathbf{C}$ and $y(T)>0$. Then it is computed that

$$\begin{aligned} d_{H_3}(v(T),\gamma v(T)) &= d_{H_3}(v(T),\alpha(T)^2 z(T)+|\alpha(T)|^2 y(T)j)\\ &= \cosh^{-1}\frac{|1-\alpha(T)^2|^2|z(T)|^2+(1+|\alpha(T)|^4)y(T)^2}{2y(T)^2|\alpha(T)|^2}\\ &\geq \cosh^{-1}\frac{|\alpha(T)|^2+|\alpha(T)|^{-2}}{2}\\ &= |\log|\alpha(T)|^2|\\ &= \log N(\gamma(T)), \end{aligned}$$

where N is the classical norm of $PSL(2,\mathbf{C})$. The minimum is realized when $z(T)=0$. Consequently, the norm of a hyperbolic element γ in the loop group is written as follows using the classical norm of $\gamma(T)$ in $PSL(2, \mathbf{C})$:

$$N(\gamma) := \exp \int_{(S^1)^n} \log N(\gamma(T))dT.$$

This definition is also a natural generalization of the classical norm. But it is difficult at the present to have some properties about the zeta function by adopting this definition.

References

[1] S. Akagawa, On Selberg zeta functions for modular groups over function fields, master thesis (1978), Tokyo University.

[2] A. A. Beilinson, Higher regulators and values of L-functions, J. Soviet Math., **30** (1985), 2036-2070.

[3] S. Bloch, Lectures on algebraic cycles, Duke Univ. Math. Series, **4** (1980).

[4] K. Kato, A Hasse principle for two dimensional global fields, J. reine angew. Math., **366** (1986), 142-181.

[5] N. Katz and S. Lang, Finiteness theorems in geometric class field theory, Enseign. Math., **27** (1981), 185-314.

[6] A. Selberg, Harmonic analysis and discontinuous groups on weakly symmetric Riemannian surfaces with applications to Dirichlet series, J. Ind. Math. Soc., **20** (1956), 47–87.

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