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Eisenstein Series on Semisimple Symmetric Spaces of Chevalley Groups

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§ 0. Introduction

0.1. In recent years a remarkable progress has been made in the theory of harmonic analysis on (not necessarily Riemannian) semisimple symmetric spaces by Oshima, Flensted-Jensen and others (cf. [7], [15]). It is also interesting to investigate semisimple symmetric spaces from the arithmetic point of view. For example, in a previous paper [21], we associated with an arbitrary indefinite rational symmetric matrix a family of Dirichlet series satisfying certain functional equations which can be regarded as Eisenstein series on the non-Riemannian symmetric space $SL(n; \boldsymbol{R})/SO(p, n-p)$. This result, along with the recent development in the theory of semisimple symmetric spaces, leads us to the problem of constructing an analogue of Eisenstein series for arbitrary semisimple symmetric spaces with $\boldsymbol{Q}$-structure. Though it seems fairly difficult to solve the problem in its full generality, we are able to find a solution in some special cases. In the present paper, we treat the case of symmetric spaces of ε-involution type (which was introduced by Oshima and Sekiguchi [15]) of Chevalley groups.

0.2. Now we shall sketch the result in this paper. Let G be a connected and simply connected semisimple algebraic group defined and split over $\boldsymbol{Q}$ and σ be an involutive automorphism of G defined over $\boldsymbol{Q}$. Denote by H the fixed point group of σ. A torus T of G is said to be σ-anisotropic if $\sigma(t)=t^{-1}$ for any $t \in T$. We consider the symmetric space $X=G/H$ under the assumption that G has a $\boldsymbol{Q}$-split σ-anisotropic maximal torus. Then every $G_{\boldsymbol{R}}$-orbit in $X_{\boldsymbol{R}}$ is a semisimple symmetric space of ε-involution type in the sense of Oshima and Sekiguchi [15]. In particular, the Riemannian symmetric space $G_{\boldsymbol{R}}/K$ appears among $G_{\boldsymbol{R}}$-orbits in $X_{\boldsymbol{R}}$ for an appropriate choice of σ.

Fix a $\boldsymbol{Q}$-split σ-anisotropic maximal torus T and let B be a Borel subgroup of G such that $B \cap \sigma(B)=T$. Let Φ^+ (resp. Δ) be the set of

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positive (resp. simple) roots of (G, T) corresponding to B. Denote by W the Weyl group of G with respect to T. Fix a W-invariant inner product $\langle\ ,\ \rangle$ on $X(T)^{\boldsymbol{R}}=X(T)\otimes_{\boldsymbol{Z}}\boldsymbol{R}$, where $X(T)$ is the group of rational characters of T. We extend the inner product to a $\boldsymbol{C}$-bilinear form on $X(T)^{\boldsymbol{C}}=X(T)\otimes_{\boldsymbol{Z}}\boldsymbol{C}$ in an obvious manner. For each $\alpha\in\Phi^+$, put $\alpha^\vee=2\alpha/\langle\alpha, \alpha\rangle$. Since G is assumed to be simply connected, there exists a $\Lambda_\alpha\in X(T)$ for each simple root $\alpha\in\Delta$ such that

$$\langle\Lambda_\alpha, \beta^\vee\rangle=\begin{cases}1 & \text{if } \beta=\alpha,\\ 0 & \text{if } \beta\in\Delta,\ \beta\neq\alpha.\end{cases}$$

The characters $\{\Lambda_\alpha;\ \alpha\in\Delta\}$ forms a $\boldsymbol{Z}$-basis of the free abelian group $X(T)$ and are called the dominant fundamental weights. The group $X(B)$ of rational characters of B can be canonically identified with $X(T)$.

We are mainly concerned with the action of B on X, especially with B-relatively invariant rational functions on X. It is known by Vust [26] that X has a Zariski-open B-orbit X_Ω. Moreover, we can prove that there exist algebraically independent regular functions f_α indexed by simple roots $\alpha\in\Delta$ such that

$$f_\alpha(b\cdot x)=\Lambda_\alpha(b)^{-2}f_\alpha(x)\quad (b\in B,\ x\in X)$$

and

$$X_\Omega=\{x\in X;\ \prod_{\alpha\in\Delta}f_\alpha(x)\neq 0\}.$$

Every non-zero rational function on X relatively invariant under the action of B is written uniquely as a monomial of f_α's.

Take a $G_{\boldsymbol{R}}$-orbit X_0 in $X_{\boldsymbol{R}}$ and let

$$X_0\cap X_\Omega=X_0^{(1)}\cup\cdots\cup X_0^{(\nu)}$$

be the $B_{\boldsymbol{R}}$-orbit decomposition of $X_0\cap X_\Omega$. These orbits $X_0^{(i)}$ are open in X_0 and each of them is characterized by the signs of the values taken by f_α $(\alpha\in\Delta)$ on it.

Let $\Gamma=G_{\boldsymbol{Z}}$ be the standard unit group of G (cf. § 1) and Γ_∞ be the intersection of Γ and the unipotent radical of B. For an $x_0\in X_0\cap X_\Omega$, we consider the Dirichlet series

$$E_i(x_0;\lambda)=\sum_x\prod_{\alpha\in\Delta}|f_\alpha(x)|^{-\langle\lambda,\alpha^\vee\rangle-1/2}\quad (\lambda\in X(T)^{\boldsymbol{C}},\ 1\leqq i\leqq\nu),$$

where x runs through a complete system of representatives of all Γ_∞-orbits in $\Gamma\cdot x_0\cap X_0^{(i)}$. We call the series $E_i(x_0;\lambda)$ $(1\leqq i\leqq\nu,\ x_0\in X_0\cap X_\Omega)$ the *Eisenstein series on* X_0 *with respect to* Γ. If X_0 is a Riemannian symmetric space, then we have $\nu=1$ and our series $E_1(x_0;\lambda)$ is nothing but the usual Eisenstein series. The Eisenstein series generalized as above have analytic properties very similar to those of Eisenstein series on Riemannian symmetric spaces. In fact we shall prove the following theorem.

Theorem. (1) *For any* $x_0 \in X_0 \cap X_Q$, *the series* $E_i(x_0;\lambda)$ $(1\leqq i\leqq \nu)$ *are absolutely convergent for* $\mathrm{Re}\langle\lambda,\alpha^\vee\rangle>1/2$ $(\alpha\in\Delta)$ *and have analytic continuations to meromorphic functions of* λ *in* $X(T)^C$.

(2) *The functions*

$$\prod_{b\in\Phi+}(\langle\lambda,b^\vee\rangle-1/2)^2\zeta(2\langle\lambda,b^\vee\rangle+1)E_i(x_0;\lambda)$$

are entire functions of λ.

(3) (*Functional equations*) *Put*

$$\Lambda_i(x_0;\lambda)=\prod_{\alpha\in\Delta}c_\alpha^{-\langle\lambda,\alpha^\vee\rangle}\prod_{b\in\Phi+}\eta(2\langle\lambda,b^\vee\rangle+1)E_i(x_0;\lambda),$$

where $\eta(z)=\pi^{-z/2}\Gamma(z/2)\zeta(z)$ *and* c_α $(\alpha\in\Delta)$ *are positive real numbers depending only on the involution* σ. *Then for any* $w\in W$, *there exists a* ν *by* ν *matrix* $C(w,\lambda)$ *whose entries are meromorphic functions of* λ *with elementary expressions in terms of trigonometric functions such that*

$$\begin{pmatrix}\Lambda_1(x_0;w\lambda)\\ \vdots\\ \Lambda_\nu(x_0;w\lambda)\end{pmatrix}=C(w;\lambda)\begin{pmatrix}\Lambda_1(x_0;\lambda)\\ \vdots\\ \Lambda_\nu(x_0;\lambda)\end{pmatrix}\quad(w\in W).$$

For the reflection w_α with respect to the hyperplane orthogonal to a simple root α, the matrix $C(w_\alpha;\lambda)$ can be easily computed and has a very simple form.

As is seen in Section 6, the theorem includes the main result of [21] ([21, Theorem 7]) as a special case of $G=SL(n+1)$. In this special case, the Eisenstein series can be related to zeta functions associated with certain prehomogeneous vector spaces and we have proved the theorem for $G=SL(n+1)$ with the aid of the general theory of prehomogeneous vector spaces developed in [20]. For Chevalley groups other than $SL(n+1)$, we can not interpret the Eisenstein series as zeta functions associated with prehomogeneous vector spaces. However the same idea as employed in [20] works well also in the present general situation without any essential modification. A similar technique has been used previously by Godement [9].

In [15, § 5.3], Oshima and Sekiguchi indicated a method for constructing an analogue of Eisenstein series for a semisimple symmetric space of ε-involution type. The construction is based upon their theory of joint eigen-hyperfunctions of invariant differential operators on such a symmetric space, and their analogue of Eisenstein series is a family of Γ-automorphic hyperfunctions on the symmetric space depending meromorphically on the parameter $\lambda \in X(T)^C$. Though it satisfies the same functional equations as in the theorem above, the relation between the Oshima-Sekiguchi Eisenstein series and ours is not clear at present.

0.3. The present paper is arranged as follows. After some preliminaries on Chevalley groups in Section 1, we investigate the structure of the symmetric space $X=G/H$ in Section 2. In Section 3, we introduce the Eisenstein series and examine their convergence property. In Section 4, we give integral representations of the Eisenstein series which play a key role in the proof of the theorem. The theorem is proved in Section 5. We devote Section 6 to a discussion of examples.

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Notation. We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For any prime v of $\boldsymbol{Q}$, $\boldsymbol{Q}_v$ is the completion of $\boldsymbol{Q}$ with respect to v. For a finite prime $v=p$ of $\boldsymbol{Q}$, $\boldsymbol{Z}_p$ is the ring of p-adic integers in $\boldsymbol{Q}_p$. For an affine algebraic variety X defined over a field K, the set of K-rational points of X is denoted by X_K. Moreover, we denote by X_A the adelization of X over $\boldsymbol{Q}$. For an algebraic matrix group G defined over $\boldsymbol{Q}$, let $G_{\boldsymbol{Z}}$ (resp. $G_{\boldsymbol{Z}_p}$) be the subgroup of $G_{\boldsymbol{Q}}$ (resp. $G_{\boldsymbol{Q}_p}$) consisting of integral matrices whose determinants are units in $\boldsymbol{Z}$ (resp. $\boldsymbol{Z}_p$). Let ω be a $\boldsymbol{Q}$-rational algebraic gauge form on an affine algebraic variety X defined over $\boldsymbol{Q}$. Then, for any prime v of $\boldsymbol{Q}$, we denote by $|\omega|_v$ the measure of $X_{\boldsymbol{Q}_v}$ induced by ω. The space of compactly supported C^∞-functions on a smooth manifold M is denoted by $C_0^\infty(M)$. The space of rapidly decreasing functions on a real vector space V is denoted by $\mathscr{S}(V)$. We denote by $\Gamma(s)$ and $\zeta(s)$ the gamma function and the Riemann zeta function, respectively.

§ 1. Chevalley groups over $\boldsymbol{Q}$

As preliminaries, we recall some results on Chevalley groups and the Bruhat decomposition.

Let G be a universal Chevalley group over $\boldsymbol{Q}$, namely, a connected and simply connected semisimple algebraic group which is defined and split over $\boldsymbol{Q}$. Let T be a $\boldsymbol{Q}$-split maximal torus of G. Take a Borel subgroup B of G containing T and denote by B_u the unipotnet radical of B. The groups B and B_u are defined over $\boldsymbol{Q}$. Moreover, B is a semi-direct product of T and B_u: $B=T\cdot B_u$. Let B^- be the Borel subgroup of G containing T opposite to B. The unipotent radical of B^- is denoted by B_u^-. Then B^- and B_u^- are also defined over $\boldsymbol{Q}$ and $B^-=T\cdot B_u^-$ (semi-direct product).

Let $X(T)$ (resp. $X(B)$, $X(B^-)$) be the group of rational characters of T (resp. B, B^-). Every element in $X(T)$ can be uniquely extended to a rational character of B (resp. B^-) which takes 1 on B_u (resp. B_u^-). Conversely every character of B (resp. B^-) is obtained in this manner. So we may identify $X(T)$ with $X(B)$ and $X(B^-)$. Since T is assumed to be split over $\boldsymbol{Q}$, all the characters in $X(T)$ are defined over $\boldsymbol{Q}$.

Let Φ be the root system of G with respect to T and Δ (resp. Φ^+, Φ^-) the set of simple (resp. positive, negative) roots determined by B. For each $b \in \Phi$, there exists a unique faithful $\boldsymbol{Q}$-morphism θ_b of the additive group $\boldsymbol{G}_a$ onto a subgroup U_b of G normalized by T such that $\theta_b(b(t)x)=t\theta_b(x)t^{-1}$ ($x \in \boldsymbol{G}_a$, $t \in T$). Let W be the Weyl group of G, which is, by definition, the quotient of the normalizer $N_G(T)$ by the centralizer $Z_G(T)$ of T. The group W acts on $X(T)$, and hence on $X(T)^{\boldsymbol{R}}=X(T)\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ and on $X(T)^{\boldsymbol{C}}=X(T)\otimes_{\boldsymbol{Z}}\boldsymbol{C}$ in an obvious manner. We may identify $X(T)^{\boldsymbol{R}}$ and $X(T)^{\boldsymbol{C}}$ with the dual space of the Lie algebra of $T_{\boldsymbol{R}}$ and its complexification, respectively. Furthermore, we write the multiplication in $X(T)^{\boldsymbol{R}}$ and $X(T)^{\boldsymbol{C}}$ in additive form. Let $\langle\ ,\ \rangle$ be a W-invariant inner product on $X(T)^{\boldsymbol{R}}$. We extend the inner product to $X(T)^{\boldsymbol{C}}$ as a $\boldsymbol{C}$-bilinear form. For any $b \in \Phi$, we put $b^\vee=2b/\langle b, b\rangle$. For any $\alpha \in \Delta$, let Λ_α be the element in $X(T)^{\boldsymbol{R}}$ such that

$$\langle \Lambda_\alpha, \beta^\vee\rangle=\begin{cases}1 & \text{if } \beta=\alpha,\\ 0 & \text{if } \beta\in\Delta \text{ and } \beta\neq\alpha.\end{cases}$$

Since we are assuming that G is simply connected, Λ_α's are in $X(T)$ and they form a system of generators of the free abelian group $X(T)$. These Λ_α's are called the fundamental dominant weights. A rational character $\chi \in X(T)$ is called dominant if $\langle \chi, \alpha^\vee\rangle \geqq 0$ for all $\alpha \in \Delta$, equivalently,

$$\chi=\prod_{\alpha\in\Delta}\Lambda_\alpha^{m_\alpha}$$

for some non-negative integers m_α.

Denote by $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebra of G and T respectively. Let $\{H_\alpha, X_b;\ \alpha \in \Delta,\ b \in \Phi\}$ be a Chevalley basis of $\mathfrak{g}$. We have then

$$[H, X_b]=b(H)X_b,\quad [X_\alpha, X_{-\alpha}]=H_\alpha,\quad \beta(H_\alpha)=\langle \beta, \alpha^\vee\rangle,$$

$$[X_b, X_{b'}]=\begin{cases} 0 & \text{if } b+b' \notin \Phi,\ b+b'\neq 0, \\ N_{b,b'}X_{b+b'} & \text{if } b+b' \in \Phi, \end{cases}$$

$$N_{b,b'}=-N_{-b,-b'},\quad N_{b,b'}^2=(p+1)^2 \quad (H \in \mathfrak{t},\ b,\ b' \in \Phi,\ \alpha,\ \beta \in \Delta),$$

where p is the largest integer such that $b-pb' \in \Phi$. Moreover, H_α ($\alpha \in \Delta$) form a basis of $\mathfrak{t}$.

Let $\rho: G \to GL(V)$ be a faithful $\boldsymbol{Q}$-morphism where V is a finite dimensional vector space defined over $\boldsymbol{Q}$. A lattice L in $V_{\boldsymbol{Q}}$ is called *admissible* if it satisfies the conditions

(i) *L has a basis formed by eigenvectors of $\rho(T)$*, and

(ii) *for every $\alpha \in \Delta$ and every positive integer j, the linear transformation $(j!)^{-1}d\rho(X_\alpha)^j$ leaves L stable.*

An admissible lattice always exists (cf. [5, § 4], [25, § 2, p. 17 Corollary 1]). Taking an admissible lattice L, we set

$$\Gamma=G_{\boldsymbol{Z}}=\{g \in G;\ \rho(g)L=L\}.$$

The group Γ is a discrete subgroup of $G_{\boldsymbol{R}}$ and is independent of the choice of L (cf. [5, § 4], [2, p. 84]). We call Γ *the standard unit group of G.*

The double coset decomposition

$$G=\bigcup_{w \in W} BwB^- \quad \text{(disjoint union)}$$

is known as the Bruhat decomposition. Set $\Omega=B\cdot B^-$. Then Ω is a Zariski-open subset of G and the product map $B_u \times T \times B_u^- \to \Omega(\subset G)$ is an isomorphism. Consider the action of $B \times B^-$ on G defined by

$$g \longrightarrow b_1 g b_2^{-1} \quad (g \in G,\ b_1 \in B,\ b_2 \in B^-).$$

The set Ω is an open $B\times B^-$-orbit in G and the Bruhat decomposition gives the orbit decomposition of G under the action of $B \times B^-$.

In general, suppose that a connected linear algebraic group H operates on an algebraic variety X morphically. Then a non-zero rational function f on X is called a *relative* (H-) *invariant* if there exists a rational character χ of H such that

$$f(hx)=\chi(h)f(x) \quad (h \in H,\ x \in X).$$

The character χ is called *the character corresponding to f.*

We shall investigate relative $B\times B^-$-invariants on G. In the present case, since G has an open $B\times B^-$-orbit, a relative invariant is determined by its corresponding character uniquely up to a constant multiple. Notice that the group $X(B\times B^-)$ is canonically identified with $X(T)\times X(T)$.

For any $\chi \in X(T)$, define a regular function f_χ on Ω by setting

$$f_\chi(u_1tu_2)=\chi(t)^{-1} \quad (u_1 \in B_u,\ u_2 \in B_u^-,\ t \in T). \tag{1.1}$$

Then f_χ can be extended uniquely to a rational function on G and satisfies the invariance property

$$f_\chi(b_1gb_2^{-1})=\chi(b_1)^{-1}\chi(b_2)f_\chi(g) \quad (b_1 \in B,\ b_2 \in B^-,\ g \in G). \tag{1.2}$$

Lemma 1.1. (1) *The function f_χ ($\chi \in X(T)$) is a regular function on G if and only if χ is dominant.*

(2) *Any relative invariant f can be written uniquely as*

$$f=c\cdot \prod_{\alpha\in\Delta} f_{\Lambda_\alpha}^{\nu_\alpha} \quad (c \in \boldsymbol{C}^\times,\ \nu_\alpha \in \boldsymbol{Z}).$$

(3) $\Omega=\{g \in G;\ \prod_{\alpha\in\Delta} f_{\Lambda_\alpha}\neq 0\}$.

Proof. (1) Let $\boldsymbol{C}[G]$ be the ring of regular functions on G and we consider $\boldsymbol{C}[G]$ as a G-module by the formula: $(gf)(x)=f(g^{-1}x)$ $(g, x \in G)$. If f_χ is in $\boldsymbol{C}[G]$, then f_χ is a highest weight vector of highest weight χ. Hence χ is dominant. Conversely assume that χ is dominant. Let (ρ, V) be the irreducible representation of G of highest weight χ. Take a basis $\{v_1, \cdots, v_t\}$ $(t=\dim V)$ of V consisting of weight vectors. We may assume that v_1 is a highest weight vector. We denote by V' the subspace spaned by $v_2, \cdots, v_t$. Then V' is a $\rho(B_u^-)$-stable subspace of V and we have

$$\begin{aligned} &\rho(u)v_1\equiv v_1 \pmod{V'} && (u \in B_u^-),\\ &\rho(b)v_1=\chi(b)v_1 && (b \in B). \end{aligned}$$

Let $\rho_{ij}(g)$ $(g \in G)$ be the matrix element of $\rho(g)$ with respect to the fixed basis:

$$\rho(g)v_i=\sum_{j=1}^{t} \rho_{ij}(g)v_j.$$

It is easy to check that

$$\rho_{11}((u_1tu_2)^{-1})=\chi(t)^{-1} \quad (u_1 \in B_u,\ u_2 \in B_u^-,\ t \in T)$$

and

$$\rho_{11}((b_1gb_2^{-1})^{-1})=\chi(b_1)^{-1}\chi(b_2)\rho_{11}(g^{-1}) \quad (b_1 \in B,\ b_2 \in B^-,\ g \in G).$$

These equalities imply that $f_\chi(g)=\rho_{11}(g^{-1})$ and f_χ is a regular function on G.

(2) Let χ be the character of $B\times B^-$ corresponding to f. Consider

the morphism Δ: $T \to B \times B^-$ defined by $\Delta(t)=(t, t)$ for $t \in T$. Then the group $\Delta(T)$ is the isotropy subgroup of $B \times B^-$ at the identity element e of G: $\Delta(T)=\{(b_1, b_2) \in B \times B^-; b_1 e b_2^{-1}=e\}$. Hence χ is trivial on $\Delta(T)$ and there exists a $\chi_0 \in X(T)$ such that $\chi(b_1, b_2)=\chi_0(b_1)^{-1}\chi_0(b_2)$ $(b_1 \in B, b_2 \in B^-)$. The character χ_0 is written as

$$\chi_0 = \prod_{\alpha \in \Delta} \Lambda_\alpha^{\nu_\alpha} \quad (\nu_\alpha \in \boldsymbol{Z}).$$

The rational function $f/\prod_{\alpha \in \Delta} f_{\Lambda_\alpha}^{\nu_\alpha}$ is a relative $B \times B^-$-invariant which corresponds to the trivial character. Hence it is a non-zero constant. This proves the second assertion.

(3) In the proof of [25, § 5, Theorem 7 a)], it is shown that

$$\Omega = \{g \in G; \prod_{b \in \Phi^+} f_b \neq 0\}.$$

Since $\prod_{b \in \Phi^+} f_b = \prod_{\alpha \in \Delta} f_{\Lambda_\alpha}^2$, we get the third assertion. q.e.d.

In the rest of this paper, we simply write f_α for f_{Λ_α} $(\alpha \in \Delta)$. Since the relative invariant corresponding to a simple root α does not appear in the subsequent sections, this abbreviation will not cause any confusion.

§ 2. The structure of symmetric spaces

2.1. Let G be a universal Chevalley group over $\boldsymbol{Q}$ and σ an involutive automorphism of G defined over $\boldsymbol{Q}$. By definition, σ^2 is the identity mapping of G. Denote by H the fixed point group of σ: $H=\{g \in G; \sigma(g)=g\}$. The group H is known to be a connected reductive $\boldsymbol{Q}$-group (cf. [26, § 1], [24, 8.1]). A torus T of G is said to be *σ-anisotropic* if $\sigma(t)=t^{-1}$ for every $t \in T$. A parabolic subgroup P of G is said to be *σ-anisotropic* if $P \cap \sigma(P)$ is a Levi subgroup of both P and $\sigma(P)$. Throughout this paper, we assume that

(2.1) *G has a σ-anisotropic maximal torus.*

Lemma 2.1. *There exists a σ-anisotropic $\boldsymbol{Q}$-split maximal torus T and a σ-anisotropic Borel subgroup B defined over $\boldsymbol{Q}$ such that $T=B \cap \sigma(B)$.*

Proof. By the assumption (2.1), there exists a σ-anisotropic maximal torus T_0 of G. Take a σ-anisotropic Borel subgroup B_0 of G such that $T_0=B_0 \cap \sigma(B_0)$. Such a B_0 exists by the results in [26, § 1]. Let B_1 be a Borel subgroup defined over $\boldsymbol{Q}$ and put $\mathscr{B}=G/B_1$. We can identify $\mathscr{B}$ (resp. $\mathscr{B}_{\boldsymbol{Q}}$) with the set of all Borel subgroups (resp. all Borel subgroups defined over $\boldsymbol{Q}$) of G. Denote by $\mathscr{B}'$ the subset of $\mathscr{B}$ consisting of all σ-anisotropic Borel subgroups. By [26, Theorem 1], $\mathscr{B}'$ is a Zariski-open and Zariski-dense subset of $\mathscr{B}$. Therefore, if G has no σ-anisotropic Borel subgroup defined over $\boldsymbol{Q}$, then $\mathscr{B}_{\boldsymbol{Q}}$ is contained in the closed subset $F=\mathscr{B}-\mathscr{B}'$. Let $\pi: G\to\mathscr{B}$ be the orbit map defined by $\pi(g)=g\cdot B_1$. It is known that $\pi: G_{\boldsymbol{Q}}\to\mathscr{B}_{\boldsymbol{Q}}$ is surjective (cf. [3, Theorem 4.13 a)]). Hence $G_{\boldsymbol{Q}}$ is contained in a proper closed subset $\pi^{-1}(F)$. This contradicts the fact that $G_{\boldsymbol{Q}}$ is Zariski-dense in G. Thus we conclude that there exists a σ-anisotropic Borel subgroup B defined over $\boldsymbol{Q}$. Set $T=B\cap\sigma(B)$. Since B is σ-anisotropic, T is a maximal torus of B, and hence of G. It is clear that T is defined and split over $\boldsymbol{Q}$. By [26, Proposition 5], B is H-conjugate to B_0. This implies that T is H-conjugate to T_0. Hence T is σ-anisotropic. q.e.d.

We fix such T and B as in the lemma and put $B^-=\sigma(B)$. Whole results in Section 1 can be applied to this choice of T, B and B^-.

Now we define a twisted action of G on G by

$$g*x=gx\sigma(g)^{-1}\quad (g,\, x\in G).$$

Then we have $H=\{g\in G;\, g*e=e\}$ where e is the identity element of G and the morphism of G/H into G given by $g\cdot H\mapsto g*e$ induces a G-equivariant isomorphism of the homogeneous space G/H and the closed twisted G-orbit $G*e$ ([16, Lemma 2.4]). Put $X=G*e$ and $X_\Omega=X\cap\Omega$. Here we put $\Omega=B\cdot B^-=B\cdot\sigma(B)$ as in Section 1.

Lemma 2.2. *The set X_Ω is a Zariski-open dense subset of X and coincides with the twisted B-orbit $B*e$.*

Proof. Let x be an element in X_Ω. Then x is written as $x=utu'$ for some $u\in B_u$, $u'\in B_u^-=\sigma(B_u)$ and $t\in T$. Since $\sigma(x)^{-1}=x$ and $\sigma(t)=t^{-1}$, we get $utu'=\sigma(u')^{-1}t\sigma(u)^{-1}$. Hence $u'=\sigma(u)^{-1}$. Therefore, taking a $t'\in T$ such that $t=(t')^2$, we obtain $x=utu'=(ut')\cdot\sigma(ut')^{-1}=(ut')*e$. This proves that X_Ω is included in $B*e$. The opposite inclusion relation is obvious. It is also obvious that X_Ω is Zariski-dense and Zariski-open in X. q.e.d.

We denote the restriction of the regular function f_α (cf. the remark at the end of § 1) on G to X by the same symbol. By (1.2), we have

$$f_\alpha((tu)*x)=\Lambda_\alpha(t)^{-2}f_\alpha(x)\quad (t\in T,\, u\in B_u,\, x\in X). \tag{2.2}$$

Lemma 2.3. *Any non-zero rational function f on X relatively invariant under the twisted B-action is written uniquely as*

$$f=c\cdot\prod_{\alpha\in\Delta}f_\alpha^{\nu_\alpha}\quad(c\in\boldsymbol{C}^\times,\nu_\alpha\in\boldsymbol{Z}).$$

Proof. Let χ be the rational character of B corresponding to f: $f(b*x)=\chi(b)f(x)$ ($b\in B$, $x\in X$). Since f is a non-zero rational function on the Zariski-open set $X_\Omega=B*e$, $f(x)$ does not vanish at $x=e$. For a $t\in T$, if $t^2=1$, then we have $\chi(t)=1$ by $f(e)=f(t*e)=\chi(t)f(e)$. Therefore, χ is written as $\chi=\prod\Lambda_\alpha^{-2\nu_\alpha}$ for some integers ν_α. Then $f\cdot\prod_{\alpha\in\Delta}f_\alpha^{-\nu_\alpha}$ is a non-zero constant on the Zariski-dense subset X_Ω, and hence on X. This proves the assertion.

2.2. Let B^+ and T^+ be the identity components of the real Lie groups $B_{\boldsymbol{R}}$ and $T_{\boldsymbol{R}}$, respectively. Then B^+ is a semi-direct product of T^+ and $(B_u)_{\boldsymbol{R}}$. Since G is simply connected, $G_{\boldsymbol{R}}$ is a connected Lie group. Set $X_{\boldsymbol{R}}=X\cap G_{\boldsymbol{R}}$ and $(X_\Omega)_{\boldsymbol{R}}=X_\Omega\cap G_{\boldsymbol{R}}$. The set $X_{\boldsymbol{R}}$ (resp. $(X_\Omega)_{\boldsymbol{R}}$) is stable under the twisted action of $G_{\boldsymbol{R}}$ (resp. B^+).

The B^+-orbit structure of $(X_\Omega)_{\boldsymbol{R}}$ is fairly simple and is given by the following lemma, which is an immediate consequence of the Bruhat decomposition of $G_{\boldsymbol{R}}$.

We denote by Σ the set of all mappings of Δ to $\{\pm1\}$.

Lemma 2.4. *For $\varepsilon\in\Sigma$, put $X_\varepsilon=\{x\in X_{\boldsymbol{R}};\operatorname{sgn}f_\alpha(x)=\varepsilon_\alpha\ (\alpha\in\Delta)\}$. Then the sets X_ε ($\varepsilon\in\Sigma$) are twisted B^+-orbits in $(X_\Omega)_{\boldsymbol{R}}$ and we have*

$$(X_\Omega)_{\boldsymbol{R}}=\bigcup_{\varepsilon\in\Sigma}X_\varepsilon\quad(\textit{disjoint union}).$$

Moreover, the action of B^+ on X_ε is free for any $\varepsilon\in\Sigma$.

Next consider the $G_{\boldsymbol{R}}$-orbit structure of $X_{\boldsymbol{R}}$. Notice that each connected component of $X_{\boldsymbol{R}}$ is a twisted $G_{\boldsymbol{R}}$-orbit.

For any $x\in X_{\boldsymbol{R}}$, the isotropy subgroup H_x of $G_{\boldsymbol{R}}$ at x coincides with the fixed point group of the involutive automorphism σ_x of $G_{\boldsymbol{R}}$ defined by $\sigma_x(g)=x\sigma(g)x^{-1}$. By Lemma 2.4, each $G_{\boldsymbol{R}}$-orbit in $X_{\boldsymbol{R}}$ contains some of X_ε's. Let t_ε be the element in $T_{\boldsymbol{R}}$ such that $\Lambda_\alpha(t_\varepsilon)=\varepsilon_\alpha$ ($\alpha\in\Delta$). We can take t_ε as a representative of X_ε. We shall study the behaviour of the involution $\sigma_\varepsilon=\sigma_{t_\varepsilon}$ on the Lie algebra $\mathfrak{g}$ of G.

Let $\{H_\alpha, X_b;\ \alpha\in\Delta,\ b\in\Phi\}$ be a Chevalley basis of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}_{\boldsymbol{R}}$ of $G_{\boldsymbol{R}}$ conicides with the subspace of $\mathfrak{g}$ spanned by $\{H_\alpha, X_b\}$ over $\boldsymbol{R}$. Since the differential $d\sigma$ of σ maps $\boldsymbol{R}X_b$ onto $\boldsymbol{R}X_{-b}$, there exists a non-zero real number c_b such that $d\sigma(X_b)=c_bX_{-b}$ ($b\in\Phi$). Moreover, since T is σ-anisotropic, we have $d\sigma(H_\alpha)=-H_\alpha$ ($\alpha\in\Delta$). It is clear that $c_b\cdot c_{-b}=1$ and $c_{b+b'}=-c_b\cdot c_{b'}$ if $b, b', b+b'\in\Phi$. For the involution σ_ε, we get the relation

$$(2.3)\qquad \begin{cases} d\sigma_\varepsilon(X_b) = \prod_{\alpha\in\Delta} \varepsilon_\alpha^{\langle b,\alpha^\vee\rangle}\cdot c_b X_{-b} \\ d\sigma_\varepsilon(H_\alpha) = -H_\alpha \quad (b\in\Phi,\ \alpha\in\Delta). \end{cases}$$

Let $d\theta$ be the involution of $\mathfrak{g}$ defined by

$$\begin{cases} d\theta(H_\alpha) = -H_\alpha & (\alpha\in\Delta), \\ d\theta(X_b) = -|c_b| X_{-b} & (b\in\Phi). \end{cases}$$

Then $d\theta$ and $d\sigma_\varepsilon$ commute and the restriction of $d\theta$ to $\mathfrak{g}_{\boldsymbol{R}}$, which we denote also by $d\theta$, is a Cartan involution of $\mathfrak{g}_{\boldsymbol{R}}$. Let $\eta(b)=\operatorname{sgn}(-c_b)$ and $\eta_\varepsilon(b)=\prod_{\alpha\in\Delta}\varepsilon_\alpha^{\langle b,\alpha^\vee\rangle}\cdot\eta(b)$ for $b\in\Phi$. The mapping $\eta_\varepsilon\colon \Phi\to\{\pm 1\}$ is a signature of roots and $d\sigma_\varepsilon$ is the η_ε-involution of $\mathfrak{g}_{\boldsymbol{R}}$ in the sense of [15, § 1.2]. This shows that the connected components of $X_{\boldsymbol{R}}$ are in the class of semisimple symmetric spaces treated in [15].

Let θ be the Cartan involution of $G_{\boldsymbol{R}}$ whose differential coincides with $d\theta$ defined above. The fixed point group K of θ in $G_{\boldsymbol{R}}$ is a maximal compact subgroup of $G_{\boldsymbol{R}}$. Take a t_0 in T such that $\alpha(t_0)=\eta(\alpha)$ for all $\alpha\in\Delta$. Note that t_0 is not always in $T_{\boldsymbol{R}}$. It is easy to check that $\sigma_\varepsilon(g)=(t_\varepsilon t_0)\theta(g)(t_\varepsilon t_0)^{-1}$. In particular, we have $\sigma(g)=t_0\theta(g)t_0^{-1}$.

For the description of the $G_{\boldsymbol{R}}$-orbit structure of $X_{\boldsymbol{R}}$, we need to define an action of W on Σ. For each w in W, we choose a representative n_w of w in $N_K(T_{\boldsymbol{R}})=\{k\in K;\ kT_{\boldsymbol{R}}k^{-1}=T_{\boldsymbol{R}}\}$. Since $n_w{*}t_\varepsilon=n_w(t_\varepsilon t_0)n_w^{-1}\cdot t_0^{-1}$, the element $n_w{*}t_\varepsilon$ is in $T_{\boldsymbol{R}}$ and $f_\alpha(n_w{*}t_\varepsilon)=\Lambda_\alpha(n_w{*}t_\varepsilon)^{-1}\neq 0$. Put $(w\varepsilon)_\alpha=\operatorname{sgn} f_\alpha(n_w{*}t_\varepsilon)$. It is easy to see that $w\varepsilon$ is independent of the choice of a representative n_w. Thus we get an action of W on Σ. For a simple root α, it is easy to see that

$$(2.4)\qquad (w_\alpha\varepsilon)_\beta = \begin{cases} \varepsilon_\beta, & \text{if } \beta\neq\alpha,\ \beta\in\Delta, \\ \varepsilon_\alpha\cdot\prod_{\gamma\in\Delta}\varepsilon_\gamma^{\langle\alpha,\gamma^\vee\rangle}\cdot\eta(\alpha), & \text{if } \beta=\alpha, \end{cases}$$

where w_α is the reflection in the hyperplane orthogonal to α.

Lemma 2.5. (1) *For ε and ε' in Σ, X_ε and $X_{\varepsilon'}$ are contained in the same $G_{\boldsymbol{R}}$-orbit in $X_{\boldsymbol{R}}$ if and only if ε and ε' are in the same W-orbit in Σ.*

(2) *For an $\varepsilon\in\Sigma$, let $W_\varepsilon=\{w\in W;\ w\varepsilon=\varepsilon\}$. Then W_ε is the subgroup of W consisting of elements whose representatives can be taken from $H_\varepsilon\cap N_K(T_{\boldsymbol{R}})$, where $H_\varepsilon=\{g\in G_{\boldsymbol{R}};\ g{*}t_\varepsilon=t_\varepsilon\}$.*

(3) *For a W-orbit ω in Σ, let X_ω be the closure of $\bigcup_{\varepsilon\in\omega}X_\varepsilon$ in $X_{\boldsymbol{R}}$. Then the $G_{\boldsymbol{R}}$-orbit decomposition of $X_{\boldsymbol{R}}$ is given by*

$$X_{\boldsymbol{R}}=\bigcup_{\omega\in W\backslash\Sigma} X_\omega.$$

Proof. (1) If ε and ε' are in the same W-orbit in Σ, then, by the definition of the W-action on Σ, X_ε and $X_{\varepsilon'}$ are contained in the same $G_{\boldsymbol{R}}$-orbit. Conversely assume that X_ε and $X_{\varepsilon'}$ are contained in the same $G_{\boldsymbol{R}}$-orbit. Take a $t \in X_\varepsilon \cap T_{\boldsymbol{R}}$ and $t' \in X_{\varepsilon'} \cap T_{\boldsymbol{R}}$ such that $\alpha(t) \neq \pm 1$ and $\alpha(t') \neq \pm 1$ for any $\alpha \in \Delta$. Let g be an element in $G_{\boldsymbol{R}}$ such that $t' = g*t$. Since σ_ε and θ commute, we have the generalized Cartan decomposition $G_{\boldsymbol{R}} = K \cdot T^+ \cdot H_\varepsilon$ (cf. [6, Theorem 4.1]). Therefore, we may write $g = kt_1h$, where $k \in K$, $t_1 \in T^+$ and h satisfies $h*t = t$. Then

$$t' = k*(t \cdot t_1^2) = k(t \cdot t_1^2 \cdot t_0)k^{-1}t_0^{-1}.$$

Hence $t't_0 = k(t \cdot t_1^2 \cdot t_0)k^{-1}$. Since $\alpha(t't_0) \neq 1$ for any $\alpha \in \Delta$, k is in the normalizer of T. Write $k = n_w t_2$ with $w \in W$ and $t_2 \in T_{\boldsymbol{R}}$. Thus we get $t' = n_w * t \cdot (t_1 t_2)^2$. This shows that $\varepsilon' = w\varepsilon$.

(2) Suppose that w is in W_ε. Then $n_w * t_\varepsilon$ is in $T_{\boldsymbol{R}} \cap X_\varepsilon$. Hence $n_w * t_\varepsilon = t_\varepsilon \cdot t_1^2$ for some $t_1 \in T^+$. Since $\sigma(t_1) = t_1^{-1}$, we have $t_1^{-1} n_w * t_\varepsilon = t_\varepsilon$. This proves that $t_1^{-1} n_w$ is a representative of w in $N_K(T) \cap H_\varepsilon$. Now the second assertion is obvious.

(3) The third assertion is an immediate consequence of the first.

q.e.d.

Remark. More precise information on the orbit-structure of semisimple symmetric spaces under the action of minimal parabolic subgroups is given in [14] and [17]. (See also [15, Proposition 1.10].)

We can carry out the $G_{\boldsymbol{R}}$-orbit decomposition of $X_{\boldsymbol{R}}$ for each almost simple algebraic group G, by using the classification of symmetric spaces of ε-involution type [15, Appendix]. The result is summarized in the following table, where we denote by $(\mathfrak{g}, \mathfrak{h})$ the symmetric Lie algebra corresponding to each $G_{\boldsymbol{R}}$-orbit and the notation with respect to real simple Lie algebras is the same as employed in [10, Chapter X, § 6, Table V]. Moreover the numeral attached to each symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h})$ indicates the number of open B^+-orbits contained in the corresponding semisimple symmetric space.

G	$G_{\boldsymbol{R}} \backslash X_{\boldsymbol{R}}$
A_l (l: even)	$\bigcup_{j=1,\ j:\text{ odd}}^{l+1} (\mathfrak{sl}(l+1; \boldsymbol{R}), \mathfrak{so}(l-j+1, j))$ $\binom{l+1}{j}$

G	$G_{\boldsymbol{R}}\backslash X_{\boldsymbol{R}}$
A_l (l: odd)	$\bigcup_{\substack{j=0\\ j:\text{ even}}}^{l+1} (\mathfrak{sl}(l+1;\boldsymbol{R}),\ \mathfrak{so}(l-j+1,j))$ $\binom{l+1}{j}$
	$\bigcup_{\substack{j=1\\ j:\text{ odd}}}^{l} (\mathfrak{sl}(l+1;\boldsymbol{R}),\ \mathfrak{so}(l-j+1,j))$ $\binom{l+1}{j}$
B_l	$2\times(\mathfrak{so}(l+1,\ l),\ \mathfrak{so}(l+1)\oplus\mathfrak{so}(l))\cup$ 1 $\bigcup_{\substack{j=2\\ j:\text{ even}}}^{l} (\mathfrak{so}(l+1,\ l),\ \mathfrak{so}(l-j+1,j)\oplus\mathfrak{so}(l-j,j))$ $2\binom{l}{j}$
	$\bigcup_{\substack{j=1\\ j:\text{odd}}}^{l} (\mathfrak{so}(l+1,\ l),\ \mathfrak{so}(l-j+1,j)\oplus\mathfrak{so}(l-j,j))$ $2\binom{l}{j}$
C_l	$\bigcup_{j=0}^{l} (\mathfrak{sp}(l;\ \boldsymbol{R}),\ \mathfrak{u}(l-j,j))$ $\binom{l}{j}$
	$(\mathfrak{sp}(l;\ \boldsymbol{R}),\ \mathfrak{gl}(l;\ \boldsymbol{R}))$ 2^l
D_l (l: odd)	$2\times(\mathfrak{so}(l,\ l),\ \mathfrak{so}(l)\oplus\mathfrak{so}(l))\cup$ 1 $\bigcup_{j=1}^{(l-1)/2} (\mathfrak{so}(l,\ l),\ \mathfrak{so}(l-j,j)\oplus\mathfrak{so}(l-j,j)$ $2\binom{l}{j}$
	$2\times(\mathfrak{so}(l,\ l),\ \mathfrak{so}(l;\ \boldsymbol{C}))$ 2^{l-1}

G	$G_R\backslash X_R$
D_l (l: even)	$4\times(\mathfrak{so}(l, l), \mathfrak{so}(l)\oplus\mathfrak{so}(l))\cup$ 1 $\bigcup_{\substack{j=2\\ j:\text{ even}}}^{l-2}(\mathfrak{so}(l, l), \mathfrak{so}(l-j, j)\oplus\mathfrak{so}(l-j, j))$ $2\binom{l}{j}$
	$\bigcup_{\substack{j=1\\ j:\text{ odd}}}^{l-1}(\mathfrak{so}(l, l), \mathfrak{so}(l-j, j)\oplus\mathfrak{so}(l-j, j))$ $2\binom{l}{j}$
	$2\times(\mathfrak{so}(l, l), \mathfrak{so}(l; \boldsymbol{C}))$ 2^{l-1}
E_6	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))\cup(\mathfrak{e}_{6(6)}, \mathfrak{sp}(2, 2))\cup(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4; \boldsymbol{R}))$ 1 27 36
E_7	$2\times(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))\cup 2\times(\mathfrak{e}_{7(7)}, \mathfrak{su}(4, 4))$ 1 63
	$(\mathfrak{e}_{7(7)}, \mathfrak{sl}(8; \boldsymbol{R}))\cup(\mathfrak{e}_{7(7)}, \mathfrak{su}^*(8))$ 72 56
E_8	$(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))\cup(\mathfrak{e}_{8(8)}, \mathfrak{so}^*(16))\cup(\mathfrak{e}_{8(8)}, \mathfrak{so}(8, 8))$ 1 120 135
F_4	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3)\oplus\mathfrak{su}(2))\cup(\mathfrak{f}_{4(4)}, \mathfrak{sp}(2, 1)\oplus\mathfrak{su}(2))$ 1 3 $\cup(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3; \boldsymbol{R})\oplus\mathfrak{sl}(2; \boldsymbol{R}))$ 12
G_2	$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2)\oplus\mathfrak{su}(2))\cup(\mathfrak{g}_{2(2)}, \mathfrak{sl}(2; \boldsymbol{R})\oplus\mathfrak{sl}(2; \boldsymbol{R}))$ 1 3

§ 3. Eisenstein series

In the following we fix a W-orbit ω in Σ. Let X_ω be the twisted G_R-orbit in X_R corresponding to ω: X_ω=the closure of $\bigcup_{\varepsilon\in\omega} X_\varepsilon$.

Let Γ be the standard unit group of the Chevalley group G (for the definition, see § 1) and put $\Gamma_\infty=\Gamma\cap B_u$. For an $x\in(X_\omega)_Q=X_\omega\cap G_Q$, denote by $\Gamma_\infty\backslash(\Gamma*x\cap X_\varepsilon)$ the set of twisted Γ_∞-orbits in $(\Gamma*x)\cap X_\varepsilon$ ($\varepsilon\in\omega$). Define Dirichlet series $E(x,\varepsilon;\lambda)$ ($\varepsilon\in\omega$) by

$$(3.1)\qquad E(x,\varepsilon;\lambda)=\sum_{\gamma}\prod_{\alpha\in\Delta}|f_\alpha(\gamma)|^{-\langle\lambda+\delta/2,\alpha^\vee\rangle}\quad(\lambda\in X(T)^C),$$

where $\delta=\sum_{\alpha\in\Delta}\Lambda_\alpha$ and the summation with respect to γ is taken over a complete system of representatives of $\Gamma_\infty\backslash(\Gamma*x\cap X_\varepsilon)$. We call the series $E(x,\varepsilon;\lambda)$ *the Eisenstein series on the symmetric space* X_ω, since it coincides with the usual Eisenstein series if X_ω is a Riemannian symmetric space (cf. § 6 (A)).

Proposition 3.1. *The series* $E(x,\varepsilon;\lambda)$ ($\varepsilon\in\omega$) *are absolutely convergent for* $\mathrm{Re}\langle\lambda,\alpha^\vee\rangle>3/2$ ($\alpha\in\Delta$).

Now we introduce another Dirichlet series $D(g;\lambda)$ ($g\in G_Q$) which plays an important role in the proof of Proposition 3.1.

For a $g\in G_Q$, put $\Gamma_\Omega(g)=(\Gamma g\Gamma)\cap\Omega$. Then $D(g;\lambda)$ is defined by the following formula similar to (3.1):

$$(3.2)\qquad D(g;\lambda)=\sum_{\gamma}{}'\prod_{\alpha\in\Delta}|f_\alpha(\gamma)|^{-\langle\lambda+\delta,\alpha^\vee\rangle}\quad(\lambda\in X(T)^C),$$

where γ runs through a complete set of representatives of double cosets belonging to $\Gamma_\infty\backslash\Gamma_\Omega(g)/\Gamma_\infty^-$ ($\Gamma_\infty^-=\Gamma\cap\sigma(B_u)$). Set $\tilde\Gamma_\infty=\Gamma\cap\sigma(\Gamma)\cap B_u$ and $\tilde\Gamma_\infty^-=\Gamma\cap\sigma(\Gamma)\cap\sigma(B_u)$. Then $\tilde\Gamma_\infty$ (resp. $\tilde\Gamma_\infty^-$) is a subgroup of Γ_∞ (resp. Γ_∞^-) of finite index. Moreover we have $\sigma(\tilde\Gamma_\infty)=\tilde\Gamma_\infty^-$.

Lemma 3.2. *If* $g\in(X_\Omega)_Q=X_\Omega\cap G_Q$, *then* $\tilde\Gamma_\infty g\tilde\Gamma_\infty^-\cap X=\tilde\Gamma_\infty*g$.

Proof. Suppose that ngn' ($n\in\tilde\Gamma_\infty$, $n'\in\tilde\Gamma_\infty^-$) is in $\tilde\Gamma_\infty g\tilde\Gamma_\infty^-\cap X$. Then $ngn'=\sigma(ngn')^{-1}=\sigma(n')^{-1}g\sigma(n)^{-1}$. Hence $\sigma(n')ngn'\sigma(n)=g$. Since g is in X_Ω and the action of $B_u\times B_u^-$ on Ω is free, this implies that $\sigma(n')n=1$. Therefore, $ngn'=ng\sigma(n)^{-1}=n*g$. This shows that $\tilde\Gamma_\infty g\tilde\Gamma_\infty^-\cap X\subset\tilde\Gamma_\infty*g$. The opposite inclusion relation is obvious. q.e.d.

By the lemma, we easily obtain the inequality

$$E(x,\varepsilon;\lambda)\leqq[\Gamma_\infty;\tilde\Gamma_\infty]\cdot D(x;\lambda-\delta/2)\quad(x\in X_Q,\ \varepsilon\in\Sigma),$$

if $\langle\lambda,\alpha^\vee\rangle$ is real for any $\alpha\in\Delta$. Hence Proposition 3.1 follows immediately from the convergence of $D(g;\lambda)$ for $\mathrm{Re}\langle\lambda,\alpha^\vee\rangle>1$ ($\alpha\in\Delta$).

We shall check the convergence of $D(g;\lambda)$ by finding its explicit expression in terms of the Riemann zeta function and the zonal spherical functions on the p-adic group G_{Q_p} (Proposition 3.3 below). In order to formulate the result on $D(g;\lambda)$, we need some preliminaries.

For any (finite or infinite) prime v of $\boldsymbol{Q}$, put $G_v=G_{\boldsymbol{Q}_v}$, $T_v=T_{\boldsymbol{Q}_v}$, $U_v=(B_u)_{\boldsymbol{Q}_v}$, $U_v^-=(B_u^-)_{\boldsymbol{Q}_v}$ and

$$K_v=\begin{cases} G_{\boldsymbol{Z}_p} & \text{if } v=\text{a finite prime } p, \\ K & \text{if } v=\text{the infinite prime } \infty, \end{cases}$$

where K is the maximal compact subgroup of $G_{\boldsymbol{R}}$ introduced in Section 2.2. For a finite prime p, we also put $T_{p,0}=T_{\boldsymbol{Z}_p}$. Then we have the Iwasawa decomposition for the p-adic group G_p: $G_p=K_pT_pU_p^-$. For a $g\in G_p$, let $g=k(g)t(g)v(g)$ ($k(g)\in K_p$, $t(g)\in T_p$, $v(g)\in U_p^-$) be its Iwasawa decomposition. Then the coset $t(g)T_{p,0}$ is uniquely determined by g.

Let du, du^- and $d^\times t$ be the $\boldsymbol{Q}$-rational invariant gauge forms on B_u, B_u^- and T, respectively, which are normalized so that

$$\int_{U_p\cap K_p}|du|_p=\int_{U_p^-\cap K_p}|du^-|_p=(1-p^{-1})^{-\dim T}\cdot\int_{T_{p,0}}|d^\times t|_p=1.$$

We define a guage form ω on Ω by

$$\omega(utu^-)=\prod_{\alpha\in\Delta}\Lambda_\alpha(t)^{-2}du\,d^\times t\,du^-.$$

It is easy to see that ω is invariant under the $B\times B^-$-action on Ω: $\omega(b_1xb_2^{-1})=\omega(x)$ ($b_1\in B$, $b_2\in B^-$). Since the restriction of any invariant gauge form dg on G to the open set Ω is also $B\times B^-$-invariant, it is a constant multiple of ω. We assume that the restriction of dg to Ω coincides with ω.

We may define the zonal spherical function $\omega_\lambda^{(p)}$ ($\lambda\in X(T)^{\boldsymbol{C}}$) on G_p by the integral

$$\omega_\lambda^{(p)}(g)=\int_{K_p}\prod_{\alpha\in\Delta}|\Lambda_\alpha(t(g^{-1}k))|_p^{\langle\lambda+\delta,\alpha^\vee\rangle}dk\quad(g\in G_p),$$

where dk is the Haar measure on K_p normalized by $\int_{K_p}dk=1$. We put

$$v_p(g)=\int_{K_pgK_p}|dg|_p\Big/\int_{K_p}|dg|_p.$$

Proposition 3.3. *When* $\operatorname{Re}\langle\lambda,\alpha^\vee\rangle>1$ ($\alpha\in\Delta$), *the Dirichlet series* $D(g;\lambda)$ ($g\in G_{\boldsymbol{Q}}$) *is absolutely convergent and is equal to*

$$2^{\dim T}\cdot\{\prod_p v_p(g)\omega_{-\lambda}^{(p)}(g)\}\prod_{b\in\Phi+}\{\zeta(\langle\lambda,b^\vee\rangle)/\zeta(\langle\lambda,b^\vee\rangle+1)\},$$

where the product with respect to p is taken over all finite primes of $\boldsymbol{Q}$.

Remarks. (1) Since $v_p(g)\omega_{-\lambda}^{(p)}(g)$ is equal to 1 for almost all p, the infinite product with respect to p is actually a finite product.

(2) The explicit formulas for $v_p(g)$ and $\omega_{-\lambda}^{(p)}(g)$ are given in [13, Prop. 3.2.5] and [13, § 4], respectively. (See also [4].)

Proof. Let $L(G_v, K_v)$ be the set of K_v-biinvariant smooth functions on G_v with compact support. For a ϕ_v in $L(G_v, K_v)$, we consider the following local zeta function:

$$\Phi_\Omega^{(v)}(\phi_v;\lambda)=\int_{\Omega_{Q_v}}\prod_{\alpha\in\Delta}|f_\alpha(x)|_v^{\langle\lambda-\delta,\alpha^\vee\rangle}\phi_v(x)\lambda_v^{-1}|\omega(x)|_v,$$

where $\lambda_v=(1-p^{-1})^{\dim T}$ or 1 according as $v=p$ or $v=\infty$. The integral $\Phi_\Omega^{(v)}(\phi_v;\lambda)$ is absolutely convergent for $\operatorname{Re}\langle\lambda,\alpha^\vee\rangle>1$ $(\alpha\in\Delta)$. In fact, the convergence is obvious for $v=\infty$. When v is a finite prime, such an integral has been considered by Casselman [4, § 3, p. 398] in a more general setting and the convergence of the integral follows immediately from [4, Lemma 3.2]. Moreover, from [4, Theorem 3.1] and the Iwasawa decomposition of G_p, we can easily derive the formula

$$\text{(3.3)}\qquad \Phi_\Omega^{(p)}(\phi_p;\lambda)=\widehat{\omega_{-\lambda}^{(p)}}(\phi_p)\cdot\prod_{b\in\Phi^+}(1-p^{-\langle\lambda,b^\vee\rangle-1})/(1-p^{-\langle\lambda,b^\vee\rangle}),$$

where $\widehat{\omega_\lambda^{(p)}}(\phi_p)$ is the Fourier transform of ϕ_p and is defined by

$$\widehat{\omega_\lambda^{(p)}}(\phi_p)=\left\{\int_{K_p}|dg|_p\right\}^{-1}\cdot\int_{G_p}\phi_p(g)\omega_\lambda^{(p)}(g^{-1})|dg|_p.$$

For a $g\in G_p$, denote by $\phi_{g,p}$ the characteristic function of K_pgK_p in G_p. Then

$$\text{(3.4)}\qquad \widehat{\omega_{-\lambda}^{(p)}}(\phi_{g,p})=v_p(g)\omega_{-\lambda}^{(p)}(g).$$

Let $L_A=\otimes_v' L(G_v, K_v)$ be the restricted tensor product of $L(G_v, K_v)$ with respect to $\{\phi_{e,p}\}$, where e is the identity element of G_p. For a $\phi\in L_A$, we put

$$\Phi_\Omega^{(A)}(\phi;\lambda)=\int_{\Omega_A}\prod_{\alpha\in\Delta}|f_\alpha(x)|_A^{\langle\lambda-\delta,\alpha^\vee\rangle}\phi(x)|\lambda^{-1}\omega(x)|_A,$$

where Ω_A is the adelization of Ω over $\boldsymbol{Q}$ and $|\lambda^{-1}\omega(x)|_A$ is the measure on Ω_A defined by $|\lambda^{-1}\omega(x)|_A=\prod_v\lambda_v^{-1}|\omega(x)|_v$. Then, by (3.3), the integral $\Phi_\Omega^{(A)}(\phi;\lambda)$ $(\phi\in L_A)$ converges absolutely for $\operatorname{Re}\langle\lambda,\alpha^\vee\rangle>1$ $(\alpha\in\Delta)$, and, if ϕ is of the form $\phi_\infty\otimes(\otimes_p\phi_p)$ $(\phi_v\in L(G_v, K_v))$, we get

$$\Phi_{\Omega}^{(A)}(\phi; \lambda)=\prod_{v} \Phi_{\Omega}^{(v)}(\phi_v; \lambda). \tag{3.5}$$

Therefore, the proposition is an immediate consequence of the formulas (3.3), (3.4), (3.5) and Lemma 3.4 below.

Lemma 3.4. *For a $g \in G_Q$ and a $\phi_\infty \in L(G_R, K)$, put $\phi=\phi_\infty \otimes(\otimes_p \phi_{g,p})$. Then we have*

$$\Phi_{\Omega}^{(A)}(\phi; \lambda)=2^{-\dim T} D(g; \lambda) \Phi_{\Omega}^{(\infty)}(\phi_\infty; \lambda)$$

for $\mathrm{Re}\langle\lambda, \alpha^\vee\rangle>1$ $(\alpha \in \Delta)$.

Proof. Since $\Omega_A=B_{uA}\cdot T_A\cdot B_{uA}^-$ and $\Omega_Q=B_{uQ}\cdot T_Q\cdot B_{uQ}^-$, we have

$$\begin{aligned}\Phi_{\Omega}^{(A)}(\phi; \lambda)=&\int_{B_{uA}/B_{uQ}} |du|_A \int_{T_A/T_Q} \prod_{\alpha\in\Delta} |\Lambda_\alpha(t)|_A^{-\langle\lambda-\delta, \alpha^\vee\rangle} |\lambda^{-1} d^\times t|_A \\ &\times \int_{B_{uA}^-/B_{uQ}^-} \sum_{x\in\Omega_Q} \phi(tuxu_-^{-1})|du_-|_A.\end{aligned}$$

We can take the sets $(B_{uR}/\Gamma_\infty)\times\prod_p(U_p\cap K_p)$, $T^+\times\prod_p T_{p,0}$ and $(B_{uR}^-/\Gamma_\infty^-)\times\prod_p(U_p^-\cap K_p)$ as fundamental domains of B_{uA}/B_{uQ}, T_A/T_Q and B_{uA}^-/B_{uQ}^-, respectively. Here we denote by T^+ the identity component of T_R. Moreover, notice that an element x in G_Q is in $\Gamma g\Gamma$ if and only if x is in $K_p g K_p$ for all p. Hence

$$\Phi_{\Omega}^{(A)}(\phi; \lambda)=\int \prod_{\alpha\in\Delta} |\Lambda_\alpha(t)|_\infty^{-\langle\lambda-\delta, \alpha^\vee\rangle} \sum_{x\in\Gamma_\Omega(g)} \phi_\infty(tuxu_-^{-1})|du|_\infty |d^\times t|_\infty |du_-|_\infty,$$

where the integral in the right hand side is taken over $T^+\times B_{uR}/\Gamma_\infty\times B_{uR}^-/\Gamma_\infty^-$. Put $\Omega_\varepsilon=\{x\in\Omega_R;\ \mathrm{sgn} f_\alpha(x)=\varepsilon_\alpha\ (\alpha\in\Delta)\}$ for any ε in Σ. Also put $\Gamma_\varepsilon(g)=\Gamma_\Omega(g)\cap\Omega_\varepsilon$. Denote by $\sim\backslash\Gamma_\varepsilon(g)$ the $\Gamma_\infty\times\Gamma_\infty^-$-equivalence classes in $\Gamma_\varepsilon(g)$. Then the right hand side of the equality above is rewritten as

$$\sum_\varepsilon \Big\{ \sum_{x\in\sim\backslash\Gamma_\varepsilon(g)} \prod_{\alpha\in\Delta} |f_\alpha(x)|^{-\langle\lambda+\delta, \alpha^\vee\rangle}\Big\} \int_{\Omega_\varepsilon} \prod_{\alpha\in\Delta} |f_\alpha(x)|^{\langle\lambda-\delta, \alpha^\vee\rangle} \phi_\infty(x)|\omega(x)|_\infty.$$

Let t_ε be the element in T such that $\Lambda_\alpha(t_\varepsilon)=\varepsilon_\alpha$ and denote by ε^+ the element in Σ defined by $\varepsilon_\alpha^+=1$ for any $\alpha\in\Delta$. The mapping of $\Gamma_{\varepsilon^+}(g)$ onto $\Gamma_\varepsilon(g)$ defined by $x\mapsto t_\varepsilon\cdot x$ induces a one to one correspondence between $\sim\backslash\Gamma_{\varepsilon^+}(g)$ and $\sim\backslash\Gamma_\varepsilon(g)$. Hence the infinite series in the bracket above do not depend on ε and are equal to $2^{-\dim T}D(g; \lambda)$. Consequently, we obtain

$$\begin{aligned}\Phi_{\Omega}^{(A)}(\phi; \lambda)&=2^{-\dim T} D(g; \lambda) \sum_\varepsilon \int_{\Omega_\varepsilon} \prod_{\alpha\in\Delta} |f_\alpha(x)|^{\langle\lambda-\delta, \alpha^\vee\rangle} \phi_\infty(x)|\omega(x)|_\infty \\ &=2^{-\dim T} D(g; \lambda)\Phi_{\Omega}^{(\infty)}(\phi_\infty; \lambda).\end{aligned}$$

q.e.d.

§ 4. Integral representations

4.1. Fix a simple root $\alpha \in \Delta$. Let M_α be the subgroup of G generated by the subgroups U_α and $U_{-\alpha}$ (for the definition of $U_{\pm\alpha}$, see § 1.1). Then there exists a unique $\boldsymbol{Q}$-morphism τ_α: $SL(2)\to M_\alpha$ such that

$$d\tau_\alpha\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=X_\alpha,\quad d\tau_\alpha\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}=X_{-\alpha},\quad d\tau_\alpha\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=H_\alpha.$$

Since G is simply connected, τ_α is an isomorphism. Moreover, τ_α induces an isomorphism of $SL(2;\boldsymbol{Z})$ onto $M_{\alpha,\boldsymbol{Z}}=M_\alpha\cap\Gamma$. For any $m\in M_\alpha$, we put $\boldsymbol{m}=\tau_\alpha^{-1}(m)$. The group M_α is σ-stable and we define an involution σ_α of $SL(2)$ by $\sigma_\alpha=\tau_\alpha^{-1}\circ\sigma|_{M_\alpha}\circ\tau_\alpha$. Then, by (2.3), we have

$$\sigma_\alpha(h)=J_\alpha{}^t h^{-1}J_\alpha^{-1},\quad J_\alpha=\begin{pmatrix}\eta(\alpha)|c_\alpha|^{-1} & 0\\ 0 & 1\end{pmatrix}\quad (h\in SL(2)). \tag{4.1}$$

Denote by T_α the connected component of the identity element of $\operatorname{Ker}\alpha=\{t\in T;\ \alpha(t)=1\}$. Let $Z_G(T_\alpha)$ be the centralizer of T_α in G. Then $Z_G(T_\alpha)=T_\alpha\cdot M_\alpha$. We consider the parabolic subgroup $P_\alpha=Z_G(T_\alpha)\cdot B$. Denoting the unipotent radical of P_α by $U_{(\alpha)}$, we have the decomposition $P_\alpha=T_\alpha M_\alpha U_{(\alpha)}$. Let ρ: $P_\alpha\to GL(2)$ be the representation defined by

$$\rho(p)=\Lambda_\alpha(t){}^t\boldsymbol{m}^{-1},\quad p=tmu\quad (t\in T_\alpha,\ m\in M_\alpha,\ u\in U_{(\alpha)}).$$

It is easy to check that $\rho(p)$ is independent of the decomposition of p and well-defines a $\boldsymbol{Q}$-rational representation.

Let $P_\alpha^*=P_\alpha\times GL(1)$ and $X^*=X\times M(2,1;\boldsymbol{C})$. The group P_α^* acts morphically on X^* as follows:

$$(p,a)\cdot(x,y)=(p*x,\ a\rho(p)y)\quad (p\in P_\alpha,\ a\in GL(1),\ x\in X,\ y\in M(2,1;\boldsymbol{C})).$$

For an $x\in X\cap P_\alpha\cdot\sigma(U_{(\alpha)})$, we write $x=utmu'$ ($u\in U_{(\alpha)}$, $t\in T_\alpha$, $m\in M_\alpha$, $u'\in\sigma(U_{(\alpha)})$) and set $S_\alpha(x)=\Lambda_\alpha(t)^{-1}\boldsymbol{m}J_\alpha$. The matrix $S_\alpha(x)$ is a non-degenerate 2 by 2 symmetric matrix and the mapping $x\mapsto S_\alpha(x)$, which is defined originally on $P_\alpha\cdot\sigma(U_{(\alpha)})\cap X$, can be extended to the whole of X as a rational mapping. If x is in $X_{\boldsymbol{R}}\cap P_\alpha\cdot\sigma(U_{(\alpha)})$, then $S_\alpha(x)$ is real symmetric and

$$\begin{cases}\det S_\alpha(x)=-c_\alpha^{-1}\cdot\prod_{\beta\in\Delta-\{\alpha\}} f_\beta(x)^{-\langle\alpha,\beta^\vee\rangle}\\ \operatorname{sgn}\det S_\alpha(x)=\eta(\alpha)\prod_{\beta\in\Delta-\{\alpha\}}\varepsilon_\beta^{\langle\alpha,\beta^\vee\rangle},\end{cases} \tag{4.2}$$

where $\varepsilon_\beta=\operatorname{sgn} f_\beta(x)$ ($\beta\in\Delta-\{\alpha\}$). By (2.4), $S_\alpha(x)$ is definite or indefinite according as $w_\alpha\varepsilon=\varepsilon$ or $w_\alpha\varepsilon\neq\varepsilon$. Set $f_\alpha^{(\alpha)}(x,y)={}^t y S_\alpha(x)y$ for $(x,y)\in X^*$

and $\chi_\alpha^{(\alpha)}(p, a)=a^2$ for $(p, a) \in P_\alpha^*$. Also set, for $\beta \in \Delta-\{\alpha\}$, $f_\beta^{(\alpha)}(x, y)=f_\beta(x)$ $((x, y) \in X^*)$ and $\chi_\beta^{(\alpha)}(p, a)=\Lambda_\beta(t)^{-2}$ $(p=tmu,\ t \in T_\alpha,\ m \in M_\alpha,\ u \in U_{(\alpha)})$.

Lemma 4.1. (i) *For any $\beta \in \Delta$, the function $f_\beta^{(\alpha)}(x, y)$ is a relatively P_α^*-invariant regular function on X^* corresponding to the character $\chi_\beta^{(\alpha)}$:*

$$f_\beta^{(\alpha)}((p, a)\cdot(x, y))=\chi_\beta^{(\alpha)}(p, a) f_\beta^{(\alpha)}(x, y).$$

(ii) *Any relative P_α^*-invariant rational function f on X^* is written uniquely as*

$$f(x, y)=c\cdot \prod_{\beta\in\Delta} f_\beta^{(\alpha)}(x, y)^{\nu_\beta} \quad (c \in \boldsymbol{C}^\times,\ \nu_\beta \in \boldsymbol{Z}).$$

Proof. (i) For a simple root β different from α, the assertion is obvious. Now we consider the function $f_\alpha^{(\alpha)}$. It follows from the definition that $f_\alpha^{(\alpha)}$ is a relative P_α^*-invariant corresponding to the character $\chi_\alpha^{(\alpha)}(p, a)=a^2$. We need to show that $f_\alpha^{(\alpha)}$ is a regular function on X^*. First we shall prove the identity

$$f_\alpha^{(\alpha)}(x, {}^t(0, 1))=f_\alpha(x). \tag{4.3}$$

Consider the mapping $\mu\colon B\to P_\alpha^*$ defined by $\mu(tu)=(tu, \Lambda_\alpha(t)^{-1})$ $(t \in T,\ u \in B_u)$. Then it is easy to check that $\mu(b)(x, {}^t(0, 1))=(b*x, {}^t(0, 1))$. Hence $f_\alpha^{(\alpha)}(x, {}^t(0, 1))$ is a relative B-invariant on X corresponding to the character Λ_α^{-2}. This implies that $f_\alpha^{(\alpha)}(x, {}^t(0, 1))$ is a constant multiple of $f_\alpha(x)$. Since $f_\alpha^{(\alpha)}(e, {}^t(0, 1))=f_\alpha(e)=1$, we obtain the identity (4.3). It follows immediately from (4.3) that $f_\alpha^{(\alpha)}(x, y)$ is a regular function on $\{(x, y) \in X^*; y\neq 0\}$. It is obvious that $f_\alpha^{(\alpha)}(x, y)=0$ if $y=0$. This proves that $f_\alpha^{(\alpha)}$ is a regular function on X^*.

(ii) Since the proof of the second assertion is quite similar to that of Lemma 2.3, we omit it. q.e.d.

Put $X_\alpha^*=\{(x, y) \in X^*; f_\beta^{(\alpha)}(x, y)\neq 0 \text{ for all } \beta \in \Delta\}$,

$$(X_\alpha^*)_{\boldsymbol{R}}=X_\alpha^* \cap (X_{\boldsymbol{R}}\times M(2, 1; \boldsymbol{R})),$$

and for $\varepsilon \in \Sigma$,

$$X_{\alpha,\varepsilon}^*=\{(x, y) \in (X_\alpha^*)_{\boldsymbol{R}};\ \operatorname{sgn} f_\beta^{(\alpha)}(x, y)=\varepsilon_\beta \quad (\beta \in \Delta)\}.$$

Let $(T_\alpha)_{\boldsymbol{R}}^\circ$, $(P_\alpha)_{\boldsymbol{R}}^\circ$ and $(P_\alpha^*)_{\boldsymbol{R}}^\circ$ be the identity components of real Lie groups $(T_\alpha)_{\boldsymbol{R}}$, $(P_\alpha)_{\boldsymbol{R}}$ and $(P_\alpha^*)_{\boldsymbol{R}}$, respectively. Then $(P_\alpha^*)_{\boldsymbol{R}}^\circ=(P_\alpha)_{\boldsymbol{R}}^\circ\times \boldsymbol{R}_+^\times= (T_\alpha)_{\boldsymbol{R}}^\circ\cdot(M_\alpha)_{\boldsymbol{R}}\cdot U_{(\alpha)\boldsymbol{R}}\times \boldsymbol{R}_+^\times$.

Lemma 4.2. (i) *The set X_α^* is a Zariski-open P_α^*-orbit in X^*.*

(ii) *The $(P_\alpha^*)_{\boldsymbol{R}}^\circ$-orbit decomposition of $(X_\alpha^*)_{\boldsymbol{R}}$ is given by*

$$(X_\alpha^*)_{\boldsymbol{R}} = \bigcup_{\varepsilon \in \Sigma} X_{\alpha,\varepsilon}^*.$$

Proof. It is obvious that X_α^* (resp. $(X_\alpha^*)_{\boldsymbol{R}}$) is P_α^*- (resp. $(P_\alpha^*)_{\boldsymbol{R}}^\circ$-) stable. If (x, y) is in X_α^* (resp. $(X_\alpha^*)_{\boldsymbol{R}}$), there exists an m in M_α (resp. $(M_\alpha)_{\boldsymbol{R}}$) such that $(m, 1)\cdot(x, y) = (x', {}^t(0, 1))$ for some $x' \in X$. Since

$$f_\beta^{(\alpha)}(x'; {}^t(0, 1)) = f_\beta(x') \quad \text{for any } \beta \in \Delta, \tag{4.4}$$

the first (resp. second) assertion follows immediately from Lemma 2.2 (resp. Lemma 2.4). q.e.d.

4.2. In this paragraph, we fix a simple root $\alpha \in \Delta$ and a W-orbit ω in Σ. Let dx be a $G_{\boldsymbol{R}}$-invariant measure on X_ω and dy the standard Euclidean measure on $M(2, 1; \boldsymbol{R}) = \boldsymbol{R}^2$. We normalize a right invariant measure dp on $(P_\alpha^*)_{\boldsymbol{R}}^\circ$ such that

$$\begin{aligned} &\int_{(P_\alpha)_{\boldsymbol{R}}^\circ \times \boldsymbol{R}_+^\times} f((p, a)\cdot(x_0, y_0))dp\frac{da}{a} \\ &= \int_{X_{\alpha,\varepsilon}^*} f(x, y)\cdot \prod_{\beta \in \Delta} |f_\beta^{(\alpha)}(x, y)|^{-1} dx\, dy \\ &(\varepsilon \in \omega,\ (x_0, y_0) \in X_{\alpha,\varepsilon}^*,\ f \in L^1(X_{\alpha,\varepsilon}^*, \prod_\beta |f_\beta^{(\alpha)}|^{-1} dx dy)). \end{aligned} \tag{4.5}$$

Note that the normalization of dp is independent of ε and $(x_0, y_0) \in X_{\alpha,\varepsilon}^*$.

Let $\mathscr{F}(X_{\boldsymbol{R}}^*)$ be the subspace of $C^\infty(X_{\boldsymbol{R}}^*)$ consisting of all functions ϕ satisfying the following two conditions:

(1) There exists a compact subset D of $X_{\boldsymbol{R}}$ such that the support of ϕ is contained in $D \times \boldsymbol{R}^2$.

(2) For a fixed $x \in X_{\boldsymbol{R}}$, $\phi(x, y)$ is a rapidly decreasing function of y in $\boldsymbol{R}^2$.

For an $x \in X_{\boldsymbol{Q}} \cap X_\omega$ and a $\phi \in \mathscr{F}(X_{\boldsymbol{R}}^*)$, set

$$\begin{aligned} Z_\omega^{(\alpha)}(x, \phi; \lambda) = \int_{(P_\alpha)_{\boldsymbol{R}}^\circ/\Gamma_{\alpha\infty} \times \boldsymbol{R}_+^\times} &\{\prod_{\beta \in \Delta} |\chi_\beta^{(\alpha)}(p, a)|^{\langle \lambda + \delta/2, \beta^\vee \rangle} \\ &\times \sum_{(\gamma, y)} \phi((p, a)\cdot(\gamma, y))\} \frac{da}{a} dp, \end{aligned}$$

where $\Gamma_{\alpha\infty} = (P_\alpha)_{\boldsymbol{R}}^\circ \cap \Gamma$ and (γ, y) runs through all the elements in $(\Gamma * x \times \boldsymbol{Z}^2) \cap X_\alpha^*$. Also set

$$\Phi_\varepsilon^{(\alpha)}(\phi; \lambda) = \int_{X_{\alpha,\varepsilon}^*} \prod_{\beta \in \Delta} |f_\beta^{(\alpha)}(x, y)|^{\langle \lambda, \beta^\vee \rangle} \phi(x, y) dx\, dy$$

$$(\phi \in \mathscr{F}(X_{\boldsymbol{R}}^*),\ \varepsilon \in \omega).$$

The integrals $\Phi_\varepsilon^{(\alpha)}(\phi; \lambda)$ are absolutely convergent for $\mathrm{Re}\langle\lambda, \beta^\vee\rangle>0$ ($\beta \in \Delta$) and have analytic continuations to meromorphic functions of λ in $X(T)^C$ (cf. [1]).

Lemma 4.3. *When* $\mathrm{Re}\langle\lambda, \beta^\vee\rangle>3/2$ ($\beta \in \Delta$), *the integral* $Z_\omega^{(\alpha)}(x, \phi; \lambda)$ *is absolutely convergent and the following identity holds*:

$$Z_\omega^{(\alpha)}(x, \phi; \lambda)=\zeta(2\langle\lambda+\delta/2, \alpha^\vee\rangle)\cdot \sum_{\varepsilon\in\omega} E(x, \varepsilon; \lambda)\cdot\Phi_\varepsilon^{(\alpha)}(\phi; \lambda-\delta/2).$$

Proof. Note that $\langle\delta, \beta^\vee\rangle=1$ for any $\beta \in \Delta$. By (4.5), we have

$$Z_\omega^{(\alpha)}(x, \phi; \lambda)=\sum_{\varepsilon\in\omega} \xi_\varepsilon^{(\alpha)}(x; \lambda)\cdot\Phi_\varepsilon^{(\alpha)}(\phi; \lambda-\delta/2),$$

$$\xi_\varepsilon^{(\alpha)}(x; \lambda)=\sum_{(\gamma, y)} \prod_{\beta\in\Delta} |f_\beta^{(\alpha)}(\gamma, y)|^{-\langle\lambda+\delta/2, \beta^\vee\rangle},$$

where (γ, y) runs through a complete system of representatives of $\Gamma_{\alpha\infty}$-equivalence classes in $(\Gamma * x\times \boldsymbol{Z}^2)\cap X_{\alpha,\varepsilon}^*$. By (4.4), we can take

$$\{(\gamma, {}^t(0, q));\ \gamma \in \Gamma_\infty\backslash(\Gamma * x\cap X_\varepsilon),\ q \in \boldsymbol{Z},\ q>0\}$$

as a complete set of representatives. Then we see that the series $\xi_\varepsilon^{(\alpha)}(x; \lambda)$ is equal to $\zeta(2\langle\lambda+\delta/2, \alpha^\vee\rangle)E(x, \varepsilon; \lambda)$ and is absolutely convergent for $\mathrm{Re}\langle\lambda, \beta^\vee\rangle>3/2$ ($\beta \in \Delta$) (cf. Proposition 3.1). This proves the lemma.

q.e.d.

§ 5. Functional equations

5.1. For a $\phi \in \mathscr{F}(X_{\boldsymbol{R}}^*)$, we put

$$\hat{\phi}(x, y)=\int_{\boldsymbol{R}^2} \phi(x, y^*) \exp(2\pi\sqrt{-1}\ {}^t yJy^*)dy^*, \quad J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We first prove a functional equation relating $\Phi_\varepsilon^{(\alpha)}(\hat{\phi}; \lambda)$ with $\Phi_\varepsilon^{(\alpha)}(\phi; \lambda)$.

Theorem 5.1. *Let* $\varepsilon \in \Sigma$, $\phi \in \mathscr{F}(X_{\boldsymbol{R}}^*)$ *and* α *be a simple root.*

(1) *If* $w_\alpha\varepsilon=\varepsilon$, *then*

$$\Phi_\varepsilon^{(\alpha)}(\hat{\phi}; \lambda-\delta/2)=\pi^{-2\langle\lambda, \alpha^\vee\rangle-1}\Gamma(\langle\lambda, \alpha^\vee\rangle+1/2)^2|c_\alpha|^{-\langle\lambda, \alpha^\vee\rangle} \times\cos(\pi\langle\lambda, \alpha^\vee\rangle)\Phi_\varepsilon^{(\alpha)}(\phi; w_\alpha\lambda-\delta/2).$$

(2) *If* $w_\alpha\varepsilon\neq\varepsilon$, *then*

$$\begin{pmatrix}\Phi_\varepsilon^{(\alpha)}(\hat{\phi}; \lambda-\delta/2)\\ \Phi_{w_\alpha\varepsilon}^{(\alpha)}(\hat{\phi}; \lambda-\delta/2)\end{pmatrix}=|c_\alpha|^{-\langle\lambda, \alpha^\vee\rangle}\pi^{-2\langle\lambda, \alpha^\vee\rangle-1}\Gamma(\langle\lambda, \alpha^\vee\rangle+1/2)^2 \times\begin{pmatrix} 1 & -\sin(\pi\langle\lambda, \alpha^\vee\rangle) \\ -\sin(\pi\langle\lambda, \alpha^\vee\rangle) & 1\end{pmatrix}\begin{pmatrix}\Phi_\varepsilon^{(\alpha)}(\phi; w_\alpha\lambda-\delta/2)\\ \Phi_{w_\alpha\varepsilon}^{(\alpha)}(\phi; w_\alpha\lambda-\delta/2)\end{pmatrix}.$$

Proof. By the principle of analytic continuation, it suffices to prove the theorem under the assumption that $\langle\lambda, \beta^\vee\rangle > 1/2$ for all $\beta \in \Delta$. Moreover, the same argument as in the proof of [20, Lemma 5.5] allows us to reduce the theorem to the case $\phi \in C_0^\infty(X^*_{\alpha,\varepsilon} \cup X^*_{\alpha,w_\alpha\varepsilon})$. Notice that these assumptions assure the convergence of the integrals appearing in the following calculation. Set

$$X'_\varepsilon = \{x \in X_{\boldsymbol{R}};\ \operatorname{sgn} f_\beta(x) = \varepsilon_\beta \quad \text{for } \beta \in \Delta,\ \beta \neq \alpha\}$$

and

$$Y(x, \varepsilon_\alpha) = \{y \in M(2, 1; \boldsymbol{R});\ \operatorname{sgn} f_\alpha^{(\alpha)}(x, y) = \varepsilon_\alpha\} \quad (x \in X_{\boldsymbol{R}}).$$

By (2.4), we have $X'_\varepsilon = X'_{w_\alpha\varepsilon}$. Moreover, we set

$$\Psi(x, \varepsilon_\alpha, \phi; \lambda) = \int_{Y(x,\varepsilon_\alpha)} |{}^t y S_\alpha(x) y|^{\langle\lambda-\delta/2, \alpha^\vee\rangle} \phi(x, y) dy.$$

Then

$$\Phi_\varepsilon^{(\alpha)}(\hat{\phi}; \lambda-\delta/2) = \int_{X'_\varepsilon} \prod_{\beta\neq\alpha} |f_\beta(x)|^{\langle\lambda-\delta/2, \beta^\vee\rangle} \Psi(x, \varepsilon_\alpha, \hat{\phi}; \lambda) dx. \tag{5.1}$$

Recall that $S_\alpha(x)$ is definite or indefinite according as $w_\alpha\varepsilon = \varepsilon$ or $w_\alpha\varepsilon \neq \varepsilon$. By [8, Chapter III 2.6], we get the following identities:

If $w_\alpha\varepsilon = \varepsilon$, then

$$\begin{aligned}\Psi(x, \varepsilon_\alpha, \hat{\phi}; \lambda) &= |\det S_\alpha(x)|^{\langle\lambda, \alpha^\vee\rangle} \pi^{-2\langle\lambda+\delta/2, \alpha^\vee\rangle} \\ &\quad \times \Gamma(\langle\lambda+\delta/2, \alpha^\vee\rangle)^2 \cos(\pi\langle\lambda, \alpha^\vee\rangle) \Psi(x, \varepsilon_\alpha, \phi; w_\alpha\lambda).\end{aligned} \tag{5.2}$$

If $w_\alpha\varepsilon \neq \varepsilon$, then

$$\begin{aligned}\begin{pmatrix}\Psi(x, \varepsilon_\alpha, \hat{\phi}; \lambda) \\ \Psi(x, \varepsilon'_\alpha, \hat{\phi}; \lambda)\end{pmatrix} &= |\det S_\alpha(x)|^{\langle\lambda, \alpha^\vee\rangle} \pi^{-2\langle\lambda+\delta/2, \alpha^\vee\rangle} \Gamma(\langle\lambda+\delta/2, \alpha^\vee\rangle)^2 \\ &\quad \times \begin{pmatrix} 1 & -\sin(\pi\langle\lambda, \alpha^\vee\rangle) \\ -\sin(\pi\langle\lambda, \alpha^\vee\rangle) & 1 \end{pmatrix} \begin{pmatrix}\Psi(x, \varepsilon_\alpha, \phi; w_\alpha\lambda) \\ \Psi(x, \varepsilon'_\alpha, \phi; w_\alpha\lambda)\end{pmatrix}\end{aligned} \tag{5.3}$$

where $\varepsilon'_\alpha = (w_\alpha\varepsilon)_\alpha = \eta(\alpha)\varepsilon_\alpha \prod_{\beta\in\Delta} \varepsilon_\beta^{\langle\alpha, \beta^\vee\rangle}$. First assume that $w_\alpha\varepsilon = \varepsilon$. Then, by (4.2), (5.1) and (5.2), we have

$$\begin{aligned}\Phi_\varepsilon^{(\alpha)}(\hat{\phi}; \lambda-\delta/2) &= \pi^{-2\langle\lambda+\delta/2, \alpha^\vee\rangle} \Gamma(\langle\lambda+\delta/2, \alpha^\vee\rangle)^2 |c_\alpha|^{-\langle\lambda, \alpha^\vee\rangle} \cos(\pi\langle\lambda, \alpha^\vee\rangle) \\ &\quad \times \int_{X'_\varepsilon} \prod_{\beta\neq\alpha} |f_\beta(x)|^{\langle w_\alpha\lambda-\delta/2, \beta^\vee\rangle} \Psi(x, \varepsilon_\alpha, \phi; w_\alpha\lambda) dx.\end{aligned}$$

The last integral is clearly equal to $\Phi_\varepsilon^{(\alpha)}(\phi; w_\alpha\lambda-\delta/2)$. Using (4.2), (5.1) and (5.3), we can prove the functional equation for the case $w_\alpha\varepsilon \neq \varepsilon$ in the same manner. q.e.d.

Let θ be the Cartan involution of $G_{\boldsymbol{R}}$ introduced in Section 2.2 and K the fixed point group of θ in $G_{\boldsymbol{R}}$. Put

$$I_\alpha=\begin{pmatrix}|c_\alpha|^{-1/2} & 0\\ 0 & |c_\alpha|^{1/2}\end{pmatrix}.$$

Then $\tau_\alpha^{-1}(K\cap M_\alpha)=SO(I_\alpha)$. For an $f\in\mathscr{S}(\boldsymbol{R}^2)$, set

$$L_\alpha(f;s)=\int_{\boldsymbol{R}^2}|{}^t y I_\alpha y|^s f(y)dy$$

where dy is the standard Euclidean measure on $\boldsymbol{R}^2$.

Lemma 5.2. *Let ϕ_1 be a K-invariant function in $C_0^\infty(X_\omega)$ and ϕ_2 a function in $\mathscr{S}(\boldsymbol{R}^2)$. Then $\phi(x,y)=\phi_1(x)\phi_2(y)$ is in $\mathscr{F}(X_{\boldsymbol{R}}^*)$ and we have*

$$\Phi_\varepsilon^{(\alpha)}(\phi;\lambda-\delta/2)=|c_\alpha|^{-\langle\lambda,\alpha^\vee\rangle/2}L_\alpha(\phi_2;\langle\lambda-\delta/2,\alpha^\vee\rangle)\Psi_\varepsilon(\phi_1;\lambda),$$

where

$$\Psi_\varepsilon(\phi_1;\lambda)=\int_{X_\varepsilon}\prod_{\beta\in\Delta}|f_\beta(x)|^{\langle\lambda-\delta/2,\beta^\vee\rangle}\phi_1(x)dx.$$

Proof. Let X'_ε and $Y(x,\varepsilon_\alpha)$ be the same as in the proof of Theorem 5.1. For $k\in K\cap M_\alpha$, let $\boldsymbol{k}=\tau_\alpha^{-1}(k)$. We normalize a Haar measure dk on $K\cap M_\alpha$ such that

$$dy=r\,dr\,dk,\quad y=\boldsymbol{k}\cdot{}^t(0,r),\quad r>0.$$

Since $S_\alpha(k*x)=\boldsymbol{k}S_\alpha(x)\,{}^t\boldsymbol{k}$, we have by (5.1)

$$\begin{aligned}\Phi_\varepsilon^{(\alpha)}(\phi;\lambda-\delta/2)=&\int_{X'_\varepsilon}\prod_{\beta\neq\alpha}|f_\beta(x)|^{\langle\lambda-\delta/2,\beta^\vee\rangle}\phi_1(x)dx\\ &\times\int_{(M_\alpha\cap K)_{\varepsilon,x}}|f_\alpha(k*x)|^{\langle\lambda-\delta/2,\alpha^\vee\rangle}dk\int_0^\infty r^{2\langle\lambda,\alpha^\vee\rangle}\phi_2({}^t\boldsymbol{k}\,{}^t(0,r))dr,\end{aligned}$$

where $(M_\alpha\cap K)_{\varepsilon,x}=\{k\in M_\alpha\cap K;\ \operatorname{sgn}f_\alpha(k*x)=\varepsilon_\alpha\}$. By the assumption, the function ϕ_1 is K-invariant. Moreover, $f_\beta(x)$ $(\beta\neq\alpha)$ are $M_\alpha\cap K$-invariant. Hence we obtain

$$\begin{aligned}\Phi_\varepsilon^{(\alpha)}(\phi;\lambda-\delta/2)=&\int_{X_\varepsilon}\prod_{\beta\in\Delta}|f_\beta(x)|^{\langle\lambda-\delta/2,\beta^\vee\rangle}\phi_1(x)dx\\ &\times\int_{M_\alpha\cap K}dk\int_0^\infty r^{2\langle\lambda,\alpha^\vee\rangle}\phi_2({}^t\boldsymbol{k}\cdot{}^t(0,r))dr.\end{aligned}$$

The first and the second factors of the right hand side are equal to $\Psi_\varepsilon(\phi_1;\lambda)$ and $|c_\alpha|^{-\langle\lambda,\alpha^\vee\rangle/2}L_\alpha(\phi_2;\langle\lambda-\delta/2,\alpha^\vee\rangle)$, respectively. Thus we get the lemma. q.e.d.

Now we are able to rewrite the functional equations in Theorem 5.1 into the following form.

Theorem 5.3. *Let* $\omega=\{\varepsilon^{(1)}, \cdots, \varepsilon^{(\nu)}\}$ *be a W-orbit in* Σ.

(1) *For each* $w \in W$, *there exists a* ν *by* ν *matrix* $C_\omega(w; \lambda)$ *whose entries are meromorphic functions of* λ *such that*

$$\begin{pmatrix} \Psi_{\varepsilon^{(1)}}(\phi_1; w\lambda) \\ \vdots \\ \Psi_{\varepsilon^{(\nu)}}(\phi_1; w\lambda) \end{pmatrix} = C_\omega(w; \lambda) \begin{pmatrix} \Psi_{\varepsilon^{(1)}}(\phi_1; \lambda) \\ \vdots \\ \Psi_{\varepsilon^{(\nu)}}(\phi_1; \lambda) \end{pmatrix}$$

for any K-invariant function ϕ_1 *in* $C_0^\infty(X_\omega)$.

(2) $C_\omega(ww'; \lambda)=C_\omega(w; w'\lambda)C_\omega(w'; \lambda)$ $(w, w' \in W)$.

(3) *For the reflection* w_α *in the hyperplane orthogonal to a simple root* α, *the functional equation reads*

$$\Psi_\varepsilon(\phi_1; w_\alpha\lambda)=\Psi_\varepsilon(\phi_1; \lambda) \quad \textit{if } w_\alpha\varepsilon=\varepsilon$$

and

$$\begin{pmatrix} \Psi_\varepsilon(\phi_1; w_\alpha\lambda) \\ \Psi_{w_\alpha\varepsilon}(\phi_1; w_\alpha\lambda) \end{pmatrix} = \begin{pmatrix} \sec \pi\langle\lambda, \alpha^\vee\rangle & -\tan \pi\langle\lambda, \alpha^\vee\rangle \\ -\tan \pi\langle\lambda, \alpha^\vee\rangle & \sec \pi\langle\lambda, \alpha^\vee\rangle \end{pmatrix} \begin{pmatrix} \Psi_\varepsilon(\phi_1; \lambda) \\ \Psi_{w_\alpha\varepsilon}(\phi_1; \lambda) \end{pmatrix}$$

if $w_\alpha\varepsilon\neq\varepsilon$.

Proof. Notice that the following functional equation holds for any $\phi_2 \in \mathcal{S}(\boldsymbol{R}^2)$:

$$\begin{aligned} L_\alpha(\hat{\phi}_2; \langle\lambda-\delta/2, \alpha^\vee\rangle) &= \pi^{-2\langle\lambda+\delta/2, \alpha^\vee\rangle}\Gamma(\langle\lambda+\delta/2, \alpha^\vee\rangle)^2 \\ &\quad \times \cos \pi\langle\lambda, \alpha^\vee\rangle L_\alpha(\phi_2; \langle w_\alpha\lambda-\delta/2, \alpha^\vee\rangle) \end{aligned}$$

where $\hat{\phi}_2(y^*)=\int_{\boldsymbol{R}^2} \phi_2(y) \exp(2\pi\sqrt{-1}\, {}^t yJy^*)\, dy$. Now the theorem follows immediately from Theorem 5.1 and Lemma 5.2. q.e.d.

Remark. Theorem 5.3 is equivalent to the functional equations of K-invariant spherical functions on the symmetric space X_ω of ε-involution type, which are proved in [15, § 4, Proposition 4.6 and Theorem 4.10].

5.2. In order to state our main theorem, we need some notational preliminaries. Let c_b $(b \in \Phi)$ be as in § 2.2. Recall that $c_b c_{-b}=1$ and $c_{b+b'}=-c_b c_{b'}$ if b, b' and $b+b'$ are all in Φ. Hence we are able to extend the mapping $|c|: \Phi \to \boldsymbol{R}_+^\times$ defined by $|c|(b)=|c_b|$ to a homomorphism of $X(T)$ into $\boldsymbol{R}_+^\times$. We denote the extension also by $|c|$. For any $\lambda \in X(T)^{\boldsymbol{C}}$, put

$$|c|(\lambda)=\prod_{\beta\in\Delta}|c|(\Lambda_\beta)^{\langle\lambda,\beta^\vee\rangle}.$$

If $\lambda=\chi$ is in $X(T)$, then the right hand side of the equality above coincides with $|c|(\chi)$. Hence the notation will not cause any confusion. Set

$$\Lambda(x,\varepsilon;\lambda)=|c|(-\lambda)\prod_{b\in\Phi+}\eta(2\langle\lambda,b^\vee\rangle+1)E(x,\varepsilon;\lambda)$$
$$(x\in X_Q,\ \varepsilon\in\Sigma,\ \lambda\in X(T)^C),$$

where $\eta(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Theorem 5.4. (1) *The Dirichlet series* $E(x,\varepsilon;\lambda)$ $(x\in X_Q,\ \varepsilon\in\Sigma)$ *are absolutely convergent for* $\mathrm{Re}\langle\lambda,\alpha^\vee\rangle>1/2$ $(\alpha\in\Delta)$ *and have analytic continuations to meromorphic functions of* λ *in* $X(T)^C$.

(2) *The functions*

$$\prod_{b\in\Phi+}(\langle\lambda,b^\vee\rangle-1/2)^2\cdot\zeta(2\langle\lambda,b^\vee\rangle+1)E(x,\varepsilon;\lambda)$$

are entire functions of λ.

(3) *Let* $\omega=\{\varepsilon^{(1)},\cdots,\varepsilon^{(\nu)}\}$ *be any W-orbit in* Σ. *Then the following functional equations hold for any* $w\in W$:

$$\begin{pmatrix}\Lambda(x,\varepsilon^{(1)};w\lambda)\\ \vdots\\ \Lambda(x,\varepsilon^{(\nu)};w\lambda)\end{pmatrix}=C_\omega(w;\lambda)\begin{pmatrix}\Lambda(x,\varepsilon^{(1)};\lambda)\\ \vdots\\ \Lambda(x,\varepsilon^{(\nu)};\lambda)\end{pmatrix}\quad(x\in X_{\omega,Q})$$

where $C_\omega(w;\lambda)$ $(w\in W)$ *are the same as in Theorem* 5.3.

Proof. Set $D^{(0)}=\{\lambda\in X(T)^C,\ \mathrm{Re}\langle\lambda,\alpha^\vee\rangle>3/2$ for all $\alpha\in\Delta\}$. Fix a simple root α and denote by D_α the convex hull of the union of $D^{(0)}$ and $w_\alpha D^{(0)}$. First we shall prove the functional equations for $w=w_\alpha$. For this purpose, we need the following lemma. Its proof is quite similar to that of [20, Lemma 6.1], hence omitted.

Lemma 5.5. *Let* ϕ *be a function in* $\mathscr{F}(X_R^*)$ *such that both* ϕ *and* $\hat{\phi}$ *vanish on the complement of* $(X_\alpha^*)_R$. *Then the integrals* $Z_\omega^{(\alpha)}(x,\phi;\lambda)$ *and* $Z_\omega^{(\alpha)}(x,\hat{\phi};\lambda)$ *have analytic continuations to holomorphic functions of* λ *in* D_α *and satisfy the functional equation* $Z_\omega^{(\alpha)}(x,\hat{\phi};\lambda)=Z_\omega^{(\alpha)}(x,\phi;w_\alpha\lambda)$.

In order to construct functions satisfying the assumption of Lemma 5.5, we use the differential operator

$$\mathscr{D}^\alpha(x,y)=\left(\frac{\partial}{\partial y_2},-\frac{\partial}{\partial y_1}\right)S_\alpha(x)\,{}^t\!\left(\frac{\partial}{\partial y_2},-\frac{\partial}{\partial y_1}\right).$$

It is easy to check the following two relations:

$$\mathscr{D}^{\alpha}(x, y) \exp(2\pi\sqrt{-1}\,{}^t yJy^*) = (2\pi\sqrt{-1})^2 f_{\alpha}^{(\alpha)}(x, y^*) \exp(2\pi\sqrt{-1}\,{}^t yJy^*), \tag{5.4}$$

$$\mathscr{D}^{\alpha}(x, y)|f_{\alpha}^{(\alpha)}(x, y)|^s = 4s^2\varepsilon'_{\alpha}|f_{\alpha}^{(\alpha)}(x, y)|^{s-1} \prod_{\beta\neq\alpha} |f_{\beta}(x)|^{-\langle\alpha, \beta^{\vee}\rangle}, \tag{5.5}$$

where $\varepsilon'_{\alpha}=(w_{\alpha}\varepsilon)_{\alpha}=\eta(\alpha)\varepsilon_{\alpha}\prod_{\beta\neq\alpha}\varepsilon_{\beta}^{\langle\alpha,\beta^{\vee}\rangle}$. By (5.4), the function $\mathscr{D}^{\alpha}\phi_0$ satisfies the assumption in Lemma 5.5 for every $\phi_0 \in C_0^{\infty}(X_{\alpha R}^*)$. Assume that ϕ_0 is in $C_0^{\infty}(X_{\alpha,\varepsilon}^*)$. Then, using Theorem 5.1 and Lemma 4.3, we can rewrite the functional equation $Z_{\omega}^{(\alpha)}(x, \widehat{\mathscr{D}^{\alpha}\phi_0}; \lambda)=Z_{\omega}^{(\alpha)}(x, \mathscr{D}^{\alpha}\phi_0; w_{\alpha}\lambda)$ in Lemma 5.5 as follows:

$$\begin{aligned}&\zeta(2\langle w_{\alpha}\lambda, \alpha^{\vee}\rangle+1)E(x, \varepsilon; w_{\alpha}\lambda)\\ &\quad=\pi^{-2\langle\lambda,\alpha^{\vee}\rangle-1}\Gamma(\langle\lambda, \alpha^{\vee}\rangle+1/2)^2\zeta(2\langle\lambda, \alpha^{\vee}\rangle+1)|c_{\alpha}|^{-\langle\lambda,\alpha^{\vee}\rangle}\\ &\quad\quad\times\begin{cases}\cos\pi\langle\lambda, \alpha^{\vee}\rangle E(x, \varepsilon; \lambda) & \text{if } w_{\alpha}\varepsilon=\varepsilon,\\ E(x, \varepsilon; \lambda)-\sin\pi\langle\lambda, \alpha^{\vee}\rangle E(x, w_{\alpha}\varepsilon; \lambda) & \text{if } w_{\alpha}\varepsilon\neq\varepsilon,\end{cases}\end{aligned} \tag{5.6}$$

where the both sides of the identity are extended to meromorphic functions of λ in D_{α}. Since $|c|(-w_{\alpha}\lambda)=|c|(-\lambda)|c_{\alpha}|^{\langle\lambda,\alpha^{\vee}\rangle}$, the identity (5.6) is easily transformed into the form

$$\begin{aligned}&|c|(-w_{\alpha}\lambda)\eta(2\langle w_{\alpha}\lambda, \alpha^{\vee}\rangle+1)E(x, \varepsilon; w_{\alpha}\lambda)=|c|(-\lambda)\eta(2\langle\lambda, \alpha^{\vee}\rangle+1)\\ &\quad\times\begin{cases}E(x, \varepsilon; \lambda) & \text{if } w_{\alpha}\varepsilon=\varepsilon,\\ \sec\pi\langle\lambda, \alpha^{\vee}\rangle E(x, \varepsilon; \lambda)-\tan\pi\langle\lambda, \alpha^{\vee}\rangle E(x, w_{\alpha}\varepsilon; \lambda) & \text{if } w_{\alpha}\varepsilon\neq\varepsilon.\end{cases}\end{aligned}$$

Since $\Phi^+-\{\alpha\}$ is w_{α}-stable, the function $\prod_{b\in\Phi^+-\{\alpha\}}\eta(2\langle\lambda, b^{\vee}\rangle+1)$ is invariant under $\lambda\mapsto w_{\alpha}\lambda$. Multiplying it to the both sides of the equality above, we get the functional equation for $w=w_{\alpha}$ (cf. Theorem 5.3 (3)).

We have, by Lemma 4.3 and (5.5),

$$\begin{aligned}Z_{\omega}^{(\alpha)}(x, \mathscr{D}^{\alpha}\phi_0; \lambda)&=4\varepsilon'_{\alpha}(\langle\lambda, \alpha^{\vee}\rangle-1/2)^2\zeta(2\langle\lambda, \alpha^{\vee}\rangle+1)\\ &\quad\times E(x, \varepsilon; \lambda)\Phi_{\varepsilon}^{(\alpha)}(\phi_0; \lambda-\delta/2+\Lambda_{\alpha}-\alpha)\end{aligned}$$

for any $\phi_0 \in C_0^{\infty}(X_{\alpha,\varepsilon}^*)$. Since there exists a $\phi_0 \in C_0^{\infty}(X_{\alpha,\varepsilon}^*)$ such that $\Phi_{\varepsilon}^{(\alpha)}(\phi_0; \lambda-\delta/2+\Lambda_{\alpha}-\alpha)\neq 0$, Lemma 5.5 implies that

$$(\langle\lambda, \alpha^{\vee}\rangle-1/2)^2\zeta(2\langle\lambda, \alpha^{\vee}\rangle+1)E(x, \varepsilon; \lambda)$$

is holomorphic in D_{α}. Since $\langle\lambda, b^{\vee}\rangle>3/2$ for any $b \in \Phi^+-\{\alpha\}$ and any λ in D_{α}, the function $(\langle\lambda, \alpha^{\vee}\rangle-1/2)^2\prod_{b\in\Phi^+}\zeta(2\langle\lambda, b^{\vee}\rangle+1)E(x, \varepsilon; \lambda)$ is also holomorphic in D_{α}. For any $\alpha, \beta \in \Delta$, $D_{\alpha}\cap D_{\beta}$ includes $D^{(0)}$. Hence

$$\prod_{\alpha \in \Delta} (\langle \lambda, \alpha^\vee \rangle - 1/2)^2 \prod_{b \in \Phi+} \zeta(2\langle \lambda, b^\vee \rangle + 1) E(x, \varepsilon; \lambda)$$

is continued holomorphically in the convex hull of $\bigcup_{\alpha \in \Delta} D_\alpha$ (cf. [11, Theorem 2.5. 10]). For any $b \in \Phi^+$, we put $l(b) = \min\{l(w); wb \in \Delta, w \in W\}$ where $l(w)$ is the length of $w \in W$ with respect to $\{w_\alpha; \alpha \in \Delta\}$. Let $D^{(k)}$ $(k = 1, 2, \cdots)$ be the convex hull of $\bigcup_{l(w) \leq k} w \cdot D^{(0)}$. It is clear that the domain $D^{(k)}$ coincides with the convex hull of $(\bigcup_{\alpha \in \Delta} w_\alpha \cdot D^{(k-1)}) \cup D^{(k-1)}$.

Lemma 5.6. *For every* k $(\geqq 1)$, *the functions*

$$\prod_{\substack{l(b) \leq k \\ b \in \Phi+}} (\langle \lambda, b^\vee \rangle - 1/2)^2 \prod_{b \in \Phi+} \zeta(2\langle \lambda, b^\vee \rangle + 1) E(x, \varepsilon; \lambda) \quad (\varepsilon \in \Sigma)$$

are holomrophic in $D^{(k)}$.

Proof. We prove the lemma by induction on k. For $k = 1$, the statement has just been proved. Assume that $k \geqq 2$. For any $\alpha \in \Delta$, $w_\alpha \cdot D^{(k-1)} \cap D^{(k-1)}$ contains the domain D_α. Hence $w_\alpha \cdot D^{(k-1)} \cap D^{(k-1)}$ is a non-empty connected convex tube domain. Therefore, by (5.6), the functions $E(x, \varepsilon; \lambda)$ can be continued to meromorphic functions in $w_\alpha \cdot D^{(k-1)} \cup D^{(k-1)}$. To see the holomorphy, we rewrite the functional equation (5.6) as follows:

$$\tilde{E}(x, \varepsilon; \lambda) = \pi^{2\langle \lambda, \alpha^\vee \rangle - 1} \Gamma(-\langle \lambda, \alpha^\vee \rangle + 1/2)^2 |c_\alpha|^{\langle \lambda, \alpha^\vee \rangle} \tag{5.7}$$
$$\times \begin{cases} \cos \pi \langle \lambda, \alpha^\vee \rangle \tilde{E}(x, \varepsilon; w_\alpha \lambda) & \text{if } w_\alpha \varepsilon = \varepsilon, \\ \tilde{E}(x, \varepsilon; w_\alpha \lambda) + \sin \pi \langle \lambda, \alpha^\vee \rangle \tilde{E}(x, w_\alpha \varepsilon; w_\alpha \lambda) & \text{if } w_\alpha \varepsilon \neq \varepsilon, \end{cases}$$

where we put $\tilde{E}(x, \varepsilon; \lambda) = \prod_{b \in \Phi+} \zeta(2\langle \lambda, b^\vee \rangle + 1) E(x, \varepsilon; \lambda)$. By the induction hypothesis, the left hand side of (5.7) multiplied by

$$\prod_{\substack{l(b) \leq k-1 \\ b \in \Phi+}} (\langle \lambda, b^\vee \rangle - 1/2)^2$$

is holomorphic in $D^{(k-1)}$. On the other hand, the right hand side of (5.7) multiplied by

$$\prod_{\substack{l(b) \leq k-1 \\ b \in \Phi+}} (\langle \lambda, (w_\alpha b)^\vee \rangle - 1/2)^2 \cdot (\langle \lambda, \alpha^\vee \rangle - 1/2)^2$$

is holomorphic in $w_\alpha D^{(k-1)}$. Since $w_\alpha(\Phi^+ - \{\alpha\}) = \Phi^+ - \{\alpha\}$ and the left hand side of (5.7) is holomorphic in $\langle \lambda, \alpha^\vee \rangle + 1/2 = 0$, these observations imply that

$$\prod_{\substack{l(b) \leq k \\ b \in \Phi+}} (\langle \lambda, b^\vee \rangle - 1/2)^2 \tilde{E}(x, \varepsilon; \lambda)$$

is holomorphic in $D^{(k-1)} \cup (\bigcup_{\alpha \in \Delta} w_\alpha D^{(k-1)})$. Since any holomorphic function on a connected tube domain D can be continued holomorphically to the convex hull of D ([11, Theorem 2.5.10]), this proves the lemma.

Now we are able to complete the proof of Theorem 5.4. Since the convex hull of $\bigcup_{w \in W} w \cdot D^{(0)}$ is $X(T)^C$, the second assertion is implied by the lemma above. The first assertion is an immediate consequence of the second. We have already proved the functional equations for the reflections with respect to simple roots. Since the Weyl group W is generated by them, the functional equations hold for any $w \in W$. q.e.d.

It frequently happens that the function $E(x, \varepsilon; \lambda)$ has a simple pole in the hyperplane $\langle \lambda, b^\vee \rangle = 1/2$ for some $b \in \Phi^+$. In this connection, we can easily derive the following proposition from the functional equations.

Proposition 5.7. *For an $\varepsilon \in \Sigma$, let $\Delta_\varepsilon = \{\alpha \in \Delta;\ w_\alpha \varepsilon = \varepsilon\}$ and denote by $W_{(\varepsilon)}$ the subgroup of W generated by $\{w_\alpha;\ \alpha \in \Delta_\varepsilon\}$. If a positive root b is written as $b = w\alpha$ for some $w \in W_{(\varepsilon)}$ and some $\alpha \in \Delta_\varepsilon$, then the Eisenstein series $E(x, \varepsilon; \lambda)$ has a simple pole in the hyperplane $\langle \lambda, b^\vee \rangle = 1/2$.*

Remarks. 1. Theorems 5.3 and 5.4 reveal an intimate relation between spherical functions and Eisenstein series on X_ω. Some special cases of this relation already appeared in the study of zeta functions of quadratic forms. In [23], Siegel proved the functional equation of his zeta functions of indefinite quadratic forms by reducing it to the functional equation of certain hypergeometric functions. Those hypergeometric functions are obtained as a special case of the spherical functions on the pseudo Riemannian symmetric space $SL(n; \boldsymbol{R})/SO(p, n-p)$ (cf. [18]). We can also understand Siegel's zeta functions of indefinite quadratic forms as Eisenstein series on $SL(n; \boldsymbol{R})/SO(p, n-p)$ corresponding to some maximal parabolic subgroup of $SL(n; \boldsymbol{R})$. In the proof of [22, Lemma 1], Shintani used the functional equations of the Legendre functions of 1st and 2nd kind, which are the spherical functions on $SL(2; \boldsymbol{R})/SO(2)$ and $SL(2; \boldsymbol{R})/SO(1, 1)$, respectively. He derived from [22, Lemma 1] certain functional equations of zeta functions in two variables related to binary quadratic forms.

2. It is an interesting problem to extend our result to not necessarily minimal parabolic subgroups. For example, Eisenstein series corresponding to the parabolic subgroup P_α will be obtained as the residue of $E(x, \varepsilon; \lambda)$ at $\langle \lambda, \alpha^\vee \rangle = 1/2$. However the calculation will become rather complicated because of the same difficulty as encounterd in the study of zeta functions of ternary zero forms (cf. [19]).

3. Recall that the group G itself can be viewed as a semisimple symmetric space in the following manner. Consider the involutive automorphism σ of $G\times G$ defined by $\sigma(g_1, g_2)=(g_2, g_1)$. Then the quotient space of $G\times G$ by the fixed point group of σ can be identified with G and we are able to consider the Dirichlet series $D(g; \lambda)$ ($g \in G_Q$) introduced in (3.2) as the Eisenstein series for $(G\times G, \sigma)$. The following theorem can be easily proved by Proposition 3.3, the functional equation of the Riemann zeta function and the functional equations of the zonal spherical functions $\omega_\lambda^{(p)}(g)$ on G_{Q_p}.

Theorem. *Put*

$$D^*(g; \lambda)=D(g; \lambda)\times \prod_{b\in\Phi+} \{(1-\langle\lambda, b^\vee\rangle^2)\cdot\langle\lambda, b^\vee\rangle^2 \times(2\pi)^{-\langle\lambda, b^\vee\rangle}\Gamma(\langle\lambda, b^\vee\rangle)\zeta(1+\langle\lambda, b^\vee\rangle)^2\}.$$

Then the function $D^*(g; \lambda)$ is a W-invariant entire function of λ in $X(T)^C$:

$$D^*(g; \lambda)=D^*(g; w\lambda) \quad (w \in W).$$

Conversely, if the theorem can be proved directly, the functional equations of $\omega_\lambda^{(p)}(g)$ are its immediate consequences. Actually, by imitating the argument in Section 4 and Section 5, we can give a proof of the theorem independent of the explicit expression in Proposition 3.3. Thus we obtain a proof of the functional equations of the zonal spherical functions on universal Chevalley groups over $\boldsymbol{Q}_p$.

§ 6. Examples

We retain the notation in the preceding sections.

(A) *Riemannian symmetric spaces.*

Assume that σ induces a Cartan involution on $G_{\boldsymbol{R}}$. Then $\eta(\alpha)=1$ for any $\alpha \in \Delta$. Let ε^+ be the element in Σ such that $\varepsilon_\alpha^+=1$ for all $\alpha \in \Delta$. By (2.4), $\omega_+=\{\varepsilon^+\}$ is a single W-orbit in Σ and $X_{\omega+}$ is the Riemannian symmetric space $G_{\boldsymbol{R}}/K$. Our Eisenstein series $E(x, \varepsilon^+; \lambda)$ ($x \in X_{\omega+,Q}$) is nothing but the usual Eisenstein series on the Riemannian symmetric space and the functional equations take the following simple form:

$$\Lambda(x, \varepsilon^+; w\lambda)=\Lambda(x, \varepsilon^+; \lambda) \quad (w \in W).$$

Let $\Delta_{\varepsilon+}$ and $W_{(\varepsilon+)}$ be as defined in Proposition 5.7. It is clear that $\Delta_{\varepsilon+}=\Delta$ and $W_{(\varepsilon+)}=W$. Hence, by Proposition 5.7, the Eisenstein series $E(x, \varepsilon^+; \lambda)$ multiplied by $\prod_{b\in\Phi+}(\langle\lambda, b^\vee\rangle-1/2)\zeta(2\langle\lambda, b^\vee\rangle+1)$ is an entire function of λ in $X(T)^C$. Notice that our assumption that x is a rational point is needed only for ensuring the convergence of $E(x, \varepsilon^+; \lambda)$. It is known that the Eisenstein series $E(x, \varepsilon^+; \lambda)$ ($x \in X_{\omega+}$) is absolutely convergent for $\mathrm{Re}\,\langle\lambda, \alpha^\vee\rangle>1/2$, even if x is not a rational point. Hence the result above is valid also for any $x \in X_{\omega+}$. (For the general theory of Eisenstein series on Riemannian symmetric spaces, see Langlands [12].)

(B) $SL(n; \boldsymbol{R})/SO(p, n-p)$.

Let $G=SL(n)$. Assume that σ is the involution defined by

$$\sigma(g)=\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} {}^t g^{-1} \begin{pmatrix} a_1^{-1} & & \\ & \ddots & \\ & & a_n^{-1} \end{pmatrix}$$

where $a_1, \cdots, a_n$ are non-zero rational numbers. Let B be the subgroup of lower triangular matrices in G. Since $T=B \cap \sigma(B)$ is the group of diagonal matrices in G, the group B is a σ-anisotropic Borel subgroup of G. The root system $\Phi=\Phi(G, T)$ is as follows:

$$\Phi=\{\alpha_{ij} \in X(T);\ 1 \leqq i, j \leqq n,\ i \neq j\},$$

$$\alpha_{ij}\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\right)=t_i/t_j.$$

The positive system Φ^+ and the simple system Δ corresponding to B are given by

$$\Phi^+=\{\alpha_{ij} \in X(T);\ 1 \leqq j < i \leqq n\}$$

and

$$\Delta=\{\alpha_i=\alpha_{i+1,i};\ i=1, 2, \cdots, n-1\}.$$

The fundamental dominant weights $\Lambda_i=\Lambda_{\alpha_i}$ $(1 \leqq i \leqq n-1)$ are the characters of T defined by

$$\Lambda_i\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\right)=(t_1 \cdots t_i)^{-1} \quad (1 \leqq i \leqq n-1).$$

The Weyl group W of G can be identified with the symmetric group $\mathfrak{S}_n$ in n letters in such a manner that w_{α_i} corresponds to the transposition $(i, i+1)$. The symmetric space $X=G*e$ is isomorphic to the space S of n by n symmetric matrices with determinant $a_1 \cdots a_n$ via the mapping

$$X \ni x \overset{\tau}{\longmapsto} \tau(x)=x \cdot \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in S.$$

We have $\tau(g*x)=g\tau(x){}^t g$ $(g \in G,\ x \in X)$.

For any matrix A, denote by $d_i(A)$ the determinant of upper left i by i block of A. Then

$$f_{\alpha_i}(x)=d_i(x)=(a_1\cdots a_i)^{-1}d_i(\tau(x)) \quad (x\in X,\ 1\leqq i\leqq n-1).$$

The $G_{\boldsymbol{R}}$-orbit decomposition of $X_{\boldsymbol{R}}$ is given by

$$X_{\boldsymbol{R}}=\coprod_{\substack{(-1)^q=\operatorname{sgn} a_1\cdots a_n\\ p+q=n}} X^{(p,q)}, \qquad X^{(p,q)}=\tau^{-1}(S^{(p,q)}),$$

where $S^{(p,q)}$ stands for the set of n by n non-degenerate real symmetric matrices with signature (p,q) and determinant $a_1\cdots a_n$. The identity component B^+ of $B_{\boldsymbol{R}}$ is the group of real matrices in B with positive diagonal entries. The B^+-orbit decomposition of $X^{(p,q)}\cap X_\Omega$ is given by

$$X^{(p,q)}\cap X_\Omega=\bigcup_\varepsilon X_\varepsilon, \qquad X_\varepsilon=\tau^{-1}(S_\varepsilon),$$

$$S_\varepsilon=\{Y\in S;\ \operatorname{sgn} d_i(Y)=\varepsilon_1\cdots\varepsilon_i\ (1\leqq i\leqq n-1)\},$$

where ε runs through all n-tuples $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_n)$ of ± 1 such that exactly q of ε_i's are equal to -1.

The standard unit group of G coincides with $\Gamma=SL(n:\boldsymbol{Z})$, and Γ_∞ is given by the group of all n by n lower triangular integral matrices with diagonal entries 1. For any $Y\in S^{(p,q)}\cap M(n;\boldsymbol{Q})$, put

$$\Gamma(Y,\varepsilon)=\{\gamma\in\Gamma;\ \gamma Y{}^t\gamma\in S_\varepsilon\} \quad\text{and}\quad \Gamma_Y=\Gamma\cap SO(Y).$$

Then the Eisenstein series at $\tau^{-1}(Y)$ are given by

$$E(\tau^{-1}(Y),\varepsilon;\lambda)=\prod_{i=1}^{n-1}|a_i|^{\langle\lambda,\alpha_i^\vee+\cdots+\alpha_{n-1}^\vee\rangle+(n-i+1)/2}\sum_\gamma\prod_{i=1}^{n-1}|d_i(\gamma Y{}^t\gamma)|^{-\langle\lambda,\alpha_i^\vee\rangle-1/2}$$

where the summation is taken over a complete set of representatives of $\Gamma_\infty\backslash\Gamma(Y,\varepsilon)/\Gamma_Y$. The Dirichlet series in the right hand side are just the Eisenstein series for Y considered in [21].

We put $z_{i+1}-z_i=\langle\lambda,\alpha_i^\vee\rangle$ $(1\leqq i\leqq n-1)$. Then $|c|(-\lambda)=\prod_{i=1}^n|a_i'|^{z_i}$, where $a_i'=a_i/|a_1\cdots a_n|^{1/n}$. Hence the function $\Lambda(\tau^{-1}(Y),\varepsilon;\lambda)$ is equal to

$$\prod_{i=1}^n|a_i|^{-(z_1+\cdots+z_n)/n+(n-i+1)/2}\cdot|\det Y|^{z_n}$$
$$\times\prod_{1\leqq i<j\leqq n}\eta(2z_j-2z_i+1)\sum_\gamma\prod_{i=1}^{n-1}|d_i(\gamma Y{}^t\gamma)|^{-(z_{i+1}-z_i+1/2)}.$$

The action of $s\in\mathfrak{S}_n\simeq W$ on λ (resp. ε) is given by

$$\langle s\cdot\lambda,\alpha_i^\vee\rangle=z_{s(i+1)}-z_{s(i)} \quad (1\leqq i\leqq n-1)$$
$$(\text{resp. } s\cdot\varepsilon=(\varepsilon_{s(1)},\cdots,\varepsilon_{s(n)})).$$

Now it is easy to check that Theorem 5.4 agrees with [21, Theorem 7, p. 207].

(C) $Sp(n; \boldsymbol{R})/U(p, n-p)$.

Let $G=Sp(n)=\{g \in GL(2n);\ gJ{}^tg=J\}$ with

$$J=\left(\begin{array}{c|c} 0 & 1_n \\ \hline -1_n & 0 \end{array}\right).$$

Consider the involution σ defined by $\sigma(g)={}^tg^{-1}$ $(g \in G)$. Put, for $t_1, \cdots, t_n \in \boldsymbol{C}^\times$,

$$D(t_1, \cdots, t_n)=\left[\begin{array}{ccc|ccc} t_1 & & & & & \\ & \ddots & & & 0 & \\ & & t_n & & & \\ \hline & & & t_1^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & t_n^{-1} \end{array}\right].$$

We may take $T=\{D(t_1, \cdots, t_n);\ t_1, \cdots, t_n \in \boldsymbol{C}^\times\}$ as a $\boldsymbol{Q}$-split σ-anisotropic maximal torus of G. The subgroup

$$B=\left\{\left[\begin{array}{ccc|ccc} t_1 & & * & & & \\ & \ddots & & & * & \\ 0 & & t_n & & & \\ \hline & & & t_1^{-1} & & 0 \\ & 0 & & & \ddots & \\ & & & * & & t_n^{-1} \end{array}\right] \in G\right\}$$

is a σ-anisotropic Borel subgroup of G satisfying $B\cap\sigma(B)=T$. Under the choice of (B, T), the sets Φ, Φ^+, Δ are given by

$$\Phi=\{\alpha_{ij}^{\pm1}, \beta_{kl}^{\pm1};\ 1\leqq i<j\leqq n,\ 1\leqq k\leqq l\leqq n\},$$
$$\Phi^+=\{\alpha_{ij}, \beta_{kl};\ 1\leqq i<j\leqq n,\ 1\leqq k\leqq l\leqq n\},$$
$$\Delta=\{\alpha_1=\alpha_{1,2},\ \alpha_2=\alpha_{2,3}, \cdots, \alpha_{n-1}=\alpha_{n-1,n},\ \alpha_n=\beta_{n,n}\},$$

where $\alpha_{ij}(D(t_1, \cdots, t_n))=t_i/t_j$ and $\beta_{kl}(D(t_1, \cdots, t_n))=t_kt_l$. Moreover we have $\Lambda_{\alpha_i}(D(t_1, \cdots, t_n))=t_1\cdots t_i$ $(1\leqq i\leqq n)$.

The symmetric space X determined by σ is realized as the space of symmetric matrices contained in G:

$$X=G*e=\{g \in G;\ g={}^tg\}.$$

For any p $(0 \leq p \leq n)$, denote by X_p the intersection of G_R and the set of $2n$ by $2n$ real non-degenerate symmetric matrices with signature $(2p, 2n-2p)$. Then the G_R-orbit decomposition of $X_R = X \cap G_R$ is given by $X_R = X_0 \cup \cdots \cup X_n$. Put

$$x^{(p)} = \left(\begin{array}{c|c} 1_{p,n-p} & 0 \\ \hline 0 & 1_{p,n-p} \end{array}\right), \quad 1_{p,n-p} = \left(\begin{array}{c|c} 1_p & 0 \\ \hline 0 & -1_{n-p} \end{array}\right).$$

Also put $U(p, n-p) = \{h \in GL(n; C);\ h 1_{p,n-p} {}^t\bar{h} = 1_{p,n-p}\}$. The element $x^{(p)}$ is in X_p and the isotropy subgroup $G_{x^{(p)},R}$ of G_R at $x^{(p)}$ is isomorphic to $U(p, n-p)$. The isomorphism is given by

$$G_{x^{(p)},R} \ni g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto A + B \cdot 1_{p,n-p}\sqrt{-1} \in U(p, n-p).$$

Hence $X_p \simeq Sp(n; R)/U(p, n-p)$.

Set

$$X'_p = \left\{ x = \begin{pmatrix} \overbrace{x_1}^{n} & \overbrace{x_2}^{n} \\ {}^t x_2 & x_3 \end{pmatrix} \begin{matrix} \}n \\ \}n \end{matrix} \in X_p;\ \det x_3 \neq 0 \right\}$$

and

$$\mathfrak{H}^{(p)} = \{Z \in M(n; C);\ {}^t Z = Z,\ \mathrm{sgn}\,(\mathrm{Im}\, Z) = (p, n-p)\},$$

where $\mathrm{sgn}\,(\mathrm{Im}\, Z)$ denotes the signature of the symmetric matrix $\mathrm{Im}\, Z$. We can define a diffeomorphism $\tau\colon X'_p \to \mathfrak{H}^{(p)}$ by

$$\tau(x) = \tau\left(\begin{pmatrix} x_1 & x_2 \\ {}^t x_2 & x_3 \end{pmatrix}\right) = x_2 x_3^{-1} + \sqrt{-1}\, x_3^{-1}.$$

For $x \in X'_p$ and for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_R$, $g * x$ is in X'_p if and only if

$$\det\,(C \cdot \tau(x) + D) \neq 0,$$

and then we have

$$\tau(g * x) = (A \cdot \tau(x) + B)(C \cdot \tau(x) + D)^{-1}.$$

For $Z \in \mathfrak{H}^{(p)}$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_R$ such that $\det\,(CZ + D) \neq 0$, we put $g\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$. Note that $g\langle Z \rangle$ is always defined if g is in B.

Remark. If $p=0$, then $\mathfrak{H}^{(0)}$ is the Siegel upper halfplane of degree n and $X_0=X_0'=\tau^{-1}(\mathfrak{H}^{(0)})$.

Let $f_i=f_{\alpha_i}$ be the relative B-invariant on X such that

$$\begin{cases} f_i(b*x)=\Lambda_{\alpha_i}(b)^{-2}f_i(x) & (b \in B,\ x \in X), \\ f_i(e)=1 & (1\leqq i\leqq n). \end{cases}$$

For $Z \in \mathfrak{H}^{(p)}$, we have $f_i(\tau^{-1}(Z))=d_i(Y^{-1})$ $(Y=\operatorname{Im} Z,\ 1\leqq i\leqq n)$. The B^+-open orbits in X_p are contained in X_p' and τ maps them to

$$\mathfrak{H}_\varepsilon^{(p)}=\{Z \in \mathfrak{H}^{(p)};\ \operatorname{sgn} d_i(Y^{-1})=\varepsilon_1\cdots\varepsilon_i\ (1\leqq i\leqq n)\},$$

where $\varepsilon=(\varepsilon_1, \cdots, \varepsilon_n)$ are n-tuples of ± 1 such that exactly p (resp. $n-p$) of ε_i's are equal to 1 (resp. -1).

In the present case, the standard unit group Γ of G coincides with the Siegel modular group $Sp(n; \boldsymbol{Z})$ and Γ_∞ is the subgroup of $B\cap\Gamma$ consisting of all elements with diagonal entries 1. The set $X_{p,\boldsymbol{Q}}'$ of rational points in X_p' is in one to one correspondence to

$$\mathfrak{H}^{(p)}_{\boldsymbol{Q}(\sqrt{-1})}=\mathfrak{H}^{(p)}\cap M(n; \boldsymbol{Q}(\sqrt{-1}))$$

through the mapping τ. For $Z \in \mathfrak{H}^{(p)}_{\boldsymbol{Q}(\sqrt{-1})}$, put

$$\Gamma_Z=\left\{\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma;\ AZ+B=Z\cdot(CZ+D)\right\}.$$

Then Γ_Z is an arithmetic subgroup of the isotropy subgroup of G at $\tau^{-1}(Z)$. Also put

$$\Gamma(Z, \varepsilon)=\left\{\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma;\ \det(CZ+D)\neq 0,\ \gamma\langle Z\rangle \in \mathfrak{H}_\varepsilon^{(p)}\right\}.$$

Now the Eisenstein series at $\tau^{-1}(Z)$ $(Z \in \mathfrak{H}^{(p)}_{\boldsymbol{Q}(\sqrt{-1})})$ are given by

$$E(\tau^{-1}(Z), \varepsilon; \lambda)=E(Z, \varepsilon; \lambda)=\sum_\gamma \prod_{i=1}^n |d_i(Y_\gamma^{-1})|^{-\langle\lambda, \alpha_i^\vee\rangle-1/2},$$

where $Y_\gamma=\operatorname{Im}(\gamma\langle Z\rangle)$ and the summation is taken over a complete system of representatives of $\Gamma_\infty\backslash\Gamma(Z, \varepsilon)/\Gamma_Z$.

Remark. For any $x \in X_p\cap X_{\boldsymbol{Q}}$, there exists a $\gamma \in \Gamma$ such that $\gamma*x$ is in X_p'. Since the Eisenstein series depend only on Γ-equivalence class of x, we may restrict our consideration to the Eisenstein series corresponding to the elements in $X_{p,\boldsymbol{Q}}'=\tau^{-1}(\mathfrak{H}^{(p)}_{\boldsymbol{Q}(\sqrt{-1})})$.

As in the example (B), we introduce a new variable $z=(z_1, \cdots, z_n)$ which is connected to λ by the formula

$$\begin{cases} z_i - z_{i+1} = \langle \lambda, \alpha_i^\vee \rangle & (1 \leqq i \leqq n-1), \\ z_n = \langle \lambda, \alpha_n^\vee \rangle. \end{cases}$$

The Weyl group W with respect to (G, T) is isomorphic to the semi-direct product of $\{\pm 1\}^n$ and $\mathfrak{S}_n$. Here we consider $\{\pm 1\}$ as a multiplicative group. The action of $w=(e, s)=(e_1, \cdots, e_n, s) \in \{\pm 1\}^n \rtimes \mathfrak{S}_n$ $(\simeq W)$ on λ is expressed in terms of z as follows:

$$z=(z_1, \cdots, z_n) \overset{w}{\longmapsto} w\cdot z=(e_1 z_{s^{-1}(1)}, \cdots, e_n z_{s^{-1}(n)}).$$

Moreover

$$w\cdot\varepsilon=(e, s)\cdot\varepsilon=(\varepsilon_{s^{-1}(1)}, \cdots, \varepsilon_{s^{-1}(n)}).$$

In particular, for simple roots $\alpha_1, \cdots, \alpha_n$, we have

$$w_{\alpha_i}\cdot z=(z_1, \cdots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \cdots, z_n),$$
$$w_{\alpha_i}\cdot \varepsilon=(\varepsilon_1, \cdots, \varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_i, \varepsilon_{i+2}, \cdots, \varepsilon_n)$$

for $i=1, \cdots, n-1$, and

$$w_{\alpha_n}\cdot z=(z_1, \cdots, z_{n-1}, -z_n), \qquad w_{\alpha_n}\cdot\varepsilon=\varepsilon.$$

Set

$$\begin{aligned} \Lambda(Z, \varepsilon; z) &= \Lambda(\tau^{-1}(Z), \varepsilon; \lambda) \\ &= \prod_{1\leqq i<j\leqq n} \{\eta(2z_i-2z_j+1)\cdot\eta(2z_i+2z_j+1)\} \\ &\quad \times \prod_{i=1}^{n} \eta(2z_i+1) \times E(Z, \varepsilon; \lambda). \end{aligned}$$

Now we can write down the explicit form of the functional equations in Theorem 5.4 for each w_{α_i} $(1 \leqq i \leqq n)$.

Theorem. (1) (*Functional equations for* w_{α_n}) *The functions* $\Lambda(Z, \varepsilon; z)$ *are invariant under* $z_n \mapsto -z_n$.

(2) (*Functional equations for* w_{α_i} $(1 \leqq i \leqq n-1)$)

(i) If $\varepsilon_i=\varepsilon_{i+1}$, *then* $\Lambda(Z, \varepsilon; z)$ *is invariant under the transposition of* z_i *and* z_{i+1}.

(ii) *If* $\varepsilon_i \neq \varepsilon_{i+1}$, *then we have*

$$\begin{pmatrix} \Lambda(Z, \varepsilon; \hat{z}) \\ \Lambda(Z, \hat{\varepsilon}; \hat{z}) \end{pmatrix} = \begin{pmatrix} \sec \pi(z_{i+1}-z_i) & \tan \pi(z_{i+1}-z_i) \\ \tan \pi(z_{i+1}-z_i) & \sec \pi(z_{i+1}-z_i) \end{pmatrix} \begin{pmatrix} \Lambda(Z, \varepsilon; z) \\ \Lambda(Z, \hat{\varepsilon}; z) \end{pmatrix},$$

where $\hat{\varepsilon}=(\varepsilon_1, \cdots, \varepsilon_{i-1}, \varepsilon_{i+1}, \varepsilon_i, \varepsilon_{i+2}, \cdots, \varepsilon_n)$ *and* $\hat{z}=(z_1, \cdots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \cdots, z_n)$.

Remark. Let K be an arbitrary imaginary quadratic field. Then, by modifying the involution σ suitably, we can obtain the same result as above for any $Z \in \mathfrak{H}_K^{(p)}=\mathfrak{H}^{(p)} \cap M(n; K)$.

References

[1] Bernstein, I. N., Gelfand, S. I., Meromorphic property of the functions P^λ, Funct. Anal. Appl., **3** (1969), 68–69.

[2] Borel, A., Density and maximality of arithmetic subgroups, J. Reine Angew. Math., **224** (1966), 78–89.

[3] Borel, A. and Tits, J., Groupes réductifs, Publ. Math I. H. E. S., **27** (1965), 55–152.

[4] Casselman, W., The unramified principal series of p-adic groups I, The spherical function, Compositio Math., **40** (1980), 387–406.

[5] Chevalley, C., Certains schemas de groupes semi-simples, Sém. Bourbaki (1960/61), Exp. **219**, New York: W. A. Benjamin 1966.

[6] Flensted-Jensen, M., Spherical functions on a real semisimple Lie group, A method of reduction to the complex case, J. Funct. Anal., **30** (1978), 106–146.

[7] ——, Discrete series for semisimple symmetric spaces, Ann. of Math., **111** (1980), 253–311.

[8] Gelfand, I. M. and Shilov, G. E., Generalized functions, Vol. 1. New York: Academic press 1964.

[9] Godement, R., Séries d'Eisenstein des groupes de Chevalley, Unpublished manuscript.

[10] Helgason, S., Differential geometry, Lie groups, and symmetric spaces. New York: Academic Press 1978.

[11] Hörmander, L., An introduction to complex analysis in several variables. Princeton, N. J.: Van Nostrand 1966.

[12] Langlands, R. P., Functional equations satisfied by Eisenstein series, Lecture notes in Math., No. **544**, Berlin-Heiderberg-New York: Springer-Verlag 1976.

[13] Macdonald, I. G., Spherical functions on a group of p-adic type, Publ. Ramanujan Institute No. 2. Madras: 1971.

[14] Matsuki, T., The orbits of affine symmetric spaces under the action of minimal parabolic subgroups. J. Math. Soc. Japan, **31** (1979), 331–357.

[15] Oshima, T. and Sekiguchi, J., Eigenspaces of invariant differential operators on an affine symmetric space, Invent. Math., **57** (1980), 1–81.

[16] Richardson, R. W., Orbits, invariants and representations associated to involutions of reductive groups, Invent. Math., **66** (1982), 287–312.

[17] Rossmann, W., The structure of semisimple symmetric spaces, Canad. J. Math., **31** (1979), 157–180.

[18] Sato, F., Zeta functions of quadratic forms and special functions, Research on micro local ananlysis. Proc. Symp. RIMS Kyoto Univ., (1977) 46–59. (Japanese) (Zbl. 414 #10019).

[19] ——, On zeta functions of ternary zero forms, J. Fac. Sci. Univ. Tokyo Sect. IA, **28** (1982), 585–604.

[20] ——, Zeta functions in several variables associated with prehomogeneous vector spaces I, Functional equations, Tôhoku Math. J., **34** (1982), 437–483.

[21] ——, Zeta functions in several variables associated with prehomogeneous

vector spaces III, Eisenstein series for indefinite quadratic forms, Ann. of Math., **116** (1982), 177–212.
[22] Shintani, T., On zeta functions associated with the vector space of quadratic forms, J. Fac. Sci. Univ. Tokyo Sect. IA, **22** (1975), 25–65.
[23] Siegel, C. L., Über die Zetafunktionen indefiniter quadratischer Formen II, Math. Z., **44** (1939), 398–426.
[24] Steinberg, R., Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc., **80** 1968.
[25] ——, Lectures on Chevalley groups, Notes by J. Faulkner and R. Wilson, Yale Univ., 1967.
[26] Vust, T., Opération de groupes réductifs dans un type de cône presque homogènes, Bull. Soc. Math. France, **102** (1974), 317–334.

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