Advanced Studies in Pure Mathematics 3, 1984
Geometry of Geodesics and Related Topics
pp. 231–238

On Topological Blaschke Conjecture I

Cohomological Complex Projective Spaces

Hajime Sato

By a Blaschke manifold, we mean a Riemannian manifold (M, g) such that, for any point $m \in M$, the tangential cut locus C_m of m in $T_m M$ is isometric to the sphere of constant radius. There are some equivalent definitions (see Besse [2, 5.43]). The Blaschke conjecture is that any Blaschke manifold is isometric to a compact rank one symmetric space. If the integral cohomology ring of M is equal to the sphere S^k, or the real projective space RP^k, this conjecture is proved by Berger with other mathematicians [2, Appendix D]). We consider the case where the cohomology ring of M is equal to that of the complex projective space CP^k.

We obtain the following theorem.

Theorem. *Let (M, g) be a $2k$-dimensional Blaschke manifold such that the integral cohomology ring is equal to that of CP^k. Then M is PL-homeomorphic to CP^k for any k.*

Blaschke manifolds with other cohomology rings will be treated in subsequent papers.

If (M, g) is a Blaschke manifold and $m \in M$, Allamigeon [1] has shown that the cut locus $C(m)$ of m in M is the base manifold of a fibration of the tangential cut locus C_m by great spheres. We study the base manifold of such fibration by great circles. We apply the Browder-Novikov-Sullivan's theory in the classification of homotopy equivalent manifolds (see Wall [4]). Calculation of normal invariants gives our theorem. In Appendix, we give examples of non-trivial fibrations of S^3 by great circles. The author thanks to T. Mizutani and K. Masuda for the discussion of results in Appendix.

Detailed proof will appear elsewhere.

§ 1. Projectable bundles

In the paper [3], we have obtained a method of a calculation of the

Received January 8, 1983.

tangent bundle of the base space of an S^1-principal bundle. We will briefly recall that.

Let X be a smooth manifold and let $\pi: L \to X$ be the projection of an S^1-principal bundle.

Definition. A vector bundle $p: E \to L$ over L is projectable onto X, if there exists a vector bundle $\hat{p}: \hat{E} \to X$ over X such that $\pi^* \hat{E} = E$. The map π induces the bundle map $\pi_!: E \to \hat{E}$, which we call the projection. The bundle $\hat{E}$ is called the projected bundle.

Let x be a point in X. For any $a, b \in \pi^{-1}(x) = S^1$, we have a linear isomorphism

$$\Phi_{ab}: p^{-1}(a) \longrightarrow p^{-1}(b)$$

of vector spaces defined by $\Phi_{ab}(u) = v$, where $\pi_!(u) = \pi_!(v)$. Then we have, for $a, b, c \in \pi^{-1}(x)$,

$$\Phi_{bc} \Phi_{ab} = \Phi_{ac}. \tag{1}$$

Let $\pi^* L = \{(a, b) \in L \times L, \pi(a) = \pi(b)\}$ be the induced S^1-bundle over L from L. We have two projections $\pi_1, \pi_2: \pi^* L \to L$ defined by $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. Let $\pi_i^* E$ $(i = 1, 2)$ be the induced vector bundle. The map $\Phi: \pi^* L \to \mathrm{Iso}(\pi_1^* E, \pi_2^* E)$ defined by $\Phi(a, b) = \Phi_{ab}$ is a continuous cross section of the bundle $\mathrm{Iso}(\pi_1^* E, \pi_2^* E)$ over $\pi^* L$.

We call Φ the projecting isomorphism associated with the projectable bundle E.

Proposition 1. *Suppose given a vector bundle E over L and a cross section Φ of the bundle* $\mathrm{Iso}(\pi_1^* E, \pi_2^* E)$ *satisfying* (1). *Then we have a vector bundle $\hat{E}$ over X such that $\pi^* \hat{E} = E$ and the projecting isomorphism is equal to Φ.*

Now let TL and TX be the tangent bundles of L and X respectively. Let $\rho: S^1 \times L \to L$ be the free S^1-action. For each $t \in S^1$, the diffeomorphism $\rho(t) = \rho(t, \cdot)$ induces a bundle isomorphism $\rho(t)_*: TL \to TL$.

Proposition 2. *The collection $\bigcup_{t \in S^1} \rho(t)_*$ induces a projecting isomorphism on the bundle TL such that the projected bundle TL is isomorphic to $TX \oplus 1$.*

Proof. Choose a bundle metric on TL. Let TL_1 be the subbundle of TL consisting of tangent vectors normal to the S^1-action. Then TL_1 is projected to TX. The line bundle tangent to the S^1-action is projected to the trivial line bundle on X.

§ 2. Pontrjagin classes

Let S^{2k-1} be the unit sphere in $\boldsymbol{R}^{2k}$ and let $\pi\colon S^{2k-1}\to B$ be a fibration of S^{2k-1} by great circles. Thus, for each $b\in B$, $\pi^{-1}(b)$ is the intersection of S^{2k-1} with a 2-plane in $\boldsymbol{R}^{2k}$. We write the 2-plane by $P(b)$. Let $\rho\colon S^1\times S^{2k-1}\to S^{2k-1}$ denote the free S^1-action.

Let $V(2k,2)$ and $G(2k,2)$, respectively, be the Stiefel and the Grassmann manifold consisting of orthogonal 2 frames or oriented 2-planes in $\boldsymbol{R}^{2k}$. Then the natural mapping $\lambda\colon V(2k,2)\to G(2k,2)$ defines a principal S^1-bundle.

The mapping $\theta\colon B\to G(2k,2)$ defined by $\theta(b)=P(b)$ is a smooth embedding. Let $\theta^*(\lambda)$ denote the induced bundle of λ by θ. Since π is also the induced bundle of λ by θ, there exists a natural bundle isomorphism between π and $\theta^*(\lambda)$ inducing the identity on B. Thus we obtain;

Lemma 3. *We may suppose that the free S^1-action ρ on S^{2k-1} is equal to the restriction on $\pi^{-1}(b)$ of the linear action on $P(b)$ for every $b\in B$.*

In the following, we always assume that ρ is the linear action on each fibre. For each $x\in S^{2k-1}$, let Kx denote the point $\rho(1/4)x$ in S^{2k-1}, where we identify S^1 with $[0,1]/[0]\sim[1]$. Define a mapping $\tilde{\theta}\colon S^{2k-1}\to V(2k,2)$ by $\tilde{\theta}(x)=(x,-Kx)$. This is a smooth embedding and is a bundle map inducing θ on the base manifolds. For an orthogonal 2-frame $w=(x,y)$, let $\tilde{\psi}(w)$ denote the vector $(x/\sqrt{2},y/\sqrt{2})$ in $\boldsymbol{R}^{2k}\oplus\boldsymbol{R}^{2k}$. Then the map $\tilde{\psi}\colon V(2k,2)\to\boldsymbol{R}^{4k}$ is a smooth embedding of $V(2k,2)$ in $S^{4k-1}\subset\boldsymbol{R}^{4k}$. We identify $\boldsymbol{R}^{2k}\oplus\boldsymbol{R}^{2k}$ with $\boldsymbol{C}^{2k}$ such that the first summand $\boldsymbol{R}^{2k}$ is the real part and the second pure imaginary. On $\boldsymbol{C}^{2k}-0$, we have the free action ρ_0 of S^1 as the multiplication by the complex number of norm one. Then $\tilde{\psi}$ is S^1-equivariant and we write by ψ the induced map $\psi\colon G(2k,2)\to\boldsymbol{C}P^{2k-1}$.

Let $\tilde{f}\colon S^{2k-1}\to S^{4k-1}$ be the composition $\tilde{f}=\tilde{\psi}\tilde{\theta}$ and $f=\psi\theta\colon B\to\boldsymbol{C}^{2k-1}$. The map $\tilde{f}$ is given by $\tilde{f}(x)=(x/\sqrt{2},-Kx/\sqrt{2})$ for $x\in S^{2k-1}$.

We define a map $\tilde{F}\colon\boldsymbol{R}^{2k}-0\to\boldsymbol{C}^{2k}-0$ by $\tilde{F}(tx)=t\tilde{f}(x)$ for $t>0$ and $x\in S^{2k-1}$. The map $\tilde{F}$ is a smooth embedding. Let E denote the restriction of the tangent bundle $T(\boldsymbol{R}^{2k}-0)$ of $\boldsymbol{R}^{2k}-0$ on S^{2k-1}, and we write p for the projection $E\to S^{2k-1}$. Then $\tilde{F}$ induces an injective bundle map $\tilde{F}_*\colon E\to\tilde{F}_*(E)\subset T(\boldsymbol{C}^{2k}-0)|_{\tilde{F}(S^{2k-1})}$.

Now define a map $\tilde{G}\colon\boldsymbol{R}^{2k}-0\to\boldsymbol{C}^{2k}-0$ by $\tilde{G}(tX)=(tx/\sqrt{2},tK/\sqrt{2})$ for $t>0$ and $x\in S^{2k-1}$. Then $\tilde{G}$ is also an embedding and $\tilde{G}$ induces an injective bundle map

$$\tilde{G}_*\colon E\longrightarrow\tilde{G}_*(E)\subset T(\boldsymbol{C}^{2k-0}-0)|_{\tilde{G}(S^{2k-1})}.$$

If $\bar{\rho}_0$ denote the conjugate action of S^1 on $\boldsymbol{C}^{2k}-0$. Then G is S^1-equivariant concerning to this conjugate action.

For any $y \in C^{2k}$, we naturally identify the tangent space $T_y C^{2k}$ with C^{2k} itself. For $x \in S^{2k-1}$, let E_x denote the fiber $p^{-1}(x)$. Then $\tilde{F}_*(E_x)$ and $\tilde{G}_*(E_x)$ are subvector spaces of C^{2k}.

Since K: $S^{2k-1} \to S^{2k-1}$ is a diffeomorphism, we obtain;

Lemma 4. *The vector spaces $\tilde{F}_*(E_x)$ and $\tilde{G}_*(E_x)$ are transversal. Thus they span C^{2k}.*

Let T denote the restriction of the tangent bundle $T(C^{2k})$ on $\tilde{F}(S^{2k-1})$. Then we have the direct sum decomposition by trivial vector bundles

$$T = \tilde{F}_*(E) \oplus \tilde{G}_*(E).$$

Notice that $\tilde{G}_*(E)$ on $\tilde{G}(S^{2k-1})$ is identified with the conjugate $\overline{\tilde{F}_*(E)}$ in T over $\tilde{F}(S^{2k-1})$.

For any $t \in S^1$, we have the induced bundle isomorphisms $\rho_*(t)$: $E \to E$ and $\rho_{0*}(t)$: $T \to T$.

Lemma 5. *The isomorphism $\rho_{0*}(t)$ is equal to the direct sum*

$$\rho_*(t) + \rho_*(t).$$

By Proposition 1, we obtain that the projected bundle $\hat{T}$, defined by the projecting isomorphism $\rho_*(t)$, is isomorphic to the Whitney sum;

$$\hat{T} \cong \hat{E} \oplus \hat{E}.$$

On the other hand, by Proposition 2, we obtain the following.

Lemma 6. *The bundle $\hat{T}$ has the complex structure. As a complex vector bundle, $\hat{T}$ is isomorphic to the Whitney sum $T(CP^{2k-1})|_{f(B)} \oplus 1$.*

Lemma 7. *As a real vector bundle, $\hat{E}$ is isomorphic to the bundle $T(B) \oplus 2$.*

Consequently, we obtain that

$$T(B) \oplus T(B) \oplus 4 \cong (T(CP^{2k-1})|_{f(B)} \oplus 1)_R.$$

Since the cohomology groups $H^*(B; Z)$ has no torsion element, by the product formula of Pontrjagin classes, we obtain the following.

Proposition 8. *The Pontrjagin classes of the smooth manifold B is equal to that of CP^{k-1}, for any k.*

§ 3. Z_2-invariants and proof of Theorem

Let $\mathscr{S}(CP^{k-1})$ denote the set of PL-homeomorphism classes of closed PL-manifolds homotopy equivalent to CP^{k-1}. The following results are due to Sullivan (cf. [4, § 14C]).

Proposition 9. Suppose that $k>3$. For any $N \in \mathscr{S}(CP^{k-1})$, there are invariants $s_{4i+2}(N) \in Z_2$ and $s_{4j}(N) \in Z$, for all integers i, j satisfying $6 \leq 4i+2<2(k-1)$, $4 \leq 4j<2(k-1)$. The invariants define a bijection of $\mathscr{S}(CP^{k-1})$ with

$$(\bigoplus_i Z_2) \oplus (\bigoplus_j Z).$$

The invariants s_{4j} satisfy the following relations.

Proposition 10. *If all the Pontrjagin classes of N in $\mathscr{S}(CP^{k-1})$ coincide with that of CP^{k-1}, then $s_{4j}(N)=0$ for all j.*

Concerning Z_2-invariants s_{4i+2}, the following holds. Let $\mathscr{S}(RP^{2k-1})$ denote the set of PL-homeomorphism classes of closed PL-manifolds homotopy equivalent to RP^{2k-1}. This set is known to be equal to the isomorphism classes of homotopy triangulations of RP^{2k-1}. Any $N \in \mathscr{S}(CP^{k-1})$ is the base manifold of a PL free S^1-action on S^{2k-1}. By restricting the action to $Z_2=S^0 \subset S^1$, we obtain a manifold homotopy equivalent to RP^{2k-1}. This defines a map

$$\pi^b : \mathscr{S}(CP^{k-1}) \longrightarrow \mathscr{S}(RP^{2k-1}).$$

The following holds ([4, § 14D.3]).

Proposition 11. *Let N be an element in $\mathscr{S}(CP^{k-1})$ such that $\pi^b(N)$ is PL-homeomorphic to RP^{2k-1}. Then*

$$s_{4i+2}(N)=0,$$

for all i.

Now let $B \in \mathscr{S}(CP^{k-1})$ be the base manifold of the fibration of S^{2k-1} by great circles. Then, obviously, the image $\pi^b(B) \in \mathscr{S}(RP^{2k-1})$ is PL-homeomorphic to RP^{2k-1}.

Combining the result of Section 2 with Propositions, we obtain;

Proposition 12. *The base manifold B of a fibration of S^{2k-1} by great circles is PL-homeomorphic to CP^{k-1} if $k \neq 3$.*

Now let us prove Theorem. Since the integral cohomology ring of

M is equal to that of CP^k, M is simply connected ([2, 7.23]). Thus M is homotopy equivalent to CP^k. By Allamigeon's theorem, we know that M is PL-homeomorphic to the union of the disc D^{2k} with the D^2-bundle associated with the fibration of S^{2k-1} by great circles. We write B for the base manifold of the fibration. If $k=3$, by Proposition 9, M is PL-homeomorphic to CP^3 if and only if $s_4(M)=0$. The invariant $s_4(M)$ is calculated from the first Pontrjagin class $p_1(B)$ of B. By Proposition 8 of Section 2, $p_1(B)$ is equal to $p_1(CP^2)$. Thus we have $s_4(M)=0$ and M is PL-homeomorphic to CP^3. Now suppose that $k\neq 3$. According to Proposition 12, B is PL-homeomorphic to CP^{k-1}. The Euler class of the S^1-bundle is equal to a generator of $H^2(CP^{k-1}; Z)=Z$. Thus the total space of the D^2-bundle is PL-homeomorphic to the tubular neighborhood of CP^{k-1} in CP^k. Any orientation preserving PL-homeomorphism of S^{2k-1} is isotopic to the identity. The attached manifold M is PL-homeomorphic to CP^k, which completes the proof of Theorem.

§ 4. Appendix

If $\pi: S^{2k-1}\to B$ is a fibration by great circles, we get the embedding $\theta: B\to G(2k, 2)$. Since the planes $\theta(b)$ for all $b\in B$ give a foliation of S^{2k-1}, we have the following property.

(*) For two different points b and b' in B, the planes $\theta(b)$ and $\theta(b')$ are transverse.

The converse holds.

Lemma 13. *Let $\pi: S^{2k-1}\to B$ be a principal S^1-bundle induced from the S^1-bundle $\lambda: W(2k, 2)\to G(2k, 2)$ by a smooth embedding $\theta: B\to G(2k, 2)$. Suppose that, for any different points b and b' in B, the planes $\theta(b)$ and $\theta(b')$ are transversal. Then the bundle π is a fibration of S^{2k-1} by great circles.*

Proof. Consider the union $\bigcup_b (\theta(b)\cap S^{2k-1})$. Then it covers S^{2k-1} and gives a fibration by great circles.

Now we consider the case where $k=2$. For the following discussion, see [2, p. 55]. Let $\Lambda^2\boldsymbol{R}^4$ denote the space of skew-symmetric 2-tensors. The Grassmann manifold $G(4, 2)$ is naturally identified with the set of decomposable elements of norm one in $\Lambda^2\boldsymbol{R}^4$. We have the Hodge operator $*$ from $\Lambda^2\boldsymbol{R}^4$ onto itself. The space $\Lambda^2\boldsymbol{R}^4$ is decomposed to two orthogonal subsets E_1 and E_{-1} associated to the eigenvalue 1 and -1 of $*$. let S^2_1 and S^2_{-1} be the sphere in E_1 and E_{-1} of radius $1/\sqrt{2}$. Then $G(4, 2)$ is equal to the product $S^2_1\times S^2_{-1}$. Define a bilinear map $\zeta: \Lambda^2\boldsymbol{R}^4\times\Lambda^2\boldsymbol{R}^4\to\boldsymbol{R}$ by $\zeta(a, b)=\|a_\Lambda b\|$, where $\|\ \|$ is the norm on $\Lambda^2\boldsymbol{R}^4\cong\boldsymbol{R}$. Two planes P_1 and P_2 in $G(4, 2)$ are transversal if and only if $\zeta(P_1, P_2)=0$. Represent P_1 and

P_2 by (x_1, x_2) and (y_1, y_2), where $x_1, y_1 \in S_1^2$ and $x_2, y_2 \in S_{-1}^2$. Then we have

$$\zeta(P_1, P_2)=\langle x_1, y_1\rangle-\langle x_2, y_2\rangle,$$

where $\langle\ \rangle$ is the inner product of the vector space E_1 or E_{-1}.

For a smooth map $\theta: S^2 \to G(4, 2)$, we define a smooth function $Z(\theta)$ on S^2 by $Z(\theta)(x)=\zeta(\theta(x), \theta(x'))$, by fixing x' in S^2. Thus the principal S^1-bundle $\pi: S^3 \to S^2$ induced by an embedding $\theta: S^2 \to G(4, 2)$ is a fibration by great circles if $Z(\theta)(x)=0$ only when $x=x'$. Obviously $Z(\theta)(x)=0$ at $x=x'$. We have;

Lemma 14. *For a smooth map $\theta: S^2 \to G(4, 2)$, the function $Z(\theta)$, for fixed $x' \in S^2$, is critical at $x=x'$.*

Proof. Fix P_2 in $G(4, 2)$. The function $\zeta(P_1, P_2)$ on $G(4, 2)$ is critical at $P_1=P_2$. Thus $Z(\theta)$ is also critical at $x=x'$.

Now consider the Hopf fibration $\pi_0: S^3 \to S^2$. The associated map $\theta_0: S^2 \to G(4, 2)=S_1^2 \times S_{-1}^2$ is given by $\theta_0(x)=(1/\sqrt{2}x, \alpha_0)$, where $\alpha_0=(1/\sqrt{2}, 0, 0)$. For two points $x=(x_1, x_2, x_3)$ and $x'=(x_1', x_2', x_3')$ in S^2, we have

$$\zeta(\theta_0(x), \theta_0(x'))=\langle x, x'\rangle-1/2=-1/2 \sum (x_i-x_i')^2.$$

Thus the function $Z(\theta_0)$ is critical if and only if $x=x'$. The symmetric matrix $(\partial^2 Z(\theta_0)/\partial x_i \partial x_j)$ is positive definite.

Let $\mathrm{Emb}(S^2, G(4, 2))$ denote the set of smooth embeddings of S^2 in $G(4, 2)$ with C^2-topology. Since S^2 is compact, we obtain the following.

Proposition 15. *There exists a neighborhood U of θ_0 in* $\mathrm{Emb}(S^2, G(4, 2))$ *such that the function $Z(\theta)(x, x')=\zeta(\theta(x), \theta(x'))$ is equal to zero if and only if $x=x'$, for any $x, x' \in S^2$* and *$\theta \in U$.*

Corollary 16. *In each direction in* $\mathrm{Emb}(S^2, G(4, 2))$, *there is a deformation of fibrations of S^3 by great circles starting from the Hopf fibration.*

The group of diffeomorphisms of S^2, denoted by $\mathrm{Diff}\ S^2$, acts naturally on $\mathrm{Emb}(S^2, G(4, 2))$. We denote by $\mathrm{Diff}\ S^2 \backslash \mathrm{Emb}(S^2, G(4, 2))$ the quotient space. Let $\pi: S^3 \to B$ be a fibration of S^3 by great circles. The B is diffeomorphic to S^2. Thus we have the class $\{\theta\}$ in $\mathrm{Diff}\ S^2 \backslash \mathrm{Emb}(S^2, G(4, 2))$.

Let π_1 and π_2 be two fibrations of S^3 by great circles, and let $\{\theta_1\}, \{\theta_2\} \in \mathrm{Diff}\ S^2 \backslash \mathrm{Emb}(S^2, G(4, 2))$ be the associated classes. We say that π_1 and π_2 are isometric if there exists a bundle map F from π_1 to π_2 such that F is an isometry of S^3 onto itself.

The group $O(4)$ acts naturally on $G(4, 2))$ and on $\mathrm{Diff}\, S^2\backslash\mathrm{Emb}(S^2, G(4, 2))$. We denote by $\mathrm{Diff}\, S^2\backslash\mathrm{Emb}(S^2, G(4, 2))/O(4)$ the quotient space.

Proposition 17. *Two fibrations π_1 and π_2 of S^3 by great circles are isometric if and only if the classes $\{\theta_1\}$ and $\{\theta_2\}$ in* $\mathrm{Diff}\, S^2\backslash\mathrm{Emb}(S^2, G(4, 2))/O(4)$ *are equal.*

Remark that we can choose the neighborhood U in Proposition 15 such that U is invariant by the actions of $\mathrm{Diff}\, S^2$ and $O(4)$. The space $\mathrm{Diff}\, S^2\backslash U/O(4)$ is of infinite "dimension".

References

[1] A. Allamigeon, Propriété globales des espaces de Riemann harmoniques, Ann. Inst. Fourier, **15** (1965), 91–132.
[2] A. Besse, Manifolds all of whose geodesics are closed, Ergebnisse der Math., **93** (1978), Springer.
[3] H. Sato, On the manifolds of periodic geodesics, this volume.
[4] C. T. C. Wall, Surgery on compact manifolds, Academic press (1970), London.

Department of Mathematics
Tôhoku University
Sendai 980
Japan