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Foliations
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Every 3-Manifold Admits a Transverse Pair of Codimension One Foliations Which Cannot be Raised to a Total Foliation

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§ 1. Introduction

Let M be an n-dimensional C^∞ manifold with or without boundary and let $\mathscr{F}$ be a C^r foliation of codimension k of M ($r \geq 1$). If $\partial M \neq \emptyset$, then for each connected component $(\partial M)_i$ of ∂M, each leaf of $\mathscr{F}$ is assumed to be *transverse* to $(\partial M)_i$, that is, $T_x\mathscr{F} + T_x\partial M = T_xM$, $x \in (\partial M)_i$, or assumed to be *tangent* to $(\partial M)_i$, that is, $T_x\mathscr{F} \subset T_x\partial M$, $x \in (\partial M)_i$, where $T_x\mathscr{F}$ denotes the tangent space of $\mathscr{F}$ at x. In the former case, the restriction of $\mathscr{F}$ to $(\partial M)_i$,

$$\mathscr{F}|_{(\partial M)_i} = \{L \cap (\partial M)_i ; L \in \mathscr{F}\}$$

is a C^r foliation of codimension k (in this case we say $\mathscr{F}$ is *transverse* to $(\partial M)_i$), and in the latter case, $\mathscr{F}|_{(\partial M)_i}$ is a C^r foliation of codimension $k-1$ (in this case we say $\mathscr{F}$ is *tangent* to $(\partial M)_i$). Let $\mathscr{G}$ be another C^r foliation of codimension l. We say $\mathscr{G}$ is *transverse* to $\mathscr{F}$ if at every point $x \in M$, $\dim(T_x\mathscr{F} \cap T_x\mathscr{G}) = \max\{n-k-l, 0\}$. In this case we say $\mathscr{G}$ is a *transverse foliation* for $\mathscr{F}$ or $(\mathscr{F}, \mathscr{G})$ is a *transverse pair* of M. If $(\mathscr{F}, \mathscr{G})$ is a transverse pair, let $\mathscr{F} \Cap \mathscr{G}$ denote $\{F \cap G; F \in \mathscr{F}, G \in \mathscr{G}\}$. Then $\mathscr{F} \Cap \mathscr{G}$ is a C^r foliation of codimension m, where $m = \min\{k+l, n\}$. For each leaf F of $\mathscr{F}$ (resp. G of $\mathscr{G}$), the restriction of $\mathscr{F} \Cap \mathscr{G}$ to F (resp. G) is a C^r foliation of codimension $k' = \min\{n-k, l\}$ (resp. $l' = \min\{k, n-l\}$).

In [13] we classified codimension one foliations transverse to the Reeb component of $S^1 \times D^2$, and using this result we proved that the 3-sphere S^3 has a codimension one foliation which does not admit a transverse foliation of codimension one. Following this result, in [8] and [9] Nishimori investigated foliations transverse to a wider class of foliations of 3-manifolds containing the Reeb component, and he showed many other examples of foliations which admit no transverse foliations. Using a result in [8], Tamura showed that every 3-manifold has a codimension

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one foliation which admits no transverse foliations ([12, Theorem D]).

Let $\mathscr{F}_1, \cdots, \mathscr{F}_n$ be C^r foliations of codimension one of an n-dimensional manifold M such that for each $j=1, \cdots, n$, $\mathscr{F}_j$ is either tangent or transverse to each connected component $(\partial M)_i$ of ∂M. Then we call $(\mathscr{F}_1, \cdots, \mathscr{F}_n)$ a *total foliation* of M if for every point $x \in M$, $\dim(T_x\mathscr{F}_1 \cap \cdots \cap T_x\mathscr{F}_n)=0$. In contrast with the results stated above, for every closed orientable 3-manifold Hardorp [5] constructed a total foliation.

In these contexts the purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let M be a C^∞ closed orientable 3-manifold. Then there exists a transverse pair $(\mathscr{F}, \mathscr{G})$ of transversely orientable C^∞ foliations of codimension one of M satisfying the following condition;*

(*) *There does not exist a C^1 foliation $\mathscr{H}$ of codimension one such that $(\mathscr{F}, \mathscr{G}, \mathscr{H})$ is a total foliation of M.*

We say a transverse pair $(\mathscr{F}, \mathscr{G})$ of a compact 3-manifold M with or without boundary *cannot be raised to a total foliation* if $(\mathscr{F}, \mathscr{G})$ satisfies the condition (*) of Theorem 1.1. Then we see easily that a transverse pair $(\mathscr{F}, \mathscr{G})$ cannot be raised to a total foliation if and only if the one dimensional foliation $\mathscr{F} \cap \mathscr{G}$ admits no transverse foliations of codimension one.

In Section 2 we construct a model transverse pair. In Section 3 we present a criterion which will be used in Section 4 to show that the model in Section 2 cannot be raised to a total foliation. Section 5 is devoted to the proof of Theorem 1.1.

In the following sections, manifolds are assumed to be orientable and of class C^∞, and foliations are assumed to be transversely orientable and of class C^∞ unless otherwise stated. We sometimes call a leaf of a one dimensional foliation an *orbit*.

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§ 2. A construction of a model

Let $A_k \colon T^2 \to T^2$, $k \in \boldsymbol{Z}$, be an orientation preserving C^∞ diffeomorphism of the two dimensional torus $T^2 = \boldsymbol{R}^2/\boldsymbol{Z}^2$ defined by

$$A_k((x, y)) = (x + k \cdot y, y), \quad (x, y) \in \boldsymbol{R}^2/\boldsymbol{Z}^2.$$

Let ξ be the one dimensional foliation of T^2 all leaves of which are parallel to x-axis:

$$\xi=\{\lambda_c;\, c \in S^1\},$$

where $\lambda_c=\{(x, y) \in T^2;\, y=c\}$.

We see A_k preserves ξ:

$$A_k^*\xi=\xi.$$

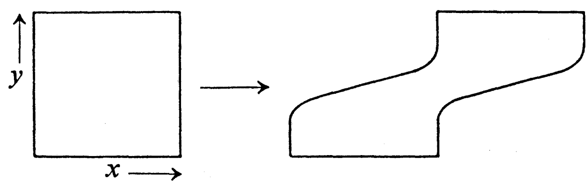
Then by constructing an isotopy which preserves ξ, we easily obtain a C^∞ diffeomorphism $A'_k\colon T^2 \to T^2$ satisfying the following conditions:

(1) A'_k is isotopic to A_k,

(2) A'_k preserves ξ, and

(3) the support of A'_k is $\{(x, y) \in T^2;\, 1/4 \leq y \leq 3/4,\, x \in S^1\}$,

where the support is the closure of the open set $\{(x, y) \in T^2;\, A'_k(x, y) \neq (x, y)\}$ (see Fig. 2.1).



A'_k for $k=1$

Fig. 2.1.

Let P_k be the total space of the T^2-bundle over S^1 with A'_k as monodromy:

$$P_k=T^2\times[0, 1]/(x, y, 1)\sim(A'_k(x, y), 0), \quad (x, y) \in T^2.$$

In what follows, when there is no confusion, we use the coordinate (x, y, t) of $T^2\times[0, 1]$ to specify a point of P_k. Let $\pi\colon P_k \to S^1=\boldsymbol{R}/\boldsymbol{Z}$ be the projection map defined by

$$\pi((x, y, t))=t \pmod 1,$$

and $q\colon T^2\times[0, 1]\to P_k$ be the quotient map. Let $\mathscr{F}_0$ be the bundle foliation of P_k:

$$\mathscr{F}_0=\{\pi^{-1}(t);\, t \in S^1\},$$

and let $\mathscr{G}_0$ be the codimension one foliation of P_k obtained by the suspension of ξ by A'_k:

$$\mathscr{G}_0=\{q(\lambda_c\times[0, 1]);\, c \in S^1\}.$$

Since A'_k preserves ξ, $\mathscr{G}_0$ is well defined. Then all leaves of $\mathscr{F}_0$ and $\mathscr{G}_0$ are diffeomorphic to T^2, and $\mathscr{G}_0$ is transverse to $\mathscr{F}_0$. Furthermore, all leaves of $\mathscr{F}_0 \cap \mathscr{G}_0$ are circles. Since $\mathscr{F}_0$ and $\mathscr{G}_0$ are foliations without holonomy, we have a locally trivial S^1-bundle η over T^2 whose fibers are the leaves of $\mathscr{F}_0 \pitchfork \mathscr{G}_0$. Then by calculating the obstruction to the existence of a cross section of η, we easily have the following lemma.

Lemma 2.1. *The Euler class $\chi(\eta)$ of η is equal to $k\cdot[T^2] \in H^2(T^2; \boldsymbol{Z})$ with a suitable orientation.*

Remark. From Lemma 2.1 and a result of Milnor [7] and Wood [14], we see that for $k \neq 0$, the transverse pair $(\mathscr{F}_0, \mathscr{G}_0)$ cannot be raised to a total foliation.

In the following, we will modify $\mathscr{F}_0$ and $\mathscr{G}_0$. Let b and c: $S^1 \to P_k$ be oriented simple closed curves in P_k defined by

$$b(y)=(1/8, y, 1/8),\ y \in S^1, \quad \text{and} \quad c(t)=(7/8, 1/8, t),\ t \in S^1.$$

We denote the image $b(S^1)$ (resp. $c(S^1)$) also by b (resp. c). We see b is transverse to $\mathscr{G}_0$ and c is transverse to $\mathscr{F}_0$ (Fig. 2.2). Let

$$N(b)=\{(x, y, t) \in P_k;\ (x-(1/8))^2+(t-(1/8))^2 \leq (1/16)^2\}$$

and

$$N(b)'=\{(x, y, t) \in P_k;\ (x-(1/8))^2+(t-(1/8))^2 \leq (3/32)^2\}$$

be two solid tori containing b as their core circle. Let

$$N(c)=\{(x, y, t) \in P_k;\ (x-(7/8))^2+(y-(1/8))^2 \leq (1/16)^2\}$$

and let

$$N(c)'=\{(x, y, t) \in P_k;\ (x-(7/8))^2+(y-(1/8))^2 \leq (3/32)^2\}.$$

By the condition (3) of A_k', these are also solid tori containing c as their

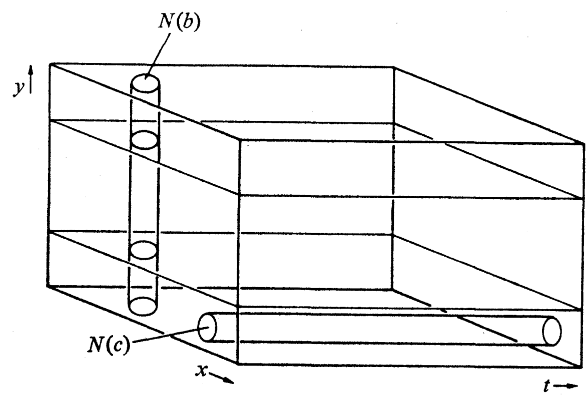


Fig. 2.2.

core circle (Fig. 2.2).

Now we modify $\mathscr{F}_0$ and $\mathscr{G}_0$ in two steps as follows.

Step 1. First we recall the notion of turbulization briefly according to [8]. Let $K_1: S^1\times S^1\times[0,1]\to P_k-\operatorname{int}N(b)$ (resp. $K_2: S^1\times S^1\times[0,1]\to N(b)$) be a collar of $\partial N(b)$ defined by

$$K_1(\theta, y, s)=(((1/32)\cdot s+(1/16))\cdot\cos 2\pi\theta+(1/8), y,$$
$$((1/32)\cdot s+(1/16))\cdot\sin 2\pi\theta+(1/8))\in P_k-\operatorname{int}N(b)$$
$$(\text{resp. } K_2(\theta, y, s)=(((1/16)-(1/32)\cdot s)\cdot\cos 2\pi\theta+(1/8), y,$$
$$((1/16)-(1/32)\cdot s)\cdot\sin 2\pi\theta+(1/8))\in N(b)).$$

Let f: $]0,1]\to[0,\infty[$ be a C^∞ function satisfying the following conditions:

(f1) $f(t)=0$ for all $t\in[1/2,1]$,

(f2) $\lim_{t\downarrow 0}f(t)=+\infty$,

(f3) $df/dt<0$ in $]0,1/2]$, and

(f4) the submanifolds $\boldsymbol{R}\times\{0\}$ and

$$F_c(f)=\{(f(t)+c, t);\ t\in]0,1]\},\ c\in\boldsymbol{R},\ \text{of } \boldsymbol{R}\times[0,1]$$

form a C^∞ foliation of $\boldsymbol{R}\times[0,1]$.

Let $\mathscr{H}_1$ (resp. $\mathscr{H}_2$) be a codimension one foliation of $K_1(S^1\times S^1\times[0,1])$ (resp. $K_2(S^1\times S^1\times[0,1])$) defined as follows: $\mathscr{H}_1$ consists of a compact leaf $K_1(S^1\times S^1\times\{0\})=\partial N(b)$ and non-compact leaves

$$\{K_1(\theta, [f(t)]+w, t);\ \theta\in S^1,\ t\in]0,1]\}$$

for $w\in S^1=\boldsymbol{R}/\boldsymbol{Z}$, where $[z]$ means z mod 1. $\mathscr{H}_2$ consists of a compact leaf $K_2(S^1\times S^1\times\{0\})=\partial N(b)$ and non-compact leaves

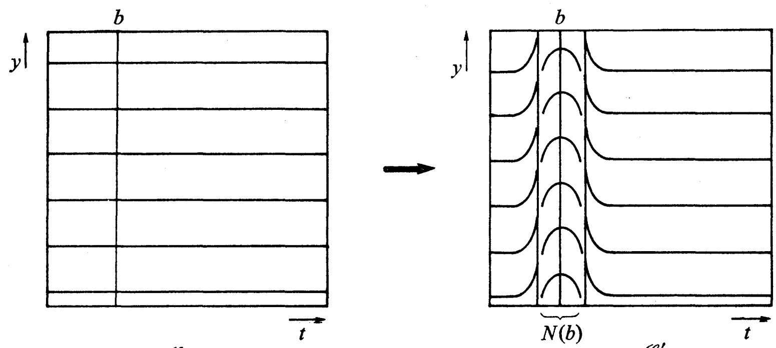
$$\{K_2(\theta, -[f(t)]+w, t);\ \theta\in S^1,\ t\in]0,1]\}$$

for $w\in S^1=\boldsymbol{R}/\boldsymbol{Z}$.

Now let us remove $\mathscr{G}_0|_{K_1(S^1\times S^1\times[0,1])\cup K_2(S^1\times S^1\times[0,1])}$, and put $\mathscr{H}_1$ and $\mathscr{H}_2$ instead. Then we have a codimension one foliation $\mathscr{G}_0'$ described in Fig. 2.3. We call $\mathscr{G}_0'$ a foliation obtained by turbulizing $\mathscr{G}_0$ around $\partial N(b)$. Note that $\mathscr{G}_0'|_{N(b)}$ is a Reeb component.

We can easily see that there are points on $\partial N(b)$ at which $\mathscr{F}_0$ is not transverse to $\mathscr{G}_0'$. Next we wish to modify $\mathscr{F}_0$ so that it becomes transverse to $\mathscr{G}_0'$. Let $S_1, S_2: S^1\times S^1\times[0,1]\to P_k$ be embeddings of $S^1\times S^1\times[0,1]$ defined by

$$S_1(x, y, s)=(x, y, (1/8)-(3/32)\cdot s),$$
$$S_2(x, y, s)=(x, y, (1/8)+(3/32)\cdot s),$$
$$(x, y, s)\in S^1\times S^1\times[0,1].$$



slice at $x=1/8$

Fig. 2.3.

Let g: $[0, 1]\to[0, 1]$ be a C^∞ diffeomorphism of $[0, 1]$ satisfying the following conditions:

(g1) $t<g(t)$ for all $t \in]0, 1[$,

(g2) g is infinitely tangent to the identity at $t=0$ and $t=1$.

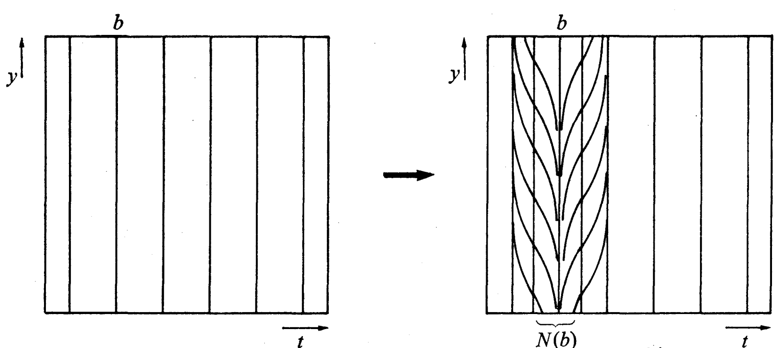
Let a: $S^1\to P_k$ be an oriented simple closed curve defined by

$$a(x)=(x, 0, 1/8) \in P_k.$$

Let

$$F_b=\{(x, y, 1/8) \in P_k; x \in S^1, y \in S^1\}$$

be a leaf of $\mathscr{F}_0$. Let $\mathscr{S}_1$ (resp. $\mathscr{S}_2$) be a codimension one foliation of $S_1(S^1\times S^1\times[0, 1])$ (resp. $S_2(S^1\times S^1\times[0, 1])$) such that $\mathscr{S}_1$ (resp. $\mathscr{S}_2$) is transverse to all $[0, 1]$-factors $S_1(\{(x, y)\}\times[0, 1])$ (resp. $S_2(\{(x, y)\}\times[0, 1])$), $(x, y) \in S^1\times S^1$, and the total holonomy homomorphism h_1: $\pi_1(F_b)$



slice at $x=1/8$

Fig. 2.4.

$\to\mathrm{Diff}\,[0, 1]$ (resp. $h_2\colon \pi_1(F_b)\to\mathrm{Diff}\,[0, 1]$) is given by $h_1([a])=\mathrm{id}$, $h_1([b])=g$, $h_2([a])=\mathrm{id}$, and $h_2([b])=g$. Now remove $\mathscr{F}_0|_{S_1(S^1\times S^1\times[0,1])\cup S_2(S^1\times S^1\times[0,1])}$ and put $\mathscr{S}_1$ and $\mathscr{S}_2$ instead. Then we have a codimension one foliation $\mathscr{F}_0'$ described in Fig. 2.4 such that $\mathscr{F}_0'$ is transverse to $\mathscr{G}_0'$.

Step 2. Since $\mathscr{F}_0'|_{N(c)'}$ is a product foliation by 2-disks, we obtain a codimension one foliation $\mathscr{F}_0''$ by turbulizing $\mathscr{F}_0'$ around $\partial N(c)$ as in Fig. 2.5. Next we modify $\mathscr{G}_0'$. From the construction of the step 1, we see $\mathscr{G}_0'|_{P_k-\mathrm{int}\,N(b)'}=\mathscr{G}_0|_{P_k-\mathrm{int}\,N(b)'}$. Let

$$D=\{(x, t)\in T^2;\ (x-(1/8))^2+(t-(1/8))^2\le(3/32)^2\}$$

be a 2-disk in T^2. Let S_1' and $S_2'\colon (T^2-\mathrm{int}\,D)\times[0, 1]\to P_k-\mathrm{int}\,N(b)'$ be embeddings defined by

$$S_1'(x, t, y)=(x, (1/8)-(3/32)\cdot y, t),$$
$$S_2'(x, t, y)=(x, (1/8)+(3/32)\cdot y, t),$$
$$(x, t)\in T^2-\mathrm{int}\,D \text{ and } y\in[0, 1].$$

Then we can define a codimension one foliation $\mathscr{S}_1'$ (resp. $\mathscr{S}_2'$) of $S_1((T^2-\mathrm{int}\,D)\times[0, 1])$ (resp. $S_2((T^2-\mathrm{int}\,D)\times[0, 1])$) similar to the one in the step 1 as described in Fig. 2.6. Let $\mathscr{G}_0''$ be the codimension one foliation obtained by removing $\mathscr{G}_0'|_{S_1((T^2-\mathrm{int}\,D)\times[0,1])\cup S_2((T^2-\mathrm{int}\,D)\times[0,1])}$ and putting $\mathscr{S}_1'$ and $\mathscr{S}_2'$ instead. Let $a'\colon S^1\to P_k-\mathrm{int}\,N(b)'$ be an oriented simple closed curve defined by

$$a'(x)=(x, 1/8, 0),\quad x\in S^1.$$

Then the foliations $\mathscr{S}_1'$ and $\mathscr{S}_2'$ have trivial total holonomy along a' and

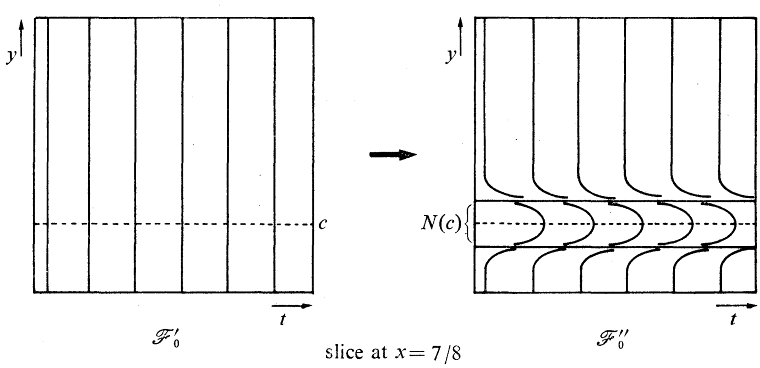
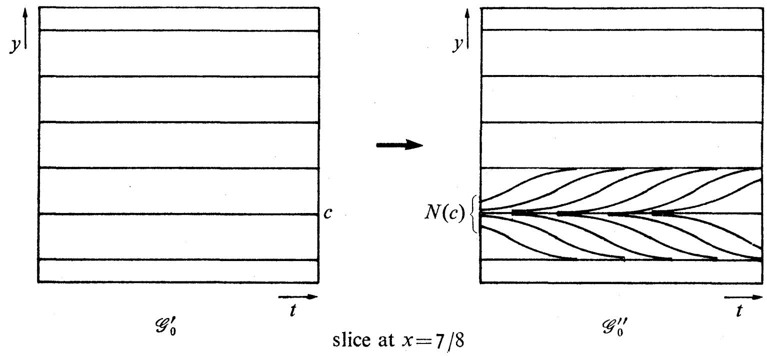


Fig. 2.5.



slice at $x=7/8$

Fig. 2.6.

have infinite cyclic total holonomy along c. Hence $\mathcal{S}'_1$ and $\mathcal{S}'_2$ have trivial total holonomy along the circle

$$S'_1(\partial(T^2-\operatorname{int} D)\times\{0\})=S'_2(\partial(T^2-\operatorname{int} D)\times\{0\}).$$

In fact using a suitable isotopy of $\mathcal{S}'_1$ (resp. $\mathcal{S}'_2$) as in Fig. 2.7, we can assume

$$\mathcal{S}'_1|_{S'_1(\partial D\times[0,1])}=\mathcal{G}'_0|_{S'_1(\partial D\times[0,1])},$$

and

$$\mathcal{S}'_2|_{S'_2(\partial D\times[0,1])}=\mathcal{G}'_0|_{S'_2(\partial D\times[0,1])}.$$

Thus we can glue $\mathcal{G}''_0$ to $\mathcal{G}'_0|_{N(b)'}$ along $\partial N(b)'$ to obtain a codimension one foliation $\mathcal{G}'''_0$ of P_k. We see easily that $\mathcal{G}'''_0$ is transverse to $\mathcal{F}''_0$. Let

$$P=P_k-(\operatorname{int} N(b)\cup\operatorname{int} N(c))$$

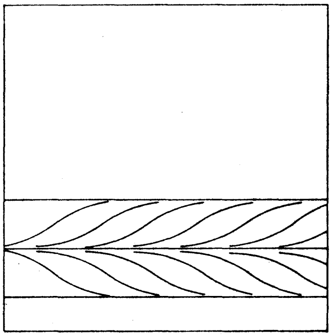
and let $\mathcal{F}_1$ (resp. $\mathcal{G}_1$) be the restriction of $\mathcal{F}''_0$ (resp. $\mathcal{G}'''_0$) to P. Let T_b be the leaf of $\mathcal{F}_1$ containing the circle

$$\{(0, y, 1/8)\in P;\ y\in S^1\}$$

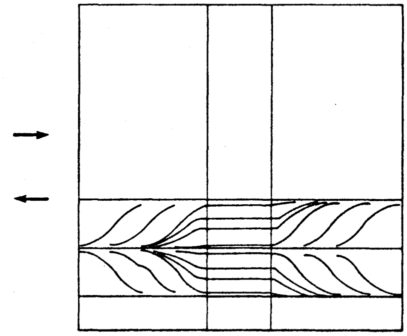
and T_c the leaf of $\mathcal{G}_1$ containing the circle

$$\{(0, 1/8, t)\in P;\ t\in S^1\}.$$

T_b and T_c are diffeomorphic to the 1-punctured annulus $S^1\times[0, 1]-pt$.
In the following sections, we only consider P for $k=1$.



slice at $x=x_0$,
$25/32 \leq x_0 \leq 31/32$



$t=1/32$ $t=7/32$
slice at $x=x_1$,
$1/32 \leq x_1 \leq 7/32$

Fig. 2.7.

§ 3. The infinitely approximating null closed orbit property

The purpose of this section is to give a criterion for the non-existence of transverse foliations for a given one dimensional foliation (Theorem 3.3) which is a modified version of the one used in [12].

Let N be a (possibly non-compact) 3-manifold and let ϕ be an oriented C^1 foliation of dimension one tangent to ∂N. We choose a Riemannian metric g on N. Let ε be an arbitrary positive number.

Definition 3.1. Let (N, ϕ, g) be as above. Let C: $[0, 1] \rightarrow N$ be an oriented smooth curve (resp. an oriented smooth closed curve). Then C is called an *ε-orbit segment* (resp. *ε-closed orbit*) of (N, ϕ, g) if the following condition holds:

(*) For every point x of $C([0, 1])$, we have

$$|1-g_x(u, v)|<\varepsilon,$$

where g_x is the metric on the tangent space T_xN at x, u and v are the positively directed unit vectors tangent to ϕ and C at x respectively.

An ε-closed orbit C is called an *ε-null closed orbit* if C is homotopic to zero in N.

Definition 3.2. (N, ϕ, g) has the *infinitely approximating null closed orbit property* (0-N.C.O.P. for short) if for every positive number ε, there exists an ε-null closed orbit C of (N, ϕ, g).

Theorem 3.3. *Let M be a compact 3-manifold and let $\mathscr{F}$ be an oriented C^1 foliation of dimension one tangent to ∂M. Let $E(\phi)$ denote the* *union of all closed orbits with infinite holonomy of* ϕ. *If* $(M-E(\phi), \phi|_{M-E(\phi)}, g|_{M-E(\phi)})$ *has* 0-*N.C.O.P., then* ϕ *does not admit a transverse* C^1 *foliation of codimension one.*

Remark. Since M is compact, the statement that $(M-E(\phi), \phi|_{M-E(\phi)}, g|_{M-E(\phi)})$ has 0-N.C.O.P. is independent of a choice of g.

For the proof, we need some lemmas.

Lemma 3.4. *Let* $(S^1\times D^2, \mathcal{F}_R)$ *be a Reeb component of* $S^1\times D^2$ *and let* ϕ *be an oriented* C^1 *foliation of dimension one transverse to* $\mathcal{F}_R$ *and pointed inward along* $\partial(S^1\times D^2)$. *Then there exists a closed orbit with infinite holonomy of* ϕ.

Proof. Let T be a solid torus in $\operatorname{int}(S^1\times D^2)$ such that ∂T is parallel to $\partial(S^1\times D^2)$, ∂T is transverse to $\mathcal{F}_R$, and ∂T is near enough to $\partial(S^1\times D^2)$ such that ϕ is still pointed inward along ∂T. Since $\mathcal{F}_R|_T$ is a product foliation by 2-disks, we have a projection map $p: T\to S^1$ such that $\{p^{-1}(t); t\in S^1\}$ is $\mathcal{F}_R|_T$. For each closed orbit l of ϕ, we associate an integer k such that $p_*[l]=k\cdot[S^1]$, where $[l]$ is the homology class of l. We call k the *degree* of l. Choosing an orientation suitably, we can assume the degree is a positive integer. Let $D_0=p^{-1}(0)$, $0\in S^1$, be a leaf of $\mathcal{F}_R|_T$. By considering the first return map of ϕ on D_0, we have a diffeomorphism f of D_0 into itself. By applying the Brouwer fixed point theorem for f, we have a closed orbit l_0 of ϕ whose degree is equal to 1. If the holonomy group of l_0 is infinite, we are done.

Consider the case when the holonomy group of l_0 is finite. Let U be the union of all closed orbits with finite holonomy and let U_0 be the connected component of U containing l_0. Since the holonomy group of an orbit with finite holonomy is conjugate to a cyclic subgroup of $\boldsymbol{SO}(2)$ (see for example [1]), we have

(i) U is an open subset of int T, and

(ii) the closed orbits with non-trivial holonomy are isolated in U.

Moreover

(iii) if l is a closed orbit with holonomy of order k and the degree of l is m, then the orbit l' near l is a closed orbit without holonomy and the degree of l' is $k\cdot m$. Let d be the degree of a closed orbit without holonomy in U_0. d is well defined from (ii). Then from (iii), we have

(iv) the order of the holonomy of each closed orbit in U_0 divides d.

Let $\pi: (\tilde{\phi}, \tilde{T}, \tilde{U}_0)\to(\phi, T, U_0)$ be the d-fold covering and let $\tilde{f}: \tilde{D}_0\to\tilde{D}_0$ be the first return map of $\tilde{\phi}$, where $\tilde{D}_0$ is a connected component of $\pi^{-1}(D_0)$. Note that $\tilde{f}$ is conjugate to f^d via the covering projection. Let l_1 be an orbit through a point of $\operatorname{bd}(U_0)=\operatorname{cl}(U_0)-U_0$ and $\tilde{l}_1$ be a connected component of $\pi^{-1}(l_1)$. From (iv), $\tilde{f}|_{\tilde{U}_0 \cap \tilde{D}_0}$ is the identity map and hence $\tilde{f}$ also fixes the point $\tilde{l}_1 \cap \tilde{D}_0$. Thus l_1 is a closed orbit. Since l_1 contains a point of $\mathrm{bd}(U_0)$, the holonomy of l_1 cannot be finite. Thus l_1 is a desired orbit.

Proof of Theorem 3.3. Suppose that there exists a codimension one C^1 foliation $\mathscr{F}$ transverse to ϕ. We will have a contradiction later. We may assume $\partial M=\emptyset$. For, if $\partial M \neq \emptyset$, then we consider the double $(DM, D\mathscr{F})$ of $(M, \mathscr{F})$, where $DM=M \cup_{\partial M} M$ and $D\mathscr{F}=\mathscr{F} \cup_{\mathscr{F}|\partial M} \mathscr{F}$ which is transverse to $D\phi=\phi \cup_{\phi|\partial M} \phi$, and we can apply our argument to $(DM, D\mathscr{F})$. Since M is compact, there exists a positive number ε small enough such that every ε-closed orbit is transverse to $\mathscr{F}$. Thus the assumption that $(M-E(\phi), \phi|_{M-E(\phi)}, g|_{M-E(\phi)})$ has 0-N.C.O.P. implies that there exists an ε-null closed orbit $C: S^1 \to M-E(\phi)$ which is transverse to $\mathscr{F}$. That is, we have a continuous map $F: D^2 \to M-E(\phi)$ such that $F|_{\partial D^2}=C$. Then by a well-known method (see [3], [10], [11] and [2] for the C^1-case), for any positive number δ, we have a C^1 map $\bar{F}: D^2 \to M$ satisfying the following conditions:

($\bar{F}$1) $\bar{F}|_{\partial D^2}=F|_{\partial D^2}$.

($\bar{F}$2) $\bar{F}$ is an immersion.

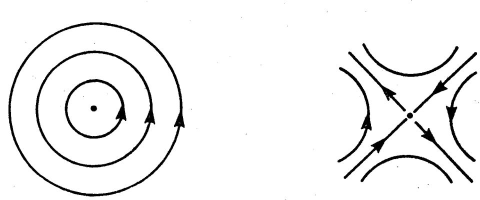
($\bar{F}$3) $\bar{F}$ is *in general position* with respect to $\mathscr{F}$; that is, for every point $x \in D^2$ there exists a foliation chart (U, π) of $\mathscr{F}$ around $\bar{F}(x)$; $\bar{F}(x) \in U$, $\pi: U \to \boldsymbol{R}$, such that $\pi \circ \bar{F}$ is a Morse function.

($\bar{F}$4) $\bar{F}$ is *δ-near* to F, that is, $d(\bar{F}(x), F(x))<\delta$ for every point $x \in D^2$, where d is the distance on M induced from g.

We choose δ so that for every Reeb component $(R, \mathscr{F}|_R)$ of $\mathscr{F}$, $R \cong S^1 \times D^2$, every point which is δ-near to ∂R is contained in a tubular neighborhood of ∂R.

By the simply-connectedness of D^2 and the condition ($\bar{F}$3), we see that the Haefliger structure $\bar{F}^*\mathscr{F}$ defines a C^1 vector field X on D^2 whose singular points are a finite number of centers and saddles (Fig. 3.1).

From ($\bar{F}$1) and the fact that $C=\bar{F}|_{\partial D^2}$ is transverse to $\mathscr{F}$, we see X is transverse to ∂D^2. We assume X is pointed inward on ∂D^2. Furthermore, by choosing $\bar{F}$ suitably, we can assume that for distinct singular



center saddle

Fig. 3.1.

points x, x' of X, $\bar{F}(x)$ and $\bar{F}(x')$ are on distinct leaves of $\mathscr{F}$. Then X has *no saddle connections*; that is, there are no orbits with the α-limit set and the ω-limit set being two distinct saddle points.

Then by the Poincaré-Bendixson theorem, the α-limit set or the ω-limit set of an orbit of X is one of the following types;

(a) a center
(b) a non-singular closed orbit
(c) a union of a saddle and a non-compact orbit
(d) a union of a saddle and two non-compact orbits.

We divide (d) into (d_1) and (d_2) according to Fig. 3.2; that is, (d_2) is the case that a non-compact orbit is surrounded by the circle consisting of the saddle and another non-compact orbit, and (d_1) is otherwise.

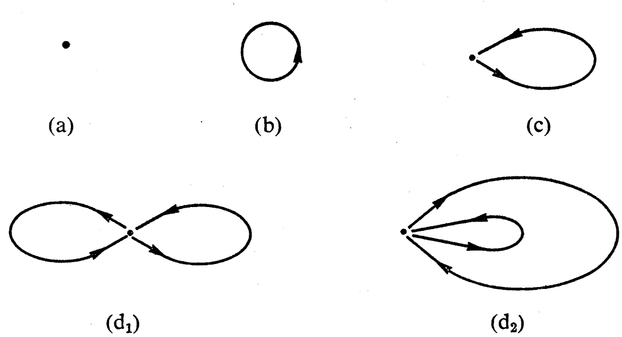


Fig. 3.2.

Then from the Novikov compact leaf theorem ([10]), we have the following statements:

(N1) There exists a *vanishing cycle*, that is, a continuous family of maps f_t: $S^1 \to M$, $0 \leq t \leq 1$, such that

(i) $f_t(S^1)$ is contained in a leaf $L_t \in \mathscr{F}$,
(ii) f_0: $S^1 \to L_0$ is not null homotopic, and
(iii) f_t: $S^1 \to L_t$, $0 < t \leq 1$, is null homotopic.

Furthermore $f_t(S^1)$ $(0 \leq t \leq 1)$ is contained in $\bar{F}(D^2)$ and $\bar{F}^{-1}(f_t(S^1))$ is a union of a finite number of circles of type (a), (b), (c) and (d) above.

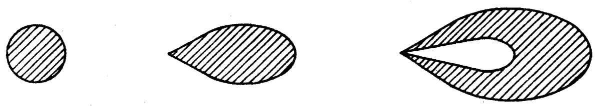
(N2) For every vanishing cycle f_t: $S^1 \to L_t \in \mathscr{F}$, $0 \leq t \leq 1$, such that f_0 is not null homotopic in L_0 and $f_t (0 < t \leq 1)$ is null homotopic in L_t, the leaf L_0 is the boundary leaf of a Reeb component $(R, \mathscr{F}|_R)$ and L_t is an interior leaf of $(R, \mathscr{F}|_R)$.

We will show that the immersed disk $\bar{F}(D^2)$ intersects $E(\phi)$. For this, we need the following assertion which is proved by considering an "inner-most" vanishing cycle on $(D^2, \bar{F}^*\mathscr{F})$ (see [4] and [10]).

Assertion. *There exists a subset D_0 of D^2 which is a disk or a disk glued at two points on its boundary* (*Fig.* 3.3) *satisfying the following conditions*:

(i) ∂D_0 *is a circle of type* (b), (c) *and* (d_2), *and* $\mathrm{int}\, D_0 = \bigcup_\lambda l_\lambda$, *where* l_λ *is a circle of type* (a), (b) *and* (d).

(ii) $\bar{F}(\partial D_0)$ *is not null homotopic in* L_0, *and* $\bar{F}(l_\lambda)$, $l_\lambda \subset \mathrm{int}\, D_0$, *is null homotopic in* L_λ, *where* L_0 (*resp.* L_λ) *denotes the leaf of* $\mathscr{F}$ *containing* $\bar{F}(\partial D_0)$ (*resp.* $\bar{F}(l_\lambda)$).



disk glued at two points
on its boundary

Fig. 3.3.

We show that for D_0 in the assertion, the image $\bar{F}(D_0)$ intersects $E(\phi)$. By the condition (ii) of the assertion, there exists a collar N of ∂D_0 in D_0 such that $N - \partial D_0$ is a union of circles of type (b) and the restriction $\bar{F}|_N : N \to M$ gives us a vanishing cycle. Hence by ($N2$) of the statements of Novikov's theorem, there exists a solid torus $R\, (\cong S^1 \times D^2)$ in M such that $\mathscr{F}|_R$ is a Reeb component and $\bar{F}(\partial D_0)$ is contained in ∂R. Furthermore we easily see that the homology class $[\bar{F}(\partial D_0)]$ in $H_1(\partial R; Z)$ represented by $\bar{F}(\partial D_0)$ is equal to a multiple of the meridian;

$$[\bar{F}(\partial D_0)] = k \cdot [\{*\} \times \partial D^2] \in H_1(\partial R; Z),$$

where we identify R with $S^1 \times D^2$ and $k \in Z - \{0\}$. By Lemma 3.4, there exists a closed orbit l of ϕ such that $l \subset \mathrm{int}\, R \cap E(\phi)$. Moreover the homology class $[l] \in H_1(R; Z)\ (\cong Z)$ of l is not zero.

Assume $\bar{F}(D_0)$ does not intersect l. First consider the case $\bar{F}(D_0) \subset R$. Then since $[\bar{F}(\partial D_0)]$ is a non-zero multiple of the meridian in $H_1(\partial R; Z)$, $\bar{F}_*[D_0, \partial D_0]$ represents a non-zero element in $H_1(R, \partial R; Z)\ (\cong Z)$. Thus the intersection number of $[l]$ and $\bar{F}_*[D_0, \partial D_0]$ is non-zero and this contradicts the assumption. Next consider the case $\bar{F}(D_0) \not\subset R$. Take the connected component S of $\bar{F}^{-1}(\bar{F}(D_0) \cap R)$ which contains ∂D_0. Let ∂S denote $S - \mathrm{int}\, S$. $\bar{F}(\partial S)$ is contained in ∂R. By the condition (i) of the assertion, $\partial S - \partial D_0$ is a union of a finite number of circles of type (b), (d). By (ii) of the assertion, $\bar{F}$ maps these circles to null homotopic circles on ∂R. Then we can extend $\bar{F}|_S : S \to R$ to a continuous map $\bar{F}' : D_0 \to R$ such that $\bar{F}'|_S = \bar{F}|_S$ and $\bar{F}'(D_0 - S)$ is contained in ∂R. If $\bar{F}'(D_0)$ does not intersect l, then we have a contradiction by the same way as in the case $\bar{F}(D_0) \subset R$. This shows that $\bar{F}(S)$ intersects l. Hence $\bar{F}(D_0)$ intersects l.

By $(\bar{F}4)$ and the choice of δ, the same argument shows that $F(D_0)$ or $F(S)$ intersects l. Since F is a map into $M-E(\phi)$ and $l\subset E(\phi)$, we have a contradiction. This completes the proof.

Proposition 3.5. *Let (M, ϕ, g) be a triad of a (possibly non-compact) 3-manifold, a C^1 foliation of dimension one tangent to ∂M and a Riemannian metric on M. Let p; $\tilde{M}\to M$ be a finite covering and let $\tilde{\phi}=p^*\phi$ (resp. $\tilde{g}=p^*g$) be the induced one dimensional foliation (resp. the induced metric) by p. If (M, ϕ, g) has 0-N.C.O.P., then $(\tilde{M}, \tilde{\phi}, \tilde{g})$ also has 0-N.C.O.P.*

Proof. If C_ε is an ε-closed orbit of (M, ϕ, g), then each lift $\tilde{C}_\varepsilon$ of C_ε is an ε-closed orbit. If C_ε is null homotopic, then each lift $\tilde{C}_\varepsilon$ is also null homotopic. Thus for every positive number ε, we have an ε-null closed orbit $\tilde{C}_\varepsilon$ of $(\tilde{M}, \tilde{\phi}, \tilde{g})$.

§ 4. Main lemma

Let $P=P_1-(\operatorname{int} N(b)\cup \operatorname{int} N(c))$ in Section 2 and let ϕ be the one dimensional foliation determined by the intersection of leaves of $\mathscr{F}_1$ and $\mathscr{G}_1$:

$$\phi=\mathscr{F}_1\cap\mathscr{G}_1.$$

We will choose an orientation of ϕ later. Let $E(\phi)$ be the union of closed orbits with infinite holonomy of ϕ. Choose a Riemannian metric g of P. The following lemma is important to the proof of our result.

Lemma 4.1. $(P-E(\phi), \phi|_{P-E(\phi)}, g|_{P-E(\phi)})$ *has 0-N.C.O.P.*

In order to prove Lemma 4.1, we need to observe the behavior of ϕ near $\partial N(b)$ and near $\partial N(c)$. We observe it according to five cases (O-1) to (O-5) (Fig. 4.1).

(O-1) Let L_1 be a leaf of $\mathscr{F}_1$ containing a point $(1/8, 0, t)\in P$, $0\leq t\leq 1/32$. Then L_1 is diffeomorphic to a 1-punctured torus T^2-pt, and the foliation $\mathscr{G}_1|_{L_1}$ of L_1 is described in Fig. 4.2.

(O-2) Let L_2 be a leaf of $\mathscr{F}_1$ containing a point $(1/8, 0, t)$, $1/32<t<1/8$. Then L_2 is diffeomorphic to ∞-punctured $S^1\times\boldsymbol{R}-U$, where U is diffeomorphic to $]0, 1[\times\boldsymbol{R}$, and $\mathscr{G}_1|_{L_2}$ is described in Fig. 4.3.

(O-3) Let $L_3=T_b$ be the leaf containing the point $(1/8, 0, 1/8)$. L_3 is diffeomorphic to 1-punctured annulus $S^1\times[0, 1]-pt$ and $\mathscr{G}_1|_{L_3}$ is described in Fig. 4.4.

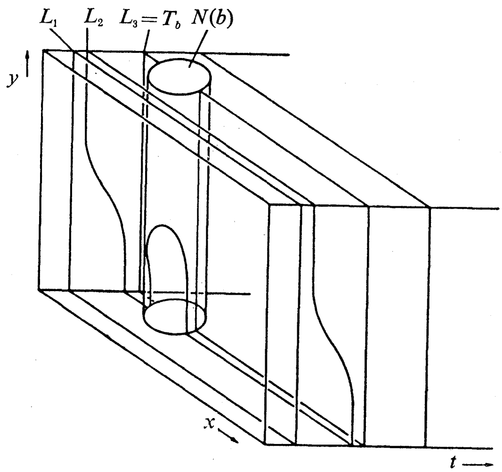


Fig. 4.1.

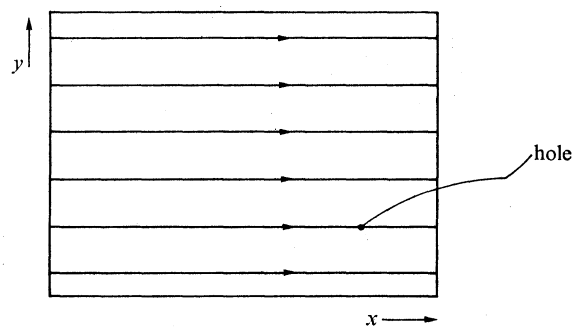


Fig. 4.2.

(O-4) Let L_4 be a leaf of $\mathscr{F}_1$ containing a point $(1/8, 0, t)$, $1/8<t<7/32$. L_4 is diffeomorphic to ∞-punctured $S^1\times \boldsymbol{R}-U$ as in (O-2) and $\mathscr{G}_1|_{L_4}$ is similar to $\mathscr{G}_1|_{L_2}$.

(O-5) Let L_5 be a leaf of $\mathscr{F}_1$ containing a point $(1/8, 0, t)$, $7/32\leq t<1$. L_5 is diffeomorphic to 1-punctured torus T^2-pt and $\mathscr{G}_1|_{L_5}$ is similar to $\mathscr{G}_1|_{L_1}$.

The behavior of ϕ near $\partial N(c)$ is similar to (O-1) to (O-5).

Lemma 4.2. *$E(\phi)=(T_b\cap\partial N(b))\cup(T_c\cap\partial N(c))=\{(x, y, 1/8)\in P;\ x=1/16, 3/16, y\in S^1\}\cup\{(x, 1/8, t)\in P;\ x=13/16, 15/16, t\in S^1\}$. Therefore $E(\phi)$ is a union of four closed orbits on $\partial N(b)\cup\partial N(c)$. In particular, $\pi_1(P-E(\phi))$ is isomorphic to $\pi_1(P)$.*

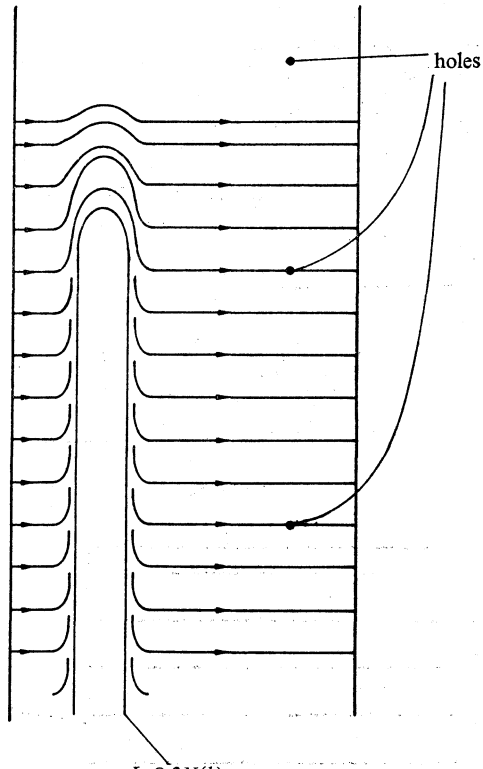


Fig. 4.3.

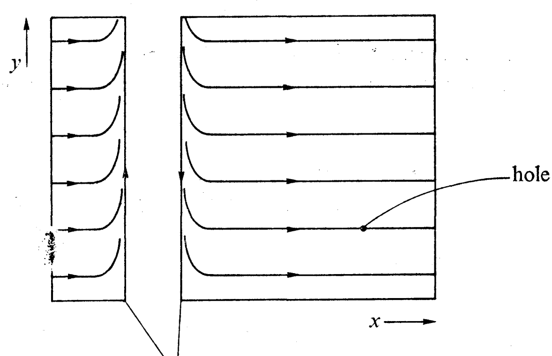


Fig. 4.4.

Proof. Leaves of $\mathscr{F}_1$ (resp. $\mathscr{G}_1$) with non-trivial holonomy are $\partial N(c)$ (resp. $\partial N(b)$), T_b (resp. T_c) and two leaves which are diffeomorphic to $T^2 - pt$. By the observation (O-1), (O-3) and (O–5) of this section, we see that closed orbits with non-trivial holonomy are only on $T_b \cup T_c$, and then we have $E(\phi) = (T_b \cap \partial N(b)) \cup (T_c \cap \partial N(c))$. This completes the proof.

The following is a presentation of the fundamental group of P. For two loops α and β, $\alpha\beta$ denotes the loop β followed by α.

Lemma 4.3. *Let $p_0 = (0, 1/8, 1/8) \in T_b \cap T_c$ be a base point of $\pi_1(P)$. Let α, β, γ, μ and ν be the homotopy classes of loops based at p_0 in Fig.* 4.5. *Then we have a presentation of $\pi_1(P)$ as follows*:

(1) *$\{\alpha, \beta, \gamma, \mu, \nu\}$ is a set of generators.*

(2) *the fundamental relations are as follows*:

(I) $\beta^{-1}\alpha^{-1}\beta\alpha = \nu$,

(II) $\mu\beta = \beta\mu$,

(III) $\gamma^{-1}\alpha^{-1}\gamma\alpha = \mu$,

(IV) $\gamma^{-1}\alpha\beta\gamma = \beta$.

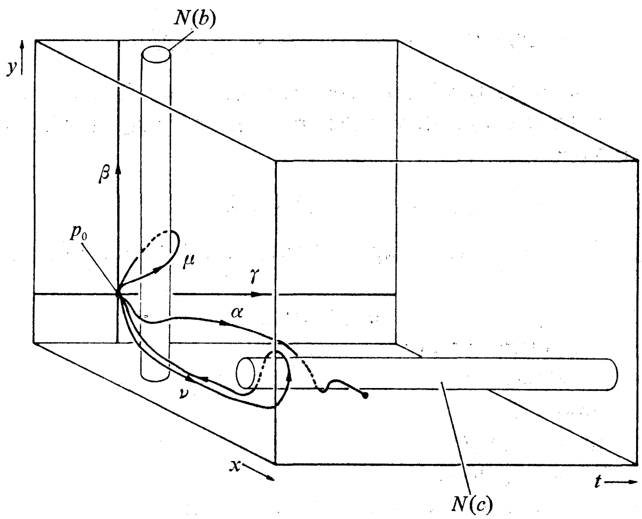


Fig. 4.5.

Proof. Let P' be the compact 3-manifold with corner obtained by cutting P along $T = \{(x, y, 0) \in P\}$, and let $T_0 = \{(x, y, 0) \in P'\}$ and $T_1 = \{(x, y, 1) \in P'\}$. T is diffeomorphic to $T^2 - \operatorname{int} D^2$. By moving p_0 along a path to $p_0' = (0, 0, 0)$ and applying the HNN construction (see [6]) repeatedly, we have the following presentation of $\pi_1(P)$. Let α, β, μ and ν be the homotopy classes of loops in P' described in Fig. 4.5. Then,

(1) $\{\alpha, \beta, \mu, \nu\}$ is a set of generators.
(2) the fundamental relations are;
(I) $\beta^{-1}\alpha^{-1}\beta\alpha=\nu$,
(II) $\mu\beta=\beta\mu$.

We see that the generators of $\pi_1(T_0)$ are α and β, and that the generators of $\pi_1(T_1)$ are the homotopy classes $\alpha'=[a']$ and $\beta'=[b']$, where $a', b': S^1\to T_1$ are defined by

$$a'(s)=(s, 0, 1),$$
$$b'(s)=(0, s, 1), \qquad s\in S^1=\boldsymbol{R}/\boldsymbol{Z}.$$

Let c: $[0, 1]\to P'$ be a path defined by

$$c(s)=(0, 0, s), \qquad s\in[0, 1].$$

Then we have

$$c_{\#}\alpha'=\alpha\mu^{-1}, \quad \text{and} \quad c_{\#}\beta'=\beta,$$

where $c_{\#}$ denotes the isomorphism $c_{\#}$: $\pi_1(P', p_0'')\to\pi_1(P', p_0')$, $p_0''=c(1)=(0, 0, 1)$, defined as follows:

$$c_{\#}[a]=[c^{-1}\circ a\circ c], \qquad [a]\in\pi_1(P', p_0''),$$

where $c^{-1}\circ a\circ c$ denotes the loop based at p_0' obtained by the conjugation of a loop a by the path c. Since the monodromy map is A_1': $T^2\to T^2$ of Section 2, we have

$$f_{\#}\alpha'=\alpha, \qquad f_{\#}\beta'=\alpha\beta,$$

where f: $T_1\to T_0$ is the gluing map. Then by the HNN construction, we have;

(1) $\{\alpha, \beta, \mu, \nu\}\cup\{\gamma\}$ is a set of generators.
(2) the fundamental relations are;
(I) $\beta^{-1}\alpha^{-1}\beta\alpha=\nu$,
(II) $\mu\beta=\beta\mu$,
(III)′ $f_{\#}\alpha'=\gamma(c_{\#}\alpha')\gamma^{-1}$,
(IV)′ $f_{\#}\beta'=\gamma(c_{\#}\beta')\gamma^{-1}$.
(III)′ is rewritten as $\alpha=\gamma\alpha\mu^{-1}\gamma^{-1}$, and then
(III) $\gamma^{-1}\alpha^{-1}\gamma\alpha=\mu$.
(IV)′ is rewritten as $\alpha\beta=\gamma\beta\gamma^{-1}$, and then
(IV) $\gamma^{-1}\alpha\beta\gamma=\beta$.

This completes the proof.

Lemma 4.4. *For every positive number ε, the following homotopy classes in $\pi_1(P-E(\phi))=\pi_1(P)$ can be represented by ε-closed orbits, where m and n are arbitrary integers*;

(0) α,
(1) $\alpha(\gamma\mu)^m\beta^n$,
(2) $\alpha(\gamma\mu)^m\mu^{-1}\beta^n$,
(3) $\alpha\nu^{-1}(\gamma\mu)^m\beta^n$,
(4) $\alpha\nu^{-1}(\gamma\mu)^m\mu^{-1}\beta^n$.

Proof. (I) First we prove (0). Let

$$\sigma=\{(x, 1/8, 1/8)\in P;\ 1/4\leq x\leq 3/4\}$$

be a segment with its end points $p_1=(1/4, 1/8, 1/8)$ and $p_2=(3/4, 1/8, 1/8)$. By the construction of $\mathscr{F}_1$ and $\mathscr{G}_1$ in Section 2, we have $\sigma\subset T_b\cap T_c$. We construct an ε-closed orbit representing α by joining the following five ε-orbit segments.

(i) Choose a point

$$p_{b,\varepsilon}=(1/32, 1/8, t_{b,\varepsilon}),\quad 0<(1/8)-t_{b,\varepsilon}<(1/2)\cdot(1/32)\cdot\varepsilon,$$

so that the linear segment $p_0p_{b,\varepsilon}$ is an $\varepsilon/2$-orbit segment. Let L be the leaf of $\mathscr{F}_1$ containing the point $p_{b,\varepsilon}$. $L=L_0-U$, where L_0 is ∞-punctured $S^1\times\boldsymbol{R}$ and U is an open disk in L_0 (Fig. 4.3). Let $\lambda_{0,\varepsilon}$ be an ε-orbit segment joining p_0 to $p_{b,\varepsilon}$ obtained by smoothing $p_0p_{b,\varepsilon}$ near p_0 and $p_{b,\varepsilon}$ so that $\lambda_{0,\varepsilon}$ is contained in the slice $\{(x, 1/8, t)\in P\}$ and is tangent to T_b (resp. L) at p_0 (resp. $p_{b,\varepsilon}$) (Fig. 4.6 (i)).

(ii) Let $\lambda_{b,\varepsilon}$ be the orbit segment of ϕ on L which starts from $p_{b,\varepsilon}$ and hits the slice $\{(7/32, y, t)\in P\}$, and let $q_{b,\varepsilon}=(7/32, 1/8, t'_{b,\varepsilon})$ be the other end point of $\lambda_{b,\varepsilon}$ (Fig. 4.6 (ii)). By the symmetry of the construction in Section 2, we have $t'_{b,\varepsilon}=t_{b,\varepsilon}$.

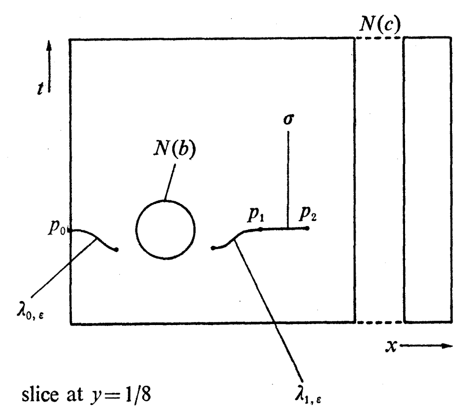


Fig. 4.6 (i).

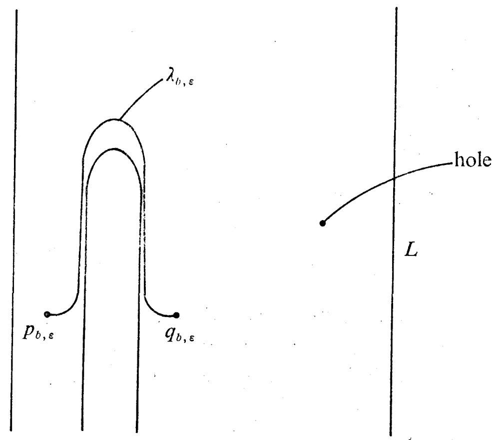


Fig. 4.6 (ii).

(iii) By (i) and (ii), we can assume the linear segment $q_{b,\varepsilon}p_1$ is also an $\varepsilon/2$-orbit segment. Let $\lambda_{1,\varepsilon}$ be an ε-orbit segment joining $q_{b,\varepsilon}$ to p_1 similar to $\lambda_{0,\varepsilon}$ so that $\lambda_{1,\varepsilon}$ is contained in the slice $\{(x, 1/8, t) \in P\}$ and is tangent to L (resp. T_b) at $q_{b,\varepsilon}$ (resp p_1) (Fig. 4.6 (i)).

(iv) We join p_1 to p_2 by σ.

(v) We construct an ε-orbit segment joining p_2 to p_0 by using $\mathscr{G}_1$ near $\partial N(c)$ instead of $\mathscr{F}_1$. Choose points $p_{c,\varepsilon}=(25/32, y_{c,\varepsilon}, 1/8)$, and $q_{c,\varepsilon}=(31/32, y_{c,\varepsilon}, 1/8)$, $0<(1/8)-y_{c,\varepsilon}<(1/2)\cdot(1/32)\cdot\varepsilon$. Let $\lambda_{2,\varepsilon}$ and $\lambda_{3,\varepsilon}$ be ε-orbit segments obtained similarly to (i) so that $\lambda_{2,\varepsilon}$ (resp. $\lambda_{3,\varepsilon}$) is contained in the slice $\{(x, y, 1/8) \in P\}$ and is tangent to the leaves of $\mathscr{G}_1$ at its end points. Let $\lambda_{c,\varepsilon}$ be the orbit segment joining $p_{c,\varepsilon}$ to $q_{c,\varepsilon}$ similar to (ii).

By joining the ε-orbit segments $\lambda_{0,\varepsilon}$, $\lambda_{b,\varepsilon}$, $\lambda_{1,\varepsilon}$, σ, $\lambda_{2,\varepsilon}$, $\lambda_{c,\varepsilon}$ and $\lambda_{3,\varepsilon}$ in that order, we have an ε-closed orbit representing the homotopy class α.

(II) Next we construct an ε-closed orbit representing $\alpha\beta^n$ for arbitrary $n \in \boldsymbol{Z}$. Fix n and ε. Let $p_b=(1/32, 1/8, 1/8)$ be a point on T_b and let $\beta'\colon S^1\to T_b$ be a loop on T_b defined by

$$\beta'(s)=(1/32, s+(1/8), 1/8) \in P, \quad s \in S^1.$$

Let $J=\{(1/32, 1/8, t) \in P; 3/32\leq t\leq 1/8\}$ be an arc one of whose end point is p_b. By the construction of $\mathscr{F}_1$ in Section 2, we see that the leaf $T_b \in \mathscr{F}_1$ has holonomy along β' and that the holonomy map $f_{\beta'}\colon (f_{\beta'})^{-1}(J)\to J$ associated with β' is an expanding diffeomorphism:

$$f_{\beta'}(1/32, 1/8, t)=(1/32, 1/8, t'),$$
$$t'<t \text{ for } (1/32, 1/8, t) \in (f_{\beta'})^{-1}(J)-\{p_b\}, \quad \text{and}$$
$$f_{\beta'}(p_b)=p_b.$$

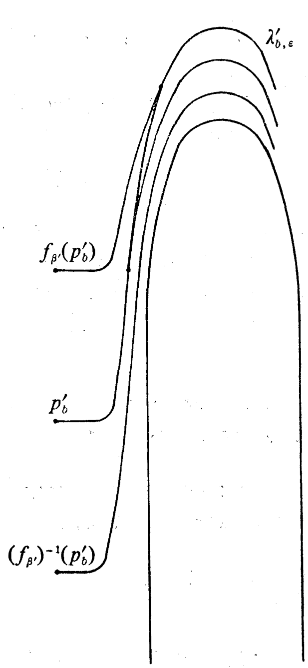


Fig. 4.7.

(i) If we choose a point $p'_b \in J-\{p_b\}$ near enough to p_b, then the point $(f_{\beta'})^k(p'_b)$, for every $k \in Z$ satisfying $|k| \leq |n|$, is contained in J.

(ii) Let L be the leaf of $\mathscr{F}_1$ containing $p'_b \in J$. If we choose $p'_b = (1/32, 1/8, t_b) \in J-\{p_b\}$ near enough to p_b, then a point of $o(p'_b)$ and a point of $o((f_{\beta'})^n(p'_b))$ is joined by an ε-orbit segment on L, where $o(p)$ denotes the orbit of ϕ through p (Fig. 4.7).

(iii) Let $\lambda'_{b,\varepsilon}$ be an ε-orbit segment on L constructed as above so that $\lambda'_{b,\varepsilon}$ starts at p'_b, moves on $o((f_{\beta'})^n(p'_b))$ and hits the slice $\{(7/32, y, t) \in P\}$ on $o((f_{\beta'})^n(p'_b))$. Let $q'_{b,\varepsilon} = (7/32, 1/8, t'_{b,\varepsilon})$ be the end point of $\lambda'_{b,\varepsilon}$ different from p'_b. If $n \neq 0$, then $t'_{b,\varepsilon} \neq t_b$. But if we choose t_b near enough to 1/8, we have $t'_{b,\varepsilon}$ arbitrary near to 1/8. Thus, if p'_b is near enough to p_b, then there exists an ε-orbit segment $\lambda_{1,\varepsilon}$ in (iii) of (I) joining $q'_{b,\varepsilon}$ to p_1.

(iv) Let $p_{b,\varepsilon} = (1/32, 1/8, t_{b,\varepsilon})$, $t_{b,\varepsilon} < 1/8$, be a point of J near enough to p_b satisfying the following conditions:

(a) there exists an ε-orbit segment $\lambda_{0,\varepsilon}$ in (i) of (I) joining p_0 to $p_{b,\varepsilon}$,

(b) $p_{b,\varepsilon}$ satisfies the conditions of p'_b in (i), (ii) and (iii) of (II).

Let σ, $\lambda_{c,\varepsilon}$ be orbit segments and let $\lambda_{2,\varepsilon}$, $\lambda_{3,\varepsilon}$ be ε-orbit segments in (I). By joining the ε-orbit segments $\lambda_{0,\varepsilon}$, $\lambda'_{b,\varepsilon}$, $\lambda_{1,\varepsilon}$, σ, $\lambda_{2,\varepsilon}$, $\lambda_{c,\varepsilon}$ and $\lambda_{3,\varepsilon}$ in that order, we have an ε-closed orbit representing $\alpha\beta^n$.

(III) By changing an orbit segment $\lambda_{c,\varepsilon}$ for an ε-orbit segment $\lambda'_{c,\varepsilon}$

similar to $\lambda'_{b,\varepsilon}$ in (iii) of (II), we have an ε-closed orbit representing $\alpha(\gamma\mu)^m\beta^n$ for an arbitrary pair $(m, n) \in Z^2$.

(IV) In order to represent $\alpha\mu^{-1}$, we choose $\bar{p}_{b,\varepsilon}=(1/32, 1/8, \bar{t}_{b,\varepsilon})$, $0<\bar{t}_{b,\varepsilon}-(1/8)<(1/2)\cdot(1/32)\cdot\varepsilon$ instead of $p_{b,\varepsilon}$. Then we construct an ε-orbit segment on the opposite side of T_b to (I), and we have an ε-closed orbit representing $\alpha\mu^{-1}$.

By a construction similar to (II) and (III), we have an ε-closed orbit representing $\alpha(\gamma\mu)^m\mu^{-1}\beta^n$. If we construct an ε-orbit segment on the opposite side of T_c to (I), then we have $\alpha\nu^{-1}$ and then $\alpha\nu^{-1}(\gamma\mu)^m\beta^n$. Lastly composing these constructions, we have $\alpha\nu^{-1}(\gamma\mu)^m\mu^{-1}\beta^n$. This completes the proof.

Note that a homotopy class represented by composing those in Lemma 4.4 is also represented by an ε-closed orbit.

Proof of Lemma 4.1. By the relations of Lemm 4.3, the classes of (1) and (4) of Lemma 4.4 are rewritten as follows:

$$\begin{aligned}
(1)\quad \alpha(\gamma\mu)^m\beta^n &= \alpha(\alpha^{-1}\gamma\alpha)^m\beta^n && \text{by (III)}\\
&= \gamma^m\alpha\beta^n.
\end{aligned}$$

$$\begin{aligned}
(4)\quad \alpha\nu^{-1}(\gamma\mu)^m\mu^{-1}\beta^n &= \beta^{-1}\alpha\beta(\gamma\mu)^m\mu^{-1}\beta^n && \text{by (I)}\\
&= \beta^{-1}\alpha\beta(\gamma\mu)^{m-1}\gamma\beta^n\\
&= \beta^{-1}\alpha\beta\alpha^{-1}\gamma^{m-1}\alpha\gamma\beta^n && \text{by (III).}
\end{aligned}$$

We set $m=1, n=-2$ for (4). Then $\beta^{-1}\alpha\beta\gamma\beta^{-2}$ is represented by an ε-closed orbit. Set $m=-1, n=2$ for (1). Then $\gamma^{-1}\alpha\beta^2$ is represented by an ε-closed orbit. Hence the composition

$$\begin{aligned}
\gamma^{-1}\alpha\beta^2\cdot\beta^{-1}\alpha\beta\gamma\beta^{-2} &= \gamma^{-1}\alpha\beta\alpha\beta\gamma\beta^{-2}\\
&= (\gamma^{-1}\alpha\beta\gamma)^2\beta^{-2}\\
&= \beta^2\cdot\beta^{-2} && \text{by (IV)}\\
&= 1
\end{aligned}$$

is represented by an ε-closed orbit. This completes the proof.

Theorem 4.5. *$(\mathscr{F}_1, \mathscr{G}_1)$ cannot be raised to a total foliation.*

Proof. This follows from Lemma 4.1 and Theorem 3.3.

§ 5. Proof of the main theorem

In this section we prove Theorem 1.1. For this purpose we need some constructions used in [5].

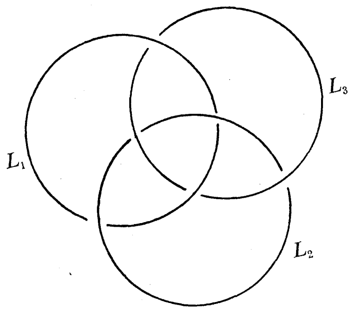


Fig. 5.1.

Let $\mathscr{L}=L_1\cup L_2\cup L_3$ be the Borromean rings in S^3 described in Fig. 5.1, where L_1, L_2 and L_3 denote the connected components of $\mathscr{L}$. Let $N(\mathscr{L})=N(L_1)\cup N(L_2)\cup N(L_3)$ denote a closed tubular neighborhood of $\mathscr{L}$. Let μ_i (resp. λ_i) be a simple closed curve on $\partial N(L_i)$ which represents the *meridian* (resp. the *longitude*) for $i=1, 2, 3$, that is,

$$\iota_{i*}[\mu_i]=0 \qquad \text{in } H_1(N(L_i); \boldsymbol{Z}),$$
$$\kappa_{i*}[\lambda_i]=0 \qquad \text{in } H_1(S^3-\operatorname{int} N(L_i); \boldsymbol{Z}),$$

where $[\mu_i]$ (resp. $[\lambda_i]$) $\in H_1(\partial N(L_i); \boldsymbol{Z})$ denotes the homology class of μ_i (resp. λ_i), and ι_{i*} (resp. κ_{i*}) denotes the induced homomorphism of the natural inclusion

$$\iota_i\colon \partial N(L_i)\to N(L_i),$$
$$\kappa_i\colon \partial N(L_i)\to S^3-\operatorname{int} N(L_i).$$

Let P_1 and $P=P_1-(\operatorname{int} N(b)\cup \operatorname{int} N(c))$ be the 3-manifolds constructed in Section 2 for $k=1$ and let

$$U=\{(x, y, t)\in P_1; (y-(1/2))^2+t^2\leq (1/4)^2\}.$$

Then we see that $P-\operatorname{int} U=P_1-(\operatorname{int} U\cup \operatorname{int} N(b)\cup \operatorname{int} N(c))$ is diffeomorphic to $T^3-(\operatorname{int} N(u)\cup \operatorname{int} N(v)\cup \operatorname{int} N(w))$, where u, v and w are mutually disjoint circles parallel to three coordinate axes of the 3-dimensional torus T^3. Let $\mu_U\colon S^1\to\partial U$ and $\lambda_U\colon S^1\to\partial U$ be oriented simple closed curves defined by

$$\mu_U(\theta)=(0, (1/4)\cdot\sin 2\pi\theta+(1/2), (1/4)\cdot\cos 2\pi\theta),$$
$$\lambda_U(\theta)=((1/4)\cdot\theta, 0, 0), \quad \theta\in S^1=\boldsymbol{R}/\boldsymbol{Z}.$$

Let $[\mu]$ and $[\beta]$ (resp. $[\nu]$ and $[\gamma\mu]$) be the homology classes of $H_1(\partial N(b); \boldsymbol{Z})$ (resp. $H_1(\partial N(c); \boldsymbol{Z})$) defined by the homotopy classes μ and β (resp. ν and

$\gamma\mu$) in Lemm 4.3. Then the following proposition holds. For a proof, see [5].

Proposition 5.1 ([5, Proposition of Part Three of Chapter 5]). *There exists a diffeomorphism*

$$h\colon P-\operatorname{int} U \to S^3-(\operatorname{int} N(L_1)\cup \operatorname{int} N(L_2)\cup \operatorname{int} N(L_3))$$

satisfying the following conditions:

(a) $h(\partial U)=\partial N(L_1)$, $h(\partial N(b))=\partial N(L_2)$ *and* $h(\partial N(c))=\partial N(L_3)$,

(b) *if we choose suitable orientations of* μ_i *and* λ_i *for* $i=1, 2, 3$, *then the following* (i) *and* (ii) *hold*:

(i) $(h|_{\partial U})_*([\mu_U])=[\lambda_1]$, $\quad (h|_{\partial U})_*(-[\lambda_U])=[\mu_1]$,
$(h|_{\partial N(b)})_*([\mu])=[\lambda_2]$, $\quad (h|_{\partial N(b)})_*(-[\beta])=[\mu_2]$,
$(h|_{\partial N(c)})_*([\nu])=[\lambda_3]$, $\quad (h|_{\partial N(c)})_*(-[\gamma\mu])=[\mu_3]$,

(ii) *the linking numbers*

$$lk(\lambda_i, \mu_i)=+1, \qquad \textit{for } i=1, 2, 3,$$

with the right hand rule.

The following lemma is also contained in [5].

Lemma 5.2. (1) *The closed* 3-*manifold obtained by the Dehn surgery with the coefficient* $+1$ *along each connected component of the Borromean rings* $\mathscr{L}=L_1\cup L_2\cup L_3$ *is diffeomorphic to the Poincaré homology* 3-*sphere* Q^3:

$$\left(S^3-\bigcup_{i=1}^{3}\operatorname{int} N(L_i)\right)\bigcup_{\partial}\bigcup_{i=1}^{3} S_i^1\times D_i^2=Q^3,$$

where $[\{*\}\times\partial D_i^2]=[\lambda_i]+[\mu_i]$ *in* $H_1(\partial N(L_i); \boldsymbol{Z})$.

(2) *Let* $l_i=S_i^1\times\{0\}$ *be a core circle of* $S_i^1\times D_i^2$ $(i=1, 2, 3)$, *and let* $p\colon S^3\to Q^3$ *be the universal covering of* Q^3 *which is of* 120 *sheets. Then* $p^{-1}(l_i)$ $(i=1, 2, 3)$ *is a union of* 12 *fibers of the Hopf fibration of* S^3:

$$p^{-1}(l_i)=\bigcup_{j=1}^{12}\tilde{l}_{ij} \qquad \textit{for } i=1, 2, 3,$$

$\tilde{l}_{ij}$ *is a trivial knot, and* $lk(\tilde{l}_{ij}, \tilde{l}_{ik})=+1$ $(j\neq k)$.

(3) *Let* $\mu_i\subset\partial N(L_i)$ *be the meridian curve of* L_i *chosen in* (b) *of Proposition* 5.1 $(i=1, 2, 3)$. *Then* $p^{-1}(\mu_i)$ *is also a union of* 12 *fibers of the fibration, and each connected component* $\tilde{\mu}_{ij}$ $(j=1, \cdots, 12)$ *together with* $\tilde{l}_{ij'}$ $(j'=1, \cdots, 12)$ *forms a Hopf link such that*

$$lk(\tilde{\mu}_{ij}, \tilde{l}_{ij'})=\pm 1.$$

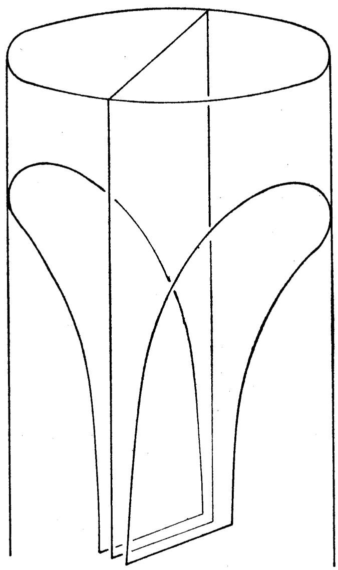


Fig. 5.2 (i).

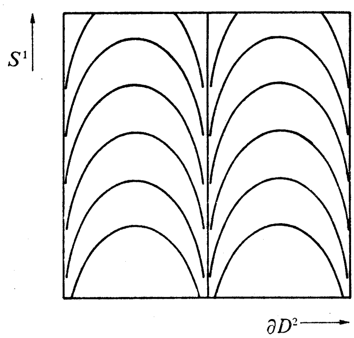


Fig. 5.2 (ii).

Theorem 5.3 ([5, Main Theorem]). *Let M be a closed orientable 3-manifold. Then M has a total foliation $(\mathscr{F}, \mathscr{G}, \mathscr{H})$ satisfying the following conditions*:

(1) *$\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$ are transversely orientable and C^∞.*

(2) *There exists a compact codimension* 0 *submanifold R of M such that*

(a) *R is diffeomorphic to $S^1 \times D^2$,*

(b) *$\mathscr{F}|_R$ is a Reeb component, and*

(c) *$\mathscr{G}|_R$ (resp. $\mathscr{H}|_R$) consists of two half-Reeb components* (Fig. 5.2 (i)) (*for the definition see* [13]) *and $\mathscr{G}|_{\partial R}$ (resp. $\mathscr{H}|_{\partial R}$) is a foliation described* *in Fig.* 5.2 (ii), *where the orientation of* S^1 *is suitably chosen.*

Though the condition (2) is not stated in the main theorem of [5], we can construct such a total foliation by using a method in [5]. We omit a proof of Theorem 5.3.

Theorem 5.4. *There exist* C^∞ *codimension one transversely orientable foliations* $\mathcal{F}'$ *and* $\mathcal{G}'$ *of* $S^1 \times D^2$ *satisfying the following conditions*:

(a) $(\mathcal{F}', \mathcal{G}')$ *is a transverse pair,*

(b) $\mathcal{F}'$ *is tangent to* $\partial(S^1 \times D^2)$,

(c) $\mathcal{G}'|_{\partial(S^1 \times D^2)}$ *is a foliation as in* Fig. 5.2. (ii), *and*

(d) $(\mathcal{F}', \mathcal{G}')$ *cannot be raised to a total foliation.*

Proof. Let $\mathcal{F}_1$ and $\mathcal{G}_1$ be codimension one foliations of P with $k=1$ constructed in Section 2. First we extend $\mathcal{F}_1$ and $\mathcal{G}_1$ to the Poincaré homology 3-sphere Q^3. We easily verify the simple closed curve

$$C = \{(\theta, (1/4)\cdot\cos 2\pi\theta + (1/2), (1/4)\cdot\sin 2\pi\theta);\ \theta \in \boldsymbol{R}/\boldsymbol{Z}\}$$

on ∂U is homotopic to zero on U. By the definition of the oriented closed curves $\lambda_U, \mu_U : S^1 \to \partial U$ in the beginning of this section, we have

$$[C] = [\lambda_U] - [\mu_U] \qquad \text{in } H_1(\partial U; \boldsymbol{Z}).$$

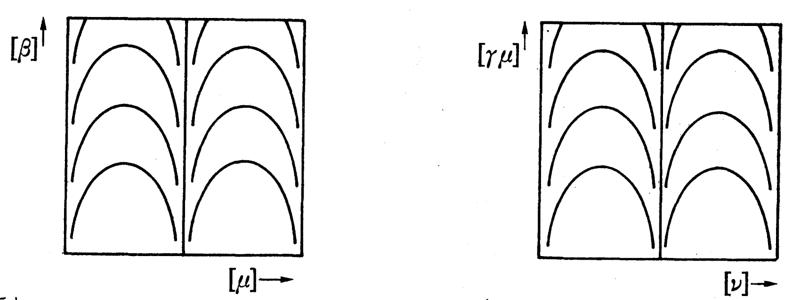
From Proposition 5.1, we have

$$h_*[C] = -[\mu_1] - [\lambda_1] \qquad \text{in } H_1(\partial N(L_1); \boldsymbol{Z}).$$

Hence P is diffeomorphic to the manifold

$$(S^3 - (\operatorname{int} N(L_1) \cup \operatorname{int} N(L_2) \cup \operatorname{int} N(L_3)) \bigcup_{\partial N(L_1)} S^1 \times D^2$$

obtained by performing the $+1$ Dehn surgery along L_1 and then deleting $\operatorname{int} N(L_2)$ and $\operatorname{int} N(L_3)$. In particular we can consider $P \subset Q^3$. By the construction in Section 2, we see $\mathcal{F}_1|_{\partial N(b)}$ and $\mathcal{G}_1|_{\partial N(c)}$ are as in Fig. 5.3.



$\mathcal{F}_1|_{\partial N(b)}$ $\qquad$ $\mathcal{G}_1|_{\partial N(c)}$

Fig. 5.3.

From Proposition 5.1, we see $\mathscr{F}_1|_{\partial N(L_2)}$ and $\mathscr{G}_1|_{\partial N(L_3)}$ are as in Fig. 5.4.

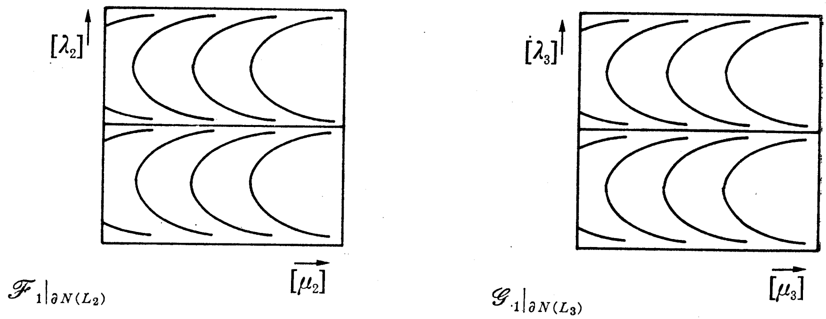


Fig. 5.4.

Let l_i $(i=2, 3)$ be the core circles in Q^3 defined in (2) of Lemma 5.2. Since $[\lambda_i]+[\mu_i]=0$ in $H_1(N(l_i); Z)$, by choosing on $\partial N(l_i)$ a *meridian* which is homotopic to zero in $N(l_i)$, and a *longitude* which is homotopy equivalent to $N(l_i)$ suitably, we see $\mathscr{F}_1|_{\partial N(l_2)}$ and $\mathscr{G}_1|_{\partial N(l_3)}$ are as in Fig. 5.5.

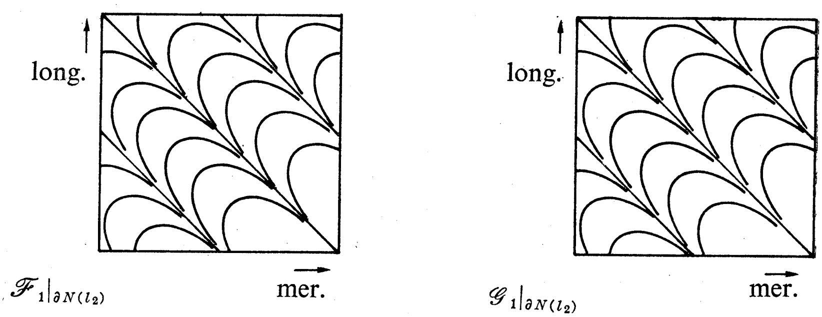


Fig. 5.5.

Let $\mathscr{F}_R$ and $\mathscr{G}_R$ be codimension one foliations of $S^1\times D^2$ such that

(α) $\mathscr{G}_R$ is transverse to $\mathscr{F}_R$,

(β) $\mathscr{F}_R$ is a Reeb component, and

(γ) $\mathscr{G}_R$ consists of two half-Reeb components (Fig. 5.2 (i)).

Since both of foliations on the boundaries are coincident, we can glue $\mathscr{F}_R$ (resp. $\mathscr{G}_R$) to $\mathscr{G}_1$ (resp. $\mathscr{F}_1$) along $\partial N(l_2)$ and also glue $\mathscr{F}_R$ (resp. $\mathscr{G}_R$) to $\mathscr{F}_1$ (resp. $\mathscr{G}_1$) along $\partial N(l_3)$, and then we obtain codimension one foliations $\mathscr{F}_2$ and $\mathscr{G}_2$ of Q^3 satisfying that $\mathscr{G}_2$ is transverse to $\mathscr{F}_2$, $\mathscr{F}_2|_P=\mathscr{F}_1$ and $\mathscr{G}_2|_P=\mathscr{G}_1$. Then by Theorem 4.6, we see that $(\mathscr{F}_2, \mathscr{G}_2)$ cannot be raised to a total foliation. Let $\mathscr{F}_3=p^*\mathscr{F}_2$ and $\mathscr{G}_3=p^*\mathscr{G}_2$ be the foliations induced by $p\colon S^3\to Q^3$. Let $\phi=\mathscr{F}_1\pitchfork\mathscr{G}_1$ be the one dimensional foliation in Section 4. Let $\tilde{P}$ denote $p^{-1}(P)$ and let $\tilde{\phi}$ (resp. $\tilde{g}$) be $p^*\phi$ (resp. p^*g). Since p is a finite covering, $E(\tilde{\phi})=p^{-1}(E(\phi))$. Since $(P-E(\phi), \phi|_{P-E(\phi)}, g|_{P-E(\phi)})$ has 0-N.C.O.P. by Lemma 4.1, we see by Proposition 3.5 that $(\tilde{P}-E(\tilde{\phi}), \tilde{\phi}|_{\tilde{P}-E(\tilde{\phi})}, \tilde{g}|_{\tilde{P}-E(\tilde{\phi})})$ has 0-N.C.O.P.. Hence by Theorem 3.3 and the fact that $\tilde{P}\subset S^3$, we see that $(\mathscr{F}_3, \mathscr{G}_3)$ cannot be raised to a total foliation. From (2) of Lemma 5.2, $p^{-1}(N(l_3))$ is a tubular neighborhood of a union of 12 fibers of the Hopf fibration of S^3. Let $\tilde{N}(l_3)$ be one of 12 components of $p^{-1}(N(l_3))$. Then since $\tilde{N}(l_3)$ is unknotted, $S^3-\mathrm{int}\,\tilde{N}(l_3)$ is diffeomorphic to $S^1\times D^2$. Consider $\mathscr{F}'=\mathscr{F}_3|_{S^3-\mathrm{int}\,\tilde{N}(l_3)}$ and $\mathscr{G}'=\mathscr{G}_3|_{S^3-\mathrm{int}\,\tilde{N}(l_3)}$. Since $\partial N(c)$ is a compact leaf of $\mathscr{F}_1$ of P, $\mathscr{F}'$ is tangent to $\partial(S^3-\mathrm{int}\,\tilde{N}(l_3))$, so the condition (b) is verified. The condition (a) is obvious. Since $S^3-\mathrm{int}\,\tilde{N}(l_3)$ contains $\tilde{P}$, the condition (d) is verified. Now verify the condition (c).

Note that the homology class represented by a compact leaf of $\mathscr{G}_1|_{\partial N(c)}$ oriented as in Section 4 is $[\gamma\mu]\in H_1(\partial N(c); \boldsymbol{Z})$. Hence by Proposition 5.1, this is equal to $-[\mu_3]$ in $H_1(\partial N(L_3); \boldsymbol{Z})$. Hence by (3) of Lemma 5.2, a compact leaf of $\mathscr{G}'|_{\partial\tilde{N}(l_3)}$ and the core circle of $\tilde{N}(l_3)$ form a Hopf link. This shows that a compact leaf of $\mathscr{G}'|_{\partial\tilde{N}(l_3)}$ and a core circle of the solid torus $S^3-\mathrm{int}\,\tilde{N}(l_3)$ also form a Hopf link. This shows that if we choose a longitude of $\partial(S^3-\mathrm{int}\,\tilde{N}(l_3))$ suitably, then $\mathscr{G}'|_{\partial(S^3-\mathrm{int}\,\tilde{N}(l_3))}$ is as in Fig. 5.2 (ii), and we have (c). This completes the proof.

Proof of Theorem 1.1. Let M be a closed orientable 3-manifold. By Theorem 5.3, we have a total foliation $(\mathscr{F}, \mathscr{G}, \mathscr{H})$ of M satisfying (1) and (2). Let $\mathscr{F}'$, $\mathscr{G}'$ be codimension one foliations of $S^1\times D^2$ of Theorem 5.4. Since $\mathscr{F}$ (resp. $\mathscr{F}'$) is tagnent to ∂R (resp. $\partial(S^1\times D^2)$) and $\mathscr{G}|_{\partial R}$ coincides with $\mathscr{G}'|_{\partial(S^1\times D^2)}$, we can replace $\mathscr{F}|_R$ (resp. $\mathscr{G}|_R$) by $\mathscr{F}'$(resp. $\mathscr{G}'$), and then we have the desired foliations.

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