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The Chern Classes and Kodaira Dimension of a Minimal Variety

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§ 1. Introduction

This paper deals with a sort of inequality for the first and second Chern classes of normal projective varieties with numerically effective canonical classes (Theorem 1.1); to some extent it is a continuation of the author's previous paper [Mil] in which the surface case was discussed. Our generalized inequality will be, however, farther-reaching in connexion with the classification theory of algebraic varieties developed by S. Iitaka, K. Ueno, M. Reid, E. Viehweg, S. Mori, Y. Kawamata and many others. For instance, we can derive the non-negativity of the Kodaira dimension for certain "*minimal*" threefolds (Theorem 1.2), which is a crucial step in the classification of threefolds after the construction of minimal models of non-uniruled varieties (the so-called "minimal model conjecture", see (6.5) below).

The precise statements of our results are as follows:

Theorem 1.1 (characteristic 0). *Let k be an algebraically closed field of characteristic* 0 *and X a normal projective $\boldsymbol{Q}$-Gorenstein variety of dimension $n \geqq 2$ over k with singular locus of codimension $\geqq 3$. Assume that the canonical divisor $K_X \in \operatorname{Pic}(X) \otimes \boldsymbol{Q}$ is numerically effective. Let $\rho\colon Y \to X$ be any resolution of the singularities. Then, for arbitrary ample divisors $H_1, \cdots, H_{n-2}$ on X, the inequality*

$$(3c_2(Y) - c_1^2(Y))(\rho^* H_1 \cdots \rho^* H_{n-2}) \geqq 0$$

holds. In particular, if $n=3$, the 1*-cycle $\rho_*(3c_2(Y) - c_1^2(Y))$ is pseudo-effective, i.e., its numerical class is a limit of those of effective rational* 1*-cycles.*

Theorem 1.2 (characteristic 0). *Let X be a normal projective threefold with only canonical singularities. Assume that the canonical divisor $K_X \in \operatorname{Pic}(X)_{\boldsymbol{Q}}$ is numerically effective. If X is Gorenstein or K_X^2 is numerically*

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non-trivial (in $A^2(X)_Q$), then there exists a positive integer r such that rK_X is a Cartier divisor linearly equivalent to an effective divisor, i.e., the Kodaira dimension of X is non-negative. (For the definition of canonical singularities, we refer to [R2].)

The proof of Theorem 1.1 heavily depends on two things that were not necessary (at least explicitly) in the surface case [Mi1]: the "generic semipositivity theorem" for cotangent bundles and the theory of semistable sheaves, especially the Bogomolov-Gieseker inequality.

The generic semipositivity theorem, first proved in [Mi3], asserts that the cotangent bundle of a smooth projective variety X is semipositive on a "generic curve" in X unless X is uniruled. The result will be quoted in Section 6 with minor modifications necessary for later use.

The notion of semistable sheaves was introduced by Mumford [Mu1] (on curves) and F. Takemoto (in higher dimension) for the study of the moduli problem of vector bundles. But it was Bogomolov who realized that semistability imposes certain constraints on geometric invariants ([B]; also compare Mumford's vanishing theorem [R1] as an ingenious application). This aspect of semistability does not seem to have been fully exploited so far, hence rather lengthy descriptions below.

Let us sketch the outline of this paper.

The subsequent three sections are devoted to discussions on semistable sheaves. The results there are more or less known or direct consequences of known facts, yet the author could not find a suitable reference in which they are presented in forms relevant to our purpose.

In Section 2, defining $\mathfrak{A}$-semistable sheaves which are generalizations of H-semistable sheaves in Takemoto's sense, we shall state a modified version of a theorem due to Mumford-Mehta-Ramanathan (without proof). In addition, we shall describe the behaviour of $\mathfrak{A}$-stability under the variation of $\mathfrak{A}$.

Section 3 contains a somewhat non-standard treatment. We shall give a numerical characterization for the semistable vector bundles on a curve defined over a field of characteristic 0 (the result was independently found by S. Mukai). This criterion, though quite easily proved, involves strikingly various connotations which will constitute the prerequisites to our arguments.

Section 4 deals with the so-called Bogomolov (-Gieseker) inequality for $\mathfrak{A}$-semistable sheaves. Our approach not only gives an elementary alternative to original proofs [B], [G2] but also provides a statement applicable to more general situations.

After a short digression to positive and mixed characteristics (§ 5), Section 6 will be a discussion on the "generic semipositivity theorem" for the cotangent sheaves of non-uniruled varieties in characteristic 0. One of the results proved here is the "*non-negativity of the second Chern class of minimal varieties*".

In Section 7, we show Theorem 1.1, where the results in the preceding sections are extensively used together with the techniques which have already appeared in the surface case.

Theorem 1.2 will be proved in Section 8. The proof in the Gorenstein case follows an idea of P.M.H. Wilson.

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Notation and Convention

In this article, *the ground field k will be of characteristic* 0 *except in* Sections 2 *and* 5. A *variety* over k will refer to a geometrically irreducible and reduced k-scheme of finite type. A *curve*, a *surface* or a *threefold* means a variety of dimension 1, 2 or 3 over the ground field. A *vector bundle* will be understood to be a locally free sheaf of finite rank. The rank of a *torsion free sheaf* means the rank at the generic point. All sheaves will be coherent unless otherwise mentioned.

A (coherent) subsheaf $\mathscr{F}$ of a torsion free sheaf $\mathscr{E}$ is said to be *saturated* if the quotient $\mathscr{E}/\mathscr{F}$ is again torsion free. $\mathscr{F}$ is a *subbundle* if both $\mathscr{F}$ and $\mathscr{E}/\mathscr{F}$ are locally free (of course this makes sense only in case $\mathscr{E}$ is a vector bundle). On a normal variety, a saturated subsheaf is a subbundle in codimension 1.

The Chern class $c_i(\mathscr{E})$ of a coherent sheaf is regarded as an element in the Chow group $A^i(X)$ generated by the cycles of codimension i (cf. [BS]).

By ***Q**-Weil* and ***Q**-Cartier divisors* we mean elements in $A^1(X)\otimes \boldsymbol{Q}$ and $\mathrm{Pic}(X)\otimes\boldsymbol{Q}$, respectively.

Besides standard notation in algebraic geometry, the following symbols will be freely used throughout this article:

$A^i(X)$ (resp. $A_i(X)$); the Chow group of codimension i (resp. dimension i).

$N^1(X)=\{\mathrm{Pic}(X)/\text{numerical equivalence}\}\otimes\boldsymbol{R}$.

$N_1(X)=\{A_1(X)/\text{numerical equivalence}\}\otimes\boldsymbol{R}$.

$NA(X)$: the ample cone $\subset N^1(X)$, viz. the open convex cone generated by the classes of real ample Cartier divisors.

$\overline{NA}(X)$: the closure of the ample cone; a Cartier divisor which represents its element is said to be *numerically effective*.

$NE(X)$: the effective cone $\subset N_1(X)$, viz. the convex cone generated by the real effective 1-cycles.

$\mathcal{O}(\mathbf{1}_{\mathcal{E}})$: the tautological line bundle on the projective bundle $\boldsymbol{P}(\mathcal{E})$ associated with a torsion free sheaf $\mathcal{E}$ on X.

$\lambda_{\mathcal{E}}=[c_1(\mathcal{O}(\mathbf{1}_{\mathcal{E}}))-(1/\mathrm{rank}\,\mathcal{E})\pi^*c_1(\mathcal{E})]\in N^1(\boldsymbol{P}(\mathcal{E}))$: the *normalized hyperplane class* of $\boldsymbol{P}(\mathcal{E})$ (when $c_1(\mathcal{E})$ is $\boldsymbol{Q}$-Cartier).

$\mathcal{S}^i\mathcal{E}$: the i-th symmetric tensor power of $\mathcal{E}$.

$\mathcal{T}_X$: the tangent sheaf.

Ω_X^1: the cotangent sheaf.

K_X: the canonical divisor $\in A^1(X)$ of a normal variety X; by definition, the closure of the canonical divisor of the smooth part of X.

$\boldsymbol{Q}_+^\times=\{x\in\boldsymbol{Q}\mid x>0\}$.

$\boldsymbol{R}_+=\{x\in\boldsymbol{R}\mid x\geqq 0\}$.

$h^i(X,\mathcal{E})=\dim_k H^i(X,\mathcal{E})$.

$\chi(X,\mathcal{E})=\sum(-1)^ih^i(X,\mathcal{E})$: the Euler characteristic.

$V_{\boldsymbol{Z}}$, $V_{\boldsymbol{Q}}$, etc.: an underlying $\boldsymbol{Z}$-module, $\boldsymbol{Q}$-vector space, etc. of an $\boldsymbol{R}$-vector space V.

§ 2. $\mathfrak{A}$-semistable sheaves

Let X be a normal projective variety of dimension n over an algebraically closed (or perfect) field k of arbitrary characteristic. Let $\mathcal{E}$ be a torsion free sheaf on X. The Chern classes $c_i(\mathcal{E})$ are defined as elements sitting in the Chow group $A^i(X)$, which is by definition the group of cycles of codimension i divided by rational equivalence. In particular, the first Chern class $c_1(\mathcal{E})$ sits in $A^1(X)$, the group of Weil divisors (modulo rational equivalence). When X is $\boldsymbol{Q}$-factorial (e.g. smooth) or $\det\mathcal{E}$ is $\boldsymbol{Q}$-Cartier (e.g. $\mathcal{E}$ is a vector bundle), $c_1(\mathcal{E})$ is a $\boldsymbol{Q}$-Cartier divisor and its numerical class in $N^1(X)$ is well-defined.

The *averaged first Chern class* of a torsion free sheaf $\mathcal{E}$ will refer to $\delta(\mathcal{E})=c_1(\mathcal{E})/(\mathrm{rank}\,\mathcal{E})\in A^1(X)_{\boldsymbol{Q}}$. For a given $(n-1)$-tuple $\mathfrak{A}=(h_1,\cdots,h_{n-1})$ of numerically effective $\boldsymbol{Q}$-Cartier divisors, the *averaged degree* with respect to $\mathfrak{A}$ will be the rational number $\delta_{\mathfrak{A}}(\mathcal{E})=\delta(\mathcal{E})h_1\cdots h_{n-1}$; it is well-defined since X is non-singular outside a closed subset of codimension at least two. $\mathcal{E}$ is said to be $\mathfrak{A}$-*semistable* if

$$\delta_{\mathfrak{A}}(\mathcal{F})\leqq\delta_{\mathfrak{A}}(\mathcal{E})$$

for every non-zero subsheaf $\mathcal{F}$ of $\mathcal{E}$. If $\mathcal{E}$ is $\mathfrak{A}$-semistable, then $\mathcal{E}\otimes\mathcal{L}$ as well as $\mathcal{E}^*$ is $\mathfrak{A}$-semistable, where $\mathcal{L}$ is a line bundle and $\mathcal{E}^*$ denotes the dual sheaf $\mathcal{H}om(\mathscr{E}, \mathcal{O}_X)$.

If $\mathfrak{A}=([H], \cdots, [H])$ (or $\phi, n=1$), we use the terminology "H-semistable" (or simply "semistable") instead of "$\mathfrak{A}$-semistable".

The semi-direct product of the symmetric group $\mathfrak{S}_{n-1}$ with the multiplicative group $(\boldsymbol{Q}_+^\times)^{n-1}$ acts on $\overline{NA}(X)_{\boldsymbol{Q}}^{n-1}$ in a natural way. The $\mathfrak{A}$-semistability and the $\mathfrak{B}$-semistability are mutually equivalent if $\mathfrak{A}$ and $\mathfrak{B}$ lie in the same orbit.

The following theorem is easily shown by a standard argument [HN]:

Theorem 2.1 ($\mathfrak{A}$-semistable or Harder-Narasimhan filtration). *Let $\mathscr{E}$ be a torsion free sheaf on X and $\mathfrak{A} \in \overline{NA}(X)_{\boldsymbol{Q}}^{n-1}$ as above. Then there exists a unique filtration*

$$\sum\nolimits_{\mathfrak{A}}: 0=\mathscr{E}_0 \subsetneqq \mathscr{E}_1 \subsetneqq \cdots \subsetneqq \mathscr{E}_s=\mathscr{E}$$

that has the following properties:

a) $\boldsymbol{Gr}_i(\sum_{\mathfrak{A}})=\mathscr{E}_i/\mathscr{E}_{i-1}$ *is a torsion free $\mathfrak{A}$-semistable sheaf;*

b) $\delta_{\mathfrak{A}}(\boldsymbol{Gr}_i(\sum_{\mathfrak{A}}))$ *is a strictly decreasing function in i.*

The coherent subsheaf $\mathscr{E}_1$ in the above $\mathfrak{A}$-semistable filtration is said to be the *maximal $\mathfrak{A}$-destabilizing subsheaf* of $\mathscr{E}$, which is characterized by the following two properties:

a) $\delta_{\mathfrak{A}}(\mathscr{E}_1) \geqq \delta_{\mathfrak{A}}(\mathscr{F})$ *for every coherent subsheaf $\mathscr{F}$ of $\mathscr{E}$;*

b) *If $\delta_{\mathfrak{A}}(\mathscr{F})=\delta_{\mathfrak{A}}(\mathscr{E}_1)$ for $\mathscr{F} \subset \mathscr{E}$, then $\mathscr{F} \subset \mathscr{E}_1$.*

The $\mathfrak{A}$-semistable filtration of the dual sheaf $\mathscr{E}^*$ is essentially the same as that of $\mathscr{E}$, with each entry substituted by the duals of the quotients $\mathscr{E}/\mathscr{E}_{s-i}$.

For a fixed torsion free sheaf $\mathscr{E}$, the $\mathfrak{A}$-semistable filtration depends on $\mathfrak{A}$, yet the dependence is not too bad:

Theorem 2.2. *Let $\mathscr{E}_1^{\mathfrak{A}} \subset \mathscr{E}$ denote the maximal $\mathfrak{A}$-destabilizing subsheaf for $\mathfrak{A} \in \overline{NA}(X)_{\boldsymbol{Q}}^{n-1}$.*

(1) *Let L be a (closed affine) segment joining $\mathfrak{A} \in \overline{NA}(X)_{\boldsymbol{Q}}^{n-1}$ and $\mathfrak{C} \in \overline{NA}(X)^{n-1}$. Let $\mathfrak{B}=(1-t)\mathfrak{A}+t\mathfrak{C}$ be a $\boldsymbol{Q}$-rational point on L ($t \in \boldsymbol{R}$, $0<t<1$). Then $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}})=\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ (in particular, $\mathscr{E}_1^{\mathfrak{B}} \subset \mathscr{E}_1^{\mathfrak{A}}$) whenever $0<t<\varepsilon$. Here the positive constant $\varepsilon=\varepsilon(\mathscr{E}; \mathfrak{A}, \mathfrak{C})$ depends continuously on $\mathfrak{C}$ provided $\mathscr{E}$ and $\mathfrak{A}$ are fixed.*

(2) *Let $K \subset \overline{NA}(X)^{n-1}$ be a compact subset and $\mathfrak{A} \in \overline{NA}(X)_{\boldsymbol{Q}}^{n-1}$ a $\boldsymbol{Q}$-rational point away from K. Let $\mathfrak{A}\# K$ stand for the join of $(\mathfrak{A}, K)$, i.e., the union of the segments joining $\mathfrak{A}$ and K. There exists an open neibourhood $U \subset N^1(X)^{n-1}$ of $\mathfrak{A}$ such that $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}})=\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ for every $\mathfrak{B} \in U \cap (\mathfrak{A}\# K) \cap \overline{NA}(X)_{\boldsymbol{Q}}^{n-1}$.*

(3) *If* $\mathfrak{A} \in NA(X)^{n-1}_{Q}$, *then there exists an open neibourhood* $U \subset NA(X)^{n-1}_{Q}$ *of* $\mathfrak{A}$ *such that* $\delta_{\mathfrak{A}}(\mathscr{E}^{\mathfrak{B}}_{1}) = \delta_{\mathfrak{A}}(\mathscr{E}^{\mathfrak{A}}_{1})$ *for every* $\mathfrak{B} \in U$.

Proof. For simplicity, we show the case $n=2$ only; the proof is quite similar for higher dimensions.

(1) If $\mathscr{E}^*(H)$ is generated by global sections, we have

$$\delta_{\mathfrak{C}}(\mathscr{E}^{\mathfrak{B}}_{1}) \leqq \delta_{\mathfrak{C}}(\mathscr{O}_X(H)) = c = c(\mathscr{E}, \mathfrak{C}).$$

Put $c' = c'(\mathscr{E}, \mathfrak{A}, \mathfrak{C}) = \delta_{\mathfrak{C}}(\mathscr{E}^{\mathfrak{A}}_{1})$. By the definition of the maximal destabilizing sheaves, we get

$$\delta_{\mathfrak{B}}(\mathscr{E}^{\mathfrak{A}}_{1}) \leqq \delta_{\mathfrak{B}}(\mathscr{E}^{\mathfrak{B}}_{1}).$$

$\delta_{\mathfrak{B}}(*)$ is a linear function in $\mathfrak{B} = (1-t)\mathfrak{A} + t\mathfrak{C}$ and this inequality is rewritten as follows:

$$(1-t)\delta_{\mathfrak{A}}(\mathscr{E}^{\mathfrak{A}}_{1}) + t\delta_{\mathfrak{C}}(\mathscr{E}^{\mathfrak{A}}_{1}) \leqq (1-t)\delta_{\mathfrak{A}}(\mathscr{E}^{\mathfrak{B}}_{1}) + t\delta_{\mathfrak{C}}(\mathscr{E}^{\mathfrak{B}}_{1}).$$

Hence

$$\begin{aligned}\delta_{\mathfrak{A}}(\mathscr{E}^{\mathfrak{B}}_{1}) \leqq \delta_{\mathfrak{A}}(\mathscr{E}^{\mathfrak{A}}_{1}) &\leqq \delta_{\mathfrak{A}}(\mathscr{E}^{\mathfrak{B}}_{1}) + \frac{t}{1-t}(\delta_{\mathfrak{C}}(\mathscr{E}^{\mathfrak{B}}_{1}) - \delta_{\mathfrak{C}}(\mathscr{E}^{\mathfrak{A}}_{1})) \\ &\leqq \delta_{\mathfrak{C}}(\mathscr{E}^{\mathfrak{B}}_{1}) + \frac{t}{1-t}(c-c').\end{aligned}$$

Note that $\delta(\mathscr{E}^{\mathfrak{A}}_{1}), \delta(\mathscr{E}^{\mathfrak{B}}_{1}) \in (1/r!)A^1(X)_Z$, while $\mathfrak{A} \in (1/m)N^1(X)_Z$ for some positive integer m. Therefore, if

$$\frac{t}{1-t}(c-c') < \frac{1}{r!m},$$

then $\delta_{\mathfrak{A}}(\mathscr{E}^{\mathfrak{B}}_{1}) = \delta_{\mathfrak{A}}(\mathscr{E}^{\mathfrak{A}}_{1})$. Now the assertion immediately follows (e.g., put $1/\varepsilon = r!m\{2+(c-c')\}$).

(2) Let U be the open ball centred at $\mathfrak{A}$ with radius r, where $r = \inf_{\mathfrak{C} \in K} \varepsilon(\mathscr{E}; \mathfrak{A}, \mathfrak{C}) d(\mathfrak{A}, \mathfrak{C})$, $d(\ ,\ )$ standing for a Euclidean distance.

(3) Let $K \subset NA(X)$ be a sphere centred at $\mathfrak{A}$ and apply (2).

Corollary 2.3. *Let* $K \subset \overline{NA}(X)^{n-1}$ *be a compact subset away from* $\mathfrak{A} \in \overline{NA}(X)^{n-1}_{Q}$. *Then the* $\mathfrak{B}$*-semistable filtration is a refinement of the* $\mathfrak{A}$*-semistable filtration for* $\mathfrak{B} \in (A\sharp K)_Q$ *sufficiently near* $\mathfrak{A}$.

Corollary 2.4. *Let* $\mathscr{E}$ *be a given torsion free sheaf on* X.

(1) *The* $\mathfrak{A}$*-semistability of* $\mathscr{E}$ *is a closed condition for* $\mathfrak{A} \in NA(X)^{n-1}_{Q}$.

(2) *The length of the* $\mathfrak{A}$*-semistable filtration of* $\mathscr{E}$ *is lower semicontinuous in* $\mathfrak{A} \in NA(X)^{n-1}_{Q}$, *while* rank $\mathscr{E}^{\mathfrak{A}}_{1}$ *is upper semicontinuous.*

(3) $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ *is a continuous, piecewise multilinear function on* $NA(X)_{\boldsymbol{Q}}^{n-1}$ *and continuous on any* (*$\boldsymbol{Q}$-rational*) *segment* $\subset \overline{NA}(X)_{\boldsymbol{Q}}^{n-1}$.

The following theorem is very important and illuminative for the understanding of $\mathfrak{A}$-semistable sheaves:

Theorem 2.5 (Mumford, Mehta-Ramanathan). *Let $\mathscr{E}$ be a torsion free sheaf on a normal projective variety X of dimension n over an algebraically closed field. Let $\mathfrak{A}=(h_1, \cdots, h_{n-1})$ be an $(n-1)$-tuple of ample $\boldsymbol{Q}$-Cartier divisors and H_{n-1} an integral Cartier divisor such that $\boldsymbol{Q}_+h_{n-1}=\boldsymbol{Q}_+[H_{n-1}]$. Let m be a large integer and Y a sufficiently generic member of the linear system $|mH_{n-1}|$. Put $\mathfrak{A}_Y=(h_1, \cdots, h_{n-2})|_Y \in NA(Y)_{\boldsymbol{Q}}^{n-2}$. Then the maximal $\mathfrak{A}_Y$-destabilizing subsheaf of $\mathscr{E}|_Y$ extends to a saturated subsheaf of $\mathscr{E}$. In particular, $\mathscr{E}$ is $\mathfrak{A}$-semistable if and only if $\mathscr{E}|_Y$ is $\mathfrak{A}_Y$-semistable* ($n=\dim_k X\geqq 2$).

The paper [MR] gives a proof when $\mathfrak{A}=(h, \cdots, h)$ and X is smooth, but the argument (essentially the theory of Hilbert schemes) remains valid in our situation.

Remark 2.6. Over $\boldsymbol{C}$, the adjective "*sufficiently generic*" above, which means the member concerned belongs to the complement of a countable union of proper closed subschemes of the parameter space (the linear system in this case), may be replaced by "*generic*" or "*general*" thanks to (3.13) below.

Remark 2.7. In case X, $\mathscr{E}$ and $\mathfrak{A}$ are defined over a perfect subfield k_0 of k, $\mathfrak{A}$-semistable filtration of $\mathscr{E}$ is also defined over k_0 by its unicity. Thus, in characteristic 0, the field of definition is not essential for the $\mathfrak{A}$-semistability. From geometric point of view, semistability behaves more naturally or functorially than *stability* (e.g. under the base change, tensor products, etc.); see the next section.

§ 3. A numerical criterion for semistability on curves

Throughout this section, the ground field will always be of characteristic 0 except in (3.2).

Let C be a complete curve defined over a (not necessarily algebraically closed) field k and assume that C is geometrically irreducible and smooth, i.e., $C\otimes_k\bar{k}$ is irreducible and smooth over the algebraic closure $\bar{k}$ of k. Let $\mathscr{E}$ be a vector bundle of rank r on C and $\pi: \boldsymbol{P}(\mathscr{E})\to C$ the associated projective bundle. Then the semistability of $\mathscr{E}$ is characterized by certain numerical properties of the *normalized hyperplane class* $\lambda_{\mathscr{E}}$, which is defined below.

Let $\mathcal{O}(\mathbf{1}_{\mathscr{E}})$ denote the tautological line bundle on $\boldsymbol{P}(\mathscr{E})$ and $\lambda_{\mathscr{E}}$ the numerical class of $c_1(\mathcal{O}(\mathbf{1}_{\mathscr{E}}))-\pi^*\delta(\mathscr{E}) \in N^1(\boldsymbol{P}(\mathscr{E}))_{\boldsymbol{Q}}$; $r\lambda_{\mathscr{E}}$ is the class of the relative anti-canonical divisor $-K_{\boldsymbol{P}(\mathscr{E})}+\pi^*K_C$. $\lambda_{\mathscr{E}}$ is uniquely determined by two properties: (a) $\lambda_{\mathscr{E}}^r=0$ and (b) $\lambda_{\mathscr{E}}$ on each fibre is numerically equivalent to the hyperplane.

It is easy to show that

$$N^1(\boldsymbol{P}(\mathscr{E}))=\boldsymbol{R}\lambda_{\mathscr{E}}\oplus\pi^*N^1(X), \qquad N_1(\boldsymbol{P}(\mathscr{E}))=\lambda_{\mathscr{E}}^{r-2}N^1(\boldsymbol{P}(\mathscr{E})).$$

Theorem 3.1 (characteristic 0). *The following five conditions are equivalent*:

1) $\mathscr{E}$ *is semistable*;

2) $\lambda_{\mathscr{E}}$ *is numerically effective*;

3) $\overline{NA}(\boldsymbol{P}(\mathscr{E}))=\boldsymbol{R}_+\lambda_{\mathscr{E}}+\boldsymbol{R}_+\pi^*d$, *where* d *is a positive generator of* $N^1(C)_{\boldsymbol{Z}}\simeq\boldsymbol{Z}$;

4) $\overline{NE}(\boldsymbol{P}(\mathscr{E}))=\boldsymbol{R}_+\lambda_{\mathscr{E}}^{r-1}+\boldsymbol{R}_+\lambda_{\mathscr{E}}^{r-2}\pi^*d$;

5) *Every effective divisor on* $\boldsymbol{P}(\mathscr{E})$ *is numerically effective.*

The equivalence between 3) and 4) is straightforward by Kleiman's criterion for ampleness [K1]. To prove the implication 1)$\Rightarrow$2), we need the following:

Proposition 3.2. *Let f be a separable surjective k-morphism of a smooth complete curve C' onto C. $\mathscr{E}$ is semistable if and only if $f^*\mathscr{E}$ is semistable.*

Proof. The "if" part is trivial. Let us prove the "only if" part. Let $\mathscr{E}$ be a semistable bundle on C. Without loss of generality, we may assume that f is a Galois morphism, with G being the Galois group. G acts on $f^*\mathscr{E}$ in a natural way. Let $\mathscr{F}_1$ be the maximal destabilizing subbundle of $f^*\mathscr{E}$. For any $g\in G$, $g^*\mathscr{F}_1=\mathscr{F}_1$ thanks to the unicity of $\mathscr{F}_1$. Hence there exists a subbundle $\mathscr{E}_1$ of $\mathscr{E}$ such that $f^*\mathscr{E}_1=\mathscr{F}_1$. By the semistability, $\mathscr{E}_1=\mathscr{E}$, $\mathscr{F}_1=f^*\mathscr{E}$. This proves the assertion.

Proof of Theorem 3.1.

1)$\Rightarrow$2): Suppose that $\lambda_{\mathscr{E}}$ is not numerically effective; i.e., there exists an irreducible curve $C'\subset\boldsymbol{P}(\mathscr{E})$ with $C'\lambda_{\mathscr{E}}<0$. Clearly C' is mapped surjectively onto C. Then, by some base change $f\colon C''\to C$, the multi-section C' becomes a union of cross sections C_i'' on the projective bundle $\boldsymbol{P}(f^*\mathscr{E})$ over C''. The intersection number $C_i''\lambda_{f^*\mathscr{E}}$ is evidently negative. There is a natural surjection $f^*\mathscr{E}=\pi''_*\mathcal{O}_{\boldsymbol{P}(f^*\mathscr{E})}(\mathbf{1}_{f^*\mathscr{E}})\to\pi''_*\mathcal{O}_{C_i''}(\mathbf{1}_{f^*\mathscr{E}})$. The line bundle $\pi''_*\mathcal{O}_{C_i''}(\mathbf{1}_{f^*\mathscr{E}})\simeq\mathcal{O}_{C_i''}(\mathbf{1}_{f^*\mathscr{E}})$ has degree $C_i''\lambda_{f^*\mathscr{E}}+\delta(f^*\mathscr{E})<\delta(f^*\mathscr{E})$ so that $f^*\mathscr{E}$ is unstable; hence $\mathscr{E}$ is unstable by (3.2).

2) $\Rightarrow$ 3), 4). If $\lambda_{\mathscr{E}}^{r-2}(a\lambda_{\mathscr{E}}+b\pi^*d)$ is pseudoeffective and $\lambda_{\mathscr{E}}$ is numerically effective, then $b=\lambda_{\mathscr{E}}^{r-1}(a\lambda_{\mathscr{E}}+b\pi^*d)\geqq 0$.

3), 4) $\Rightarrow$ 5). Since $\lambda_{\mathscr{E}}$ is numerically effective, $\lambda_{\mathscr{E}}+\varepsilon\pi^*d$ is ample for any positive real number ε. Assume that $a\lambda_{\mathscr{E}}+b\pi^*d$ is an effective divisor. Then the 1-cycles $(a\lambda_{\mathscr{E}}+b\pi^*d)(\lambda_{\mathscr{E}}+\varepsilon\pi^*d)^{r-2}$ and their limit $(a\lambda_{\mathscr{E}}+b\pi^*d)\lambda_{\mathscr{E}}^{r-2}$ sit in the closure of the effective cone $\mathrm{NE}(\boldsymbol{P}(\mathscr{E}))$. Hence $a, b\geqq 0$ by 4) and $a\lambda_{\mathscr{E}}+b\pi^*d$ is numerically effective by 3).

5) $\Rightarrow$ 1). Suppose that $\mathscr{E}$ is unstable and let $\mathscr{E}_1$ be the maximal destabilizing subbundle. Let α be a rational number with $\delta(\mathscr{E}_1)>\alpha>\delta(\mathscr{E})$. Then, by the Riemann-Roch theorem,

$$\begin{aligned} H^0(C, \mathscr{S}^N\mathscr{E}_1(-N\alpha d)) &\subset H^0(C, \mathscr{S}^N\mathscr{E}(-N\alpha d)) \\ &\simeq H^0(\boldsymbol{P}(\mathscr{E}), \mathcal{O}(N_{\mathscr{E}})\otimes\pi^*\mathcal{O}_C(-N\alpha d)) \end{aligned}$$

is non-trivial for sufficiently large N. Hence $N\{\lambda_{\mathscr{E}}+(\delta(\mathscr{E})-\alpha)\pi^*d\}$ is effective but clearly not numerically effective.

For convenience' sake, let us introduce new concepts. Let D be a $\boldsymbol{Q}$-Cartier divisor on a normal projective variety X and $\mathscr{E}$ a vector bundle (or torsion free sheaf). The symbol $\mathscr{E}(D)$ is called a *$\boldsymbol{Q}$-vector bundle* (or *$\boldsymbol{Q}$-torsion free sheaf*). A $\boldsymbol{Q}$-torsion free sheaf $\mathscr{F}=\mathscr{E}(D)$ is said to be *ample* or *semipositive* if $\mathbf{1}_{\mathscr{E}}+\pi^*D$ is ample or numerically effective. Since there exists a finite covering $f\colon X'\to X$ such that the pull-back f^*D is an integral Cartier divisor [BG], $f^*\mathscr{F}=f^*\mathscr{E}(f^*D)$ is a well-defined sheaf on X'; $\mathscr{F}$ is ample or semipositive if and only if so is $f^*\mathscr{F}$ in the usual sense [H1].

If $\mathscr{F}=\mathscr{E}(D)$ is $\boldsymbol{Q}$-torsion free, the dual $\boldsymbol{Q}$-torsion free sheaf $\mathscr{F}^*$ of course refers to $\mathscr{E}^*(-D)$. If $\mathscr{F}^*$ is ample or semipositive, $\mathscr{F}$ is said to be *negative* or *seminegative*. The tensor product of $\mathscr{E}_i(D_i)$ will stand for $(\mathscr{E}_1\otimes\cdots\otimes\mathscr{E}_s)(D_1+\cdots+D_s)$. Similarly, we can naturally define symmetric products, exterior products, direct sums, etc. of $\boldsymbol{Q}$-torsion free sheaves.

The following assertion is a matter of triviality:

Proposition 3.3. *The direct sums, tensor products, symmetric tensor products and exterior products of ample* (*or semipositive*) *$\boldsymbol{Q}$-torsion free sheaves are all ample* (*or semipositive*).

Let us return to curves. Theorem 3.1 will be rephrased via the notion of $\boldsymbol{Q}$-vector bundles:

Theorem 3.4. *Let $\mathscr{E}$ be a vector bundle on C. Then $\mathscr{E}$ is semistable if and only if $\mathscr{E}(-\delta(\mathscr{E}))$ is semipositive.*

Since $\mathscr{E}$ is semistable if and only if $\mathscr{E}^*$ is semistable, this also implies:

Theorem 3.4′. *$\mathscr{E}$ is semistable if and only if $\mathscr{E}(-\delta(\mathscr{E}))$ is seminegative.*

Let $\mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = \mathscr{E}$ be the semistable filtration of $\mathscr{E}$. Since $\mathscr{G}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$ is semistable and $\deg \delta(\mathscr{G}_i)$ is decreasing in i, $\mathscr{G}_i(-\delta(\mathscr{E}_1))$ is seminegative. Conversely, if $\deg D$ is smaller than $\deg \delta(\mathscr{E}_1)$ for a $\boldsymbol{Q}$-divisor D, then $\mathscr{E}(-D)$, containing an ample $\boldsymbol{Q}$-bundle $\mathscr{E}_1(-D)$, is never seminegative. Hence:

Corollary 3.5. *$\mathscr{E}(-D)$ is seminegative (or negative) if and only if $\deg D \geqq \deg \delta(\mathscr{E}_1)$ (or $>$), where $\mathscr{E}_1$ is the maximal destabilizing subbundle of $\mathscr{E}$. Similarly, $\mathscr{E}(D)$ is semipositive (or positive) if and only if $\deg D \geqq \deg \delta((\mathscr{E}^*)_1)$ (or $>$).*

Corollary 3.6 (cf. [H1]). *A semistable vector bundle $\mathscr{E}$ on C is ample (resp. semipositive, seminegative, negative) if and only if its degree is positive (resp. non-negative, non-positive, negative).*

Corollary 3.7. *Let $\mathscr{E}$ and $\mathscr{F}$ be semistable bundles on C. Then $\mathscr{E} \otimes \mathscr{F}$ and $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ are also semistable.*

Proof. $\mathscr{E} \otimes \mathscr{F}(-\delta(\mathscr{E} \otimes \mathscr{F})) = (\mathscr{E}(-\delta(\mathscr{E}))) \otimes (\mathscr{F}(-\delta(\mathscr{F})))$ is semipositive and we have the first assertion. The second is trivial because $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ is isomorphic to $\mathscr{E}^* \otimes \mathscr{F}$.

Corollary 3.8. *Let $\mathscr{E}$ and $\mathscr{F}$ be two vector bundles. $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ is negative if and only if $\deg \delta(\mathscr{F}_1) + \deg \delta((\mathscr{E}^*)_1) < 0$. In particular, $\mathscr{H}om(\mathscr{E}_1, \mathscr{E}/\mathscr{E}_1)$ is negative.*

Corollary 3.9. *Let ρ: $GL(r) \to GL(s)$ be a polynomial representation which maps the scalar matrices to scalars. Let $\mathscr{E}^\rho$ be the vector bundle of rank s induced by a rank r vector bundle $\mathscr{E}$ and the representation ρ. If $\mathscr{E}$ is semistable, then so is $\mathscr{E}^\rho$.*

Corollary 3.10. *A vector bundle $\mathscr{E}$ is semistable if and only if $\mathscr{S}^n\mathscr{E}$ is semistable ($n \geqq 2$).*

Proposition 3.11. *For a vector bundle $\mathscr{E}$ on C, the following conditions are equivalent:*

(1) *$\mathscr{E}$ is semistable;*

(2) *$\mathscr{E}(-D)$ is negative where D is a $\boldsymbol{Q}$-divisor of degree $\delta(\mathscr{E}) + (1/2r!)$.*

Proof. The implication (1)$\Rightarrow$(2) has been shown above. Assume (2) and let $\mathscr{E}_1$ be the maximal destabilizing subsheaf. Then $\mathscr{E}(-D)$ is negative only if $\deg D > \deg \delta(\mathscr{E}_1)$ so that $\deg \delta(\mathscr{E}) \leqq \deg \delta(\mathscr{E}_1) < \delta(\mathscr{E}) + (1/2r!)$. On the other hand, both $\deg \delta(\mathscr{E}_1)$ and $\deg \delta(\mathscr{E})$ sit in $(1/r!)\boldsymbol{Z}$. Hence we have $\deg \delta(\mathscr{E}_1) = \deg \delta(\mathscr{E})$.

Corollary 3.12. *Let $\mathscr{C} \to T$ be a proper smooth family of irreducible curves, where $\mathscr{C}$ and T are k-varieties. Let $\mathscr{E}$ be a vector bundle on $\mathscr{C}$. Then the set*

$$S(T) = \{t \in T \mid \mathscr{E} \text{ is semistable on } C_t\}$$

is a Zariski open subset of T.

Corollary 3.13 (A modified version of Mumford-Mehta-Ramanathan's theorem). *Let X be a normal projective variety of dimension n over $\boldsymbol{C}$ and $\mathscr{E}$ a torsion free sheaf. Let $H_1, \cdots, H_{n-1}$ be ample Cartier divisors. Then, for sufficiently large integers $m_1, \cdots, m_{n-1}$, the maximal destabilizing subbundle $\mathscr{F}$ of $\mathscr{E}|_C$ extends to a saturated subsheaf of $\mathscr{E}$ on X if C is a general complete intersection curve of $|m_i H_i|$'s.* (Such an extension of $\mathscr{F}$ is necessarily the maximal $(H_1, \cdots, H_{n-1})$-destabilizing subsheaf of $\mathscr{E}$ hence unique.)

Remark 3.14. Over $\boldsymbol{C}$, the results (3.7), (3.9), (3.10) are well-known as easy consequences of Narasimhan-Seshadri's deep theorem [NS] which asserts that $\mathscr{E}$ is semistable if and only if it is induced by a unitary representation of the fundamental group.

§ 4. The Bogomolov-Gieseker inequality for semistable sheaves

In this section, the ground field k is always algebraically closed of characteristic 0.

Lemma 4.1. *Let X be a normal projective variety of dimension n and $\mathfrak{A} \in NA(X)^{n-1}$. Let $\mathscr{E}$ be an $\mathfrak{A}$-semistable torsion free sheaf on X, with its first Chern class being a $\boldsymbol{Q}$-Cartier divisor. Let D be a non-zero effective Cartier divisor on X. Then*

$$H^0(X, \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E}) - D)) = 0$$

for every positive integer t such that $tc_1(\mathscr{E})$ is an integral Cartier divisor.

Since $\mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E})) = \mathscr{S}^{rt}\{\mathscr{E}(-\delta(\mathscr{E})\}$ has degree 0 on every curve, the proof is a trivial consequence of (3.6) once one uses the theorem of Mumford-Mehta-Ramanathan (3.13) (although one can do without it if one is willing to (spend another couple of pages).

Corollary 4.2. *Let things be as in* (4.1) *and L a fixed Cartier divisor. Then the dimension $h^0(X, \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E})+L))$ is bounded by a polynomial of degree $r-1$ in t.*

Proof. For simplicity of the notation, put $\mathcal{F}^t = \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E}))$. The proof is by induction on the dimension n of X. If $n=1$, let D be a reduced effective divisor of degree $d > \deg L$. We have a natural exact sequence

$$H^0(X, \mathcal{F}^t(L-D)) \longrightarrow H^0(X, \mathcal{F}^t(L)) \longrightarrow H^0(D, \mathcal{F}^t(L))$$

of which the first term vanishes by (4.1), while the last term is a k-vector space of dimension $d\binom{rt+r-1}{rt} = d\binom{rt+r-1}{r-1}$. Thus we are done. For $n \geqq 2$, let $\mathfrak{A} \equiv (H_1, \cdots, H_{n-1}) \bmod (\boldsymbol{Q}_+^\times)^{n-1}$, where H_i is integral and ample. Let Y be a general hyperplane section $\in |mH_{n-1}|$ $(m \gg 0)$ such that $\mathcal{E}|_Y$ is $(H_1, \cdots, H_{n-2})$-semistable on Y and that $Y-L$ is ample. Note that such a number m, though possibly very large, is independent of t. Consider the exact sequence

$$H^0(X, \mathcal{F}^t(L-Y)) \longrightarrow H^0(X, \mathcal{F}^t(L)) \longrightarrow H^0(Y, \mathcal{F}^t(L)).$$

The first term vanishes by (4.1) and the dimension of the last term is bounded by a polynomial of degree $r-1$ by the induction hypothesis. This completes the proof.

Theorem 4.3 (The Bogomolov-Gieseker inequality). *Let S be a smooth projective surface over k. If $\mathcal{E}$ is an H-semistable sheaf of rank r on S (H is an ample divisor), then*

$$(r-1)c_1^2(\mathcal{E}) \leqq 2rc_2(\mathcal{E}).$$

Proof. From (4.2) it follows that neither $h^0(S, \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E}))$ nor $h^2(S, \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E})) = h^0(S, \mathcal{S}^{rt}\mathcal{E}^*(-tc_1(\mathcal{E}^*)+K_S))$ grows like t^{r+1}, where $r+1 = \dim \boldsymbol{P}(\mathcal{E})$. Hence we obtain the inequality

$$\chi(S, \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E})) \leqq (\text{polynomial of degree } r \text{ in } t).$$

On the other hand, by the Riemann-Roch theorem,

$$\begin{aligned}\chi(S, \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E})) &= \frac{t^{r+1}}{(r+1)!}\{rc_1(\mathcal{O}(\mathbf{1}_{\mathcal{E}})) - \pi^* c_1(\mathcal{E})\}^{r+1} + O(t^r)\\ &= \frac{(rt)^{r+1}}{(r+1)!}\left\{-c_2(\mathcal{E}) + \frac{r-1}{2r}c_1^2(\mathcal{E})\right\} + O(t^r),\end{aligned}$$

which implies the assertion.

Remark 4.4. There have been two different proofs of the inequality due to Bogomolov [B] and Gieseker [G2]. Our proof is, however, far more elementary and geometric.

Corollary 4.5. *Let $\mathscr{E}$ be a vector bundle of rank r on a smooth surface S. Let L be an ample integral divisor on S such that $\mathscr{E}(-\delta(\mathscr{E})+L)$ is ample and $\mathscr{E}(-\delta(\mathscr{E})-L)$ is negative (as $\boldsymbol{Q}$-vector bundles). Assume the inequality $2rc_2(\mathscr{E})<(r-1)c_1^2(\mathscr{E})$ and put*

$$\alpha=\{(r-1)c_1^2(\mathscr{E})-2rc_2(\mathscr{E})\}/6r^2(r+1)L^2 \in \boldsymbol{Q}.$$

Then either $\mathscr{S}^t\mathscr{E}(-t\delta(\mathscr{E}))$ or $\mathscr{S}^t\mathscr{E}^*(-t\delta(\mathscr{E}^*))$ contains the ample line bundle $\mathscr{O}_S(t\alpha L)$, where t is any very large integer such that $t\delta(\mathscr{E})$ and $t\alpha$ are integral.

Proof. For simplicity, put $\mathscr{F}=\mathscr{E}(-\delta(\mathscr{E}))$. Then we have

$$\chi(S, \mathscr{S}^t\mathscr{F})=\frac{1}{(r+1)!}\left\{\frac{r-1}{2r}c_1^2(\mathscr{E})-c_2(\mathscr{E})\right\}t^{r+1}+O(t^r).$$

Hence, by the Serre duality, we infer that

$$\begin{aligned} &h^0(S, \mathscr{S}^t\mathscr{F}) \text{ or } h^0(S, \mathscr{S}^t\mathscr{F}^*) \\ &\qquad \geqq \frac{1}{4(r+1)!\,r}\{(r-1)c_1^2(\mathscr{E})-2rc_2(\mathscr{E})\}t^{r+1}+O(t^r). \end{aligned}$$

Assume the first case and consider the natural exact sequences

$$0\longrightarrow H^0(S, \mathscr{S}^t\mathscr{F}(-t\alpha L))\longrightarrow H^0(S, \mathscr{S}^t\mathscr{F})\longrightarrow H^0(C, \mathscr{S}^t\mathscr{F}),$$
$$0\longrightarrow H^0(C, \mathscr{S}^t\mathscr{F}(-tL))\longrightarrow H^0(C, \mathscr{S}^t\mathscr{F})\longrightarrow H^0(D, \mathscr{S}^t\mathscr{F}),$$

where C is a general curve linearly equivalent to $t\alpha L$ and D is a 0-cycle of degree $t^2\alpha L^2$. The first term of the latter sequence vanishes as $\mathscr{F}(\ L)$ is negative. Hence $h^0(C, \mathscr{S}^t\mathscr{F})$ is bounded by

$$\begin{aligned} t^2\alpha(\text{rank } \mathscr{S}^t\mathscr{F})L^2 &\equiv \frac{\alpha t^{r+1}}{(r-1)!}L^2 \\ &\equiv \frac{t^{r+1}}{3(r+1)!}\left\{\frac{r-1}{2r}c_1^2(\mathscr{E})-c_2(\mathscr{E})\right\} \quad \text{mod } O(t^r). \end{aligned}$$

This implies that $H^0(S, \mathscr{S}^t\mathscr{F}(-t\alpha L))$ is non-zero whenever t is very large in view of the first exact sequence. Similarly, the second case yields $H^0(S, \mathscr{S}^t\mathscr{F}^*(-t\alpha L))\neq 0$.

Corollary 4.6. *Let $\mathscr{E}$ be a torsion free subsheaf of rank r on a smooth surface S and L an ample divisor as in* (4.5). *Assume that $\mathscr{E}$ satisfies the inequality $2rc_2(\mathscr{E})<(r-1)c_1^2(\mathscr{E})$. Then, for any numerically effective divisor D, the maximal D-destabilizing subsheaf $\mathscr{E}_1^D$ has normalized degree not less than*

$$\delta_D(\mathscr{E})+\frac{(r-1)c_1^2(\mathscr{E})-2rc_2(\mathscr{E})}{6r^3(r+1)L^2}LD$$

with respect to D.

Proof. We may assume that $\mathscr{E}$ is a vector bundle since $\mathscr{E}^{**}$ is a vector bundle with $c_1(\mathscr{E}^{**})=c_1(\mathscr{E})$, $c_2(\mathscr{E}^{**})\leqq c_2(\mathscr{E})$. We employ the same notation as in (4.5). If $\mathscr{S}^t\mathscr{F}$ contains $t\alpha L$, then $\delta_D(\mathscr{E}_1^D)-\delta_D(\mathscr{E})\geqq \alpha LD$. When $\mathscr{S}^t\mathscr{F}^*$ contains $t\alpha L$, $\delta_D(\mathscr{E}_1^D)-\delta_D(\mathscr{E})\geqq(1/r)\{\delta_D((\mathscr{E}^*)_1)-\delta_D(\mathscr{E}^*)\}\geqq (1/r)\alpha LD$.

Thanks to (4.6) and the Mumford-Mehta-Ramanathan theorem, the Bogmolov-Gieseker inequality is generalized to higher dimensions:

Corollary 4.7. *Let $\mathscr{E}$ be a torsion free sheaf of rank r on a normal projective variety X of dimension n and $H_1, \cdots, H_{n-2}$ ample Cartier divisors. Let D be a numerically effective Cartier divisor on X. Assume that X is smooth in codimension 2 and that $H_1\cdots H_{n-2}D$ is not numerically trivial. If $\mathscr{E}$ is $(H_1, \cdots, H_{n-2}, D)$-semistable, then*

$$(r-1)c_1^2(\mathscr{E})H_1\cdots H_{n-2}\leqq 2rc_2(\mathscr{E})H_1\cdots H_{n-2}.$$

Proof. Suppose the contrary. We may assume that $\mathscr{E}$ is a vector bundle in codimension 2 by taking the double dual. Fix an ample divisor H_0 such that $\mathscr{E}(-\delta(\mathscr{E})+H_0)$ and $\mathscr{E}^*(\delta(\mathscr{E})+H_0)$ are both ample. Let H be an (arbitrary) ample divisor. Then there exist positive integers $m_1, \cdots, m_{n-2}$ depending on H such that the $H|_S$-semistable filtration of $\mathscr{E}|_S$ coincides with the restriction of the $(H_1, \cdots, H_{n-2}, H)$-semistable filtration of $\mathscr{E}$ to a generic complete intersection surface $S=(m_1H_1)\cdots(m_{n-2}H_{n-2})$. By (4.6) we have

$$\begin{aligned}\delta(\mathscr{E}_1^{(\mathfrak{B},H)})SH-\delta(\mathscr{E})SH&=\delta_H((\mathscr{E}|_S)_1^H)-\delta_H(\mathscr{E}|_S)\\ &\geqq \text{const. }\{(r-1)c_1^2(\mathscr{E}|_S)-2rc_2(\mathscr{E}|_S)\}(H, H_0)_S/(H_0^2)_S\\ &=\text{const. }[\{(r-1)c_1^2(\mathscr{E})-2rc_2(\mathscr{E})\}S]\frac{HH_0S}{H_0^2S},\end{aligned}$$

where $\mathfrak{B}=(H_1, \cdots, H_{n-2})$. Therefore, by dividing out both sides by $m_1\cdots m_{n-2}$, we obtain the inequality

$$\delta_{(\mathfrak{B},H)}(\mathscr{E}_1^{(\mathfrak{B},H)})\geqq\delta_{(\mathfrak{B},H)}(\mathscr{E})+\text{const. }HH_0H_1\cdots H_{n-2}.$$

By the continuity of the function $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ on a segment joining $(\mathfrak{B}, D)$ and $(\mathfrak{B}, H)$ (Corollary 2.4 (3)), we have

$$\delta_{(\mathfrak{B},D)}(\mathscr{E}_1^{(\mathfrak{B},D)}) \geqq \delta_{(\mathfrak{B},D)}(\mathscr{E}) + \text{const. } DH_0 \cdots H_{n-2} > \delta_{(\mathfrak{B},D)}(\mathscr{E}),$$

a contradiction.

Remark 4.8. If D is ample, (4.7) will be a direct consequence of (4.3) and (3.13).

Corollary 4.9. *Let $H_1, \cdots, H_{n-2}$ be ample Cartier divisors. If*

$$\{(r-1)c_1^2(\mathscr{E}) - 2rc_2(\mathscr{E})\}H_1 \cdots H_{n-2} > 0,$$

then $\mathscr{E}$ is $(H_1, \cdots, H_{n-2}, D)$-unstable for any non-zero numerically effective divisor D.

§ 5. Semistability in positive and mixed characteristics

Over a field of positive characteristic, our argument in Sections 3, 4 needs amendments simply because the key lemma (3.2) may fail when $k(C')$ is an inseparable extension of $k(C)$ (i.e., $f\colon C' \to C$ is a Frobenius k-morphism; for an example, see [G1]).

Let C be a smooth complete curve over an algebraically closed field k of characteristic $p>0$. A vector bundle $\mathscr{E}$ on C is said to be *strongly semistable* if, for every positive integer s, $F^{s*}\mathscr{E}$ is semistable, where $F^s\colon F^{-s}C \to C$ is the Frobenius k-morphism of degree $q=p^s$. If C is an elliptic curve, it is known that semistable bundles are strongly semistable [O], but that is not the case when $g(C) \geqq 2$.

Proposition 5.1. *If $\mathscr{E}$ is strongly semistable on C, then $f^*\mathscr{E}$ is semistable for any surjective k-morphism $f\colon C' \to C$.*

Proof. Let C'' be a smooth model of the separable closure of $k(C)$ in $k(C')$. The natural projection $C' \to C''$ is purely inseparable and hence a Frobenius F^s for some non-negative integer [H2, p. 302, Proposition 2.5]. Thus we get the commutative diagram

$$\begin{array}{ccc} C' = F^{-s}C'' & \xrightarrow{F^s} & C'' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ F^{-s}C & \xrightarrow{F^s} & C. \end{array}$$

$F^{s*}\mathscr{E}$ is semistable on $F^{-s}C$. Since g is separable, $f^*\mathscr{E} = g^*F^{s*}\mathscr{E}$ is also semistable by (3.2).

From this we infer that the results (3.1), (3.4), (3.4)′, (3.6)–(3.11) can be saved in positive characteristics if "semistability" is substituted by "strong semistability". Similarly, we have analogous assertions to those in Section 4 by changing the assumption of $\mathfrak{A}$-semistability to that of strong $\mathfrak{A}$-semistability.

On an ordinary abelian variety, the semistability behaves quite nicely. (By definition, X is "ordinary" if the number of the points of order p on $X^*=\mathrm{Pic}^0(X)$ is equal to p^g, $g=\dim X$.) In fact, we have:

Proposition 5.2. *Let X be an ordinary abelian variety of dimension g over an algebraically closed field of positive characteristics. Let $\mathscr{E}$ be a torsion free sheaf which is locally free on an open subset $U=X-Y$ of X* ($\dim Y\leqq g-2$). *Let C be a smooth irreducible complete curve in U. Then $\mathscr{E}|_C$ is strongly semistable if and only if $\mathscr{E}|_C$ is semistable. In particular, the Bogomolov-Gieseker inequality is valid for every $\mathfrak{A}$-semistable torsion free sheaf on X* ($\mathfrak{A}\in NA(X)_{\boldsymbol{Q}}^{g-1}$).

If one recalls that $F_*F^*\mathcal{O}_X=\bigoplus_{\tau\in X_p^*}\mathcal{O}_X(\tau)$, the proof readily follows.

In mixed characteristics, the semistability is an open condition on the base scheme. Let X be a smooth projective scheme over a noetherian integral domain R of characteristic 0 and $\mathscr{E}$ a torsion free sheaf on X. Fix $\mathfrak{A}\in \overline{NA}(X/R)_{\boldsymbol{Q}}^{n-1}$, where n is the relative dimension of X. Then the set of geometric points $t\in \mathrm{Spec}\, R$ such that $\mathscr{E}_t/X_t$ is $\mathfrak{A}$-semistable forms an open subset of Spec R by (3.11) and (3.13) together with the following:

Lemma 5.3. *Let C be a smooth complete curve defined over an algebraically closed field k of arbitrary characteristic. If $\mathscr{E}(-D)$ is negative for a $\boldsymbol{Q}$-divisor D of degree $\delta(\mathscr{E})+(1/2r!)$, then $\mathscr{E}$ is semistable.*

The proof is implicitly contained in that of (3.11).

This lemma proves that the semistability behaves rather nicely under the deformations in mixed characteristics. On the contrary, we know very little about the strong semitability of the reductions of a semistable sheaf.

Problem 5.4. *Let C be an irreducible smooth projective curve over a noetherian integral domain R of characteristic* 0. *Assume that a locally free sheaf $\mathscr{E}$ on C is $\mathfrak{A}$-semistable on the generic fibre C_*. Let S be the set of primes of char.>0 on* Spec R *such that $\mathscr{E}$ is strongly semistable. Is S a dense subset of* Spec R? *Equivalently: Is $\lambda_{\mathscr{E}}$ numerically effective over a dense subset* $\subset$Spec $R-\{$*primes of char.* 0$\}$?

This question is closely related to the following natural but extremely hard problem:

Problem 5.5. *Let C be a smooth projective curve defined over an algebraic number field k. Is* Jac(C) *ordinary over infinitely many primes of* $\mathfrak{o}_k$?

The answer is affirmative if $g(C)=1$ (the density of such primes is $\geqq 1/2$ and the equality is attained if and only if C is of CM-type [D]) or $g(C)=2$ (A. Ogus).

§ 6. Generic semipositivity theorem for cotangent bundles

In this and the subsequent sections, all varieties are defined over an algebraically closed field k of characteristic 0.

Let $\mathfrak{B} \in \overline{NA}(X)^{n-2}_{\boldsymbol{Q}}$, where X is normal projective and n is the dimension of X. A torsion free sheaf $\mathscr{E}$ on X is said to be *generically* $\mathfrak{B}$-*seminegative* if, for every numerically effective $\boldsymbol{Q}$-Cartier divisor D on X, its maximal $(\mathfrak{B}, D)$-destabilizing subsheaf $\mathscr{E}_1$ satisfies $\delta_{(\mathfrak{B},D)}(\mathscr{E}_1)\leqq 0$. $\mathscr{E}$ will be *generically* $\mathfrak{B}$-*semipositive* if $\mathscr{E}^*$ is generically $\mathfrak{B}$-seminegative. Let

$$0=\mathscr{E}_0\subset\mathscr{E}_1\subset\cdots\subset\mathscr{E}_s=\mathscr{E}$$

be the $(\mathfrak{B}, D)$-semistable filtration and put $\alpha_i=\delta_{(\mathfrak{B},D)}(\mathscr{E}_i/\mathscr{E}_{i-1})$. Then $\alpha_1>\cdots>\alpha_s\geqq 0$ for every $D \in \overline{NA}(X)_{\boldsymbol{Q}}$ if $\mathscr{E}$ is generically $\mathfrak{B}$-semipositive.

Theorem 6.1. *Assume that X is smooth in codimension 2. Let $\mathscr{E}$ be a torsion free sheaf on X, with its first Chern class being a numerically effective $\boldsymbol{Q}$-Cartier divisor. Assume that $\mathscr{E}$ is generically $\mathfrak{B}$-semipositive, where $\mathfrak{B}=(h_1, \cdots, h_{n-2}) \in NA(X)^{n-2}_{\boldsymbol{Q}}$ (N.B.: not $\overline{NA}(X)^{n-2}_{\boldsymbol{Q}}$). Then the inequality*

$$c_2(\mathscr{E})h_1\cdots h_{n-2}\geqq 0$$

holds.

Proof. Let $0=\mathscr{E}_0\subset\mathscr{E}_1\subset\cdots\subset\mathscr{E}_s=\mathscr{E}$ be the $(\mathfrak{B}, D)$-semistable filtration. Put $\mathscr{G}_i=\mathscr{E}_i/\mathscr{E}_{i-1}$, $r_i=\operatorname{rank}\mathscr{G}_i$, $r=\operatorname{rank}\mathscr{E}$. We have

$$\begin{aligned}2c_2(\mathscr{E})|\mathfrak{B}|&=\{2\textstyle\sum_i c_2(\mathscr{G}_i)+2\sum_{i<j}c_1(\mathscr{G}_i)c_1(\mathscr{G}_j)\}|\mathfrak{B}|\\&=\{2\textstyle\sum_i c_2(\mathscr{G}_i)+c_1^2(\mathscr{E})-\sum_i c_1^2(\mathscr{G}_i)\}|\mathfrak{B}|,\end{aligned}$$

where $|\mathfrak{B}|$ denotes $h_1\cdots h_{n-2}$. Since $\mathscr{G}_i$ is $(\mathfrak{B}, D)$-semistable, the inequality (4.7) shows:

$$\begin{aligned}2c_2(\mathscr{E})|\mathfrak{B}|&\geqq\left[\textstyle\sum_i\frac{r_i-1}{r_i}c_1^2(\mathscr{G}_i)+\{c_1^2(\mathscr{E})-\sum_i c_1^2(\mathscr{G}_i)\}\right]|\mathfrak{B}|\\&=\left\{c_1^2(\mathscr{E})-\textstyle\sum_i\frac{1}{r_i}c_1^2(\mathscr{G}_i)\right\}|\mathfrak{B}|.\end{aligned}$$

In case $c_1^2(\mathscr{E})|\mathfrak{B}|>0$, let $D=c_1(\mathscr{E})$ and put $\alpha_i c_1^2(\mathscr{E})|\mathfrak{B}|=\delta(\mathscr{G}_i)c_1(\mathscr{E})|\mathfrak{B}|$. Then we have $\alpha_1>\cdots>\alpha_s\geqq 0$. The Hodge index theorem yields

$$c_1^2(\mathscr{G}_i)|\mathfrak{B}|=r_i^2\delta(\mathscr{G}_i)^2|\mathfrak{B}|\leqq r_i^2\alpha_i^2c_1^2(\mathscr{E})|\mathfrak{B}|$$

so that

$$\begin{aligned}2c_2(\mathscr{E})|\mathfrak{B}|&\geqq(1-\textstyle\sum_i r_i\alpha_i^2)c_1^2(\mathscr{E})|\mathfrak{B}|\\&\geqq\{1-(\textstyle\sum_i r_i\alpha_i)\alpha_1\}c_1^2(\mathscr{E})|\mathfrak{B}|\\&=(1-\alpha_1)c_1^2(\mathscr{E})|\mathfrak{B}|\geqq 0,\end{aligned}$$

since $\sum r_i\alpha_i=1$, $\alpha_1\leqq 1$. When $c_1^2(\mathscr{E})|\mathfrak{B}|=0$, let $h=c_1(\mathscr{E})+th_0$ be an ample $\boldsymbol{Q}$-Cartier divisor, where h_0 is ample and t is a small positive rational number. Again by the Hodge index theorem, we have

$$c_1^2(\mathscr{G}_i)|\mathfrak{B}|\leqq r_i^2\beta_i^2h^2|\mathfrak{B}|,$$

where $\beta_ih^2|\mathfrak{B}|=\delta(\mathscr{G}_i)h|\mathfrak{B}|$. Hence, in this case,

$$\begin{aligned}2c_2(\mathscr{E})|\mathfrak{B}|&\geqq-\textstyle\sum_i r_i\beta_i^2h^2|\mathfrak{B}|\geqq-(\textstyle\sum_i r_i\beta_i)^2h^2|\mathfrak{B}|\\&=-(\textstyle\sum_i c_1(\mathscr{G}_i)h|\mathfrak{B}|)^2/h^2|\mathfrak{B}|=-(c_1(\mathscr{E})h|\mathfrak{B}|)^2/h^2|\mathfrak{B}|\\&=-t^2\{c_1(\mathscr{E})h_0|\mathfrak{B}|\}^2/\{2tc_1(\mathscr{E})h_0|\mathfrak{B}|+t^2h_0^2|\mathfrak{B}|\}.\end{aligned}$$

If $c_1(\mathscr{E})h_0|\mathfrak{B}|=0$, the last term vanishes. On the other hand, if $c_1(\mathscr{E})h_0|\mathfrak{B}|>0$, it converges to 0 as t tends to 0. This completes the proof.

Remark 6.2. If $\mathscr{E}$ is ample on a generic complete intersection surface cut out by $|m_1H_1|,\cdots,|m_{n-2}H_{n-2}|$, we can easily show the inequality $c_2(\mathscr{E})H_1\cdots H_{n-2}>0$ by a standard argument (cf. [BG]). Yet such a property is far stronger than the generic $(H_1,\cdots,H_{n-2})$-semipositivity. For instance, the cotangent bundle of a projective $K3$ surface is generically semipositive (see below), while it is *never* semi-positive since a projective $K3$ surface *always* contains a rational curve (D. Mumford; some elliptic $K3$ surface carries even infinitely many rational curves.)

Theorem 6.3 [Mi3, Theorem 8.4]. *Let X be a smooth projective variety over k. Let $\mathscr{F}$ be a saturated subsheaf of $\mathscr{T}_X$ and $\{C_t\}$ a family of closed irreducible curves on X sweeping out an open subset of X. Assume that C_t and $\mathscr{F}$ have the following three properties for generic t:*

1) *$\mathscr{F}|_{C_t}\subset\mathscr{T}_X|_{C_t}$ is a subbundle;*
2) *$\mathscr{F}|_{C_t}$ is ample;*
3) *$\mathscr{H}om(\mathscr{F},\mathscr{T}_X/\mathscr{F})|_{C_t}$ is negative.*

Then, for a given geometric point $x\in C_t$, there exists a rational curve $L\ni x$ on X such that its tangent space is contained in $\mathscr{F}$ at each geometric point *on L. In particular, X is uniruled. If $\mathscr{F}$ has rank 1 in addition, X is birationally equivalent to a conic bundle; namely, there is a dominant rational map $X \to W$ whose general fibre is an irreducible rational curve and $\mathscr{F} = \mathscr{T}_{X/W}$ on an open subset of X.*

Let $\oplus \mathscr{G}_{i,t}$ and $\oplus \mathscr{H}_{j,t}$ be the graded modules associated with the semistable filtrations of $\mathscr{F}|_{C_t}$ and $(\mathscr{T}_X/\mathscr{F})|_{C_t}$, respectively. Then the above conditions 2) and 3) are rephrased to:

2′) $\deg \delta(\mathscr{G}_{i,t})>0$ *for every* i;

3′) $\deg \delta(\mathscr{G}_{i,t})>\deg \delta(\mathscr{H}_{j,t})$ *for every pair* (i, j).

Therefore, in case $\mathscr{F}|_{C_t}$ has positive degree and coincides with the maximal destabilizing subbundle of $\mathscr{T}_X|_{C_t}$, the conditions 2) 3) are satisfied.

Corollary 6.4 (generic semipositivity of cotangent sheaves). *Let X be a normal projective variety of dimension n and let $H_1, \cdots, H_{n-2}$ be ample Cartier divisors on X. Then the torsion free sheaf $\rho_* \Omega^1_{X'}$ is generically $(H_1, \cdots, H_{n-2})$-semipositive unless X is uniruled. Here ρ: $X' \to X$ denotes an arbitrary resolution.*

Proof. Assume the contrary. Then there will be a numerically effective Cartier divisor D such that the maximal $(H_1, \cdots, H_{n-2}, D)$-destabilizing subsheaf $\mathscr{F}'$ of $\rho_* \mathscr{T}_{X'}$ satisfies $\delta(\mathscr{F}')H_1 \cdots H_{n-2}D>0$. Therefore, when an ample $\boldsymbol{Q}$-Cartier divisor H is sufficiently near D, then $\delta(\mathscr{F}')H_1 \cdots H_{n-2}H>0$. Hence the maximal $(H_1, \cdots, H_{n-2}, H)$-destabilizing subsheaf $\mathscr{F}''$ of $\rho_* \mathscr{T}_{X'}$ must also satisfy the inequality $\delta(\mathscr{F}'')H_1 \cdots H_{n-2}H>0$. Consequently, the saturation $\mathscr{F}$ of the natural image of $\rho^* \mathscr{F}''$ in $\mathscr{T}_{X'}$ and $\{C_t\}$ will have the three properties in (6.3), where $\{C_t\}$ stands for the family of complete intersection curves of very high multiples of $\rho^* H_i$ and $\rho^* H$. This shows that X is uniruled. (Note that $C_t \simeq \rho(C_t)$ and $\rho_* \mathscr{T}_{X'}$ is isomorphic to $\mathscr{T}_X$ around C_t provided C_t is generic. In fact, X is isomorphic to X' outside a subset of codimension 2.)

Let X be a normal projective variety with only *terminal singularities* (for the definition, see [R3]). X is smooth in codimension 2 and has $\boldsymbol{Q}$-Cartier canonical divisor K_X. X is said to be *minimal* if K_X is numerically effective. A uniruled variety does not have a minimal model. The following conjecture, due to many people including S. Mori, M. Reid, Y. Kawamata and J. Kollár, is of extreme importance for the contemporary classification theory of complex algebraic varieties:

Minimal model conjecture 6.5 (cf. [Ka2], [KMM], [Mo]). *Let X be an algebraic variety defined over the complex number field* $\boldsymbol{C}$. *If X is not uniruled, then X will have a normal projective model X_0 which has only terminal singularities and is minimal.*

When $\dim X=2$, the answer is affirmative by the classical theorem of G. Castelnuovo: there is a *unique smooth* minimal model X_0. If dimension $\geqq 3$, the situation is quite different. First, a minimal model, if any, is by no means unique. Secondly, we cannot avoid introducing some mild singularities (at least terminal ones) to construct a minimal model [U2, Theorem 1.6], which is not Gorenstein in general but only $\boldsymbol{Q}$-Gorenstein, i.e., its canonical divisor is a $\boldsymbol{Q}$-Cartier divisor. Thus, for the study of the cotangent sheaves on minimal models, we have to analyze torsion free sheaves with $\boldsymbol{Q}$-Cartier first Chern classes as was discussed above.

As a direct consequence of (6.1) and (6.4), we have:

Theorem 6.6. *Let X be a normal projective variety of dimension n and $\rho\colon X'\to X$ an (arbitrary) resolution. Assume that X is smooth in codimension* 2 *and that the canonical divisor $K_X \in A^1(X)$ is $\boldsymbol{Q}$-Cartier and numerically effective. Then, for any numerically effective Cartier divisors $H_1, \cdots, H_{n-2}$, the inequality*

$$c_2(X')(\rho^*H_1)\cdots(\rho^*H_{n-2})\geqq 0$$

holds.

Proof. We prove the case where X is isomorphic to X' in codimension 2, to which general cases are easily reduced. Thus

$$c_2(X')(\rho^*H_1)\cdots(\rho^*H_{n-2})=c_2(\rho_*\Omega^1_{X'})H_1\cdots H_{n-2}$$

by dimension count. Apply (6.1) to the generically $(H_1, \cdots, H_{n-2})$-semipositive torsion free sheaf $\rho_*\Omega^1_{X'}$.

Corollary 6.7. *Let X be a projective minimal model and $\rho\colon X'\to X$ a resolution. Then $\rho_*c_2(X')H_1\cdots H_{n-2}$ is non-negative for any numerically effective Cartier divisors $H_1, \cdots, H_{n-2}$ on X. When $n=\dim X=3$, the* 1*-cycle $\rho_*c_2(X')$ is pseudo-effective, i.e., its numerical class is a limit of effective $\boldsymbol{Q}$-cycles.*

Proof. Terminal singularities have codimension at least 3 in X [R3]. The final statement follows from Kleiman's criterion for ampleness.

§ 7. "Semi-positivity" of $3c_2-c_1^2$ (characteristic 0)

Let X be a normal projective variety of dimension n with a $\boldsymbol{Q}$-Cartier canonical divisor K_X. Assume that X is non-uniruled and smooth in codimension 2. Let $H_1, \cdots, H_{n-2}$ be ample Cartier divisors on X. Take a resolution $\rho\colon X'\to X$ such that ρ is an isomorphism over the smooth locus of X. The torsion free sheaf $\mathscr{E}=\rho_*\Omega^1_{X'}$ is generically $\mathfrak{B}$-semipositive on X, where $\mathfrak{B}$ denotes $(H_1, \cdots, H_{n-2})$. Since X is non-singular in codimension 2, the intersection number $K_X^2|\mathfrak{B}|=K_X^2H_1\cdots H_{n-2}$ is integer valued and

$$K_X^2|\mathfrak{B}|=c_1^2(\mathscr{E})H_1\cdots H_{n-2}=K_{X'}^2(\rho^*H_1)\cdots(\rho^*H_{n-2}).$$

Proposition 7.1. *Assume that K_X is numerically effective and that $K_X^2H_1\cdots H_{n-2}$ is positive. Then the inequality*

$$\{3c_2(\mathscr{E})-c_1^2(\mathscr{E})\}H_1\cdots H_{n-2}\geqq 0$$

holds.

Proof. Let $0=\mathscr{E}_0\subset\mathscr{E}_1\subset\cdots\subset\mathscr{E}_s=\mathscr{E}$ be the $(\mathfrak{B}, K_X)$-semistable filtration. Put

$$\mathscr{G}_i=\mathscr{E}_i/\mathscr{E}_{i-1},$$
$$r_i=\operatorname{rank}\mathscr{G}_i,$$
$$\alpha_i=\delta(\mathscr{G}_i)K_X|\mathfrak{B}|/K_X^2|\mathfrak{B}|.$$

Then we have $r_1\alpha_1+\cdots+r_s\alpha_s=1$ and $\alpha_1>\cdots>\alpha_s\geqq 0$ by (6.4). The total Chern class $c(\mathscr{E})$ is the product of the $c(\mathscr{G}_i)$ [BS]. In particular,

$$\begin{aligned}P:&=\{6c_2(\mathscr{E})-2c_1^2(\mathscr{E})\}|\mathfrak{B}|\\&=\{6\textstyle\sum_i c_2(\mathscr{G}_i)+6\sum_{i<j}c_1(\mathscr{G}_j)c_1(\mathscr{G}_j)-2c_1^2(\mathscr{E})\}|\mathfrak{B}|\\&=\{6\textstyle\sum_i c_2(\mathscr{G}_i)-3\sum_i c_1^2(\mathscr{G}_i)+c_1^2(\mathscr{E})\}|\mathfrak{B}|.\end{aligned}$$

Applying the Bogomolov-Gieseker inequality to $\mathscr{G}_2, \cdots, \mathscr{G}_s$, we get

$$\begin{aligned}P&\geqq\Big[3\textstyle\sum_{i>1}\Big\{\frac{r_i-1}{r_i}c_1^2(\mathscr{G}_i)-c_1^2(\mathscr{G}_i)\Big\}+6c_2(\mathscr{E}_1)-3c_1^2(\mathscr{E}_1)+c_1^2(\mathscr{E})\Big]|\mathfrak{B}|.\\&=\Big\{-3\textstyle\sum_{i>1}\frac{1}{r_i}c_1^2(\mathscr{G}_i)+6c_2(\mathscr{E}_1)-3c_1^2(\mathscr{E}_1)+c_1^2(\mathscr{E})\Big\}|\mathfrak{B}|.\end{aligned}\tag{7.1.1}$$

By the Hodge index theorem, this yields

$$P\geqq\{(1-3\textstyle\sum_{i>1}r_i\alpha_i^2)K_X^2+6c_2(\mathscr{E}_1)-3c_1^2(\mathscr{E}_1)\}|\mathfrak{B}|.\tag{7.1.2}$$

There are three possibilities: $r_1\geqq 3$, $r_1=2$ and $r_1=1$.

Case A: $r_1\geqq 3$. Apply the Bogomolov-Gieseker inequality and then the Hodge index theorem to (7.1.2):

$$\begin{aligned} P &\geqq \left\{(1-3\textstyle\sum_{i>1} r_i\alpha_i^2)K_X^2 - 3\dfrac{1}{r_1}c_1^2(\mathscr{E}_1)\right\}|\mathfrak{B}| \\ &\geqq (1-3\textstyle\sum_i r_i\alpha_i^2)K_X^2|\mathfrak{B}| \geqq (1-3\alpha_1\textstyle\sum_i r_i\alpha_i)K_X^2|\mathfrak{B}| \\ &= (1-3\alpha_1)K_X^2|\mathfrak{B}| \geqq 0, \end{aligned}$$

since $3\alpha_1 \leqq r_1\alpha_1 \leqq \sum_i r_i\alpha_i = 1$. If $P=0$, then $\alpha_1 = 1/3$, $r_1 = 3$ so that $\alpha_i = 0$ for $i>1$. Moreover, $c_1(\mathscr{E}_1)H_1\cdots H_{n-2}$ must be numerically equivalent to $K_X H_1\cdots H_{n-2}$ by the Hodge index theorem. This implies that $c_1(\mathscr{E}_1)$ is actually numerically equivalent to K_X and that $c_1(\mathscr{E}/\mathscr{E}_1)$ is numerically trivial.

Case B: $r_1 = 2$. Let S be a general complete intersection surface of the linear systems $|m_i H_i|$. Then the natural composite map

$$\mathscr{E}_1|_S \longrightarrow \Omega_X^1|_S \longrightarrow \Omega_S^1$$

is injective (all sheaves are locally free because S lies on the smooth locus of X). Hence, by [Mi2, Remark (4.18)], we have either $\kappa(S, c_1(\mathscr{E}_1|_S)) \leqq 0$ or $3c_2(\mathscr{E}_1|_S) \geqq c_1^2(\mathscr{E}_1|_S)$.

Subcase B$_-$: $\kappa(S, c_1(\mathscr{E}_1|_S)) \leqq 0$. Since $(c_1(\mathscr{E}_1|_S), K_X|_S) > 0$, we have $c_1^2(\mathscr{E}_1|_S) = \text{const.}\ c_1^2(\mathscr{E}_1)|\mathfrak{B}| \leqq 0$. Now apply the Bogomolov-Gieseker inequality to (7.1.2):

$$\begin{aligned} P &\geqq \left\{(1-3\textstyle\sum_{i>1} r_i\alpha_i^2)K_X^2 - \dfrac{3}{2}c_1^2(\mathscr{E}_1)\right\}|\mathfrak{B}| \\ &\geqq (1-3\textstyle\sum_{i>1} r_i\alpha_i^2)K_X^2|\mathfrak{B}| \\ &\geqq (1-3\alpha_2\textstyle\sum_{i>1} r_i\alpha_i)K_X^2|\mathfrak{B}| \\ &= \{1-3\alpha_2(1-2\alpha_1)\}K_X^2|\mathfrak{B}| \\ &\geqq \{1-3\alpha_1(1-2\alpha_1)\}K_X^2|\mathfrak{B}| = \left\{6\left(\alpha_1 - \dfrac{1}{4}\right)^2 + \dfrac{5}{8}\right\}K_X^2|\mathfrak{B}| > 0. \end{aligned}$$

Subcase: B_+: $3c_2(\mathscr{E}_1|_S) \geqq c_1^2(\mathscr{E}_1|_S)$. From (7.1.2), we get

$$P \geqq \{(1-3\textstyle\sum_{i>1} r_1\alpha_i^2)K_X^2 - c_1^2(\mathscr{E}_1)\}|\mathfrak{B}|.$$

Again by the Hodge index theorem,

$$\begin{aligned} P &\geqq (1-4\alpha_1^2 - 3\textstyle\sum_{i>1} r_i\alpha_i^2)\ K_X^2|\mathfrak{B}| \\ &\geqq \{1-4\alpha_1^2 - 3\alpha_2\textstyle\sum_{i>1} r_i\alpha_i\}K_X^2|\mathfrak{B}| \\ &= \{1-4\alpha_1^2 - 3\alpha_2(1-2\alpha_1)\}K_X^2|\mathfrak{B}| \\ &= (1-2\alpha_1)(1+2\alpha_1 - 3\alpha_2)K_X^2|\mathfrak{B}|. \end{aligned}$$

On the other hand, we have $3\alpha_2 < r_1\alpha_1 + r_2\alpha_2 \leqq 1$. Hence

$$P \geqq 2(1-2\alpha_1)\alpha_1 K_X^2|\mathfrak{B}| \geqq 0.$$

If $P=0$, then $\alpha_1 = 1/2$, $c_1(\mathscr{E}_1)$ is numerically equivalent to K_X and $c_1(\mathscr{E}/\mathscr{E}_1) \equiv 0$.

Case C: $r_1 = 1$. In this case, $c_2(\mathscr{E}_1)|\mathfrak{B}| \geqq c_2((\mathscr{E}_1)^{**})|\mathfrak{B}| = 0$, while Bogomolov's lemma [Mil, Theorem 2] asserts that $c_1^2(\mathscr{E}_1) \leqq 0$. Hence from (7.1.2) we derive

$$\begin{aligned} P &\geqq (1-3\Sigma_{i>1} r_i\alpha_i^2)K_X^2|\mathfrak{B}| \\ &> (1-3\alpha_1\Sigma_{i>1} r_i\alpha_i)K_X^2|\mathfrak{B}| \\ &= \{1-3\alpha_1(1-\alpha_1)\}K_X^2|\mathfrak{B}| \\ &\geqq \left\{1-\frac{3}{2}\left(1-\frac{1}{2}\right)\right\}K_X^2|\mathfrak{B}| = \frac{1}{4}K_X^2|\mathfrak{B}| > 0. \end{aligned}$$

This completes the proof of (7.1) and at the same time of Theorem 1.1 if combined with (6.6).

Proposition 7.2. *Assume that K_X is numerically effective and that $\{3c_2(\rho_*\Omega_{X'}^1) - K_X^2\}|\mathfrak{B}| = 0$. Then one of the following four cases occurs*:*

a) *$K_X^2|\mathfrak{B}| > 0$, and the maximal $(\mathfrak{B}, K_X)$-destabilizing subsheaf $\mathscr{E}_1$ of $\mathscr{E} = \rho_*\Omega_{X'}^1$ satisfies* rank $\mathscr{E}_1 = 3$, $c_1(\mathscr{E}_1) \equiv K_X$, $3c_2(\mathscr{E}_1) = K_X^2$, *where $\equiv$ denotes the numerical equivalence. The quotient $\mathscr{F} = \mathscr{E}/\mathscr{E}_1$ has numerically trivial c_1 and is $\mathfrak{A}$-semistable for any $\mathfrak{A} \in NA(X)_Q^{n-1}$. $c_2(\mathscr{F}) \equiv 0$.*

b) *$K_X^2|\mathfrak{B}| > 0$, and the maximal $(\mathfrak{B}, K_X)$-destabilizing subsheaf $\mathscr{E}_1$ of $\mathscr{E}$ satisfies* rank $\mathscr{E}_1 = 2$, $c_1(\mathscr{E}_1) \equiv K_X$, $3c_2(\mathscr{E}_1) = K_X^2$. *$\mathscr{F} = \mathscr{E}/\mathscr{E}_1$ is $\mathfrak{A}$-semistable ($\mathfrak{A} \in NA(X)_Q^{n-1}$) with numerically trivial c_1. Moreover, $K_X^3 \equiv 0$.*

c) *$K_X^2|\mathfrak{B}| = 0$ but $K_X|\mathfrak{B}|$ is not numerically trivial.*

d) *K_X is numerically trivial and $\mathscr{E}$ is always $\mathfrak{A}$-semistable for $\mathfrak{A} \in NA(X)_Q^{n-1}$.*

Proof. Almost everything has already been proved. It remains to verify the numerical triviality of K_X^3 in Case b) and the semistability of $\mathscr{F}$ in a), b) or of $\mathscr{E}$ in d). The latter is an obvious consequence of the following easy

Lemma 7.3. *A torsion free quotient $\mathscr{F}$ of a generically $\mathfrak{B}$-semipositive torsion free sheaf is again generically $\mathfrak{B}$-semipositive. In particular, if*

*) Case a) is presumably impossible, while we have simple examples of the remaining three cases.

$c_1(\mathscr{F})$ is numerically trivial, then $\mathscr{F}$ is $(\mathfrak{B}, D)$-semistable for every $D \in \overline{NA}(X)_{\boldsymbol{Q}}$.

The equality $K_X^3 \equiv 0$ in Case b) follows from:

Lemma 7.4 (a generalization of Bogomolov's lemma). *Let $\mathscr{F}$ be a vector bundle of rank r contained in the cotangent bundle Ω_X^1 of a Kähler manifold X. Then $\kappa(X, c_1(\mathscr{F})) \leqq r$.*

The proof is similar to that in the case $r=1$.

Remark 7.5. The pseudo-effectivity of c_2 is classical in surface case (see, for example, [BPV]). Although it is unknown in dimension $\geqq 4$, yet there is a partial result due to S.T. Yau [Y] which says:

a) *If X is smooth and K_X is numerically trivial, then $c_2(X)$ is represented by a non-negative $(2, 2)$-form. If $c_2(X)=0$ in $H^4(X, \boldsymbol{R})$, then X is a complex torus up to finite étale cover.*

b) *If X is smooth and K_X is ample, then $2(n+1)c_2(X)K_X^{n-2} \geqq nK_X^n$. If the equality holds, then X is covered by the open unit ball.*

a) actually implies the pseudo-effectivity of c_2, whereas b) shows only the positivity of $c_2(X)$ measured by the power of a specific ample divisor K_X. It does not exclude the possibility of $\{2(n+1)c_2(X)-nK_X^2\}|\mathfrak{B}|$ being negative for general $\mathfrak{B} \in NA(X)_{\boldsymbol{Q}}^{n-2}$. For instance, let S be a surface with $3c_2(S)=c_1^2(S)$ and X be a product $S \times V$, where V is a manifold with ample canonical divisor. If $(ac_2(X)-c_1^2(X))$ has non-negative intersection with every H^{n-2} (H is ample), then we have $a \geqq 3$. This example shows that our result (7.1) is best possible as such even in case X is smooth with ample canonical class.

Remark 7.6. Theorem 1.1 is redundant for the proof of Theorem 1.2 which can be derived from (6.6) only. It is, however, not only interesting in its own right but also of some use for the analysis of canonical rings of threefolds with ample canonical classes [A].

§ 8. Non-negativity of the Kodaira dimension of minimal threefolds: the case where X is Gorenstein or K_X^2 is numerically non-trivial

A) The Gorenstein case

Let X be a normal projective Gorenstein threefold with only canonical singularities (X is Gorenstein if and only if K_X is a Cartier divisor because canonical singularities are Cohen-Macauley [E]). Then we have the following:

Theorem 8.1. *Assume that the canonical divisor $K_X \in \mathrm{Pic}(X)$ is numerically effective. Then the Euler characteristic $\chi(X, \mathcal{O}_X)$ is non-negative. In particular, either $p_g(X) = h^0(X, \mathcal{O}_X(K_X))$ or $q(X) = h^1(X, \mathcal{O}_X)$ is non-zero and $\kappa(X) \geqq 0$.*

Lemma 8.2 [R2, Corollary 2.12]. *If X is a threefold with only Gorenstein canonical singularities, then there exists a partial resolution $f\colon X' \to X$, which is proper birational, such that*

(1) $f^*K_X = K_{X'}$;

and that

(2) *X' has only hypersurface singularities, i.e., the Zariski tangent space is of dimension $\leqq 4$ at every closed point.*

Let X' be as in the above lemma. Then, by successive monoidal transformations along smooth curves and closed points, we obtain a smooth resolution $g\colon Y \to X'$. The condition that X has only canonical singularities means that the fibre of each monoidal transformation is a cubic or conic surface in $\boldsymbol{P}^3$ or a conic curve in $\boldsymbol{P}^2$. In particular, if one represents K_Y in the form $g^*K_{X'} + \Delta$, the image $g(\Delta)$ of the correction term is a *finite* subset of double points on X'. Now one easily checks:

Lemma 8.3 (cf. [R, p. 305]). *Under the same hypothesis as in* (8.2), $\Delta c_2(Y) = 0$.

Remark 8.4. A more elementary (though less informative) proof of (8.3) is as follows: By duality, we have the identity

$$\chi(Y, \mathcal{O}_Y(K_Y)) = -\chi(Y, \mathcal{O}_Y) = -\chi(X, \mathcal{O}_X) = \chi(X, \mathcal{O}_X(K_X)) = \chi(Y, \mathcal{O}_Y(K_X)).$$

Writing down the Euler characteristics explicitly by the Riemann-Roch formula, we obtain (8.3).

Proof of Theorem 8.1. Let $g\colon Y \to X'$, $f\colon X' \to X$ be as above and $\rho\colon Y \to X$ the composite map. Then $K_Y = \rho^*K_X + \Delta$, so that the Euler characteristic of Y is given by

$$\begin{aligned}\chi(\mathcal{O}_Y) &= (1/24)c_1(Y)c_2(Y)\\ &= -(1/24)(\rho^*K_X) + \Delta c_2(Y).\end{aligned}$$

Now, by Lemma 8.3,

$$\chi(\mathcal{O}_Y) = -(1/24)(\rho^*K_X)c_2(Y).$$

Since K_X is numerically effective, Theorem 1.1 or 6.6 implies that

$$p_g(X) + q(X) - 1 \geqq -\chi(\mathcal{O}_Y) = (1/24)\rho^*K_X c_2(Y) \geqq 0$$

and *a fortiori* either $p_g(X)$ or $q(X)$ is non-zero. If $p_g(X)>0$, then $\kappa(X)\geqq 0$ by the very definition of the Kodaira dimension. In case $q(X)$ is positive, the Albanese morphism $\alpha: Y\to \mathrm{Alb}(Y)$ gives rise to a surjective morphism $\alpha': Y\to Z$ with $\kappa(Z)\geqq 0$ and connected fibres. As Y is not uniruled, the general fibre Y_z is neither ruled nor rational and hence $\kappa(Y_z)\geqq 0$ by the classification of curves and surfaces. Thus the assertion $\kappa(X)\geqq 0$ is reduced to Iitaka's famous conjectures $C_{3,2}$ and $C_{3,1}$ (see [U1]), which have been affirmatively solved by Viehweg [V1] and Viehweg-Ueno [V2], [U3].

B) The Case where K_X^2 is numerically non-trivial

Let X be a normal projective threefold with only isolated canonical singularities. Assume that the $\boldsymbol{Q}$-Cartier divisor K_X is numerically effective and that $K_X^2 \in A^2(X)_{\boldsymbol{Q}}=A_1(X)_{\boldsymbol{Q}}$ is numerically non-zero. Let $r\in \boldsymbol{N}$ be the *index* of X, the least positive integer such that rK_X is an integral Cartier divisor. Let $\rho: Y\to X$ be a resolution of the singularities. By the Riemann-Roch theorem,

$$\begin{aligned}\chi(X, \mathcal{O}_X(mrK_X))&=\chi(Y, \rho^*\mathcal{O}_X(mrK_X))\\ &=(mr/12)(2m^2r^2-3mr+1)K_X^3+(mr/12)(\rho^*K_X, c_2(Y))+\chi(Y, \mathcal{O}_Y).\end{aligned}$$

In case $\chi(Y, \mathcal{O}_Y)\leqq 0$, the proof of (8.1) gives the non-negativity of the Kodaira dimension. Assume that $\chi(Y, \mathcal{O}_Y)$ is positive. Then, from Theorem 1.1 or 6.6, we infer that

$$\begin{aligned}&h^0(Y, \rho^*\mathcal{O}_X(mrK_X))+h^2(Y, \rho^*\mathcal{O}_X(mrK_X))\\ &\qquad\geqq\chi(Y, \rho^*\mathcal{O}_X(mrK_X))\geqq\chi(Y, \mathcal{O}_Y)>0.\end{aligned}$$

Thus, in this case, either $H^0(Y, \rho^*\mathcal{O}_X(mrK_X))$ or $H^2(Y, \rho^*\mathcal{O}_X(mrK_X)$ is non-trivial.

Lemma 8.5. *$H^2(Y, \rho^*\mathcal{O}_X(mrK_X))$ vanishes* ($mr\geqq 2$).

Proof. Let S be a smooth sufficiently ample surface in X and consider the long exact sequence

$$H^1(S, \mathcal{O}_S(mrK_X+S))\longrightarrow H^2(X, \mathcal{O}_X(mrK_X))\longrightarrow H^2(X, \mathcal{O}_X(mrK_X+S))$$

associated with the short exact sequence

$$\mathcal{O}_X(mrK_X)\longrightarrow \mathcal{O}_X(mrK_X+S)\longrightarrow \mathcal{O}_S(mrK_X+S).$$

Since S lies in the smooth locus of X, $K_X|_S$ is a numerically effective *integral* divisor with *positive* self-intersection and $K_S=(K_X+S)|_S$ by the adjunction formula. Hence

$$h^1(S, \mathcal{O}_S(mrK_X+S))=h^1(S, \mathcal{O}_S(-(mr-1)(K_X|_S)))=0$$

by the Mumford or Kawamata vanishing theorem [R1], [Ka1]. So the first term vanishes, while the third term $H^2(X, \mathcal{O}_X(mrK_X+S))$ is trivial whenever S is sufficiently ample [FAC]. Then the rationality of the canonical singularities [E] yields our claim.

Corollary 8.6. *Let r be the index of K_X and m a positive integer with $mr \geqq 2$. If $\chi(X, \mathcal{O}_X)>0$, then mrK_X is linearly equivalent to an effective divisor. In particular, the mr-genus P_{mr} of X is positive so that $\kappa(X) \geqq 0$.*

Remark 8.7. When X is not Gorenstein, $\chi(X, \mathcal{O}_X)$ can be positive even if K_X is ample. The simplest example is the following: Let C_i be a hyperelliptic curve with the canonical involution ι_i $(i=1, 2, 3)$. Let ι be the involution on $C_1 \times C_2 \times C_3$ acting by $\iota(x_1, x_2, x_3)=(\iota_1(x_1), \iota_2(x_2), \iota_3(x_3))$ and let X be the quotient $(C_1 \times C_2 \times C_3)/\iota$. Then, an easy computation shows:

$$h^1(X, \mathcal{O}_X)=h^3(X, \mathcal{O}_X)=0,$$

$$h^2(X, \mathcal{O}_X)=g(C_1)g(C_2)+g(C_2)g(C_3)+g(C_3)g(C_1).$$

The index of K_X is of course 2.

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