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Regular Holonomic Systems and their Minimal Extensions I

Masatoshi Noumi

This note, together with J. Sekiguchi [13], is intended to be an introduction to Professor Kashiwara's lectures at RIMS in 1981. At that time, he lectured on three topics as follows:

(i) *Gabber's theorem on the involutiveness of the characteristic variety of a coherent $\mathscr{D}_X$-Module.*

(ii) *Some fundamental results on regular holonomic systems (holonomic systems with regular singularities).*

(iii) *An application of regular holonomic systems to the representation theory of a semisimple Lie algebra.*

Based on his lectures, we will make here a survey of (i) and (ii) above. As to (iii), the reader is referred to J. Sekiguchi [13].

Throughout this note, X stands for a complex manifold. We denote by T^*X the cotangent bundle of X with canonical projection $\pi: T^*X \to X$. If Y is a submanifold of X, the conormal bundle of Y in X will be denoted by $T^*_Y X$. We also use the notations $\mathring{T}^*X = T^*X \setminus T^*_X X$ and $\mathring{\pi} = \pi|_{\mathring{T}^*X}$. As usual, we denote by $\mathscr{D}_X$ the Ring over X of linear differential operators of finite order and by $\mathscr{E}_X$ the Ring over T^*X of microdifferential operators of finite order, respectively. In Section 2 and Section 3, we will freely use the terminology of derived categories. For a Ring $\mathscr{A}$ on X, we denote by $\mathrm{D}(\mathscr{A})$ the derived category of the category of (left) $\mathscr{A}$-Modules.

§ 1. Regular holonomic systems

Let Ω be an open subset of $\mathring{T}^*X = T^*X \setminus T^*_X X$ and V a conic involutive closed analytic subset of Ω. Then we define $\mathscr{J}_V$ to be the sub-Module of $\mathscr{E}_X(1)|_\Omega$ consisting of all microdifferential operators P whose symbols $\sigma_1(P)$ vanish on V. We denote by $\mathscr{A}_V = \bigcup_{k \geqslant 1} \mathscr{J}_V^k$ the sub-Algebra of $\mathscr{E}_X|_\Omega$ generated by $\mathscr{J}_V$. Note that $\mathscr{J}_V$ is a bilaterally coherent $\mathscr{E}_X(0)|_\Omega$-Module.

Proposition 1.1. *For a coherent $\mathscr{E}_X|_\Omega$-Module M, the following conditions are equivalent:*

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i) *For any point p of Ω, there exist an open neighborhood U of p and a coherent $\mathscr{E}_X(0)|_U$-sub-Module L of $M|_U$ such that $\mathscr{E}_X L = M|_U$ and $\mathscr{J}_V L = L$.*

ii) *For any open subset U of Ω and for any coherent $\mathscr{E}_X(0)|_U$-sub-Module N of $M|_U$, the sub-Module $\mathscr{A}_V N$ of $M|_U$ is coherent over $\mathscr{E}_X(0)|_U$.*

Definition 1.2. A coherent $\mathscr{E}_X|_\Omega$-Module M is said to *have R.S.* (*regular singularities*) *along V in Ω* if it satisfies the equivalent conditions of Proposition 1.1.

Let M be a coherent $\mathscr{E}_X|_\Omega$-Module. Then we denote by $\mathrm{IR}(M, V)$ the set of all points p of Ω such that M does *not* have R.S. along V in any neighborhood of p. Recall that, for any point p in Ω, there are an open neighborhood U of p and a coherent $\mathscr{E}_X(0)|_U$-sub-Module N of $M|_U$ such that $\mathscr{E}_X N = M|_U$. Then, by Proposition 1.1, one can show that

$$\mathrm{IR}(M, V) \cap U = \bigcap_{k \geqslant 0} \mathrm{Supp}(\mathscr{J}_V^{k+1} N / \mathscr{J}_V^k N)$$

($\mathscr{J}_V^0 = \mathscr{E}_X(0)|_\Omega$). Note that $(\mathrm{Supp}(\mathscr{J}_V^{k+1} N / \mathscr{J}_V^k N))_{k \geqslant 0}$ defines a decreasing sequence of conic closed analytic subsets of U, hence is locally stationary. From this it follows that $\mathrm{IR}(M, V)$ is a conic closed analytic subset of Ω and that, for any point p of Ω, there are an open neighborhood U of p and a coherent $\mathscr{E}_X(0)|_U$-sub-Module L of $M|_U$ such that $\mathscr{E}_X L = M|_U$ and $\mathrm{IR}(M, V) \cap U = \mathrm{Supp}(\mathscr{J}_V L / L)$. Furthermore, we have

Proposition 1.3. *With the notations as above,* $\mathrm{IR}(M, V)$ *is a conic involutive closed analytic subset of Ω contained in* $\mathrm{Supp}(M)$.

In order to prove that $\mathrm{IR}(M, V)$ is involutive, we recall an unpublished result of O. Gabber concerning the extension of coherent $\mathscr{E}_X(0)|_\Omega$-sub-Modules of M.

Theorem 1.4 (O. Gabber). *Let M be a coherent $\mathscr{E}_X|_\Omega$-Module and L a coherent $\mathscr{E}_X(0)|_\Omega$-sub-Module of M. Let Z be a conic closed analytic subset of Ω and j the inclusion mapping $\Omega \backslash Z \hookrightarrow \Omega$. Assume that Z does not contain any non-empty conic involutive analytic subset. Then the sub-Module*

$$L' = j_* j^{-1}(L) \cap M = \{u \in M;\ u|_{\Omega\backslash Z} \in L|_{\Omega\backslash Z}\}$$

of M is coherent over $\mathscr{E}_X(0)|_\Omega$.

This theorem is reduced to a result in O. Gabber [3].

We will explain here how one can derive Proposition 1.3 from Theorem 1.4. Setting $Z = \mathrm{IR}(M, V)$, we prove by contradiction that Z is involutive. Suppose that Z is not involutive. Then, replacing Ω by an open subset of Ω, we are faced with the following situation:

1) $Z\neq\phi$ and M has a coherent $\mathscr{E}_X(0)|_\Omega$-sub-Module L such that $\mathscr{E}_X L=M$ and $Z=\mathrm{Supp}(\mathscr{J}_V L/L)$.

2) There are holomorphic functions f and g defined on Ω such that $f|_Z=g|_Z=0$ and $\{f,g\}(p)\neq 0$ for any p in Ω.

The condition 2) implies that Z cannot contain any non-empty involutive subset. Hence, by Theorem 1.4, we find that $L'=j_*j^{-1}(L)\cap M$ is coherent over $\mathscr{E}_X(0)|_\Omega$. On the other hand, from the condition 1), it follows that $\mathscr{E}_X L'=M$ and $\mathscr{J}_V L'=L'$. This shows that M has R.S. along V in Ω, hence $Z=\phi$, which contradicts the hypothesis that Z is not involutive.

Remark 1.5. In the case where $V=\phi$, one has $\mathrm{IR}(M, V)=\mathrm{Supp}(M)$. Proposition 1.3 thus implies that the support of a coherent $\mathscr{E}_X|_\Omega$-Module is an involutive analytic set.

From now on, our attention will be directed to holonomic systems. As to the regular singularity of a holonomic $\mathscr{E}_X|_\Omega$-Module, we have

Theorem 1.6. *Let M be a holonomic $\mathscr{E}_X|_\Omega$-Module with $\Lambda=\mathrm{Supp}(M)$. Then the following four conditions are equivalent:*

i) *M has R.S. along Λ.*

ii) *M has R.S. along any conic involutive analytic set V containing Λ.*

iii) *M has R.S. along some conic Lagrangean analytic set Λ' containing Λ.*

iv) *M has R.S. along Λ in an open neighborhood of a dense open subset of Λ.*

The crucial point of Theorem 1.6 lies in the implication iv)$\Rightarrow$i), which is readily a consequence of Proposition 1.3. In fact, any Lagrangean analytic set cannot contain a non-empty nowhere dense involutive analytic subset. Thus we arrive at

Definition 1.7. A holonomic $\mathscr{E}_X|_\Omega$-Module M is called a *regular holonomic $\mathscr{E}_X|_\Omega$-Module* if it has R.S. along $\mathrm{Supp}(M)$.

Remark 1.8. The above definition of a regular holonomic $\mathscr{E}_X|_\Omega$-Module is different from Definition 1.1.16 of M. Kashiwara and T. Kawai [9]. In fact, the condition iv) of Theorem 1.6 is adopted there to define a "holonomic $\mathscr{E}_X|_\Omega$-Module with R.S.". In the context of [9], the implication iv)$\Rightarrow$i) is proved as a corollary to Theorem 5.1.6 which asserts that any holonomic $\mathscr{E}_X|_\Omega$-Module M with R.S. has a *globally defined* coherent $\mathscr{E}_X(0)|_\Omega$-sub-Module L such that $\mathscr{E}_X L=M$ and $\mathscr{J}_V L=L$.

Definition 1.9. A holonomic $\mathscr{D}_X$-Module M is called a *regular* *holonomic $\mathscr{D}_X$-Module* if its microlocalization $\mathscr{E}_X \otimes_{\pi^{-1}(\mathscr{D}_X)} \pi^{-1}(M)|_{\mathring{T}^*X}$ is a regular holonomic $\mathscr{E}_X|_{\mathring{T}^*X}$-Module.

Now we try to paraphrase Definition 1.9 into an expression proper for $\mathscr{D}_X$-Modules. Let M be a coherent $\mathscr{D}_X$-Module. Then an increasing sequence $(M_k)_{k\in Z}$ of coherent $\mathcal{O}_X$-sub-Modules of M is called a *good filtration* if the following conditions are satisfied:

1) $\cup_k M_k = M$. 2) $M_k = 0$ for $k \ll 0$.
3) $\mathscr{D}_X(m)M_k \subset M_{k+m}$ for $k \in Z$ and $m \in N$.
4) For $k \gg 0$, $\mathscr{D}_X(m)M_k = M_{k+m} (m \in N)$.

The following theorem plays as a dictionary for our purpose.

Theorem 1.10. *Let M be a coherent $\mathscr{D}_X$-Module and $\tilde{M}$ its microlocalization $\mathscr{E}_X \otimes_{\pi^{-1}(\mathscr{D}_X)} \pi^{-1}(M)|_{\mathring{T}^*X}$. Denote by sp the canonical homomorphism $M \to \mathring{\pi}_*(\tilde{M})$ or $\mathring{\pi}^{-1}(M) \to \tilde{M}$.*

a) *For a good filtration $(M_k)_{k\in Z}$ of M, define a sub-Module L of $\tilde{M}$ by*

$$L = \sum_{k\in Z} \mathscr{E}_X(-k)sp(\mathring{\pi}^{-1}(M_k)).$$

Then L is a coherent $\mathscr{E}_X(0)|_{\mathring{T}^*X}$-sub-Module of $\tilde{M}$ with $\mathscr{E}_X L = \tilde{M}$. Furthermore, one has

1) *$M_k/M_{k-1} \cong \mathring{\pi}_*(L(k)/L(k-1))$ for $k \gg 0$ and*
2) *$M_k = sp^{-1}(\mathring{\pi}_*(L(k))$ for $k \gg 0$,*

where $L(k) = \mathscr{E}_X(k)L$ for $k \in Z$.

b) *Conversely, let L be a coherent $\mathscr{E}_X(0)|_{\mathring{T}^*X}$-sub-Module of $\tilde{M}$ with $\mathscr{E}_X L = \tilde{M}$. Set $M_k = sp^{-1}(\mathring{\pi}_*(L(k))$ for $k \in Z$. Then $(M_k)_{k\in Z}$ defines a good filtration of M, leaving the condition that $M_k = 0$ for $k \ll 0$ out of consideration.*

(Theorem 1.10 is essentially proved in [8], Lemma 4.1.3.)

By virtue of Theorem 1.10, one can show

Corollary 1.11. *Let M be a holonomic $\mathscr{D}_X$-Module and Λ a conic Lagrangean analytic subset of T^*X containing the characteristic variety* Ch(M) *of M. Then the following conditions are equivalent:*

i) *M is a regular holonomic $\mathscr{D}_X$-Module.*
ii) *Locally on X, M has a good filtration $(M_k)_{k\in Z}$ such that, for any operator P in $\mathscr{D}_X(m)$ $(m \in N)$ satisfying $\sigma_m(P)|_\Lambda = 0$, one has $PM_k \subset M_{k+m-1}$ for all $k \in Z$.*

It should be noted here that, for any Lagrangean analytic set Λ of T^*X, one has

$$\mathscr{J}_\Lambda = \sum_{P\in\mathscr{D}_X(m),\, \sigma_m(P)|_\Lambda = 0} \mathscr{E}_X(1-m)P$$

on $\mathring{T}^*X$.

Remark 1.12. By Theorem 5.1.6 [9] combined with Theorem 1.10, one knows that any regular holonomic $\mathscr{D}_X$-Module M has a good filtration $(M_k)_{k\in Z}$ *defined globally on* X such that, if the symbol $\sigma_m(P)$ of an operator P in $\mathscr{D}_X(m)$ ($m \in N$) vanishes on $\mathrm{Ch}(M)$, then $PM_k \subset M_{k+m-1}$ for all $k \in Z$.

§ 2. Regular holonomic $\mathscr{D}_X$-Module and perverse complexes

As we have seen in Section 1, a regular holonomic $\mathscr{D}_X$-Module can be defined as follows:

Definition 2.1. Let M be a holonomic $\mathscr{D}_X$-Module with characteristic variety $\Lambda = \mathrm{Ch}(M) \subset T^*X$. Then M is called a *regular holonomic* $\mathscr{D}_X$-*Module* if, locally on X, M has a good filtration $(M_k)_{k\in Z}$ with the property

$$(*)\quad P \in \mathscr{D}_X(m),\quad \sigma_m(P)|_\Lambda = 0 \Longrightarrow PM_k \subset M_{k+m-1} \quad (k \in Z).$$

As a matter of fact, it is known that any regular holonomic $\mathscr{D}_X$-Module M has a globally defined good filtration $(M_k)_{k\in Z}$ with the above property ($*$). (Remark 1.12.)

We denote by $\mathrm{RH}(\mathscr{D}_X)$ the category of regular holonomic $\mathscr{D}_X$-Modules. For an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of holonomic $\mathscr{D}_X$-Modules, M is regular holonomic if and only if so are M' and M''. Recall that, for a holonomic $\mathscr{D}_X$-Module M, the dual system M^* of M is defined by

$$M^* = \mathscr{E}xt^n_{\mathscr{D}_X}(M, \mathscr{D}_X) \underset{\mathcal{O}_X}{\otimes} (\Omega^n_X)^{\otimes(-1)},$$

where $n = \dim X$, and that one has $M \xrightarrow{\sim} M^{**}$. Then M is regular holonomic if and only if so is the dual M^*. Thus we obtain an exact functor $*: \mathrm{RH}(\mathscr{D}_X)^\circ \xrightarrow{\sim} \mathrm{RH}(\mathscr{D}_X)$. (For a category $\mathscr{C}$, $\mathscr{C}^\circ$ denotes the opposed category of $\mathscr{C}$.)

Examples 2.2. a) Let X be an open subset of C, containing the origin, with canonical coordinate x. Let $P = \sum_{j=0}^m a_j(x) D_x^j$ be an ordinary differential operator such that $a_m(x) \neq 0$ for $x \neq 0$. Then the $\mathscr{D}_X$-Module $\mathscr{D}_X/\mathscr{D}_X P$ is regular holonomic if and only if $m - \nu_m \geqslant j - \nu_j$ for $0 \leqslant j \leqslant m$, where ν_j stands for the order of zero of $a_j(x)$ at $x = 0$.

b) Let X be an open subset of C^n with canonical coordinate system $x = (x_1, \cdots, x_n)$. Consider an *integrable* differential system for the vector $\vec{u} =$

${}^t(u_1, \cdots, u_m)$ of unknown functions

$$\begin{cases} x_i D_{x_i}\vec{u} = A_i(x)\vec{u} & (i \leqslant l) \\ D_{x_j}\vec{u} = B_j(x)\vec{u} & (j > l), \end{cases}$$

where $0 \leqslant l \leqslant n$ and $A_i(x)$, $B_j(x) \in \mathrm{M}(m, \mathcal{O}_X(X))$. Then the $\mathcal{D}_X$-Module associated with the above system is a regular holonomic $\mathcal{D}_X$-Module.
c) The De Rham system $\mathcal{O}_X$ is a regular holonomic $\mathcal{D}_X$-Module. If Y is a submanifold of X, the system $\mathcal{B}_{X|Y}$ of multiple layers with support in Y is a regular holonomic $\mathcal{D}_X$-Module. Note that one has $\mathcal{O}_X = \mathcal{O}_X^*$ and $\mathcal{B}_{Y|X} = \mathcal{B}_{Y|X}^*$.
d) If f is a holomorphic function defined on X, $\mathcal{D}_X f^\alpha$ is a regular holonomic $\mathcal{D}_X$-Module for any $\alpha \in \boldsymbol{C}$.

This notion of regular singularity gives a natural extension of that of P. Deligne [2] to holonomic $\mathcal{D}_X$-Modules. To see this, we quote a comparison theorem concerning the local cohomology of $\mathcal{D}_X$-Modules. (For the algebraic local cohomology of a $\mathcal{D}_X$-Module, see [5] or [12].)

Theorem 2.3. *Let Y be a closed analytic subset of X and set $Z = Y$ or $X \backslash Y$.*
a) *If M is a regular holonomic $\mathcal{D}_X$-Module, then the algebraic local cohomology sheaves $\mathcal{H}^j_{[Z]}(M)$ $(j \geqslant 0)$ are regular holonomic $\mathcal{D}_X$-Modules.*
b) *If M and N are regular holonomic $\mathcal{D}_X$-Modules, then one has a natural isomorphism*

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(M, \boldsymbol{R}\Gamma_{[Z]}(N)) \xrightarrow{\sim} \boldsymbol{R}\Gamma_Z \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(M, N)$$

in the derived category $\mathrm{D}(\boldsymbol{C}_X)$.

Theorem (2.3.b) is a generalization of the comparison theorem of A. Grothendieck and P. Deligne to regular holonomic $\mathcal{D}_X$-Modules. (The assertion a) is proved in [9], Theorem 5.4.1. The assertion b) can be proved by combining Theorem 6.1.1 and Theorem 5.4.1 of [9].) In what follows, we denote by $\mathrm{D}^b_{rh}(\mathcal{D}_X)$ the full subcategory of $\mathrm{D}(\mathcal{D}_X)$ whose objects are the cohomologically bounded complexes with regular holonomic cohomologies.

Here we recall the notion of a constructible $\boldsymbol{C}_X$-Module.

Definition 2.4. A $\boldsymbol{C}_X$-Module F is said to be *constructible* if there exists a decreasing sequence

$$(X_j)_{j \in \boldsymbol{N}}:\ X = X_0 \supset X_1 \supset X_2 \supset \cdots$$

of closed analytic subsets of X such that

1) $\bigcap_{j\geqslant 0} X_j = \phi$.

2) For each $j \geqslant 0$, the restriction $F|_{X_j \backslash X_{j+1}}$ of F is a local system on $X_j \backslash X_{j+1}$.

Hereafter, by a "local system on X", we mean a locally constant $\boldsymbol{C}_X$-Module of finite rank. We denote by $\mathrm{D}^b_c(\boldsymbol{C}_X)$ the full subcategory of $\mathrm{D}(\boldsymbol{C}_X)$ whose objects are the cohomologically bounded complexes with constructible cohomologies. For each complex $F^\cdot$ in $\mathrm{D}^b_c(\boldsymbol{C}_X)$, we define the dual $F^{\cdot *}$ of $F^\cdot$ by

$$F^{\cdot *} = \boldsymbol{R}\,\mathscr{H}om_{\boldsymbol{C}_X}(F^\cdot, \boldsymbol{C}_X).$$

Then one knows that $F^{\cdot *}$ is an object of $\mathrm{D}^b_c(\boldsymbol{C}_X)$ and that $F^\cdot \rightleftarrows F^{\cdot **}$ (due to J.-L. Verdier).

For a complex $M^\cdot$ in $\mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X)$, we define $\mathscr{D}\mathscr{R}_X(M^\cdot) = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, M^\cdot)$ and $\mathscr{S}ol_X(M^\cdot) = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(M^\cdot, \mathscr{O}_X)$. Then $\mathscr{D}\mathscr{R}_X(M^\cdot)$ and $\mathscr{S}ol_X(M^\cdot)$ are objects of $\mathrm{D}^b_c(\boldsymbol{C}_X)$ (M. Kashiwara [4]). Thus we obtain the two functors

$$\mathscr{D}\mathscr{R}_X : \mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X) \to \mathrm{D}^b_c(\boldsymbol{C}_X) \quad \text{and} \quad \mathscr{S}ol_X : \mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X)^\circ \to \mathrm{D}^b_c(\boldsymbol{C}_X).$$

These two functors are connected by the relation $\mathscr{S}ol_X(M^\cdot) = \mathscr{D}\mathscr{R}_X(M^{\cdot *})$, so we pay our attention mainly to $\mathscr{D}\mathscr{R}_X$.

Proposition 2.5. a) $\mathscr{D}\mathscr{R}_X(M^{\cdot *}) = \mathscr{D}\mathscr{R}_X(M^\cdot)^*$ *for any* $M^\cdot$ *in* $\mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X)$. b) *For a closed analytic subset* Y *of* X, *set* $Z = Y$ *or* $X \backslash Y$. *Then one has*

$$\mathscr{D}\mathscr{R}_X(\boldsymbol{R}\Gamma_{[Z]}(M^\cdot)) = \boldsymbol{R}\Gamma_Z \mathscr{D}\mathscr{R}_X(M^\cdot)$$

for any $M^\cdot$ *in* $\mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X)$.

(The assertion a) is a version of Proposition 1.4.6, [9].)

Example 2.6. a) For any submanifold Y of X of codimension l, one has

$$\begin{aligned} \mathscr{D}\mathscr{R}_X(\mathscr{B}_{Y|X}) &= \mathscr{D}\mathscr{R}_X(\boldsymbol{R}\Gamma_{[Y]}(\mathscr{O}_X))[l] = \boldsymbol{R}\Gamma_Y \mathscr{D}\mathscr{R}_X(\mathscr{O}_X)[l] \\ &= \boldsymbol{R}\Gamma_Y(\boldsymbol{C}_X)[l] = \boldsymbol{C}_Y[-l]. \end{aligned}$$

b) Let Y be a hypersurface of X defined by a holomorphic function f and j the inclusion mapping $X \backslash Y \hookrightarrow X$. Then, one has $\mathscr{D}\mathscr{R}_X(\mathscr{O}_X[f^{-1}]) = \boldsymbol{R}j_* j^{-1}(\boldsymbol{C}_X)$ and $\mathscr{D}\mathscr{R}_X(\mathscr{O}_X[f^{-1}]^*) = \mathscr{D}\mathscr{R}_X(\mathscr{O}_X[f^{-1}])^* = j_! j^{-1}(\boldsymbol{C}_X)$. In the case where X is the complex n-space $\boldsymbol{C}^n$ with canonical coordinate system $x = (x_1, \cdots, x_n)$ and $f(x) = x_1 \cdots x_r$ $(1 \leqslant r \leqslant n)$, the cohomology sheaves of the complex $F^\cdot = \mathscr{D}\mathscr{R}_X(\mathscr{O}_X[f^{-1}])$ are computed as follows:

$$\mathscr{H}^0(F^\cdot)=C_X \quad \text{and} \quad \mathscr{H}^j(F^\cdot)=\bigoplus_{1\leqslant k_1<\cdots<k_j\leqslant r} C_{Y_{k_1}\cap\cdots\cap Y_{k_j}} \quad \text{for} \quad j>0\,,$$

where $Y_k=\{x_k=0\}$.

Now it is natural to ask if, for a given complex $F^\cdot$ in $\mathrm{D}^b_c(C_X)$, there exists $M^\cdot$ in $\mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X)$ such that $\mathscr{DR}_X(M^\cdot)\simeq F^\cdot$. (So-called the Riemann-Hilbert problem.) In this direction, we give first

Proposition 2.7. *For any $M^\cdot$ and $N^\cdot$ in $\mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X)$, one has*

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(M^\cdot, N^\cdot)\xrightarrow{\sim}\boldsymbol{R}\mathscr{H}om_{C_X}(\mathscr{DR}_X(M^\cdot), \mathscr{DR}_X(N^\cdot)).$$

(Proposition 2.7 follows from Theorem 6.1.1 and Theorem 1.4.9 of [9]). This proposition implies that the functor $\mathscr{DR}_X$: $\mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X)\to\mathrm{D}^b_c(C_X)$ is fully faithful. Furthermore,

Theorem 2.8 (M. Kashiwara [7] and Z. Mebkhout [11]). *The De Rham functor $\mathscr{DR}_X$ gives the equivalence of categories*

$$\mathscr{DR}_X\colon \mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X)\xrightarrow{\sim}\mathrm{D}^b_c(C_X)\,.$$

Theorem 2.8 can be regarded as an affirmative answer to the Riemann-Hilbert problem for constructible C_X-Modules. The above theorem can be proved by reducing it to the following fact: Let Y be a hypersurface with normal crossings in X and F the constructible C_X-Module obtained as the extension by zero of a local system on $X\backslash Y$. Then there exists a regular holonomic $\mathscr{D}_X$-Module M such that $\mathscr{DR}_X(M)\simeq F$. In the course of reduction, we use Hironaka's desingularization theorem and the stability of regular holonomic systems under the integration along the fibres of a proper holomorphic mapping (M. Kashiwara [7], Theorem 8.1). In [7], this theorem is proved by constructing the inverse functor of $\mathscr{DR}_X$.

The category $\mathrm{RH}(\mathscr{D}_X)$ of regular holonomic $\mathscr{D}_X$-Modules is identified with the full subcategory of $\mathrm{D}^b_{\mathrm{rh}}(\mathscr{D}_X)$ consisting of all complexes $M^\cdot$ with $\mathscr{H}^j(M^\cdot)=0$ for $j\neq 0$. Then how can one characterize such a complex $F^\cdot$ in $\mathrm{D}^b_c(C_X)$ that is expressed as the De Rham complex $\mathscr{DR}_X(M)$ of a regular holonomic $\mathscr{D}_X$-Module M?

Definition 2.9. An object $F^\cdot$ of $\mathrm{D}^b_c(C_X)$ is called a *perverse* complex if it satisfies the following conditions:

1) codim Supp $(\mathscr{H}^j(F^\cdot))\geqslant j$ for all $j\in \boldsymbol{Z}$.

1*) codim Supp $(\mathscr{H}^j(F^{\cdot *}))\geqslant j$ for all $j\in \boldsymbol{Z}$.

We denote by Perv (C_X) the full subsategory of $\mathrm{D}^b_c(C_X)$ whose objects are the perverse complexes on X. Then Theorem 2.8 can be refined as follows:

Theorem 2.10. *The De Rham functor $\mathscr{DR}_X$ induces the equivalence of categories*

$$\mathscr{DR}_X\colon \mathrm{RH}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{Perv}(C_X)\,.$$

Let us show here that, if M is a regular holonomic $\mathscr{D}_X$-Module, then the De Rham complex $F^{\boldsymbol{\cdot}} = \mathscr{DR}_X(M)$ of M is a perverse complex. Since $F^{\boldsymbol{\cdot}*} = \mathscr{DR}_X(M^*)$, we have only to show that the condition 1) is satisfied. For a fixed j, set $Y = \mathrm{Supp}(\mathscr{H}^j(F^{\boldsymbol{\cdot}}))$ and $l = \mathrm{codim}\, Y$. So as to prove $l \geqslant j$, one can replace X by an open neighborhood of a generic point of Y so that Y is smooth of codimension l and that all $\mathscr{H}^k(F^{\boldsymbol{\cdot}})$ $(k \in Z)$ are locally constant on Y. (It is possible since $\mathscr{H}^k(F^{\boldsymbol{\cdot}})$ $(k \in Z)$ are all constructible.) In this setting, we compute the local cohomology sheaves $R^k\Gamma_Y(F^{\boldsymbol{\cdot}*})$ $(k \in Z)$ in two manners. Since $F^{\boldsymbol{\cdot}*} = R\mathscr{H}om_{\mathscr{D}_X}(M, \mathscr{O}_X)$, one has $R\Gamma_Y(F^{\boldsymbol{\cdot}*}) = R\mathscr{H}om_{\mathscr{D}_X}(M, R\Gamma_{[Y]}(\mathscr{O}_X)) = R\mathscr{H}om_{\mathscr{D}_X}(M, \mathscr{B}_{Y|X})[-l]$, hence $R^k\Gamma_Y(F^{\boldsymbol{\cdot}*}) = 0$ for $k < l$. On the other hand, one has $R\Gamma_Y(F^{\boldsymbol{\cdot}*}) = R\mathscr{H}om_{C_X}(F^{\boldsymbol{\cdot}}_Y, C_X)$. Since $\mathscr{H}^i(F^{\boldsymbol{\cdot}})$ $(i \in Z)$ are all locally constant on Y, one has $\mathscr{E}xt^k_{C_X}(\mathscr{H}^j(F^{\boldsymbol{\cdot}})_Y, C_X) = 0$ for $k \neq 2l$. Hence, $R^k\Gamma_Y(F^{\boldsymbol{\cdot}*}) = \mathscr{E}xt^{2l}_{C_X}(\mathscr{H}^{2l-k}(F^{\boldsymbol{\cdot}})_Y, C_X)$ for all $k \in Z$. Here one has $R^{2l-j}\Gamma_Y(F^{\boldsymbol{\cdot}*}) \neq 0$ since $\mathscr{H}^j(F^{\boldsymbol{\cdot}})|_Y$ has positive rank. Comparing this with the above computation, we have $2l - j \geqslant l$, i.e., $l \geqslant j$, as desired.

§ 3. Minimal extension of a regular holonomic $\mathscr{D}_X$-Module.

Let Y be a closed analytic subset of X and j the inclusion mapping $X \setminus Y \hookrightarrow Y$. Then we obtain the functor $j^{-1}\colon \mathrm{RH}(\mathscr{D}_X) \to \mathrm{RH}(\mathscr{D}_{X\setminus Y})$ of restriction. Let us begin with a characterization, in terms of the De Rham complex, of a regular holonomic $\mathscr{D}_{X\setminus Y}$-Module which can be extended to a regular holonomic $\mathscr{D}_X$-module.

Proposition 3.1. *For a regular holonomic $\mathscr{D}_{X\setminus Y}$-Module N, set $G^{\boldsymbol{\cdot}} = \mathscr{DR}_{X\setminus Y}(N)$. Then the following conditions are equivalent:*

i) *There is a regular holonomic $\mathscr{D}_X$-Module M such that $j^{-1}(M) \xrightarrow{\sim} N$.*

i′) *There is a holonomic $\mathscr{D}_X$-Module M such that $j^{-1}(M) \xrightarrow{\sim} N$.*

ii) *There is a perverse complex $F^{\boldsymbol{\cdot}}$ on X such that $j^{-1}(F^{\boldsymbol{\cdot}}) \xrightarrow{\sim} G^{\boldsymbol{\cdot}}$ in* $\mathrm{Perv}(C_{X\setminus Y})$.

ii′) *There is an object $F^{\boldsymbol{\cdot}}$ of $\mathrm{D}^b_c(C_X)$ such that $j^{-1}(F^{\boldsymbol{\cdot}}) \xrightarrow{\sim} G^{\boldsymbol{\cdot}}$ in $\mathrm{D}^b_c(C_{X\setminus Y})$.*

ii″) *The extension $j_!(G^{\boldsymbol{\cdot}})$ by zero of $G^{\boldsymbol{\cdot}}$ has constructible cohomology sheaves, i.e., $j_!(G^{\boldsymbol{\cdot}}) \in \mathrm{D}^b_c(C_X)$.*

We denote by $\mathrm{RH}^{\mathrm{ext}}(\mathscr{D}_{X\setminus Y})$ (resp. $\mathrm{Perv}^{\mathrm{ext}}(C_{X\setminus Y})$) the full subcategory of $\mathrm{RH}(\mathscr{D}_{X\setminus Y})$ (resp. $\mathrm{Perv}(C_{X\setminus Y})$) consisting of all objects *extendable with respect to j* in the sense of Proposition 3.1. Then the De Rham functor $\mathscr{DR}_{X\setminus Y}$ induces the equivalence of categories $\mathrm{RH}^{\mathrm{ext}}(\mathscr{D}_{X\setminus Y}) \xrightarrow{\sim} \mathrm{Perv}^{\mathrm{ext}}(C_{X\setminus Y})$.

Note that any local system on $X\backslash Y$ is extendable with respect to j.

Theorem 3.2. *For any extendable regular holonomic $\mathscr{D}_{X\backslash Y}$-Module N, there is a regular holonomic $\mathscr{D}_X$-Module M with the properties*

1) $j^{-1}(M)\simeq N$ *and* 2) $\Gamma_Y(M)=\Gamma_Y(M^*)=0$.

Moreover, such an M is determined uniquely up to isomorphism.

Note first that, for any coherent $\mathscr{D}_X$-Module M, one has $\Gamma_Y(M)=\Gamma_{[Y]}(M)$. Before proving Theorem 3.2, we propose

Lemma 3.3. *Let M' and M'' be regular holonomic $\mathscr{D}_X$-Modules such that $\Gamma_Y(M'^*)=0$ and $\Gamma_Y(M'')=0$. Then one has*

$$\mathscr{H}om_{\mathscr{D}_X}(M', M'')\xrightarrow{\sim} j_*\mathscr{H}om_{\mathscr{D}_{X\backslash Y}}(j^{-1}M', j^{-1}M'')\,.$$

Proof. Since $\Gamma_{[Y]}(M'')=0$, one has an exact sequence

$$0\longrightarrow M''\longrightarrow \Gamma_{[X\backslash Y]}(M'')\longrightarrow \mathscr{H}^1_{[Y]}(M'')\longrightarrow 0\,.$$

Since Supp $(\mathscr{H}^1_{[Y]}(M''))\subset Y$ and $\Gamma_Y(M'^*)=0$, one has

$$\mathscr{H}om_{\mathscr{D}_X}(M', \mathscr{H}^1_{[Y]}(M''))=\mathscr{H}om_{\mathscr{D}_X}(\mathscr{H}^1_{[Y]}(M'')^*, M'^*)=0\,.$$

Hence, by applying $\mathscr{H}om_{\mathscr{D}_X}(M', .)$ to the above exact sequence, one has an isomorphism

$$\mathscr{H}om_{\mathscr{D}_X}(M', M'')\xrightarrow{\sim}\mathscr{H}om_{\mathscr{D}_X}(M', \Gamma_{[X\backslash Y]}(M''))\,.$$

On the other hand, one has

$$\begin{aligned}\mathscr{H}om_{\mathscr{D}_X}(M', \Gamma_{[X\backslash Y]}(M''))&=j_*j^{-1}\mathscr{H}om_{\mathscr{D}_X}(M', M'')\\&=j_*\mathscr{H}om_{\mathscr{D}_{X\backslash Y}}(j^{-1}M', j^{-1}M'')\end{aligned}$$

by Theorem 2.3. Thus one obtains the isomorphism of Lemma. Q.E.D.

Proof of Theorem 3.2. For the given N in $\mathrm{RH}^{\mathrm{ext}}(\mathscr{D}_{X\backslash Y})$, take an M' in $\mathrm{RH}(\mathscr{D}_X)$ such that $j^{-1}(M')\simeq N$. Then $M''=M'/\Gamma_Y(M')$ has the property $\Gamma_Y(M'')=0$. Again, set $M=(M''^*/\Gamma_Y(M''^*))^*$. Then M is a regular holonomic $\mathscr{D}_X$-Module with the desired property. Uniqueness of such an M follows from Lemma 3.3 immediately. Q.E.D.

Definition 3.4. For an extendable regular holonomic $\mathscr{D}_{X\backslash Y}$-Module N, the regular holonomic $\mathscr{D}_X$-Module M determined by Theorem 3.2 is called the *minimal extension of* N and denoted by ${}^{\pi}N$.

By means of Lemma 3.3, we obtain the functor ${}^{\pi}$: $\mathrm{RH}^{\mathrm{ext}}(\mathscr{D}_{X\backslash Y})\rightarrow$

$\mathrm{RH}(\mathscr{D}_X)$ of minimal extension, which gives the equivalence between $\mathrm{RH}^{\mathrm{ext}}(\mathscr{D}_{X\setminus Y})$ and the full subcategory of $\mathrm{RH}(\mathscr{D}_X)$ consisting of all regular holonomic $\mathscr{D}_X$-Modules M with the property $\Gamma_Y(M)=\Gamma_Y(M^*)=0$. Furthermore, by the equivalence of categories of Theorem 2.10, we obtain the functor of minimal extension ${}^{\pi}$: $\mathrm{Perv}^{\mathrm{ext}}(\boldsymbol{C}_{X\setminus Y})\to\mathrm{Perv}(\boldsymbol{C}_X)$ for extendable perverse complexes on $X\setminus Y$. It should be noted that the minimal extention is compatible with the dualizing operation:

$$({}^{\pi}N)^*={}^{\pi}(N^*)\quad\text{for any}\quad N\in\mathrm{RH}^{\mathrm{ext}}(\mathscr{D}_{X\setminus Y})$$
$$\text{and}\quad ({}^{\pi}G^{\boldsymbol{\cdot}})^*={}^{\pi}(G^{\boldsymbol{\cdot}*})\quad\text{for any}\quad G^{\boldsymbol{\cdot}}\in\mathrm{Perv}^{\mathrm{ext}}(\boldsymbol{C}_{X\setminus Y}).$$

(This can be shown easily by Theorem 3.2.)

By an argument similar to that of Theorem 2.10, one can show

Theorem 3.5. *For an extendable perverse complex $G^{\boldsymbol{\cdot}}$ on $X\setminus Y$, the minimal extension $F^{\boldsymbol{\cdot}}={}^{\pi}G^{\boldsymbol{\cdot}}$ is characterized as a unique perverse complex on X such that*

1) $j^{-1}(F^{\boldsymbol{\cdot}})\xrightarrow{\sim}G^{\boldsymbol{\cdot}}$.
2) codim $Y\cap\mathrm{Supp}(\mathscr{H}^j(F^{\boldsymbol{\cdot}}))>j$ *for all* $j\in\boldsymbol{Z}$.
2*) codim $Y\cap\mathrm{Supp}(\mathscr{H}^j(F^{\boldsymbol{\cdot}*}))>j$ *for all* $j\in\boldsymbol{Z}$.

Recall that, if Y is an l-codimensional submanifold of X, then the regular holonomic $\mathscr{D}_X$-Module $\mathscr{B}_{Y|X}=\mathscr{H}^l_{[Y]}(\mathscr{O}_X)$ has the following properties:

1) $\mathscr{B}^*_{Y|X}=\mathscr{B}_{Y|X}$.
2) For any point y of Y, the stalk $\mathscr{B}_{Y|X,y}$ is a simple $\mathscr{D}_{X,y}$-module.

Now we propose to apply the above arguments to defining "$\mathscr{B}_{Y|X}$" for a closed analytic subset Y of X.

Definition 3.6. Let Y be a closed analytic subset of X, purely of codimension l. Set $Y'=Y\setminus Y_{\mathrm{sing}}$ and $X'=X\setminus Y_{\mathrm{sing}}$. Then we denote by $\mathscr{L}(Y,X)$ (or ${}^{\pi}\mathscr{B}_{Y|X}$) the minimal extension ${}^{\pi}\mathscr{B}_{Y'|X'}$ of $\mathscr{B}_{Y'|X'}$ with respect to the inclusion mapping $X'\hookrightarrow X$.

Since the formation of minimal extensions is compatible with the dualizing operation, one has immediately $\mathscr{L}(Y,X)^*=\mathscr{L}(Y,X)$.

Proposition 3.7. *If Y is irreducible at a point y of Y, then the stalk $\mathscr{L}(Y,X)_y$ is a simple $\mathscr{D}_{X,y}$-module.*

Proof. We denote $\mathscr{L}$ for $\mathscr{L}(Y,X)$. Fix a $\mathscr{D}_{X,y}$-submodule of $\mathscr{L}_y$. Then, one can find a $\mathscr{D}_X$-sub-Module M of $\mathscr{L}$, defined in an open neighborhood of y, whose stalk M_y at y coincides with the given submodule of $\mathscr{L}_y$. On the assumption that Y is irreducible at y, one can replace X by an open neighborhood of y so that $Y'=Y\setminus Y_{\mathrm{sing}}$ is connected and that M

is defined on X. Note that, if z is a smooth point of Y, then one has either $M_z=0$ or $M_z=\mathscr{B}_{Y|X,z}=\mathscr{L}_z$. So one has either $M|_{Y'}=0$ or $M|_{Y'}=\mathscr{B}_{Y'|X'}=\mathscr{L}|_{Y'}$, since Y' is connected. If $M|_{Y'}=0$, then one has $M\subset\Gamma_{Y_{\text{sing}}}(\mathscr{L})$, hence $M=0$. If $M|_{Y'}=\mathscr{L}|_{Y'}$, then one has $(\mathscr{L}/M)^*\subset\Gamma_{Y_{\text{sing}}}(\mathscr{L}^*)$, hence $(\mathscr{L}/M)^*=0$, i.e., $M=\mathscr{L}$. Q.E.D.

Remark 3.8. Recall that, if Y is smooth, one has $\mathscr{D}\mathscr{R}_X(\mathscr{B}_{Y|X})=\boldsymbol{C}_Y[-l]$. In the setting of Definition 3.6, the De Rham complex $F^\bullet=\mathscr{D}\mathscr{R}_X(\mathscr{L}(Y,X))$ gives an extension of $\boldsymbol{C}_{Y'}[-l]$ to a self-dual perverse complex on X: $F^\bullet|_{Y'}=\boldsymbol{C}_{Y'}[-l]$ and $F^{\bullet *}=F^\bullet$. Shifted suitably, the complex $F^\bullet=\mathscr{D}\mathscr{R}_X(\mathscr{L}(Y,X))$ coincides with π_Y of Deligne, Goresky and Mac Pherson. (See [1], [6].)

At the end of this note, we include a basic example of $\mathscr{L}(Y,X)$ for a hypersurface Y with an isolated singularity.

Let X be the complex n-space $\boldsymbol{C}^n$ with canonical coordinate $x=(x_1,\cdots,x_n)$ and Y the hypersurface defined by $f=x_1^2+\cdots+x_n^2$. Assuming that $n\geqslant 3$, we set $Y'=Y\backslash\{0\}$ and $X'=X\backslash\{0\}$. Note first that $M:=\mathscr{H}^1_{[Y]}(\mathscr{O}_X)=\mathscr{O}_X[f^{-1}]/\mathscr{O}_X$ gives a regular holonomic extension of $\mathscr{B}_{Y'|X'}$. Since $\Gamma_{[0]}(M)=\mathscr{H}^1_{[0]}(\mathscr{O}_X)=0$, the minimal extension $\mathscr{L}=\mathscr{L}(Y,X)$ can be realized by $\mathscr{L}=(M^*/\Gamma_{[0]}(M^*))^*$. In other words, $\mathscr{L}$ is the *minimal* $\mathscr{D}_X$-sub-Module of M satisfying $\operatorname{Supp}(M/\mathscr{L})\subset\{0\}$. Let us denote by u the residue class of f^{-1} in $M=\mathscr{O}_X[f^{-1}]/\mathscr{O}_X$. Then one has $\mathscr{L}\subset\mathscr{D}_Xu$ since $\operatorname{Supp}(M/\mathscr{D}_Xu)\subset\{0\}$.

Claim. On the condition $n\geqslant 3$, one has $\mathscr{L}=\mathscr{D}_Xu\subset M$.

Proof. The assertion is equivalent to $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_Xu,\mathscr{D}_Xu/\mathscr{L})=0$. Note that $\mathscr{D}_Xu/\mathscr{L}$ is isomorphic to a copy of $\mathscr{B}_{\{0\}|X}$ since $\operatorname{Supp}(\mathscr{D}_Xu/\mathscr{L})\subset\{0\}$. So it is enough to show that $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_Xu,\mathscr{B}_{\{0\}|X})=0$. Here we have

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_Xu,\mathscr{B}_{\{0\}|X})=\{\varphi\in\mathscr{B}_{\{0\}|X}: P\varphi=0\quad \text{if } Pf^{-1}\in\mathscr{O}_X\}.$$

For the operator $P=\sum_{i=1}^n x_iD_{x_i}+2$, we have $Pf^{-1}=0$. However, any non-zero section φ of $\mathscr{B}_{\{0\}|X}$ cannot satisfy the equation $P\varphi=0$ as can be directly checked by the relation

$$\sum_{i=1}^n x_iD_{x_i}\delta^{(\alpha)}(x)=-\sum_{i=1}^n(\alpha_i+1)\delta^{(\alpha)}(x),$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)\in\boldsymbol{N}^n$. Q.E.D.

Thus we obtain an isomorphism

$$\mathscr{L}=\mathscr{D}_Xu\xleftarrow{\sim}\mathscr{D}_X/\mathscr{J}\qquad\text{where } \mathscr{J}=\{P\in\mathscr{D}_X:\ Pf^{-1}\in\mathscr{O}_X\}.$$

The structure of the system $\mathscr{L}=\mathscr{L}(Y,X)$ varies according to the parity of n.

Case where n is odd:

a) $\mathscr{L}=\mathscr{D}_X u=M$. The ideal $\mathscr{J}$ is generated by $x_i D_{x_j}-x_j D_{x_i}(i<j)$, $\sum_{i=1}^n x_i D_{x_i}+2$ and f.

b) $\mathrm{Ch}(\mathscr{L})=\overline{T_Y^*X}\cup T_{\{0\}}^*X$, where $\overline{T_Y^*X}$ stands for the closure of $T_{Y'}^*X$ in T^*X.

c) $\mathscr{D}\mathscr{R}_X(\mathscr{L})=C_Y[-1]$.

Case where n is even:

a) $\mathscr{L}=\mathscr{D}_X u\subsetneqq M$. The ideal $\mathscr{J}$ is generated by $x_i D_{x_j}-x_j D_{x_i}(i<j)$, $\sum_{i=1}^n x_i D_{x_i}+2$, f and $\varDelta^{(n-2)/2}$, where $\varDelta=\sum_{i=1}^n D_{x_i}^2$.

b) $\mathrm{Ch}(\mathscr{L})=\overline{T_Y^*X}$.

c) $$\mathscr{H}^j(\mathscr{D}\mathscr{R}_X(\mathscr{L}))=\begin{cases}C_Y & (j=1)\\ C_{\{0\}} & (j=n-1)\\ 0 & (j\neq 1, n-1).\end{cases}$$

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Department of Mathematics
Sophia University
7 Kioi-cho, Chiyoda-ku
Tokyo 102, Japan