Advanced Studies in Pure Mathematics 13, 1988
Investigations in Number Theory
pp. 237–260

Arithmetic of Some Zeta Function Connected with the Eigenvalues of the Laplace-Beltrami Operator

Akio Fujii

§ 1. Introduction

Let $\lambda_0=0\leqq\lambda_1\leqq\lambda_2\leqq\cdots$ run over the eigenvalues of the discrete spectrum of the Laplace-Beltrami operator on $L^2(H/\Gamma)$, where H is the upper half of the complex plane and we take $\Gamma=PSL(2, Z)$. It is well known that $\lambda_1>\frac{1}{4}$. We put $\lambda_j=\frac{1}{4}+r_j^2$ for $j\geqq 0$. In our previous work [7], we have introduced and studied the zeta function defined by

$$Z_\alpha(s)=\sum_{r_j>0}\frac{\sin(\alpha r_j)}{r_j^s},$$

where α is any positive number and the series is convergent for $\operatorname{Re} s>1$. Using the Selberg's trace formula, which will be stated below, we have shown that $Z_\alpha(s)$ is an entire function for any positive α. In this paper we are concerned with the arithmetical properties of the values of $Z_\alpha(s)$ at $s=1$ or $s=0$. In particular, we obtain some new expressions of the values of Dirichlet L-functions at $s=1$ and a new proof of Dirichlet's class number formula for the real quadratic number fields.

To explain a general principle, we recall a primitive situation. Let $\zeta(s)$ be the Riemann zeta function and let γ run over the positive imaginary parts of the zeros of $\zeta(s)$. We have introduced in [5] the zeta function defined by

$$\zeta_\alpha(s)=\sum_{\gamma>0}\frac{\sin(\alpha\gamma)}{\gamma^s},$$

where α is any positive number and the series is convergent for $\operatorname{Re} s>0$. We have shown under the Riemann Hypothesis that this is entire for any positive α. The value of $\zeta_\alpha(s)$ at $s=1$ has been known long before by Guinand [9]. Namely, $\zeta_\alpha(1)$ as $\alpha\to\infty$, is essentially

$$-\tfrac{1}{2}e^{-(1/2)\alpha}\Big(\sum_{n\leqq e^\alpha}\Lambda(n)-e^\alpha\Big),$$

where $\Lambda(n)$ is the von Mangoldt function defined by

Received April 24, 1986.

$$\frac{\zeta'}{\zeta}(s)=-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} \qquad \text{for Re } s>1$$

and

$$\Lambda(n)=\begin{cases}\log p & \text{if } n=p^k\\ 0 & \text{otherwise},\end{cases}$$

where p runs over the prime numbers $\geqq 2$ and k runs over the integers $\geqq 1$. On the other hand, as a by-product of the proof of the analytic continuation of $\zeta_\alpha(s)$, we can show that

$$\lim_{\alpha\to\log n}(\alpha-\log n)\zeta_\alpha(0)=-\frac{\Lambda(n)}{\sqrt{n}}.$$

The zeta function $Z(s)$ which corresponds to $\zeta(s)$ has been introduced by Selberg:

$$Z(s)=\prod_{\{P_0\}}\prod_{k=0}^{\infty}(1-N(P_0)^{-k-s}),$$

where $\{P_0\}$ runs over all primitive hyperbolic conjugacy classes in Γ and $N(P_0)$ is the square of the eigenvalue (greater than one) of a representative element P_0. The Selberg's trace formula describes the location of the zeros and the poles of $Z(s)$ and we see that $Z_\alpha(1)$ and $Z_\alpha(0)$ will describe some properties of the distribution of the von Mangoldt function $\tilde{\Lambda}(P)$ of $Z(s)$, where we put

$$\tilde{\Lambda}(P)=\frac{\log N(P_0)}{1-N(P)^{-1}}$$

for a hyperbolic conjugacy class $\{P\}$ satisfying $P=P_0^k$ with an integer $k\geqq 1$ and we put $N(P)=N(P_0)^k$.

Here we state the Selberg's trace formula as follows. Let $h(r)$ satisfy the conditions;

1) $h(r)=h(-r)$
2) $h(r)$ is analytic in the strip $|\text{Im } r|<\frac{1}{2}+\varepsilon$, $\varepsilon>0$
3) $h(r)=O((1+|r|^2)^{-1-\varepsilon})$ in this strip.

Then we have

$$\begin{aligned}\sum h(r_j)&=\frac{1}{6}\int_{-\infty}^{\infty} r\,\text{th}(\pi r)h(r)dr+\int_{-\infty}^{\infty}\left(\frac{1}{2}+\frac{2}{3\sqrt{3}}(e^{\pi r/3}+e^{-\pi r/3})\right)\frac{h(r)}{e^{\pi r}+e^{-\pi r}}dr\\&\quad+2\sum_{\{P_0\}}\sum_{k=1}^{\infty}\frac{\log N(P_0)}{N(P_0)^{k/2}-N(P_0)^{-k/2}}\cdot g(k\log N(P_0))\end{aligned}$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir)dr-\frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr$$
$$-2\log 2\cdot g(0)+\tfrac{1}{2}(1-\varphi(\tfrac{1}{2}))h(0),$$

where the left hand side is over all the solutions r_j of all the equations $\lambda_j=\frac{1}{4}+r_j^2$ for $j=0, 1, 2, \cdots$,

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru}h(r)dr \quad \text{and} \quad \varphi(s)=\sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}$$

(Cf. Selberg [16] and Hejhal [10]).

Using this formula we obtain first the following expressions of $Z_\alpha(1)$ and $Z_\alpha(0)$.

Theorem 1. *As* $\alpha\to\infty$,

$$Z_\alpha(1)=\frac{1}{4\sqrt{\pi}}\sum_{\{P\},\,\alpha\neq\log N(P)}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}\operatorname{sgn}(\alpha-\log N(P))$$
$$\times\left(\Gamma\left(\frac{1}{2}\right)-\int_0^{(1/4)(\alpha-\log N(P))^2} y^{-1/2}e^{-y}\,dy\right)$$
$$+e^{(1/2)\alpha}(e^{1/4}-1)+A\alpha\cos\alpha+O(1),$$

where

$$A=\frac{1}{48}\int_1^{\infty} e^{-x}(2-3x^{-2}-3x^{-3})dx.$$

Theorem 2.
(i) $\lim_{\alpha\to 2\log n}(\alpha-2\log n)Z_\alpha(0)=(1/\pi)(\Lambda(n)/n)$.
(ii) $\lim_{\alpha\to\log N(P_1)}(\alpha-\log N(P_1))Z_\alpha(0)$
$=(1/2\pi)\sum_{\{P\},\,N(P)=N(P_1)}(\tilde{\Lambda}(P)/\sqrt{N(P)})$,
where $\{P_1\}$ *is any hyperbolic conjugacy class.*
(iii) $\lim_{\alpha\to+0}(1/(1/\alpha)\log(1/\alpha))Z_\alpha(0)=-2/\pi$.

From Theorem 1, we can deduce the following

Corollary 1. *As* $\alpha\to\infty$,

$$Z_\alpha(1)=\tfrac{1}{2}e^{-(1/2)\alpha}\Big(\sum_{N(P)\leqq e^\alpha}\tilde{\Lambda}(P)-e^\alpha\Big)+O(\alpha).$$

Thus we see that $Z_\alpha(1)$ plays the same role in the theory of the distribution of $\tilde{\Lambda}(P)$ as does $\zeta_\alpha(1)$ in the theory of prime numbers.

By the way, we must recall here an important Kuznecov's version of the Selberg's trace formula which may be stated as follows (Cf. Kuznecov [12], [13]). Let h satisfy the following conditions;

1) $h(r)=h(-r)$
2) $h(r)$ is analytic in the strip $|\mathrm{Im}\, r|\leqq \Delta$ with $\Delta>3/4$
3) $h(r)=O(|r|^{-2-\varepsilon})$ as $|r|\to\infty$, for some $\varepsilon>0$.

Then we have

$$\begin{aligned}\sum h(r_j) &= \frac{1}{6}\int_{-\infty}^{\infty} r th(\pi r)h(r)dr \\ &+\int_{-\infty}^{\infty}\left(\frac{1}{2}+\frac{2}{3\sqrt{3}}(e^{\pi r/3}+e^{-\pi r/3})\right)\frac{h(r)}{e^{\pi r}+e^{-\pi r}}dr \\ &+\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\log\frac{\pi}{2}-\frac{\Gamma'}{\Gamma}(1+ir)-\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir)\right. \\ &\left.-\frac{\zeta'}{\zeta}(1+2ir)-\frac{\zeta'}{\zeta}(1-2ir)\right)h(r)dr \\ &+\tfrac{1}{2}h(0)+4\sum_{n=3}^{\infty}B(n)g\left(2\log\frac{n+\sqrt{n^2-4}}{2}\right),\end{aligned}$$

where g is the same as before and $B(n)$ is the residue at the point $s=1$ of the function $\zeta(2s)\sum_{c=1}^{\infty}A_n(c)/c^s$, $A_n(c)$ being the number of the solutions of the congruence

$$x^2+nx+1\equiv 0 \pmod{c}.$$

If we use this formula instead of Selberg's, we obtain different expressions for $Z_\alpha(1)$ and $Z_\alpha(0)$. Namely, we get the following results.

Corollary 1′. *As* $\alpha\to\infty$,

$$Z_\alpha(1)=\Big(\sum_{n\leqq e^{(1/2)\alpha}}B(n)-e^{(1/2)\alpha}\Big)+O(\alpha).$$

Theorem 2′. *For an integer* $n\geqq 3$,

$$\lim_{\alpha\to 2\log(n+\sqrt{n^2-4})/2}\left(\alpha-2\log\frac{n+\sqrt{n^2-4}}{2}\right)Z_\alpha(0)=\frac{1}{\pi}B(n).$$

It is important to combine (ii) of Theorem 2 and Theorem 2′. It gives first the following

Corollary 2. *For an integer* $n\geqq 3$,

$$\frac{1}{2}\sum_{|\mathrm{tr}\,P|=n}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}=B(n),$$

where tr $P=a+d$ *if* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is a representative of* $\{P\}$.

We remark that $B(n)$ can be rewritten explicitly using the value of Dirichlet L-function $L(s, \chi)$ at $s=1$. Thus Theorem 2′ and Corollary 2 implies the following theorem, which gives some new expressions of $L(1, \chi)$.

Theorem 3. *Let n be an integer* $\geqq 3$. *Suppose that* $n^2-4=Q^2D$ *and* D *is square free. Let* χ *be the character of the quadratic number field* $Q(\sqrt{n^2-4})$. *Then*

$$L(1, \chi)=\pi F(n)^{-1}\lim_{\alpha\to 2\log(n+\sqrt{n^2-4})/2}\left(\alpha-2\log\frac{n+\sqrt{n^2-4}}{2}\right)Z_\alpha(0)$$

$$=\frac{\log\dfrac{n+\sqrt{n^2-4}}{2}}{\sqrt{n^2-4}}\Phi(n)F(n)^{-1},$$

where we put

$$F(n)=\left(1-\frac{1}{2}\chi(2)\right)\prod_{\substack{p|Q\\ p>2}}\left(1-\frac{1}{p}\chi(p)\right)v_2(1, n)\left(1+\frac{1}{2}\right)^{-1}$$

$$\cdot\prod_{\substack{p|n^2-4\\ p>2}}v_p(1, n)\left(1+\frac{1}{p}\right)^{-1},$$

$$v_p(s, n)=1+\sum_{k=1}^{\infty}\frac{A_n(p^k)}{p^{ks}},\qquad \Phi(n)=\sum_{|\mathrm{tr}\,P|=n}\frac{1}{\nu(P)},$$

and $\nu(P)$ *is defined by* $P=P_0^{\nu(P)}$ *with the primitive hyperbolic class* $\{P_0\}$ *attached to* $\{P\}$.

We remark that the second expression of $L(1, \chi)$ in the above Theorem can be rewritten in terms of the quadratic number fields. For this we introduce some notations (Cf. Andrianov-Fomenko [1]). Let d be a positive integer satisfying $d^2|n^2-4$. Let $h(n^2-4, 2d)$ be the number of the classes of the quadratic forms $ax^2+2bxy+cy^2$ with $b^2-ac=n^2-4$ and $(a, 2b, c)=2d$. Let η_n be the fundamental unit of $Q(\sqrt{n^2-4})$. Let k be the integer such that $(n+\sqrt{n^2-4})/2=\eta_n^k$. Let $\nu(n, d)$ be defined by

$$\nu(n, d)=\mathrm{Max}\left\{\nu;\ \nu|k\quad\text{and}\quad\frac{\eta_n^k-\eta_n^{-k}}{\eta_n^{k/\nu}-\eta_n^{-k/\nu}}\,\middle|\,d\right\}.$$

Then we can rewrite Theorem 3 as follows.

Theorem 3′. *Under the same notations and assumptions as in Theorem 3, we have*

$$L(1,\chi)=\frac{\log\dfrac{n+\sqrt{n^2-4}}{2}}{\sqrt{n^2-4}}\Bigl(\sum_{\substack{d^2\mid n^2-4\\ d>0}}\frac{h(n^2-4,2d)}{\nu(n,d)}\Bigr)F(n)^{-1}.$$

From this Theorem we can derive the following corollary.

Corollary to Theorem 3′ (Dirichlet). *Let D be a square free integer $\geqq 2$. Let χ be the character of $Q(\sqrt{D})$. Let $\tilde{D}$ be the discriminant of this field. Let $\varepsilon(\tilde{D})$ be the fundamental unit and $h(\tilde{D})$ be the class number* (*in the wider sense*) *of this field. Then*

$$L(1,\chi)=\frac{2h(\tilde{D})\log\varepsilon(\tilde{D})}{\sqrt{\tilde{D}}}.$$

Thus we have obtained Dirichlet's class number formula using the spectral analysis.

We shall prove Theorem 1 in the section 2, Corollary 1 in the section 3, Theorem 2 in the section 4, Corollary 1′ in the section 5 Theorems 3 and 3′ and its corollary in the sections 6 and 7.

We remark that the results of the present paper have been announced in "Zeros, Eigenvalues and Arithmetic" [6].

Finally, the author wishes to express his thanks to Professor Bruggeman who has kindly sent the author a Kuzencov's [13] which was hard to obtain.

§ 2. Proof of Theorem 1

Here we evaluate $Z_\alpha(1)$. We remark that for $\mathrm{Re}\, s>3/2$,

$$\begin{aligned}Z_\alpha(2s-1)\Gamma(s)&=\int_0^\infty x^{s-1}\bigl(\sum_{r>0}e^{-r^2x}\sin(\alpha r)r\bigr)dx\\&=\Bigl(\int_0^1+\int_1^\infty\Bigr)\bigl(x^{s-1}\sum_{r>0}e^{-r^2x}\sin(\alpha r)r\bigr)dx\\&=I_1(s)+I_2(s),\ \text{say, where } r \text{ runs over } r_j.\end{aligned}$$

$$I_2(1)=\sum_{r>0}\frac{\sin(\alpha r)}{r}e^{-r^2}.$$

Using Selberg's trace formula with

$$h(r)=e^{-((1/4)+r^2)x}\sin(\alpha r)r$$

and

$$g(u)=-\frac{(u-\alpha)}{8\sqrt{\pi}\,x^{3/2}}e^{-(u-\alpha)^2/4x}e^{-(1/4)x}+\frac{(u+\alpha)}{8\sqrt{\pi}\,x^{3/2}}e^{-(u+\alpha)^2/4x}e^{-(1/4)x},$$

$I_1(s)$ can be written as

$$I_1(s)=I_3(s)+I_4(s)+I_9(s)+I_{10}(s)+I_{11}(s)+I_{19}(s),$$

where each term is defined below and of the form

$$I(s,f(x))=\int_0^1 x^{s-1}f(x)\,dx.$$

Our evaluation of each term at $s=1$ is as follows.

$$\begin{aligned}
I_3(1)&\equiv I(1,\ -\tfrac{1}{2}\sum_{(1/4)+r_0^2=0}h(r_0)e^{(1/4)x})\\
&=(e^{\alpha/2}-e^{-\alpha/2})(e^{1/4}-1).\\
I_4(1)&\equiv I\left(1,\ \frac{1}{2}e^{(1/4)x}\frac{1}{6}\int_{-\infty}^{\infty}r\,th(\pi r)h(r)dr\right)\\
&=I\left(1,\ \frac{1}{6}\int_0^{\infty}r^2\sin(\alpha r)e^{-r^2x}dr\right)\\
&\quad+I\left(1,\ -\frac{1}{3}\int_0^{\infty}\frac{r^2\sin(\alpha r)e^{-r^2x}}{e^{2\pi r}+1}dr\right)\\
&=I_5(1)+I_6(1),\ \text{say}.\\
I_5(1)&=I\left(1,\ \frac{1}{6}\int_0^1 r^2\sin(\alpha r)e^{-r^2x}dr\right)\\
&\quad+I\left(1,\ \frac{1}{6}\int_1^{\infty}r^2\sin(\alpha r)e^{-r^2x}dr\right)\\
&=I_7(1)+I_8(1),\ \text{say}.\\
I_7(1)&=\frac{1}{6}\int_0^1\sin(\alpha r)G(r^2,\ 1)dr,\ \text{where we put}
\end{aligned}$$

$$G(x,\ s)=\int_0^x y^{s-1}e^{-y}dy.$$

We remark that

$$6I_8(s)=\frac{\sin\alpha}{2}G(1,s-1)+\left(\frac{\alpha\cos\alpha}{4}+\frac{\sin\alpha}{4}\right)G(1,s-2)$$
$$+\left(\frac{\alpha\cos\alpha}{8}-\frac{\alpha^2\sin\alpha}{8}\right)G(1,s-3)$$
$$-\frac{\alpha^3\cos\alpha}{16}G(1,s-4)-\frac{1}{4}F_1(s-3)-\frac{1}{4}\alpha F_2(s-4)$$
$$+\frac{\alpha^4}{16}F_1(s-5)+\frac{\alpha^3}{8}F_2(s-5).$$

where we put

$$F_1(s-3)=\int_0^1 x^{s-3}\left(\int_1^\infty r^{-2}\sin(\alpha r)e^{-r^2x}dr\right)dx=\int_0^1 x^{s-3}\beta_1(x)dx, \text{ say,}$$

and

$$F_2(s-4)=\int_0^1 x^{s-4}\left(\int_1^\infty r^{-3}\cos(\alpha r)e^{-r^2x}dr\right)dx=\int_0^1 x^{s-4}\beta_2(x)dx, \text{ say,}$$

By evaluating the Laurent coefficients at $s=1$ of each term in the right hand side, we get

$$6I_8(1)=\frac{\sin\alpha}{2}(-C_0-E(0))+\left(\frac{\alpha\cos\alpha}{4}+\frac{\sin\alpha}{4}\right)(C_0-1-E(-1))$$
$$+\left(\frac{\alpha\cos\alpha}{8}-\frac{\alpha^2\sin\alpha}{8}\right)\left(-\frac{1}{2}C_0+\frac{3}{4}-E(-2)\right)$$
$$-\frac{\alpha^3\cos\alpha}{16}\left(\frac{1}{6}C_0-\frac{11}{36}-E(-3)\right)-\frac{1}{4}A_1$$
$$-\frac{1}{4}\alpha\left(\frac{\cos\alpha}{2}\left(-\frac{3}{4}+\frac{1}{2}C_0+E(-1)\right)-\beta_2(1)+\frac{1}{2}\alpha A_2\right.$$
$$\left.-\frac{1}{2}\frac{\cos\alpha}{\alpha^2}-\frac{\sin\alpha}{2\alpha}\right)$$
$$+\frac{1}{16}\alpha^4A_3$$
$$+\frac{1}{8}\alpha^3\left(-\frac{1}{2}\beta_2(1)+\frac{1}{2}\cos\alpha\left(\frac{11}{72}-\frac{C_0}{12}+\frac{1}{2}E(-3)\right)\right.$$
$$\left.+\frac{1}{2\alpha}\left(-\frac{\cos\alpha}{\alpha^3}-\frac{\sin\alpha}{\alpha^2}+\frac{\cos\alpha}{2\alpha}\right)+\frac{\sin\alpha}{12\alpha}+\frac{1}{4}\alpha A_3\right)$$
$$=\frac{1}{8}\alpha\cos\alpha\cdot\int_1^\infty e^{-x}(2-3x^{-2}-3x^{-3})dx+O(1),$$

where we put

$$\alpha^4 A_1 = 8\left(-\frac{3}{2}A_4 + \beta_1(1) + \frac{1}{2}\sin\alpha(1-e^{-1}) + \frac{1}{4}\alpha\cos\alpha - \frac{1}{2}\alpha\beta_2(1)\right)$$

$$-2\alpha^2\sin\alpha + \alpha^3\cos\alpha(C_\circ - 1 - E(-1)) + 2\alpha^3\beta_2(1),$$

$$\alpha^3 A_2 = \alpha A_1 + \frac{5}{2}\cos\alpha + 2\alpha\beta_1(1) + 2\alpha\sin\alpha\left(\frac{1}{4} + \frac{1}{2}(C_\circ - 1 - E(-1))\right)$$

$$+\alpha^2\cos\alpha\left(-\frac{1}{2}C_\circ + \frac{3}{4} - E(-2)\right) + \alpha^2\beta_2(1),$$

$$\alpha^2 A_3 = \frac{8}{3}\beta_1(1) + \alpha\cos\alpha\left(\frac{1}{6}C_\circ - \frac{11}{36} - E(-3)\right) + \frac{2}{3}\alpha\beta_2(1)$$

$$-\frac{1}{3}\alpha^{-3}(-10\cos\alpha - 10\alpha\sin\alpha + 5\alpha^2\cos\alpha - 12\alpha^3 A_2)$$

$$+\frac{1}{9}\sin\alpha\left(-1 + 12\left(-\frac{1}{2}C_\circ + \frac{3}{4} - E(-2)\right)\right).$$

$$A_4 = \int_1^\infty \sin(\alpha r) r^{-4}(1 - e^{-r^2})dr,$$

$G(1, s) = \Gamma(s) - E(s)$ and $C_\circ$ is the Euler constant.

$$I_6(1) = -\frac{1}{3}\int_0^\infty \frac{\sin(\alpha r)}{e^{2\pi r}+1} G(r^2, 1)dr.$$

$$I_9(1) \equiv I\left(1, \frac{1}{2}e^{(1/4)x}\int_{-\infty}^\infty \left(\frac{1}{2} + \frac{2}{3\sqrt{3}}(e^{\pi r/3} + e^{-\pi r/3})\right)\frac{h(r)}{e^{\pi r}+e^{-\pi r}}dr\right)$$

$$= \int_{-\infty}^\infty \left(\frac{1}{4} + \frac{1}{3\sqrt{3}}(e^{\pi r/3} + e^{-\pi r/3})\right)\frac{\sin(\alpha r)}{e^{\pi r}+e^{-\pi r}}\frac{G(r^2, 1)}{r}dr.$$

$$I_{10}(1) \equiv I\left(1, e^{(1/4)x}\sum_{\{P_0\}}\sum_{k=1}^\infty \frac{\log N(P_0)}{N(P_0)^{k/2} - N(P_0)^{-k/2}}\cdot g(k\log N(P_0))\right)$$

$$= \frac{1}{4\sqrt{\pi}}\sum_{\alpha \neq \log N(P)} \frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}\operatorname{sgn}(\alpha - \log N(P))$$

$$\times\left(\Gamma\left(\frac{1}{2}\right) - \int_0^{(1/4)(\alpha - \log(N(P))^2} y^{-(1/2)}e^{-y}dy\right)$$

$$+\frac{1}{4\sqrt{\pi}}\sum_{\{P\}}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}\int_{(1/4)(\alpha+\log N(P))^2}^\infty y^{-(1/2)}e^{-y}dy.$$

$$I_{11}(1) \equiv I\left(1, \frac{e^{(1/4)x}}{2\pi}\int_{-\infty}^\infty h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right)dr\right.$$

$$\left.-\frac{e^{(1/4)x}}{2\pi}\int_{-\infty}^\infty h(r)\frac{\Gamma'}{\Gamma}(1 + ir)dr\right)$$

$$=I\left(1, \frac{\log\pi}{2\pi}\int_{-\infty}^{\infty} e^{-r^2x}\sin(\alpha r)r\,dr\right)$$

$$+I\left(1, -\frac{1}{2\pi}\int_{0}^{\infty} e^{-r^2x}\sin(\alpha r)r\mathscr{G}(r)dr\right)$$

$$+I\left(1, -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-r^2x}\sin(\alpha r)r\cdot\frac{\zeta'}{\zeta}(1+2ir)dr\right)$$

$$=I_{12}(1)+I_{13}(1)+I_{14}(1), \text{ say, where we put}$$

$$\mathscr{G}(r)=\frac{\Gamma'}{\Gamma}(1+ir)+\frac{\Gamma'}{\Gamma}(1-ir)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-ir\right).$$

$$I_{12}(1)=I\left(1, \frac{\log\pi}{4\sqrt{\pi}}\alpha x^{-3/2}e^{-(1/4)\alpha^2x^{-1}}\right)=\frac{\log\pi}{2\sqrt{\pi}}\int_{(1/4)\alpha^2}^{\infty} x^{-(1/2)}e^{-x}dx.$$

$$I_{14}(1)=\frac{1}{2\sqrt{\pi}}\sum_{\alpha\neq 2\log n}\frac{\Lambda(n)}{n}\operatorname{sgn}(\alpha-2\log n)$$

$$\times\left(\Gamma\left(\frac{1}{2}\right)-\int_0^{(1/4)(\alpha-2\log n)^2} y^{-(1/2)}e^{-y}dy\right)$$

$$+\frac{1}{2\sqrt{\pi}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n}\int_{(1/4)(\alpha+2\log n)^2}^{\infty} y^{-1/2}e^{-y}dy$$

$$I_{13}(1)=I\left(1, -\frac{1}{2\pi}\int_0^1 e^{-r^2x}\sin(\alpha r)r\cdot\mathscr{G}(r)dr\right)$$

$$+I\left(1, -\frac{2}{\pi}\int_1^{\infty} e^{-r^2x}\sin(\alpha r)r\log r\,dr\right)$$

$$-\frac{b}{2\pi}I\left(1, \int_1^{\infty}e^{-r^2x}\sin(\alpha r)r^{-1}dr\right)$$

$$+I\left(1, -\frac{1}{2\pi}\int_1^{\infty} e^{-r^2x}\sin(\alpha r)r\cdot\mathscr{G}_1(r)dr\right)$$

$$=I_{15}(1)+I_{16}(1)-\frac{b}{2\pi}I_{17}(1)+I_{18}(1), \text{ say, where}$$

we remark that by Stirling's formula,

$$\mathscr{G}(r)=4\log r+\frac{b}{r^2}+\mathscr{G}_1(r) \quad \text{and} \quad \mathscr{G}_1(r)=O\left(\frac{1}{r^4}\right) \qquad \text{as } r\to\infty.$$

$$I_{15}(1)=-\frac{1}{2\pi}\int_0^1 \sin(\alpha r)r^{-1}G(r^2, 1)\mathscr{G}(r)dr.$$

$$I_{18}(1)=-\frac{1}{2\pi}\int_1^{\infty} \sin(\alpha r)r^{-1}\cdot\mathscr{G}_1(r)(1-e^{-r^2})dr.$$

$$I_{17}(1)=\alpha^{-4}\left(\alpha^{3}\cos\alpha(1-e^{-1})-\alpha^{3}\int_{1}^{\infty}\frac{(1-e^{-r^{2}})}{r^{4}}\cos(\alpha r)dr\right.$$
$$\left.-8\sqrt{\pi}\int_{(1/4)\alpha^{2}}^{\infty}x^{-5/2}e^{-x}dx+2\alpha^{3}\int_{0}^{1}\int_{0}^{1}xe^{-r^{2}x}\cos(\alpha r)drdx\right).$$

In a similar manner, we get

$$I_{16}(1)=-\frac{\cos\alpha}{\pi\alpha}\int_{0}^{1}\log x\cdot e^{-x}dx+\frac{1}{\alpha\sqrt{\pi}}\int_{1}^{\infty}x^{-(3/2)}e^{-(1/4)\alpha^{2}x}\left(1-\frac{1}{2}\log x\right)dx$$
$$-\frac{1}{\pi\alpha}\int_{0}^{1}\int_{0}^{1}\cos(\alpha r)e^{-xr^{2}}(2+\log x-\alpha^{2}r^{2}\log r\cdot\log x)drdx$$
$$-\frac{\alpha}{4\sqrt{\pi}}\left(\int_{1}^{\infty}x^{-(1/2)}\left(\int_{0}^{1}e^{-y}e^{-(1/4)\alpha^{2}x}-\frac{e^{-\alpha^{2}x/4(1+y)}}{\sqrt{1+y}}\right)\frac{dy}{y}dx\right.$$
$$+\int_{1}^{\infty}x^{-(1/2)}e^{-(1/4)\alpha^{2}x}\left(E(0)+\log x+\frac{\log x}{\sqrt{2}}e^{\alpha^{2}x/8}\right)dx$$
$$\left.-8\cdot\alpha^{-3}\int_{1}^{\infty}x^{-2}\int_{0}^{\alpha^{2}x/8}\frac{e^{-y}y^{1/2}}{1-(4y/\alpha^{2}x)}dydx\right).$$

Finally, we get

$$I_{19}(1)\equiv I(1,\ -\log 2\cdot g(0))$$
$$=-\frac{\log 2}{2\sqrt{\pi}}\int_{(1/4)\alpha^{2}}^{\infty}x^{-(1/2)}e^{-x}dx.$$

This completes our explicit evaluation of $Z_{\alpha}(1)$.

Taking the order of the magnitude as $\alpha\to\infty$ into account, we get our Theorem 1.

§ 3. Proof of Corollary 1

In this section we shall study further the main term in the formula of $Z_{\alpha}(1)$ in Theorem 1.

$$F(\alpha)\equiv\frac{1}{4\sqrt{\pi}}\sum_{\substack{e^{\alpha/2}<N(P)<e^{3\alpha/2}\\ \alpha\neq\log N(P)}}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}\operatorname{sgn}(\alpha-\log N(P))$$
$$\times\left(\Gamma\left(\frac{1}{2}\right)-\int_{0}^{(\alpha-\log N(P))^{2}/4}x^{-(1/2)}e^{-x}dx\right)$$
$$=-\frac{1}{4\sqrt{\pi}}\int_{e^{\alpha}}^{e^{3\alpha/2}}y^{-(1/2)}\left(\Gamma\left(\frac{1}{2}\right)-\int_{0}^{(\alpha-\log y)^{2}/4}x^{-(1/2)}e^{-x}dx\right)d(y+R(y))$$

$$+\frac{1}{4\sqrt{\pi}}\int_{e^{\alpha/2}}^{e^{\alpha}} y^{-(1/2)}\left(\Gamma\left(\frac{1}{2}\right)-\int_0^{(\alpha-\log y)^2/4} x^{-(1/2)}e^{-x}dx\right)d(y+R(y))$$

$=(\Sigma_1+\Sigma_2)+(\Sigma_3+\Sigma_4)$, say, where we put

$$\sum_{N(P)\leqq y}\tilde{\Lambda}(P)-y=R(y).$$

$$\Sigma_1=-\frac{1}{4\sqrt{\pi}}e^{\alpha/2}\int_0^{(1/2)\alpha} e^{(1/2)y}\left(\Gamma\left(\frac{1}{2}\right)-\int_0^{(1/4)y^2} x^{-(1/2)}e^{-x}dx\right)dy.$$

$$\Sigma_3=\frac{1}{4\sqrt{\pi}}e^{\alpha/2}\int_0^{(1/2)\alpha} e^{-(1/2)y}\left(\Gamma\left(\frac{1}{2}\right)-\int_0^{(1/4)y^2} x^{-(1/2)}e^{-x}dx\right)dy.$$

$$\begin{aligned}\Sigma_1+\Sigma_3&=-\frac{1}{2\sqrt{\pi}}e^{\alpha/2}(e^{(1/4)\alpha}+e^{-(1/4)\alpha})\left(\Gamma\left(\frac{1}{2}\right)-\int_0^{\alpha^2/16} x^{-(1/2)}e^{-x}dx\right)\\&\quad+\frac{1}{\sqrt{\pi}}e^{\alpha/2}\Gamma\left(\frac{1}{2}\right)-\frac{e^{(1/4)+(1/2)\alpha}}{2\sqrt{\pi}}\int_0^{1/}(e^{-(1/4)(y-1)^2}+e^{-(1/4)(y+1)^2})dy\\&=\frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}}e^{\alpha/2}-\frac{e^{(1/4)+(1/2)\alpha}}{\sqrt{\pi}}\int_0^{\infty}e^{-(1/4)x^2}dx+O(e^{-C\alpha^2})\\&=e^{\alpha/2}(1-e^{1/4})+O(e^{-C\alpha^2}),\end{aligned}$$

where C is a positive absolute constant.

$$\begin{aligned}\Sigma_2+\Sigma_4&=\frac{R(e^{\alpha})\Gamma(\frac{1}{2})}{2\sqrt{\pi}\,e^{\alpha/2}}+O(e^{-C\alpha^2})+\frac{1}{4\sqrt{\pi}}\int_{e^{\alpha}}^{e^{3\alpha/2}}R(y)H(y)dy\\&\quad-\frac{1}{4\sqrt{\pi}}\int_{e^{(1/2)\alpha}}^{e^{\alpha}}R(y)H(y)dy,\end{aligned}$$

where we put

$$\begin{aligned}H(y)&=y^{-3/2}\left(-\frac{1}{2}\Gamma\left(\frac{1}{2}\right)+\frac{1}{2}\int_0^{(\alpha-\log y)^2/4}x^{-1/2}e^{-x}dx-e^{-(\alpha-\log y)^2/4}\right)\\&=y^{-3/2}H_1(y),\ \text{say.}\end{aligned}$$

Here we remark the following lemma which can be proved in a standard way (cf. Section 11 of Hejhal [10]).

Lemma.

$$\begin{aligned}R(x)=O(\sum_{|N(P)-x|\leqq e^{-\alpha}}\tilde{\Lambda}(P))+O(x^2e^{\alpha}T^{-1})+O(x^{1/2}(\log x)^{-1})\\+\sum_{|r|\leqq T}\frac{x^{1/2+ir}}{\frac{1}{2}+ir}+\sum_{|\mathrm{Im}(\rho/2)|\leqq T}\frac{x^{\rho/2}}{\rho/2},\end{aligned}$$

where ρ runs over the non-trivial zeros of $\zeta(s)$, $\alpha>0$, T satisfies $T>x^2\gg 1$, $T\neq r$ or $\gamma/2$ and

$$\frac{\zeta'}{\zeta}(\sigma \pm i2T) \ll \log^2 T \qquad \textit{for } -1 \leqq \sigma \leqq 2.$$

Now

$$\Sigma_5 \equiv \int_{e^\alpha}^{e^{3\alpha/2}} R(y)H(y)dy$$
$$= \int_{e^\alpha}^{e^{3\alpha/2}} O\Big(\sum_{|N(P)-y| \leqq e^{-\alpha}} \tilde{\Lambda}(P)\Big)H(y)dy + \int_{e^\alpha}^{e^{3\alpha/2}} O\left(e^\alpha \frac{y^2}{T}\right)H(y)dy$$
$$+ O\left(\int_{e^\alpha}^{e^{3\alpha/2}} y^{1/2}H(y)/\log y dy\right) + \sum_{|r| \leqq T} \frac{1}{\frac{1}{2}+ir} \int_{e^\alpha}^{e^{3\alpha/2}} y^{(1/2)+ir} H(y)dy$$
$$+ \sum_{|\mathrm{Im}\rho/2| \leqq T} \frac{2}{\rho} \int_{e^\alpha}^{e^{3\alpha/2}} y^{\rho/2} H(y)dy$$
$$= \Sigma_6 + \Sigma_7 + \Sigma_8 + \Sigma_9 + \Sigma_{10}, \text{ say.}$$
$$\Sigma_6 \ll e^{-5\alpha/2} \sum_{N(P) \leqq 2e^{3\alpha/2}} \tilde{\Lambda}(P) \ll e^{-\alpha}.$$
$$\Sigma_7 \ll e^{13\alpha/4} \cdot T^{-1}.$$
$$\Sigma_8 \ll 1.$$
$$\Sigma_9 = \sum_{|r| \leqq T} \frac{1}{\frac{1}{2}+ir} \left(\left[\frac{y^{ir} H_1(y)}{ir} \right]_{e^\alpha}^{e^{3\alpha/2}} - \int_{e^\alpha}^{e^{3\alpha/2}} \frac{y^{ir}}{ir} H_1'(y)dy \right)$$
$$\ll \sum_{|r| \leqq T} \frac{1}{|r|^2} + \sum_{|r| \leqq T} \frac{1}{|r|^3}$$
$$\ll \log T.$$

$$\Sigma_{10} \ll \sum_{|\gamma| \leqq 2T} \frac{1}{|\gamma|} \left(\left[\frac{y^{\rho/2-1/2}}{(\rho/2)-\frac{1}{2}} H_1(y) \right]_{e^\alpha}^{e^{3\alpha/2}} - \frac{1}{(\rho/2)-\frac{1}{2}} \int_{e^\alpha}^{e^{3\alpha/2}} y^{\rho/2-1/2} H_1'(y)dy \right)$$
$$\ll \sum_{|\gamma| \leqq 2T} \frac{1}{|\gamma|^2}$$
$$\ll 1.$$

Thus we get, by choosing $T = e^{C\alpha}$ with some positive constant C,

$$\Sigma_5 \ll \alpha.$$

Similarly, we get

$$\int_{e^{(1/2)\alpha}}^{e^\alpha} R(y)H(y)dy \ll \alpha.$$

Thus we have proved that

$$F(\alpha)=e^{(1/2)\alpha}(1-e^{1/4})+\frac{R(e^{\alpha})}{2e^{\alpha/2}}+O(\alpha).$$

Since

$$\sum_{N(P)\geqq e^{3\alpha/2},N(P)\leqq e^{(1/2)\alpha}}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}\left(\Gamma\left(\frac{1}{2}\right)-\int_0^{(\alpha-\log N(P))^2/4}y^{-(1/2)}e^{-y}dy\right)$$
$$\ll e^{-C\alpha^2},$$

we get our corollary.

§ 4. Proof of Theorem 2

We shall evaluate $Z_{\alpha}(0)\Gamma(\frac{1}{2})$. We use the same notations as in the section 2. We shall write down explicitly the values of $I_j(s)$ at $s=\frac{1}{2}$ for each j.

$$I_1\left(\frac{1}{2}\right)=\sum_{r>0}\sin(\alpha r)\int_{r^2}^{\infty}y^{-(1/2)}e^{-y}dy.$$

$$I_3\left(\frac{1}{2}\right)=\frac{1}{4}(e^{(1/2)\alpha}-e^{-(1/2)\alpha})\int_0^1 x^{-(1/2)}e^{(1/4)x}dx.$$

$$I_7\left(\frac{1}{2}\right)=\frac{1}{6}\int_0^1 r\sin(\alpha r)G\left(r^2,\frac{1}{2}\right)dr.$$

$$\begin{aligned}6I_8\left(\frac{1}{2}\right)=&\frac{1}{2}\sin\alpha\cdot G\left(1,-\frac{1}{2}\right)+\frac{1}{4}(\alpha\cos\alpha+\sin\alpha)G\left(1,-\frac{3}{2}\right)\\&+\frac{1}{8}(\alpha\cos\alpha-\alpha^2\sin\alpha)G\left(1,-\frac{5}{2}\right)\\&-\frac{1}{16}\alpha^3\cos\alpha G\left(1,-\frac{7}{2}\right)-\frac{1}{4}F_1\left(-\frac{5}{2}\right)-\frac{1}{4}\alpha F_2\left(-\frac{7}{2}\right)\\&+\frac{1}{16}\alpha^4F_1\left(-\frac{9}{2}\right)+\frac{1}{8}\alpha^3F_2\left(-\frac{9}{2}\right),\end{aligned}$$

where

$$\begin{aligned}F_1\left(-\frac{5}{2}\right)=&4\cdot\alpha^{-2}\left(\frac{1}{3}\beta_1(1)+\frac{\sin\alpha}{6}\left(\Gamma\left(-\frac{1}{2}\right)-E\left(-\frac{1}{2}\right)\right)\right.\\&\left.+\frac{1}{4}\alpha\cos\alpha\left(\Gamma\left(-\frac{3}{2}\right)-E\left(-\frac{3}{2}\right)\right)+\frac{\alpha}{3}\beta_2(1)\right),\end{aligned}$$

$$F_1\left(-\frac{7}{2}\right)=\frac{12}{5\alpha^2}\left(F_1\left(-\frac{5}{2}\right)+\beta_1(1)+\frac{\sin\alpha}{2}\left(\Gamma\left(-\frac{3}{2}\right)-E\left(-\frac{3}{2}\right)\right)\right.$$

$$+\frac{5}{12}\alpha\cos\alpha\cdot\left(\Gamma\left(-\frac{5}{2}\right)-E\left(-\frac{5}{2}\right)\right)-\frac{\alpha}{3}\cdot\beta_2(1)\Big),$$

$$F_1\left(-\frac{9}{2}\right)=\frac{20}{7\alpha^2}\left(2F_1\left(-\frac{7}{2}\right)+\beta_1(1)+\frac{\sin\alpha}{2}\left(\Gamma\left(-\frac{5}{2}\right)-E\left(-\frac{5}{2}\right)\right)\right.$$

$$\left.+\frac{7}{20}\alpha\cos\alpha\left(\Gamma\left(-\frac{7}{2}\right)-E\left(-\frac{7}{2}\right)\right)+\frac{\alpha}{5}\cdot\beta_2(1)\right),$$

$$F_2\left(-\frac{7}{2}\right)=-\frac{1}{3}\left(2\cdot\beta_2(1)+\cos\alpha\left(\Gamma\left(-\frac{5}{2}\right)-E\left(-\frac{5}{2}\right)\right)-\alpha F_1\left(-\frac{7}{2}\right)\right)$$

and

$$F_2\left(-\frac{9}{2}\right)=-\frac{1}{5}\left(2\cdot\beta_2(1)+\cos\alpha\cdot\left(\Gamma\left(-\frac{7}{2}\right)-E\left(-\frac{7}{2}\right)\right)\right.$$

$$\left.-\alpha F_1\left(-\frac{9}{2}\right)\right).$$

$$I_6\left(\frac{1}{2}\right)=-\frac{1}{3}\int_0^1 x^{-(1/2)}\int_0^\infty \frac{r^2\sin(\alpha r)e^{-r^2x}}{e^{2\pi r}+1}\,dr\,dx.$$

$$I_9\left(\frac{1}{2}\right)=\int_0^1 x^{-(1/2)}\int_{-\infty}^\infty\left(\frac{1}{4}+\frac{1}{3\sqrt{3}}(e^{\pi r/3}+e^{-\pi r/3})\right)\frac{e^{-r^2x}\sin(\alpha r)r}{e^{\pi r}+e^{-\pi r}}\,dr\,dx.$$

$$I_{10}\left(\frac{1}{2}\right)=\frac{1}{2\sqrt{\pi}}\sum_{\{P\}}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}\frac{1}{(\alpha+\log N(P))}e^{-((\alpha+\log N(P))^2)/4}$$

$$+\frac{1}{2\sqrt{\pi}}\sum_{\alpha\neq\log N(P)}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}\frac{1}{(\alpha-\log N(P))}\cdot e^{-((\alpha-\log N(P))^2)/4}.$$

$$I_{12}\left(\frac{1}{2}\right)=\frac{\log\pi}{\sqrt{\pi}\,\alpha}e^{-(1/4)\alpha^2}.$$

$$I_{14}\left(\frac{1}{2}\right)=\frac{1}{\sqrt{\pi}}\sum_{\alpha\neq 2\log n}\frac{\Lambda(n)}{n}\frac{1}{(\alpha-2\log n)}e^{-((\alpha-2\log n)^2)/4}$$

$$+\frac{1}{\sqrt{\pi}}\sum_{n=2}^\infty\frac{\Lambda(n)}{n}\frac{1}{(\alpha+2\log n)}e^{-((\alpha+2\log n)^2)/4}$$

$$I_{15}\left(\frac{1}{2}\right)=-\frac{1}{2\pi}\int_0^1 x^{-(1/2)}\int_0^1 e^{-r^2x}\sin(\alpha r)r\mathscr{G}(r)\,dr\,dx.$$

$$-\frac{b}{2\pi}I_{17}\left(\frac{1}{2}\right)+I_{18}\left(\frac{1}{2}\right)=-\frac{b}{2\pi\alpha}\left(\cos\alpha\int_0^1 x^{-(1/2)}e^{-x}dx\right.$$

$$-2\int_1^\infty\frac{\cos(\alpha r)}{r^2}e^{-r^2}dr-2\sqrt{\pi}\int_0^1 e^{-(1/4)\alpha^2x^{-1}}dx$$

$$\left.+4\int_0^1 x^{1/2}\int_0^1\cos(\alpha r)e^{-xr^2}dr\,dx\right)$$

$$-\frac{1}{\pi}\int_1^\infty e^{-r^2}\sin(\alpha r)r\cdot\mathscr{G}_1(r)dr$$

$$+\frac{1}{\pi}\int_0^1 x^{1/2}\int_1^\infty r^3\cdot\mathscr{G}_1(r)e^{-r^2x}\sin(\alpha r)drdx.$$

$$I_{16}\Big(\frac{1}{2}\Big)=-\frac{1}{\pi\alpha}\Big(\cos\alpha\cdot G\Big(1,\,-\frac{1}{2}\Big)+2\int_1^\infty r^{-2}\cos(\alpha r)e^{-r^2}dr\Big)$$

$$-\frac{\alpha}{4\pi}\Big(\sqrt{\pi}\int_1^\infty\Big(\int_0^1 e^{-y}e^{-(1/4)\alpha^2x}-\frac{e^{-(\alpha^2x/4(1+y))}}{\sqrt{1+y}}\Big)\frac{dy}{y}dx$$

$$+\int_1^\infty e^{-(1/4)\alpha^2x}(\sqrt{\pi}\cdot\log x+E(0))dx$$

$$+\frac{4}{\alpha}\Big(\int_0^{\alpha^2/8}e^{-y}y^{-1/2}dy+\sqrt{8}\,\alpha^{-1}\cdot e^{-\alpha^2/8}\Big)$$

$$-8\cdot\alpha^{-3}\int_1^\infty x^{-(3/2)}\int_0^{\alpha^2x/8}\frac{e^{-y}y^{1/2}}{1-(4y/(\alpha^2x))}dydx\Big)$$

$$-\frac{2\alpha}{\pi}\Big(\int_0^1 e^{-r^2}\cos(\alpha r)\log r dr$$

$$+\int_0^1 x^{-(1/2)}\int_0^1 r^2e^{-r^2x}\cos(\alpha r)\log r drdx).$$

$$I_{19}\Big(\frac{1}{2}\Big)=-\frac{\log 2}{\alpha\sqrt{\pi}}e^{-(1/4)\alpha^2}.$$

Combining all these together, we get an explicit expression of $\Gamma(\frac{1}{2})Z_\alpha(0)$. Taking into account of the dependence on α, we get our Theorem 2.

§ 5. Proof of Corollary 1′

We shall prove Corollary 1′ using Corollary 1 and Corollary 2. Let $R(x)$ be the same as in the section 3.

By Corollary 2, we get

$$\sum_{n\leqq X}B(n)=\frac{1}{2}\sum_{N(P)\leqq\mathfrak{X}}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}$$

$$=\frac{1}{2}\int_c^{\mathfrak{X}}y^{-(1/2)}d(y+R(y))+O(1)$$

$$=\mathfrak{X}^{1/2}+\frac{1}{2}\mathfrak{X}^{-(1/2)}R(\mathfrak{X})+\frac{1}{4}\int_c^{\mathfrak{X}}y^{-3/2}R(y)dy+O(1),$$

where we put

$$\mathfrak{X}=\left(\frac{X+\sqrt{X^2-4}}{2}\right)^2.$$

The last integral can be treated by the section 3 and is

$$O(\log \mathfrak{X}).$$

We put $e^\alpha=\mathfrak{X}$. Then by Corollary 1, we get

$$\begin{aligned}Z_\alpha(1)&=\tfrac{1}{2}e^{-(1/2)\alpha}R(e^\alpha)+O(\alpha)\\&=\sum_{n\leqq e^{\alpha/2}+e^{-\alpha/2}}B(n)-e^{\alpha/2}+O(\alpha)\\&=(\sum_{n\leqq e^{\alpha/2}}B(n)-e^{\alpha/2})+O(\alpha).\end{aligned}$$

§ 6. Proof of Theorems 3 and 3′ and Corollary to Theorem 3′

Suppose that $n^2-4=Q^2D$ with the square free integer D. Let $\tilde{D}$ be the discriminant of the field $Q(\sqrt{n^2-4})$. We use the same notations as in Theorem 3. We remark first that

$$\begin{aligned}&\zeta(2s)\prod_p v_p(s,n)\\&=\zeta(2s)\prod_{\substack{p\nmid n^2-4\\p>2}}\left(1+\frac{1}{p^s}\right)\left(1-\frac{1}{p^s}\left(\frac{n^2-4}{p}\right)\right)^{-1}\sum_{\substack{p\mid n^2-4\\p>2}}v_p(s,n)v_2(s,n)\\&=\zeta(s)\prod_{\substack{p\nmid n^2-4\\p>2}}\left(1-\frac{1}{p^s}\left(\frac{n^2-4}{p}\right)\right)^{-1}\prod_{\substack{p\mid n^2-4\\p>2}}\left(1+\frac{1}{p^s}\right)^{-1}v_p(s,n)\left(1+\frac{1}{2^s}\right)^{-1}v_2(s,n),\end{aligned}$$

where $\left(\frac{n^2-4}{p}\right)$ is the Legendre symbol (cf. Kuznecov [13]). Hence we get

$$\begin{aligned}B(n)&=\prod_{\substack{p\nmid n^2-4\\p>2}}\left(1-\frac{1}{p}\left(\frac{n^2-4}{p}\right)\right)^{-1}\prod_{\substack{p\mid n^2-4\\p>2}}\left(1+\frac{1}{p}\right)^{-1}v_p(1,n)\left(1+\frac{1}{2}\right)^{-1}v_2(1,n)\\&=L(1,\chi)\left(1-\frac{1}{2}\chi(2)\right)\prod_{\substack{p\mid Q\\p>2}}\left(1-\frac{1}{p}\chi(p)\right)\prod_{\substack{p\mid n^2-4\\p>2}}\left(1+\frac{1}{p}\right)^{-1}v_p(1,n)\\&\quad\times\left(1+\frac{1}{2}\right)^{-1}v_2(1,n)\\&=L(1,\chi)F(n).\end{aligned}$$

On the other hand we see that

$$\frac{1}{2}\sum_{|\mathrm{tr}\,P|=n}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}=\frac{\log\dfrac{n+\sqrt{n^2-4}}{2}}{\sqrt{n^2-4}}\,\Phi(n).$$

Hence, by Corollary 2, we get

$$L(1, \chi)F(n)=\frac{\log \dfrac{n+\sqrt{n^2-4}}{2}}{\sqrt{n^2-4}}\Phi(n).$$

This proves our Theorem 3.

Let d be an integer satisfying $d^2|n^2-4$. Let $H(n, d)$ be the number of the classes $\{P\}$ such that $|\mathrm{tr}\, P|=n$ and $\Delta(P)=(\alpha-\delta, \beta, \mu)=d$, where $\begin{pmatrix}\alpha & \beta\\ \mu & \delta\end{pmatrix}$ is a representative of $\{P\}$. Then we remark first that

$$H(n, d)=h(n^2-4, 2d) \qquad \text{(cf. Andrianov-Fomenko [1]).}$$

To prove Theorem 3′ we have only to prove the following lemma, where we use the same notations as in the introduction.

Lemma.

$$\nu(P)=\nu(|\mathrm{tr}\, P|, \Delta(P)).$$

Proof.

Suppose that $|\mathrm{tr}\, P|=n$, η_n is the fundamental unit of $Q(\sqrt{n^2-4})$ and $\eta_n^k=(n+\sqrt{n^2-4})/2$ as in the introduction. We denote $\nu(n, \Delta(P))$ by ν_0. We put $m=\eta_n^{k/\nu_0}+\eta_n^{-k/\nu_0}$. For each $\nu|k$, we put $C(\nu)=(\eta_n^k-\eta_n^{-k})/(\eta_n^{k/\nu}-\eta_n^{-k/\nu})$. Let $\begin{pmatrix}\alpha & \beta\\ \mu & \delta\end{pmatrix}$ be a representative of $\{P\}$. We put

$$A=\begin{pmatrix}\dfrac{1}{2}\left(m+\dfrac{\alpha-\delta}{C(\nu_0)}\right) & \dfrac{\beta}{C(\nu_0)}\\[2ex] \dfrac{\mu}{C(\nu_0)} & \dfrac{1}{2}\left(m-\dfrac{\alpha-\delta}{C(\nu_0)}\right)\end{pmatrix}$$

Then it is easily seen that

$$A\in SL(2, Z) \quad \text{and} \quad A^{\nu_0}=\begin{pmatrix}\alpha & \beta\\ \mu & \delta\end{pmatrix}.$$

Here we remark that if $\{A'\}^{\nu}=\{P\}$, then $\nu|k$ and

$$\Delta(P)=\Delta(A'^{\nu})=C(\nu)\Delta(A').$$

Hence, the conjugacy class $\{A\}$ must be primitive, since otherwise there exists some integer $\nu'>1$ such that

$$\nu'\nu_0|k$$

and

$$C(\nu'\nu_0)|\Delta(P),$$

and it contradicts with the definition of ν_0. Q.E.D. of Lemma

We now proceed to prove Corollary to Theorem 3′. Let D be a square free integer $\geqq 2$. We use the same notations as in Corollary. We choose the integers n and Q ($\geqq 1$) such that

$$n^2-4=Q^2D$$

and $(n+Q\sqrt{D})/2$ is the smallest among (n', Q')'s which satisfy

$$n'\geqq 1.\ Q'\geqq 1,\ n'^2-4=Q'^2D \quad \text{and} \quad \frac{n'+Q'\sqrt{D}}{2}>1.$$

We remark that if $N\varepsilon(\tilde{D})=1$, then

$$\varepsilon(\tilde{D})=\frac{n+\sqrt{n^2-4}}{2}$$

and if $N\varepsilon(\tilde{D})=-1$, then

$$\varepsilon^2(\tilde{D})=\frac{n+\sqrt{n^2-4}}{2}.$$

Under the present situation we see that $\nu(P)=1$ and

$$\Phi(n)=\sum_{\substack{d^2|n^2-4\\ d>0}} h(n^2-4,\ 2d).$$

We shall rewrite $\Phi(n)$ further as follows. For $d^2|n^2-4$, let f_d ($\geqq 1$) be defined by

$$\frac{n^2-4}{d^2}=\tilde{D}f_d^2.$$

Then

$$f_d=\frac{Q}{2d}\,\Big|\,\frac{Q}{2} \qquad \text{if } \tilde{D}=4D,$$

and

$$f_d=\frac{Q}{d}\,\Big|\,Q \qquad \text{if } \tilde{D}=D.$$

By Theorem 4 in Chapter 2 and Problem 20 in Chapter 3 of Borevich-Shafarevich [2], we have

$$h(n^2-4,\ 2d)=2^{\delta(d)}h(\tilde{D})\frac{f_d}{e_{f_d}}\prod_{p\mid f_d}\left(1-\frac{\chi(p)}{p}\right),$$

where we put

$$\delta(d)=\begin{cases}0 & \text{if } N\varepsilon_{f_d}=-1,\\ 1 & \text{if } N\varepsilon_{f_d}=1,\end{cases}$$

ε_{f_d} is the fundamental unit of the order $\mathcal{O}_{f_d}$ and e_{f_d} is the index of the unit group of $\mathcal{O}_{f_d}$ in the unit group of the maximal order (cf. Borevich-Shafarevich [2]). Hence if $\tilde{D}=D$, then

$$\Phi(n)=\begin{cases}2h(\tilde{D})\sum_{d\mid Q} d\prod_{p\mid d}\left(1-\frac{\chi(p)}{p}\right) & \text{when } N\varepsilon(\tilde{D})=1,\\ h(\tilde{D})\sum_{d\mid Q} d\prod_{p\mid d}\left(1-\frac{\chi(p)}{p}\right) & \text{when } N\varepsilon(\tilde{D})=-1.\end{cases}$$

If $\tilde{D}=4D$, then

$$\Phi(n)=\begin{cases}2h(\tilde{D})\sum_{d\mid (Q/2)} d\prod_{p\mid d}\left(1-\frac{\chi(p)}{p}\right) & \text{when } N\varepsilon(\tilde{D})=1,\\ h(\tilde{D})\sum_{d\mid (Q/2)} d\prod_{q\mid d}\left(1-\frac{\chi(p)}{p}\right) & \text{when } N\varepsilon(\tilde{D})=-1.\end{cases}$$

Thus we have shown that

$$\frac{\log\frac{n+\sqrt{n^2-4}}{2}}{\sqrt{n^2-4}}\Phi(n)$$

$$=\begin{cases}\dfrac{2h(\tilde{D})\log\varepsilon(\tilde{D})}{Q\sqrt{\tilde{D}}}\sum_{d\mid Q} d\prod_{p\mid d}\left(1-\frac{\chi(p)}{p}\right) & \text{if } \tilde{D}=D,\\ \dfrac{2h(\tilde{D})\log\varepsilon(\tilde{D})}{(Q/2)\sqrt{\tilde{D}}}\sum_{d\mid (Q/2)} d\prod_{p\mid d}\left(1-\frac{\chi(p)}{p}\right) & \text{if } \tilde{D}=4D.\end{cases}$$

Thus our problem is reduced to show that

$$\left(1-\frac{1}{2}\chi(2)\right)\prod_{\substack{p\mid Q\\ p>2}}\left(1-\frac{\chi(p)}{p}\right)v_2(1,n)\left(1+\frac{1}{2}\right)^{-1}\prod_{\substack{p\mid Q\\ p>2}}\left(1+\frac{1}{p}\right)^{-1}v_p(1,n)$$

$$=\begin{cases}\dfrac{1}{Q}\sum_{d\mid Q} d\prod_{p\mid d}\left(1-\dfrac{\chi(p)}{p}\right) & \text{if } \tilde{D}=D,\\ \dfrac{1}{Q/2}\sum_{d\mid(Q/2)} d\prod_{p\mid d}\left(1-\dfrac{\chi(p)}{p}\right) & \text{if } \tilde{D}=4D.\end{cases}$$

We shall prove this in the next section.

§ 7. Completion of the proof of Corollary to Theorem 3′

We shall prove the last formula in the last section. We suppose first that n is odd. Then $\tilde{D}=D$ and $D\equiv 5\ (8)$. Hence $1-\frac{1}{2}\chi(2)=1+\frac{1}{2}$ and $v_2(1, n)=1$. Thus we have to show that

$$\prod_{\substack{p\mid Q\\ p>2}}\left(1-\frac{\chi(p)}{p}\right)\left(1+\frac{1}{p}\right)^{-1} v_p(1, n)=\frac{1}{Q}\sum_{d\mid Q} d\prod_{p\mid d}\left(1-\frac{\chi(p)}{p}\right).$$

Suppose that $Q=p_1^{g_1}p_2^{g_2}\cdots p_t^{g_t}$ with different odd primes $p_1, p_2, \cdots, p_t$. Then the right hand side is

$$\prod_{j=1}^{t} p_j^{-g_j}\sum_{d\mid p_j^{g_j}} d\prod_{p\mid d}\left(1-\frac{\chi(p')}{p'}\right).$$

So it is enough to show that for $p^g\,\|\,Q$,

$$p^{-g}\sum_{d\mid p^g} d\prod_{p'\mid d}\left(1-\frac{\chi(p')}{p'}\right)=\left(1-\frac{\chi(p)}{p}\right)\left(1+\frac{1}{p}\right)^{-1} v_p\,(1, n).$$

If $p\nmid D$, then $p^{2g}\,\|\,n-2$ or $p^{2g}\,\|\,n+2$.
If $p\mid D$, then $p^{2g+1}\,\|\,n-2$ or $p^{2g+1}\,\|\,n+2$.

We remark that if $n\equiv\pm 2\ (p^{2g})$, then

$$v_p(1, n)=\begin{cases}\dfrac{p+1}{p-1} & \text{if } \chi(p)=1,\\ \left(1+\dfrac{1}{p}\right)\dfrac{1-p^{-g}}{1-p^{-1}}+p^{-g} & \text{if } \chi(p)=-1\end{cases}$$

and that if $n\equiv\pm 2\ (p^{2g+1})$, then

$$v_p(1, n)=\left(1+\frac{1}{p}\right)\frac{1-p^{-(g+1)}}{1-p^{-1}}.$$

Thus if $p\nmid D$, then

$$\left(1-\frac{\chi(p)}{p}\right)\left(1+\frac{1}{p}\right)^{-1} v_p(1, n)$$
$$=\begin{cases} 1 & \text{if } \chi(p)=1, \\ \left(1+\dfrac{1}{p}\right)\dfrac{1-p^{-g}}{1-p^{-1}}+p^{-g} & \text{if } \chi(p)=-1. \end{cases}$$

On the other hand, if $p \nmid D$, then

$$p^{-g}\sum_{d \mid p^g} d \prod_{p' \mid d}\left(1-\frac{\chi(p')}{p'}\right)$$
$$=p^{-g}\left(1+\left(1-\frac{\chi(p)}{p}\right)\sum_{j=1}^{g} p^j\right)$$
$$=\begin{cases} 1 & \text{if } \chi(p)=1, \\ \left(1+\dfrac{1}{p}\right)\dfrac{1-p^{-g}}{1-p^{-1}}+p^{-g} & \text{if } \chi(p)=-1. \end{cases}$$

If $p \mid D$, then

$$\left(1-\frac{\chi(p)}{p}\right)\left(1+\frac{1}{p}\right)^{-1} v_p(1, n)=\frac{1-p^{-(g+1)}}{1\quad p^{-1}}$$

and

$$p^{-g}\sum_{d \mid p^g} d \prod_{p' \mid d}\left(1-\frac{\chi(p')}{p'}\right)=p^{-g}\sum_{j=0}^{g} p^j=\frac{1-p^{-(g+1)}}{1-p^{-1}}.$$

Thus we have proved our assertion for odd n.

We suppose next that n is even. We devide the present case into the cases I-1, I-2-1, $\cdots$ and III as follows

			U	V	W
$n=8k+2$	$k=2^{2c}k'$		2	$2+c$	I–1
	$k=2^{2c+1}k'$	$k'\equiv 1\ (4)$	1	$3+c$	I–2–1
		$k'\equiv 3\ (4)$	3	$3+c$	I–2–2
$n=8k-2$	$k=2^{2c}k'$		2	$2+c$	II–1
	$k=2^{2c+1}k'$	$k'\equiv 1\ (4)$	3	$3+c$	II–2–1
		$k'\equiv 3\ (4)$	1	$3+c$	II–2–2
$4 \mid n$			3	1	III

In the above table $k \geqq 1$, $c \geqq 0$ and $2 \nmid k'$. The column U denotes the residue class of D mod 4, the column V denotes the exponent of the power of 2 in Q and the column W denotes the number of the classification which we shall use below.

As is seen in the odd case, we have only to check the 2-part of Q.

(I-1) and (II-1). $\tilde{D}=4D$. $Q/2=2^{1+c}Q'$, $2 \nmid Q'$.

$$2^{-1-c} \sum_{d \mid 2^{1+c}} d \prod_{p \mid d} \left(1-\frac{\chi(p)}{p}\right)=2-2^{-1-c}.$$

$$v_2(1, n)=\left(1+\frac{1}{2}\right)\frac{1-2^{-c-2}}{1-2^{-1}}.$$

$$(1-\tfrac{1}{2}\chi(2))v_2(1, n)(1+\tfrac{1}{2})^{-1}=2-2^{-1-c}.$$

(I-2-2) and (II-2-1). $\tilde{D}=4D$. $Q/2=2^{2+c}Q'$, $2 \nmid Q'$,

$$2^{-2-c} \sum_{d \mid 2^{2+c}} d \prod_{p \mid d} \left(1-\frac{\chi(p)}{p}\right)=2-2^{-2-c}.$$

$$v_2(1, n)=(1+\tfrac{1}{2})(2-2^{-c-2}).$$

$$(1-\tfrac{1}{2}\chi(2))v_2(1, n)(1+\tfrac{1}{2})^{-1}=2-2^{-c-2}.$$

(I-2-1) and (II-2-2). $\tilde{D}=D$. $Q=2^{3+c}Q'$, $2 \nmid Q'$.

$$2^{-3-c} \sum_{d \mid 2^{3+c}} d \prod_{p \mid d} \left(1-\frac{\chi(p)}{p}\right)=2-2^{-3-c}.$$

$$v_2(1, n)=(1+\tfrac{1}{2})(2-2^{-3-c}).$$

$$(1-\tfrac{1}{2}\chi(2))v_2(1, n)(1+\tfrac{1}{2})^{-1}=2-2^{-3-c}.$$

(III). $\tilde{D}=4D$. $2 \nmid Q/2$.

$$(1-\tfrac{1}{2}\chi(2))v_2(1, n)(1+\tfrac{1}{2})^{-1}=1$$

$$\frac{1}{2/2} \sum_{d \mid 2/2} d \prod_{p \mid d} \left(1-\frac{\chi(p)}{p}\right)=1.$$

References

[1] A. N. Andrianov and O. M. Fomenko, Distribution of the norms of the hyperbolic elements of the modular group and the class number of indefinite binary quadratic forms, Soviet Math. Dokl., **12** (1971), 217–219.

[2] Z. I. Borevich and Shafarevich, Number Theory, Academic Press New York, London. 1966.

[3] J. Delsarte, Formules de Poisson avec reste, J. Anal. Math.. **17** (1966), 419–431.

[4] A. Fujii, On the uniformity of the distribution of the zeros of the Riemann zeta function (II), Comment. Math. Univ. Sancti Pauli, **31** (1982), 99–113.

[5] ——, The zeros of the Riemann zeta function and Gibbs's phenomenon,

Comment. Math. Univ. Sancti Pauli, **32** (1983), 229–248.

[6] ——, Zeros, Eigenvalues and Arithmetic, Proc. Japan Academy, **60** (A) (1984), 22–25.

[7] ——, A zeta function connected with eigenvalues of the Laplace-Beltrami operator on the fundamental domain of the modular group, Nagoya Math. J., **96** (1984), 167–174.

[8] I. S. Grashtein and I. M. Ryzhik, Tables of integrals, Series and Products, Academic Press, 1980.

[9] A. P. Guinand, A summation formula in the theory of prime numbers, Proc. London Math. Soc. Ser. 2, **50** (1945), 107–119.

[10] D. Hejhal, The Selberg trace formula for $PSL(2, R)$, Lecture Notes in Math. vol. 548, Springer, 1976.

[11] ——, The Selberg trace formula and the Riemann zeta function, Duke Math. J. **43** (1976), 441–482.

[12] N. V. Kuznecov, The distribution of norms of primitive hyperbolic classes of the modular group and asymptotic formulas for the eigenvalues of the Laplace-Beltrami operator on a fundamental region of the modular group, Soviet Math. Dokl., **19** (1978), 1053–1056.

[13] ——, Arithmetic form of the Selberg trace formula and the distribution of the norms of primitive hyperbolic classes of the modular group, Preprint, Habarovsk Complex Res. Inst. 1978.

[14] S. Minakshisundaram and A. Pleijel, Some properties of the eigen function of the Laplace operator on Riemannian manifolds, Canad. J. Math., **1** (1949), 242–256.

[15] K. Pracher, Primzahlverteilung, Springer, 1957.

[16] A. Selberg, Harmonic analysis and discontinuous group in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc., **20** (1956), 177–189.

[17] L. A. Takhtajan and A. I. Vinogradov, The Gauss-Hasse hypothesis on real quadratic fields with class number one, Crelle J., **335** (1983), 40–86.

[18] E. C. Titchmarsh, The Theory of the Riemann Zeta function, Oxford 1951.

[19] A. B. Venkov. Selberg's trace formula for the Hecke operator generated by an involution and the eigenvalues of the Laplace Beltrami operator on the fundamental domain of the modular group $PSL(2. Z)$, Math. USSR Izv.. **12** (1978), 448–462.

[20] ——, Remainder term in the Weyl-Selberg Asymptotic formula, J. Soviet Math., **17** (1981), 2083–2097.

Department of Mathematics
Rikkyo University
Nishi-Ikebukuro
Tokyo 171, Japan