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CHAPTER 25

Numerical Solution of the 2-D Incompressible Non-stationary Navier-Stokes Equations by Adapted Projection Method, *by A.Seck, M. Ndiaye, A. Sy and D. Seck*

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Abstract. A fast numerical method to solve a two dimensional non stationary Navier-Stokes problem (projection method) is introduced and studied. (See next page for the full abstract).

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Full Abstract. In this paper, we propose a fast numerical method to solve a two dimensional non stationary Navier-Stokes problem: the projection method. The advantages of this method are, on the one hand it can be applied all types of flows (laminar, viscous and turbulent) and on the other hand, its convergence analysis is faster, compared to classical numerical methods (Newton method, Jacobian Fixed method).

1. Introduction

From a mathematical viewpoint, one of the most intriguing unsolved questions concerning the Navier-Stokes equations and closely related to turbulence phenomena is the regularity and uniqueness of the solutions to the initial value problem. More precisely, given a smooth datum at time zero, will the solution of the Navier-Stokes equations continues to be smooth and unique for all time? This question was posed in 1934 by Leray and is still without answer. Let us note that the authors would say that they are not aware that this question is solved.

The mathematical theory of the Navier Stokes equations is based on the use of functions spaces, which are at the heart of the modern theory of partial differential equations.

In the nineteenth century, finding the exact solution to the Navier Stokes equation were studied. In the twentieth century, the concept of weak solution was introduce. Only the existence of the solutions can be ensured. The uniqueness question is among the most important unsolved problems in fluid mechanics. Some particular results of existence, uniqueness, and regularity of the Navier Stokes equations are become classical. They can be found in many references on the mathematical theory of the Navier Stokes equations (see for instance Constantin and Foias (1988), Ladyzhenskaya

(1963), Lions (1969), Temam (1979), Lions (1996) and the references therein.

In the 3-dimensional evolution case, the mathematical theory is not yet complete. It is known that the weak solutions exists for all time, but it is not known whether they are unique. On the other hand, the strong solution is unique and exists on a certain finite time interval, but it is not known whether they exist for all time (see Temam (1984) for example).

In the 2-dimensional case, the mathematical theory is fairly complete. The weak solutions turn out to be more regular and are, in fact, strong solutions. Moreover, the solutions is unique for a given initial condition and exist for all time (see Temam (1984), Theorem 1.3). Even if the theoretical study of the 2D Navier Stokes is complete, the proposed numerical methods have some limits. Throughout this paper, we propose a numerical method to solve 2D incompressible non-stationary Navier-Stokes problem. The paper is organized as follows: In section 2, we recall to some theoretical results for the Navier-Stokes equation. The section 3 presents the proposed numerical method. The results of numerical experiments for two dimensional and for all kind of flows are presented, in order to show the possibilities of the proposed method.

2. About 2-D incompressible non-stationary Navier Stokes system

2.1. Some theoretical aspects of a 2-D non-stationary Navier Stokes system.

The Navier-Stokes equations describing the $n-$dimensional motion of a viscous and incompressible fluid are as follows:

$$\varrho\left(\frac{\partial u_i}{\partial t}+\sum_{j=1}^{n}u_j\frac{\partial u_i}{\partial x_j}\right)-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\sigma_{ij}=\varrho f_i,\quad 1\ \leq\ i,\ j\ \leq\ n, \tag{2.1}$$

with the incompressibility condition

$$div\ u=\sum_{i=1}^{n}D_{ii}(u)=0, \tag{2.2}$$

where

$$\left\{ \begin{array}{lcl} \sigma_{ij} & = & -P\delta_i^j + 2\mu D_{ij}(u) \\ D_{ij}(u) & = & \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \end{array} \right\} \quad 1 \leq i, j \leq n. \tag{2.3}$$

We also recall the following spaces

$$H = \{v \in (\mathcal{D}(\Omega))^n; \quad div\ v = 0\}\ and\ \ \mathrm{V} = \{v \in (H_0^1(\Omega))^n\ \ div\ v = 0\} \tag{2.4}$$

In these equations, the vector u is the velocity of the fluid, ϱ is its density (assumed to be constant), $\mu > 0$ is the viscosity (also assumed to be constant) and P is its pressure; (σ_{ij}) is the stress tensor and the vector f represents a density of body forces per unit mass (gravity for instance). We set

$$p = \frac{P}{\varrho} \quad and \quad \nu = \frac{\mu}{\varrho}.$$

Here p is the kinematic pressure and ν the kinematic viscosity, but for sake of simplicity they will be called pressure and viscosity. Let $T > 0$ a real. Thus the global incompressible and non-stationary Navier-Stokes system with Dirichlet conditions is written as follows:

$$\left\{ \begin{array}{l} \dot{u} - \nu\Delta u + \sum_{i=1}^n u_i D_i u + \nabla p = f \quad in\ \ \Omega \times (0,T); \\ div(u) = 0 \qquad in\ \ \Omega \times (0,T); \\ \gamma(u) = g, \quad on \quad \partial\Omega \times (0,T) = \Gamma_t; \\ u(0) = u_0 \qquad in\ \ \Omega \times (0,T) \end{array} \right. \tag{2.5}$$

where $\dot{u} = \frac{\partial u}{\partial t}$ and $u_0 \in H$ an initial condition (*i.e.* at time $t = 0$):

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{2.6}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a Lipschitz continuous boundary Γ and g is a regular given vector function and u_0 is a given divergence-free vector field.

In order to write (2.5) in a variational form, let us introduce a trilinear functional,

$$b(u,v,w) = \sum_{i,j=1}^{n} \int_{\Omega} u_i D_i v_j w_i dx. \tag{2.7}$$

The following lemmas give useful properties of b, there proofs can be found in Girault and RavaiartT (1968) and Temam (1979).

LEMMA 49. *The trilinear form b is defined and continuous on $H_0^1(\Omega)^n \times H_0^1(\Omega)^n \times (H_0^1(\Omega)^n \cap L^n(\Omega))$, Ω bounded or unbounded, for any dimension of $\mathbb{R}^n$.*

LEMMA 50. *Let $u \in H^1(\Omega)^n$ with $div\ u = 0$ and $\gamma(u) = 0$ and let v and $w \in H_0^1(\Omega)^n \cap L^n(\Omega)$; then*

$$b(u,v,v) = 0 \tag{2.8}$$

$$b(u,v,w) = -b(u,w,v) \tag{2.9}$$

And the following estimations holds:

$$\left| \int_{\Omega} u_i (D_i v_j) w_i dx \right| \leq |u_i|_{L^{\frac{2n}{n-2}}(\Omega)} |D_i v_j|_{L^2(\Omega)} |w_j|_{L^n(\Omega)} \tag{2.10}$$

$$\|u_i\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C(\Omega) \|u\|_{H_0^1(\Omega)^n}$$

$$\|D_i v_j\|_{L^2(\Omega)} \leq \|v_j\|_{H_0^1(\Omega)}$$

$$\|w_i\|_{L^n(\Omega)} \leq \|w_i\|_{H_0^1(\Omega) \cap L^n(\Omega)}$$

It follows (2.10) and the above estimations that:

$$|b(u,v,w)|_{\mathcal{L}_3(\Omega)} \leq C(\Omega) \|u\|_{H_0^1(\Omega)} \|v_j\|_{H_0^1(\Omega)^n} \|w_i\|_{H_0^1(\Omega)} \tag{2.11}$$

with

$$C(\Omega) = \begin{cases} \frac{2}{3}|\Omega|^{1/6}, & if \;\; n = 3 \\ \frac{|\Omega|^{1/2}}{2}, & if \;\; n = 2 \end{cases}$$

Now let

$$a(u,v)=\int_\Omega \nabla u.\nabla v dx \tag{2.12}$$

There are not information on $\dot{u}$ and p more than

$$\dot{u}-\nu\Delta u+\sum_{i=1}^{n} u_i D_i u+\nabla p=f \quad in \;\; \Omega\times(0,T) \tag{2.13}$$

Let us takes $\varphi\in V$. Thus, we have

$$(\nabla p,\varphi)=0 \text{ in } \mathcal{D}'(0,T)^n$$

and (2.13) gives

$$(\dot{u},\varphi)=-\nu a(u,\varphi)-b(u,u,\varphi)+(f,\varphi). \tag{2.14}$$

Thanks to the Lemma 50, $b(u,u,\varphi)=-b(u,\varphi,u)$, then (2.14) is equivalent to

$$(\dot{u},\varphi)=-\nu a(u,\varphi)+b(u,\varphi,u)+(f,\varphi). \tag{2.15}$$

The variational form associated to (2.5) is as follows: given $f\in L^2(0,T;V)$, $T>0$ and $u_0\in H$, the addressed question is to find $u\in L^2(0,T;V)\cap L^\infty(0,T;H)$ such that:

$$\begin{cases}(\dot{u},v)+\nu a(u,v)+b(u,u,v)=(f,v) & \text{for all } v\in V\\ u(0)=u_0\end{cases}. \tag{2.16}$$

Remark 25. *In the case of incompressible Stoke's system, the pressure can be interpreted as a Lagrange multiplier. In fact the pressure $p\in L^2_{loc}(\Omega)$.*

If (u,p) is a solution to problem (2.5), then u is a solution to problem (2.16). The converse is also true as stated next. The following theorem gives the existence and uniqueness (in two dimension) of a solution to (2.16):

Theorem 71. *If $n\leq 4$ and ν sufficiently large or f sufficiently small so that*

$$\nu^2>c(n)\|f\|_{V'}$$

then the problem (2.16) admits a unique solution.

Proof. For the proof, we refer for example, Temam (1984) Chapter II page 167, theorem 1.3. □

2.2. Presentation and discretization in time of the Navier Stokes equations . In this subsection, we present the Navier-Stokes equations in the temporal variable. We shall focus on the condition at the spatial-periodic limits and the Dirichlet types in two dimensions. H and V are defined by (2.4).

We consider on time interval $[0,\ T]$, the weak form of the Navier-Stokes equations in the following sense : for $u_0\ \in\ H$ and $f\ in\ L^2(Q\times(0,T))^2$ (or $L^2(0,T,H)$), we look for a function $t\mapsto u(t)$ from $[0,T]$ in V, satisfying:

$$\tag{2.17} \begin{cases} \frac{d}{dt}(u(t),v)_0+\nu a(u(t),v)+b(u(t),u(t),v)=(f(t),v)\,\forall v\in V, \\ u(0)=u_0 \end{cases}$$

Let N be an integer great enough and let us set $k=\delta t=T/N.$

We are going to consider a discretized version in the time of (2.17) (with the temporal discretization $k=\delta t$) that will furnish a set of elements of $V,\ u_0,\ u_n$ where u_n will be someway an approximation of u at time δt at the neighborhood of the point in Ω. We should first define the elements $f^1,\ f^2,\ `\ldots,f^n$ of $L^2(\Omega)^2$ or H. When f is continuous in time, we can define

$$\tag{2.18} f^n=f(n\delta t),\quad n=1,\cdots,N.$$

If the data are note regular, we define f^n as temporal means of f defined as

$$\tag{2.19} f^n=\frac{1}{\delta t}\int_{(n-1)\delta t}^{n\delta t}f(t)dt,\quad n=1,\cdots,N.$$

Thereafter, we suppose that the f^n are defined by (2.19), the case (2.18) is a particular case of (2.19) and can be treated in the same way.

We restrict ourselves to a unique discretized schema, the Euler complete implicit schema.

We consider the Eulerian implicit scheme and define recursively the element u_n of V as follows ($u^0 \in H$ where $u_0 \in H$):

$$u^0 = u_0 \quad \text{+ initial data} \tag{2.20}$$

where $u^0, \ \ldots, u^{n-1}$ are known, we define u^n as an element of V which satisfies

$$\frac{1}{\delta t}(u^n - u^{n-1}, v)_0 + \nu(u^n, v) + b(u^n, u^n, v) = (f^n, v), \forall v \in V. \tag{2.21}$$

Equation (2.21) is a non linear equation for u^n. The definition and properties of the trilinear form b are given by (2.7) and the lemma 50. The consistency of (2.21) is given in the following subsection.

The equation (2.21) looks like to the stationary Navier-Stokes equation and the existence of its solution can be proven exactly as this latter. Let us rewrite (2.21) as

$$\alpha(u, v)_0 + \nu((u, v)) + b(u, u, v) = (g, v)_0, \quad \forall v \in V \tag{2.22}$$

where $u = u^n$, $\alpha = 1/\delta t$ and $g = g^n = \alpha u^{n-1} + f^n$.

We construct a Galerkin approximation of (2.22). Let $(V_h)_{h \in \mathbb{N}}$ be an increasing sequence of linear subspace of V with finite dimension

$$\bigcup_{h \in \mathbb{N}} V_h \ \textbf{is dense in } V \tag{2.23}$$

Then, for any h we search u_h and V_h such that

$$\alpha(u_h, v)_0 + \nu((u_h, v)) + b(u_h, u_h, v) = (g, v)_0, \quad \forall v \in V_h. \tag{2.24}$$

The equation (2.24) is equivalent to a set of finite coupled quadratic equation for the components of u_h in a basis of V_h. The existence of u_h is not obvious but relies on the (Brouwer's (or Schauder) fixed point theorem: for mor details, (see Temam (1984), theorem 1.2 chapter III). Then when we set $v = u_h$ on (2.24), we see that the sequence u_h is bounded on V, independently of h thanks to Lemma 50. Then

$$\alpha|u_h|_0^2 + \nu \|u_h\|^2 = (g, u_h)_0 \leqslant |g|_0|u_h|_0 \leqslant \lambda_1^{-1/2}|g|_0\|u_h\| \tag{2.25}$$

$$\leqslant \frac{\nu}{2}\|u_h\|^2 + \frac{1}{2\nu\lambda_1}|g|_0^2, \tag{2.26}$$

$$2\alpha|u_h|_0^2 + \nu \|u_h\|^2 \leqslant \frac{1}{\nu\lambda_1}|g|_0^2 \tag{2.27}$$

where $\lambda_1 > 0$ is a constant.

We can extract an sub-sequence of u_h, still noted u_h that converges weakly in V. We can then go to a limit in (2.24) with v fixed in $\bigcup_{h\in\mathbb{N}} V_h$, *i.e.*, $v \in V_{h_0}$, for a certain fixed h_0, and for any $h > h_0$; particularly, for $v \in V_h$, we can show that

$$b(u_h, v_h, w_h) \to b(u, v, w).$$

Thus passing to the limit in (2.24), we obtain

$$\alpha(u, v)_0 + \nu((u, v)) + b(u, u, v) = (g, v)_0, \quad \forall v \in \bigcup V_h. \tag{2.28}$$

By density (2.28) is still valid for any $v \in V$ and u solution of (2.22).

3. The proposed method: Projection method adapted to Navier-Stokes equations

Several numerical methods are proposed in order to derive simulation of the Navier-Stokes system. We can quote Newton method, Jacobian fixed method, method using equivalent optimization solution. We recall here briefly the description of these methods in order to point out there limits.

3.1. Newton method. The Newton method for approximation of the solution of the Navier-Stokes system is described as follows. For an initial data $u^{(0)} \in V_0^h$, we generate a couple of sequences $(u^m, p^m), \quad m = 1\dots$ by solving sequences of linear problems:

$$\begin{aligned} & a(u^{(m)}, v^h) + b(u^{(m)}, u^{(m-1)}, v^h) + b(u^{(m-1)}, u^{(m)}, v^h) + a(v^h, p^{(m)}) \\ & = (f, v^h) + b(u^{(m-1)}, u^{(m-1)}, v^h) \quad \text{for all } \ v^h \in V_0^h \end{aligned} \tag{3.1}$$

and

$$a(u^{(m)}, q^h) = 0 \quad \text{for all } \ q^h \in V_0^h \tag{3.2}$$

where $a(.,.)$ and $b(.,.,.)$ denote the bilinear and the trilinear forms associated with the weak formulation of the Navier Stokes system.

The main disadvantage of this method is that, we have to solve an algebraic linear system at each iteration. Another disadvantage is the convergence of Newton method is guaranteed only when $u^{(0)}$ is sufficiently closed to the solution.

3.2. Jacobian fixed method. The fixed Jacobian method allows to keep the matrix coefficients, so we have not to solve an algebraic system at each iteration. Unfortunately, this fixed Jacobian method is at best linearly convergent, and it is required ”best” initial assumptions to ensure convergence.

In the practical implementations of the method, we note that the Jacobian matrix must be reevaluated after M iterations, where M is a specified positive integer chosen empirically to maintain convergent iterations. In this case, a linear system with a new matrix of coefficients must be solved every M iterations.

3.3. The proposed method. In order to overcome this difficulties, we propose a new method by adapting the projection method to the Navier-Stokes system, which we study in details in the following. We shall emphasize the time discretization by the operator decomposition method, since such method provide an efficient way to decouple the main difficulties of the problem, namely the in-compressibility and the non-linearity.

For a problem of the following form

$$(\mathcal{P}) \quad \begin{cases} \frac{\partial \varphi}{\partial t} + A(\varphi) = 0 \\ \\ \varphi(0) = \varphi_0 \end{cases} \tag{3.3}$$

where A is a non linear operator which can be written as : $A = A_1 + ... + A_r$, $r \geq 2$ and A_i is an operator simplest than A. The method consists in solving a succession of r problems simplest than the initial one.

There are many operators decomposition methods, but we shall focus to the one which allows us to apply projection method to the Navier Stokes. It is based on the projection of the velocity field, approached by a prediction step on a zero divergence field. The novelty of this method concerns how the projection is made, directly operating on all the components of the velocity field. A highly implicit algorithm allows us to maintain all physical boundary conditions of the problem during the solution steps.

The main difficulties for solving Navier-Stokes equations are the treatment of the non linear terms and the in-compressibility condition $div\, u = 0$.

The introductory works on numerical analysis of the Navier Stokes equation had been concreated to the treatment of the condition of incompressibility for the general discretization. The difficulties related to this condition can be overcome by reducing the solution of the problem to that of a family of problems or by using the vorticity of the flux function.

The treatment of incompressible condition on numerical analysis of Navier Sokes equations is related to the first and the second types of projections methods. The first type of projection method contains the study of penalty method for slightly compressible fluid (*see for example:* Chorin (1967a), Chorin (1967b), Chorin (1968a), Chorin (1968b), Temam (1979), Temam (1966), Temam (1968a), Temam (1968b), Temam (1969a), Temam (1969b) and Yanenko (1971)). This effort is continued in the years seventies and at the beginning of the $80s$ with the development of the finite elements. The objective was to approach the function of the null divergence vector by the general functions, in relation to the UZAWA algorithm or to build V_h spaces that approach V and take into consideration the in-compessibility condition in their construction (or to consider the condition of in-compressibility like a constraint (of local type) and the pressure as a Lagrange's multiplier) in relation with mixed finite elements and the *inf-sup* condition (see for example Brezzi and Fortin (1991), Girault and RavaiartT (1968), Gresho and Sani (1996), Gunzburger (1989), Pironneau (1988), Temam (1984), Thomasset (1981)). Now, there exists several methods for the treatment of the in-compressibility condition but difficulties related to turbulence still exist. Therefore, it needs an improvements for the use of engineering applications.

Our objective is to review some methods of treatment of the in-compressibility condition wich do not dependent of the discretization method. In fact, the proposed algorithm should be reconsidered and adapted for every specific discretization method.

Let us consider an evolution equation

$$\frac{du}{dt} + \mathcal{A}u = 0, \tag{3.4}$$

where $\mathcal{A}$ is a linear or nonlinear operator, such that

$$\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r \tag{3.5}$$

The fractional step method is a decomposition method which reduces the resolution of (3.4) to that of equations

$$\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_i \tag{3.6}$$

This is related to the so-called Trotter formula which asserts that, for suitable operators A_i, and for $t > 0$, the operators $[(I + (t/n)A_1)^{-1} \cdots (I + (t/n)A_r,)^{-1}]^n$ converge to $exp[t(A_1 + \cdots + A_r)]$ as $n \to \infty$. This decomposition method is particularly suitable when the equations (3.4) are not easier to solve or when the operators A_i are of very different types and the solution of (3.6) needs different techniques. For a general presentation of the fractional steps method, see Marchuk (1990), Yanenko (1971) and for the analysis of the method, see Temam (1968a). The fractional step method is also well suited in some parallel computing cases.

The projection method is a kind of the fractional steps method adapted to the Navier-Stokes equations. It was introduced and studied in Chorin

(1967a), Chorin (1967b), Chorin (1968a), Chorin (1968b) and Temam (1969a), Temam (1969b) see also Temam (1979), Temam (1966), Temam (1968a), Temam (1968b).

In one of its forms it consists in writing the full Navier-Stokes operator in the form (3.4)-(3.5) with $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ $(r = 2)$ where A_l where loosely speaking, A_1 is the sum of the viscosity and inertial terms while $\mathcal{A}_2$ accounts for the pressure term and the incompressibility condition. We shall recall and study in some details this form of the projection method .

We now describe this scheme. For the sake of simplicity we restrict ourselves to the nonslip (Dirichlet) boundary condition and we consider a semidiscretization (*a.e.*, discretization in time but not in space).

Let $[0, T]$ be the time interval, let N be an integer and $\delta t = t/N$. We define recursively two families of elements u^n, $u^{n+\frac{1}{2}}$, $n = 0, \cdots, N$. We start with

$$u^0 = u_0 \in H_0^1(\Omega)^2 \tag{3.7}$$

and when the u^m are known for $m = 0, \cdots, n$, we define $u^{n+\frac{1}{2}}$ as the solution in $H_0^1(\Omega)^2$ of

$$\frac{1}{\delta t}(u^{n+\frac{1}{2}} - u^n) - \nu\Delta u^{n+\frac{1}{2}} + (u^{n+\frac{1}{2}}.\nabla)u^{n+\frac{1}{2}} + \underline{\frac{1}{2}(\nabla.u^{n+\frac{1}{2}})u^{n+\frac{1}{2}}} = f^{n+1} \tag{3.8}$$

where $f = (f^1, \cdot, f^{N+1})$ is defined from $[0; T]$ to $L^2(\Omega)^d$ (or $H_0^1(\Omega)^2$) by,

$$f^n = f(n\delta t), \quad n = 1, \cdots, N+1 \tag{3.9}$$

and if the non regular data are considered we define f^n as the average of f defined by

$$f^n = \frac{1}{\delta t}\int_{(n-1)\delta t}^{n\delta t} f(x)dx \quad n = 1, \cdots, N+1 \tag{3.10}$$

Then we define $u^{n+1} \in H_0^1(\Omega)^2$ and a function $q^n \in L^2(\Omega)$ by setting

$$\frac{1}{\delta t}(u^{n+1} - u^{n+\frac{1}{2}}) + \nabla q^{n+1} = 0 \; in \; \Omega \tag{3.11}$$

$$div \; u^{n+1} = 0 \; in \; \Omega \tag{3.12}$$

$$u^{n+1}.\nu = 0 \; on \; \partial\Omega \tag{3.13}$$

where ν stands to the outward normal vector to Ω. At this point we observe that (3.8) for $u^{n+\frac{1}{2}}$ is equivalent to

$$u^{n+\frac{1}{2}} \in H_0^1(\Omega)^2 \quad \text{and}$$

$$(u^{n+\frac{1}{2}}, v)_0 + \nu\delta t((u^{n+\frac{1}{2}}, v)) + \delta t\tilde{b}(u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}, v) = (g^n, v)_0, \quad \forall v \in H_0^1(\Omega)^2. \tag{3.14}$$

where

$$g^n = u^n + \delta t f^{n+1} \tag{3.15}$$

and

$$\tilde{b}(\varphi, \psi, \theta) = b(\varphi, \psi, \theta) + b_1(\varphi, \psi, \theta) \tag{3.16}$$

$$b_1(\varphi, \psi, \theta) = \frac{1}{2}\int_\Omega (div\varphi)\psi \cdot \theta \; dx. \tag{3.17}$$

The existence of $u^{n+1/2}$ solution of (3.14) will be proved in the next section. Without the incompressibility of the flow, the skewness property (2.8) is not valid for functions belonging to $H_0^1(\Omega)^2$ and not to V. As proposed in Temam (1966), the skewness property which guarantees that the nonlinear term is conservative can be recovered by adding b_1 to b and this corresponds to adding the underlined term in the equation (3.8), the result can be obtained, because we have beside (3.16):

$$\tilde{b}(\varphi, \psi, \theta) = \frac{1}{2}\left\{b(\varphi, \psi, \theta) - b_1(\varphi, \theta, \psi)\right\}, \quad \forall \varphi, \psi, \theta \in H_0^1(\Omega)^2 \tag{3.18}$$

In fact, for any $\varphi, \psi, \theta \in H_0^1(\Omega)^2$ smooth, thanks to Stokes formula, we have

$$2b_1(\varphi,\psi,\theta) = \sum_{i,j=1}^{2}\int_\Omega \frac{\partial \varphi_i}{\partial x_i}\psi_j\theta_j \; dx = -\sum_{i,j=1}^{2}\int_\Omega \varphi_i \frac{\partial}{\partial x_i}(\psi_j\theta_j)\; dx$$

REMARK 26. *The underlined term is not always present in actual computations; this can be explained by the fact that $divu^{n+1/2}$ remains small as we shall see, and therefore the underlined term in the equation (3.8) is small. However this term is indispensable in the analysis of the method.*

Note also that, with a few changes in what follows, we can consider a linearized version of Equation (3.8) which reads:

(3.19) $$\frac{1}{\delta t}(u^{n+1/2}-u^n) - \nu\Delta u^{n+1/2} + (u^n\cdot\nabla)u^{n+1/2} = f^{n+1}$$

The analog of the underlined term in the equation 3.8 is not needed since $div\, u^n = 0$ ($u^n \in H_0^1(\Omega)^2$). Let us now consider the second step (3.11). Referring to the decomposition of $L^2(\Omega)^2$ as the sum of $H_0^1(\Omega)^2$ and its orthogonal component, we see that $u^{n+1/2}\in H_0^1(\Omega)^2$ and

(3.20) $$(u^{n+1}, v)_0 = (u^{n+1/2}, v)_0, \quad \forall v \in H_0^1(\Omega)^2,$$

which amounts to saying that

(3.21) $$u^{n+1} = P_H u^{n+1/2},$$

where $P = P_H$ is the orthogonal projector in $L^2(\Omega)^2$ onto $H_0^1(\Omega)^2$. Alternatively if we consider the Helmholtz decomposition of $u^{n+1/2}$, then

(3.22) $$u^{n+1/2} = u^{n+1} + \nabla\varphi^n,$$

for some function $\varphi^n \in L^2(\Omega)$, so that $\varphi^n = \delta t q^{n+1}$ with q^{n+1} defined up to an additive constant by solving the Neumann problem related to the projector P_H

(3.23) $$\Delta q^{n+1} = \frac{1}{\delta t} divu^{n+1/2} \text{ in } \Omega$$

(3.24) $$\frac{\partial q^{n+1}}{\partial \nu} = \left(\frac{1}{\delta t}u^{n+1/2}\cdot\nu\right) = 0 \text{ on } \partial\Omega$$

Since usually q^{n+1} is defined up to a constant which can be chosen by imposing the condition

$$\int_{\Omega} q^{n+1}\, dx = 0. \tag{3.25}$$

Comparing (3.23) to the divergence of (3.8), and since $divu^n = 0$, we see that

$$\begin{aligned} \Delta q^{n+1} \\ = \; div(\nu\Delta u^{n+1/2} + f^{n+1}) - div\left\{(u^{n+1/2}\cdot\nabla)u^{n+1/2} + \frac{1}{2}(\nabla\cdot u^{n+1/2})u^{n+1/2}\right\}. \end{aligned} \tag{3.26}$$

REMARK 27. *We will show below that the u^n and $u^{n+1/2}$ approximate the exact velocity u; it was shown also in Temam (1969b) that the q^{n+1} approximate the exact pressure. A solution to this paradox was proposed in Temam (1991), along the following lines:*

(i) The convergence of the q to the pressure which is proved in Temam (1969b) holds in a very weak sense, close to the distribution sense. Hence this allows the q^{n+1} to be different from their limit in a thin numerical boundary layer near $\partial\Omega$ (for the details see Temam (1991)).

(ii) If the pressure is poorly approximated by q^{n+1}, how could the velocity be satisfactorily approximated by the u^n and $u^{n+1/2}$? This point which is hard to understand if we look at the Navier-Stokes in their initial form involving the pressure (conservation of momentum equation) becomes easy to understand and even transparent if we look at the functional form of the Navier-Stokes equations which appears as an evolution equation for u only. In fact the resolution of the contradiction proposed in Temam (1991) consists in considering the q as a necessary computational step to determine the projection (3.21), and not as an approximation of the pressure. An accurate approximation of the pressure must be determined otherwise or one can consider one of the many modified forms of the projection scheme.

(iii) Many authors consider the quantity

$$p^{n+1} = (I - \nu\delta t\Delta)q^{n+1} \tag{3.27}$$

as a more suitable approximation of the pressure at time $t_{n+1} = (n+1)\delta t$. This is justified as follows. We infer from (3.11) that

$$u^{n+1/2} = u^{n+1} + \delta t \nabla q^{n+1} \tag{3.28}$$

and upon substituting this expression of $u^{n+1/2}$ in (3.8), we obtain

$$\frac{1}{\delta t}(u^{n+1} - u^n) - \nu \Delta u^{n+1} + (u^{n+1/2} \cdot \nabla) u^{n+1/2} + \frac{1}{2}(\nabla \cdot u^{n+1/2}) u^{n+1/2} + \nabla p^{n+1} = f^{n+1}. \tag{3.29}$$

This expression is consistent with a time discretization of the conservation of momentum equation ($\rho\left\{\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right\} - \mu\Delta u + \nabla p = f$). Also it leads, for p^{n+1}, to a Neumann problem consistent with the Neumann problem corresponding to the exact pressure:

$$\begin{aligned}
\Delta p^{n+1} &= \nabla \cdot \left\{ f^{n+1} - \mathcal{B}(u^{n+1/2}) \right\} \quad \textbf{\textit{in }} \Omega \\
\frac{\partial p^{n+1}}{\partial \nu} &= \nu \cdot \left\{ f^{n+1} - \mathcal{B}(u^{n+1/2}) \right\} \quad \textbf{\textit{on }} \partial\Omega \\
\mathcal{B}(u^{n+1/2}) &= (u^{n+1/2} \cdot \nabla) u^{n+1/2} + \frac{1}{2}(\nabla \cdot u^{n+1/2}) u^{n+1/2}.
\end{aligned} \tag{3.30}$$

PROPOSITION 28 (Stability analysis). *Let $u^{n+\frac{1}{2}}$ and u^n be defined by scheme (3.8), (3.11). Then the following estimates holds*

$$\begin{aligned}
|u^{n+1/2}|_0^2 - |u^n|_0^2 + |u^{n+1/2} - u^n|_0^2 + 2\nu\delta t \|u^{n+1/2}\|^2 \\
= 2\delta t (f^{n+1}, u^{n+1/2})_0 \leqslant 2\delta t |f^{n+1}|_0 |u^{n+1/2}|_0 \\
\leqslant 2c\delta t |f^{n+1}|_0 \|u^{n+1/2}\|_0 \leqslant \nu\delta t \|u^{n+1/2}\|^2 + \frac{c\delta t}{\nu}|f^{n+1}|_0^2
\end{aligned} \tag{3.31}$$

Proof. Replacing v by u^{n+1} in (3.20) and using Poincaré Inequality, we obtain similarly

$$|u^{n+1}|_0^2 - |u^n|_0^2 + |u^{n+1} - u^{n+1/2}|_0^2 = 0 \tag{3.32}$$

and in particular, as expected for a projection

$$|u^{n+1}|_0 \leqslant |u^{n+1/2}|_0 \tag{3.33}$$

We add (3.32) to (3.33)

$$|u^{n+1}|_0^2 - |u^n|_0^2 + |u^{n+1} - u^{n+1/2}|_0^2 + \nu\delta t \|u^{n+1/2}\|^2 \leqslant \frac{c\delta t}{\nu}|f^{n+1}|_0^2, \tag{3.34}$$

$n = 0, \cdots, N-1$. Adding these inequalities from $n = 0$ to $N-1$ leads to

$$|u^N|_0^2 + \sum_{n=0}^{N-1} \left\{ |u^{n+1} - u^n|_0^2 + |u^{n+1/2} - u^n|_0^2 \right\} + \nu \delta t \sum_{n=0}^{N-1} \|u^{n+1/2}\|^2 \leqslant |u_0|_0^2 + \frac{c\delta t}{\nu} \sum_{n=0}^{N-1} |f^{n+1}|_0^2, \tag{3.35}$$

the right-hand side of (3.35) is less than or equal to

$$K_1 = |u_0|_0^2 + \frac{c}{\nu} \int_0^T |f(t)|_0^2 \, dt \tag{3.36}$$

Also upon adding the relations (3.34) for $n = 0, \cdots, m$, for some m, $0 \leqslant m \leqslant N-1$, we see that

$$|u^m|_0^2 \leqslant K_1, \quad 0 \leqslant m \leqslant N-1 \tag{3.37}$$

We infer from (3.35)-(3.37) and (3.33) the following estimates which are independent of δt:

$$\sum_{n=0}^{N-1} \left\{ |u^{n+1} - u^{n+1/2}|_0^2 + |u^{n+1/2} - u^n|_0^2 \right\} \leqslant K_1, \nu \delta t \sum_{n=0}^{N-1} \|u^{n+1/2}\|^2 \leqslant K_1, \quad |u^{m+1/2}|_0^2 \, |u^m|_0^2 \leqslant K_1, \quad \text{for } m = 0, \cdots, N-1. \tag{3.38}$$

We introduce the approximate functions u_{1k}, u_{2k}, $\tilde{u}_k$ ($k = \delta t$) defined as follows:

$$u_{1k} = u^{n+1/2} \text{ for } t \in [n\delta t, (n+1)\delta t[\; n = 0, \cdots, N-1,$$
$$u_{2k} = u^{n+1} \text{ for } t \in [n\delta t, (n+1)\delta t[\; n = 0, \cdots, N-1,$$

$\tilde{u}_k : [0; T] \to H$ is continuous, linear on each interval $](n-1)\delta t, n\delta t[$ and equal to u^n at $n\delta t$, $n = 0, \cdots, N$.

Then (3.38) yields the lemma hereafter. □

LEMMA 51. *As $k = \delta t \to 0$, u_{1k}, u_{2k} and $\tilde{u}_k$ remain bounded in $L^\infty(0,T;L^2(\Omega)^2)$; u_{1k} and u_{2k} remain bounded in $L^2(0,T;H^1(\Omega)^2)$, and the same result holds for $\tilde{u}_k$ if $u_0 \in V$.*
Furthermore, $u_{1k} - u_{2k}$ et $u_{2k} - \tilde{u}_k$ converge to 0 in $L^\infty(0,T;L^2(\Omega)^2)$ as $k = \delta t \to 0$, their norms being bounded by $(K_1\Delta/3)^{1/2}$.

Proof. The main point which remains to be proved in Lemma 51 is the fact that u_{2k} is bounded in $L^2(0,T;H^1(\Omega)^2)$. This follows from the continuity of the projector in $H^1(\Omega)^2$ (besides its continuity in $L^2(\Omega)^2$, see Temam (1984), chapter I, (1.47)). Hence there exists a constant c depending only on Ω such that

$$\|P_H\varphi\|_{H^1(\Omega)^2} \leqslant c\|\varphi\|_{H^1(\Omega)^2}, \quad \forall \varphi \in H^1(\Omega)^2. \tag{3.39}$$

Since $u^{n+1} = P_H u^{n+1/2}$, we find with (3.38)

$$\|u^{n+1}\|_{H^1(\Omega)^2} \leqslant c\|u^{n+1/2}\| \quad n = 0, \cdots, N-1 \delta t \sum_{n=0}^{N-1} \|u^{n+1}\|^2_{H^1(\Omega)^2} = \int_0^T \|u_{2k}(t)\|^2_{H^1(\Omega)^2}\, dt \leqslant \frac{c}{\nu}K_1.$$

For $v \in V$, both equations (3.14) and (3.20) are valid; therefore adding these relations we see that

$$\begin{aligned} \frac{1}{\delta t}(u^{n+1} - u^n, v)_0 + \nu((u^{n+1/2}, v)) + \tilde{b}(u^{n+1/2}, u^{n+1/2}, v) \\ = (f^{n+1}, v)_0, \quad \forall v \in V\; n = 0, \cdots, N-1. \end{aligned} \tag{3.40}$$

This is equivalent to

$$\begin{aligned} &\forall t \in [0;T]\, \forall v \in V, \\ &\frac{\mathrm{d}}{d}(u(t), v)_0 + \nu((u_{1k}(t), v)) + \tilde{b}(u_{1k}(t), u_{1k}(t), v) = (f_k(t), v)_0, \end{aligned} \tag{3.41}$$

where f_k is defined by, $f_k(t) = f^n$ for $t \in [(n-1)\delta t; n\delta t[$, $n = 1, \cdots, N$. □

THEOREM 72 (Convergence analysis). *In two dimensions space, the projection method converges in the following sense: there exists a sub-sequence $k = \delta t \to 0$, such that the functions u_{1k}, u_{2k} and $\tilde{u}_k$, associated with the scheme (3.8), (3.11) converge to a solution u of the Navier-Stokes equations in $L^2(0,T;H^1(\Omega)^2)$ weakly. Furthermore u_{1k} converges to u in $L^2(0,T;H_0^1(\Omega)^2)$ strongly.*

Proof. The proof of convergence of the projection method given hereafter is slightly different than in Temam (1969b). For the sake of simplicity we assume that $u_0 \in V$, so that $\tilde{u}_k$ is bounded in $L^2(0,T;H^1(\Omega)^2)$.

Because of Lemma 51, there exists a sub-sequence $k' \to 0$ such that

$u_{1k'} \to u_1$ in $L^\infty(0,T;H)$ weak-star and $L^\infty(0,T;H_0^1(\Omega)^2)$ weakly
$u_{2k'} \to u_2$ in $L^\infty(0,T;H)$ weak-star and $L^\infty(0,T;H^1(\Omega)^2)$ weakly
$\tilde{u}_{k'} \to u$ in $L^\infty(0,T;H)$ weak-star and $L^\infty(0,T;H^1(\Omega)^2)$ weakly

Lemma 51 also implies that $u_l = u_2 = u$ and thus

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V) \tag{3.42}$$

Passing to the limit in (3.41) is straightforward if we know $u_{1k'}$ converges to u in $L^2(0,T;L^2(\Omega)^2)$, strongly, as $k' \to 0$. Alternatively, because of Lemma 51, it suffices to show that

$$\tilde{u}_k \to u \text{ in } L^2(0,T;L^2(\Omega)^2) \text{ strongly} \tag{3.43}$$

First, we observe that $\tilde{u}_k$ is an H-valued and hence a V'-valued function and show that

$$\tilde{u}_{k'} \to w \text{ in } L^2(0,T;V') \text{ strongly} \tag{3.44}$$

Proof of (3.44)

By integration of (3.41) between t and $t+a$, $0 < t < T$, $a > 0$, we can write

(3.45)
$$(\tilde{u}_k(t+a)-\tilde{u}_k(t),v)_0=\int_t^{t+a}\left\{-\nu((u_{1k}(s),v))-\tilde{b}(u_{1k}(s),u_{1k}(s),v)+(f_k(s),v)_0\right\}$$

Using the Schwarz and Poincaré inequalities, the right-hand side of (3.45) can be bounded by:

$$\|v\|(I_1+I_2+I_3)$$

In fact:

$$\begin{aligned}
I_1 &= \nu\int_t^{t+a}\|u_{1k}(s)\|\,ds\leqslant \nu a^{1/2}\left(\int_t^{t+a}\|u_{1k}(s)\|^2\,ds\right)^{1/2}\\
&\leqslant Ka^{1/2}\,\textbf{(with Lemma 51)}\\
I_2 &\leqslant c\int_t^{t+a}|u_{1k}(s)|_0^{1/4}\|u_{1k}(s)\|^{7/4}\,ds\\
&\leqslant K\int_t^{t+a}\|u_{1k}(s)\|^{7/4}\,ds\,\textbf{(with Lemma 51)}\\
&\leqslant Ka^{1/8}\left(\int_t^{t+a}\|u_{1k}(s)\|^2\,ds\right)^{7/8},\textbf{(with Hölder's inequality)}\\
&\leqslant Ka^{1/8};\\
I_3 &= c\int_t^{t+a}|f_k(s)|_0\,ds\leqslant ca^{1/2}\left(\int_t^{t+a}|f_k(s)|_0^2\,ds\right)^{1/2}\\
&\leqslant Ka^{1/2}
\end{aligned}$$

We then infer from (3.45) that

$$\begin{aligned}
\|u_k(t+a)-u_k(t)\|_{V'} &= \sup_{\substack{v\neq 0\\ v\in V}}\left\{\frac{1}{\|v\|}(\tilde{u}_k(t+a)-\tilde{u}_k(t),v)_0\right\}\\
&\leqslant K(a^{1/2}+a^{1/8}),
\end{aligned}$$

$$\int_0^{T-a}\|u_k(t+a)-u_k(t)\|_{V'}^2\,dt\leqslant Ka^{1/4},$$

where K depends only on the data; u_k, and (3.44). Once the strong convergence is proved we establish (3.43) by using the following well-known property: since by Rellich's theorem the injection of X in H is compact,

For every $\varepsilon > 0$ there exists a constant c_ε depending on ε and Ω such that

$$|\varphi|_0 \leqslant \varepsilon \|\varphi\|_X + c_\varepsilon \|\varphi\|_{V'}, \quad \forall \varphi \in X, \tag{3.46}$$

(see, Temam (1984), Lemme 2.1, Chapter III). Thus, for every $\varepsilon > 0$,

$$\begin{aligned}
&\int_0^T |\tilde{u}_{k'}(t) - u(t)|_0^2 \, dt \\
&\qquad \leqslant \varepsilon \int_0^T \|\tilde{u}_{k'}(t) - u(t)\|_{H^1(\Omega)^2}^2 \, dt + c_\varepsilon \int_0^T \|\tilde{u}_{k'}(t) - u(t)\|_{V'}^2 \, dt \\
&\qquad \leqslant K\varepsilon + c_\varepsilon \int_0^T \|\tilde{u}_{k'}(t) - u(t)\|_{V'}^2 \, dt \text{ with Lemma 51.}
\end{aligned}$$

Letting $k' \to 0$, we obtain

$$\limsup_{k' \to 0} \int_0^T |\tilde{u}_{k'}(t) - u(t)|_0^2 \, dt \leqslant K\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, (3.43) follows. □

We continue in this section the study of the projection method and perform its error analysis. Instead of considering (3.19) as the first step of the scheme, we shall consider (3.19) which leads to slightly simpler calculations. We follow Shen (1992). The projection scheme is found to be of order $(\delta t)^{1/2}$ for the velocity for the uniform L^2-norm, of order $(\delta t)^{1/2}$ for the root mean square (RMS) – H^1 norm, and of order δt for the (RMS) – L^2 norm. At the end of the section we recall, without performing error analysis, some of the higher order projection methods which have been derived in the literature.

THEOREM 73. *[Error analysis]*
In space dimension 2(*ou* 3), *if the exact solution* u, p *of the Navier-Stokes equations satisfies* (3.47), *then*

$$\|u_{1k} - u_k\|_{L^\infty(0,T;L^2(\Omega)^2)} \leqslant M(\delta t)^{1/2},$$
$$\|u_{2k} - u_k\|_{L^\infty(0,T;L^2(\Omega)^2)} \leqslant M(\delta t)^{1/2}$$
$$\|u_{1k} - u_k\|_{L^2(0,T;L^2(\Omega)^2)} \leqslant M(\delta t)^{1/2},$$
$$\|u_{2k} - u_k\|_{L^2(0,T;L^2(\Omega)^2)} \leqslant M(\delta t)^{1/2}$$

Proof. We assume that the space dimension is two, and that the solution to the Navier-Stokes equations is sufficiently regular, namely

(3.47)
$$u \in \mathcal{C}\left([0,T]; D(A)\right)\ u_t = \frac{\partial u}{\partial t} \in \mathcal{C}\left([0,T]; D(A)\right)$$
$$p \in L^2\left(0,T; H^2(\Omega)\right) \cap \mathcal{C}\left([0,T]; H^1(\Omega)\right),\ p_t = \frac{\partial p}{\partial t} \in L^2\left(0,T; L^2(\Omega)\right).$$

We start with equations satisfied by the exact solution, which we write at time $t_{n+1} = (n+1)\delta t$:

$$u_t(t_{n+1}) + \nu A u(t_{n+1}) + B(u(t_{n+1})) = f(t_{n+1}) \tag{3.48}$$

$$\frac{1}{\delta t}(u(t_{n+1}) - u(t_n)) + \nu A u(t_{n+1}) + B(u(t_{n+1})) = f(t_{n+1}) + R_{n+1} \tag{3.49}$$

where
$$R_n = \frac{1}{\delta t}(u(t_n) - u(t_{n-1})) - u_t(t_n).$$

Reintroducing the pressure we see that (3.48) is equivalent to

$$u_t(t_{n+1}) - \nu\Delta u(t_{n+1}) + (u(t_{n+1})\cdot\nabla)u(t_{n+1}) + \nabla p(t_{n+1}) = f(t_{n+1}), \tag{3.50}$$

so that (3.49) is equivalent to

$$\begin{aligned}(3.51) \qquad & \frac{1}{\delta t}(u(t_{n+1}) - u(t_n)) - \nu\Delta u(t_{n+1}) + (u(t_{n+1})\cdot\nabla)u(t_{n+1}) + \nabla p(t_{n+1}) \\ = \quad & f(t_{n+1}) + R_{n+1}\end{aligned}$$

We assume that

$$(3.52) \qquad \begin{aligned} &\|R_{n+1}\|^2_{H^{-1}(\Omega)^2} \leqslant \delta t \int_{t_n}^{t_{n+1}} \|u_t\|^2_{H^{-1}(\Omega)^2}\, dt, \\ &\delta t \sum_{n=0}^{N-1} \|R_{n+1}\|^2_{H^{-1}(\Omega)^2} \leqslant M(\delta t)^2. \end{aligned}$$

We now consider the scheme (3.19), (3.11) with $f^n = f(t_n)$ and we set

$$e^{n+1/2} = u^{n+1/2} - u(t_{n+1}), \quad e^{n+1} = u^{n+1} - u(t_{n+1})$$

We subtract (3.51) from (3.19) and we find

$$\begin{aligned} \frac{1}{\delta t}(e^{n+1/2} - e^n) - \nu\Delta e^{n+1/2} \quad &= \quad \nabla p(t_{n+1}) - R_{n+1} - (u^n\cdot\nabla)u^{n+1/2} + (u(t_{n+1})\cdot\nabla)u(t_{n+1}) \\ &= \quad \nabla p(t_{n+1}) - R_{n+1} - (e^n\cdot\nabla)u^{n+1/2} - ((u(t_n) \\ (3.53) \qquad\qquad &- \quad u(t_{n+1}))\cdot\nabla)u^{n+1/2} - (u(t_{n+1})\cdot\nabla)e^{n+1/2}. \end{aligned}$$

We take the scalar product in $L^2(\Omega)^2$ of (3.53) with $2\delta t e^{n+1/2}$; since $e^{n+1/2} \in H_0^1(\Omega)^2$, we can write the left-hand side of the corresponding equation

$$|e^{n+1/2}|_0^2 - |e^n|_0^2 + |e^{n+1/2} - e^n|_0^2 + 2\nu\delta t\|e^{n+1/2}\|^2 = 2\delta t\left(\nabla p(t_{n+1}), e^{n+1/2}\right)_0 - 2\delta t\left(R_{n+1}, e^{n+1/2}\right)_0$$

$$(3.54) \qquad - 2\delta t b\left(e^n, u^{n+1/2}, e^{n+1/2}\right) + 2\delta t b\left(u(t_n) - u(t_{n+1}), u^{n+1/2}, e^{n+1/2}\right).$$

where $u(t_{n+1}) \in V$.

Next we bound above the terms in the right-hand side of (3.54). As $e^n \in H$,

$$\begin{aligned}
2\delta t\left(\nabla p(t_{n+1}), e^{n+1/2}\right)_0 &= 2\delta t\left(\nabla p(t_{n+1}), e^{n+1/2}-e^n\right)_0 \leqslant \frac{1}{2}|e^{n+1/2}-e^n|_0^2 + 2(\delta t)^2|\nabla p(t_{n+1})|_0^2 \\
&\leqslant \frac{1}{2}|e^{n+1/2}-e^n|_0^2 + M(\delta t)^2 \text{ (for (3.54))}. \\
-2\delta t\left(R_{n+1}, e^{n+1/2}\right)_0 &\leqslant 2\delta t\|R_{n+1}\|_{H^{-1}(\Omega)^2}\|e^{n+1/2}\| \\
&\leqslant \frac{\nu}{2}\delta t\|e^{n+1/2}\|^2 + \frac{\nu}{2}\delta t\|R_{n+1}\|^2_{H^{-1}(\Omega)^2} \\
&\leqslant \frac{\nu}{2}\delta t\|e^{n+1/2}\|^2 + \frac{\nu}{2}(\delta t)^2\int_{t_n}^{t_{n+1}}\|u_{tt}(t)\|^2_{H^{-1}(\Omega)^2}\,dt - 2\delta t\left(e^n, u^{n+1/2}, e^{n+1/2}\right) \\
&= 2\delta t b\left(e^n, e^{n+1/2}, u^{n+1/2}\right) \\
&\leqslant c\delta t|e^n|_0\|e^{n+1/2}\||Au^{n+1/2}|_0 \\
&\leqslant M\delta t|e^n|_0\|e^{n+1/2}\| \leqslant \frac{\nu\delta t}{4}\|e^{n+1/2}\|^2 + M\delta t|e^n|_0^2,
\end{aligned}$$

$$\begin{aligned}
2\delta t b\left(u(t_n)-u(t_{n+1}), u^{n+1/2}, e^{n+1/2}\right) &= 2\delta t b\left(u(t_n)-u(t_{n+1}), u(t_{n+1}), e^{n+1/2}\right) \\
&= 2\delta t b\left(u(t_{n+1})-u(t_n), e^{n+1/2}, u(t_{n+1})\right) \\
&\leqslant c\delta t|u(t_{n+1})-u(t_n)|_0\|e^{n+1/2}\||Au(t_{n+1})|_0^2 \\
&\leqslant M\delta t\left|\int_{t_n}^{t_{n+1}} u_t(t)\,dt\right|_0 \|e^{n+1/2}\| \\
&\leqslant \frac{\nu}{4}\delta t\|e^{n+1/2}\|^2 + M(\delta t)^2\int_{t_n}^{t_{n+1}}|u_t(t)|_0^2\,dt.
\end{aligned}$$

Taking into account these inequalities (3.54), yields

$$|e^{n+1/2}|_0^2 - |e^n|_0^2 + \frac{1}{2}|e^{n+1/2}-e^n|_0^2 + \nu\delta t\|e^{n+1/2}\|^2 \leqslant M\delta t|e^n|_0^2 + r_n, \quad n = 0,\cdots,N-1,$$

(3.55) $$r_n = M(\delta t)^2\left\{1 + \int_{t_n}^{t_{n+1}}\left(\|u_{tt}(t)\|^2_{H^{-1}(\Omega)^2} + |u_t(t)|_0^2\right)\,dt\right\}.$$

We also infer from (3.20) that

(3.56) $$\left(e^{n+1}-e^{n+1/2}, v\right)_0 = 0, \quad \forall v \in H,$$

and setting $v = 2e^{n+1}$, we obtain

$$|e^{n+1}|_0^2 - |e^{n+1/2}|_0^2 + |e^{n+1} - e^{n+1/2}|_0^2 = 0. \tag{3.57}$$

We add (3.55) to (3.57), and obtain, for $n = 0, \cdot, N-1$,

(3.58)

$$|e^{n+1}|_0^2 - |e^n|_0^2 + \frac{1}{2}\left\{|e^{n+1} - e^{n+1/2}|_0^2 + |e^{n+1/2} - e^n|_0^2\right\} + \nu\delta t\|e^{n+1/2}\|^2 \leqslant M\delta t|e^n|_0^2 + r_n,$$

In particular

$$|e^{n+1}|_0^2 \leqslant (1 + M\delta t)|e^n|_0^2 + r_n, \quad n = 0, \cdots, N-1.$$

The following lemma will be used in the sequel.

LEMMA 52. *Consider two sequences of numbers $(\xi^n)_{n\in\mathbb{N}}$ and $(r^n)_{n\in\mathbb{N}}$ at positif terms such that*

$$(1-\alpha)\xi^n \leqslant (1-\beta)\xi^{n-1} + r^n. \tag{3.59}$$

For all $n \geqslant 1$ and for some $\alpha < 1$ et $\beta > -1$, Then for all n:

$$\xi^n \leqslant \left(\frac{1+\beta}{1-\alpha}\right)^n \xi^0 + \frac{(1+\beta)^{n-1}}{(1-\alpha)^n}\sum_{j=1}^{n} r_j. \tag{3.60}$$

If $r_j \leqslant r$, $\forall j$, we also have

$$\xi^n \leqslant \left(\frac{1+\beta}{1-\alpha}\right)^n \left(\xi^0 + \frac{r}{\beta+\alpha}\right). \tag{3.61}$$

Proof. For $m = 0, \cdots, n-1$, we write

$$\xi^{n-m} \leqslant \frac{1+\beta}{1-\alpha}\xi^{n-m-1} + \frac{1}{1-\alpha}r_{n-m}.$$

We multiply this relation by $((1+\beta)/(1-\alpha))^m$ and add the corresponding inequalities for $m = 0, \cdots, n-1$; Formulas (3.59) and (3.61) follow. □

We apply Lemma 52 with $\alpha = 0$, $\beta = M\delta t$ and since $e^0 = 0$, we find

$$|e^n|_0^2 \leqslant (1+M\delta t)^{n-1}\sum_{j=1}^{n} r_j \leqslant M\delta t(1+M\delta t)^{n-1} \textbf{ with } (3.55) \textbf{ and } (3.47).$$

Writing $(1+x) \leqslant 2^x$ for $0 \leqslant x = M\delta t \leqslant 1$, we obtain the bounds

(3.62) $$|e^n|_0^2 \leqslant M\delta t 2^{M\delta t n} = M\delta t, \quad n = 0, \cdots, N.$$

Also by (3.58)

(3.63) $$|e^{n+1/2}|_0^2 \leqslant M\delta t$$

We return to (3.55) and add these relations for $n = 0, \cdots, N-1$. This yields since $e^0 = 0$.

$$|e^n|_0^2 + \frac{1}{2}\sum_{n=0}^{N-1}\left\{|e^{n+1} - e^{n+1/2}|_0^2 + |e^{n+1/2} - e^n|_0^2\right\} + \nu\delta t\sum_{n=0}^{N-1}\|e^{n+1/2}\|^2 \leqslant M\delta t + \sum_{n=0}^{N-1} r_n$$

(3.64) $$\leqslant M\delta t + M(\delta t)^2\int_0^T (\|u_{tt}(t)\|^2_{H^{-1}(\Omega)^2} + |u_t(t)|_0^2)\,dt \leqslant M\delta t.$$

Finally since $u^{n+1} = P_H u^{n+1/2}$ and the projector P_H is continuous in $H^1(\Omega)^2$ (see (3.39)) we infer from (3.64) that

$$\delta t\sum_{n=0}^{N-1}\|e^{n+1}\|^2 \leqslant c\delta t\sum_{n=0}^{N-1}\|e^{n+1/2}\|^2 \leqslant M\delta t.$$

We summarize the following estimates:

LEMMA 53. *Under the assumption* (3.4), *we have*

$$|e^n|_0^2 \leqslant M\delta t, \quad n = 0, \cdots, N,$$
$$|e^{n+1/2}|_0^2 \leqslant M\delta t, \quad n = 0, \cdots, N-1,$$
$$\sum_{n=0}^{N-1} \left\{ |e^{n+1} - e^{n+1/2}|_0^2 + |e^{n+1/2} - e^n|_0^2 \right\} \leqslant M\delta t,$$
$$\delta t \sum_{n=0}^{N-1} \|e^n\|^2 \leqslant M\delta t,$$
$$\delta t \sum_{n=0}^{N-1} \|e^{n+1/2}\|^2 \leqslant M\delta t,$$

where $e^n = u^n - u(n\delta t)$, $e^{n+1/2} = u^{n+1/2} - u((n+1)\delta t)$.

LEMMA 54. *Under the assumption* (3.47), *we have*

$$\|e^n\|_{V'}^2 \leqslant M(\delta t)^2, \quad n = 0, \cdots, N,$$
$$\delta t \sum_{j=0}^{n-1} |e^n|_0^2 \leqslant M(\delta t)^2.$$

We can interpret Lemmas 53 and 54 in terms of the approximating functions u_{1k}, u_{2k}, already defined and a function u_k associated with the exact solution:

$$u_{1k} = u^{n+1/2}, \quad u_{2k} = u^{n+1}, \quad u_k = u(t_{n+1}),$$

$$\text{for } t \in [n\delta t, (n+1)\delta t] \; n = 0, \cdots, N-1, \; t_{n+1} = (n+1)\delta t.$$

■

REMARK 28. *Error estimates on the pressure can be derived if we make more regularity hypotheses on the exact solution* u, p. *However, this implies some compatibility hypotheses on the data* u_0, f, *or otherwise the estimates are not as good near* $t = 0$ *(see Shen (1992), Shen (1992), Shen (1994)).*

REMARK 29. *As it appears clearly from the previous proof there are some difficulties resulting from the combination of the boundary condition and the incompressibility condition. Let us make here some relevant remarks.*

The space $H_0^1(\Omega)^2$ is the direct sum of the space V and its orthogonal in $H_0^1(\Omega)^2$, $V^\perp$. It can be easily seen that $V^\perp$ is the space of v in $H_0^1(\Omega)^2$ such that $\Delta v = \mathbf{grad} q$ for some q in $L^2(\Omega)$. The operator $-\Delta$ which maps $H_0^1(\Omega)^2$ onto $H^{-1}(\Omega)^2$, maps V onto V' and $V^\perp$ onto the polar V^0 of V which is exactly $\mathbf{grad} L^2(\Omega)$.

When we write $H \subset V'$, we identify in fact H with the space i^*H where i the canonical injection of V into H and i^* is its adjoint. The statement in (3.48), $u_{tt} \in L^2(0,T;V')$ should be understood as

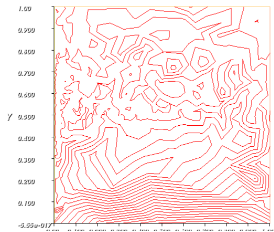
$$\frac{d^2}{dt^2}(i^* u) \in L^2(0,T;V').$$

This statement is different from that made in Remark 73, namely

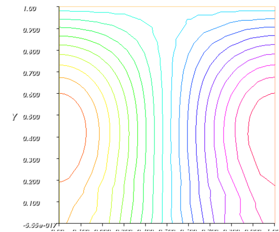
$$u_{tt} \in L^2(0,T;H^{-1}(\Omega)^2).$$

The latter statement is a mere consequence of the hypothesis $p_t \in L^2(0,T;L^2(\Omega))$, and of the other assumptions in (3.47).

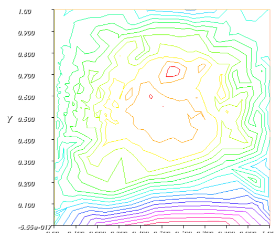
3.4. Numerical simulations. In this section, we present some numerical simulations obtained with the proposed method. The numerical simulations presented are obtained for differents Reynauld numbers.



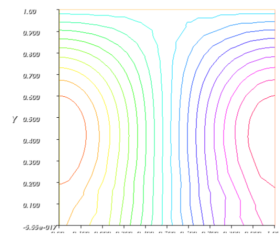
$(a): \ u(x,y); \ Re = 10^{-3}.$



$(b): \ p(x,y); \ Re = 10^{-3}.$

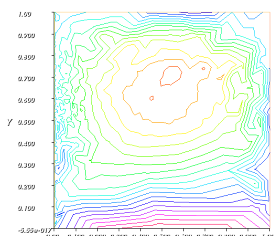


$(c): \ u(x,y); \ R = 10^{-2}.$

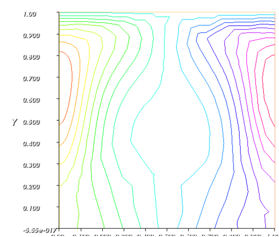


$(d): \ p(x,y); \ R = 10^{-2}.$

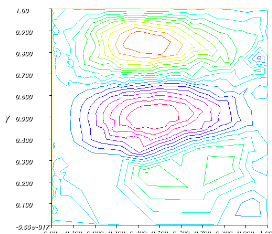
Viscous flows



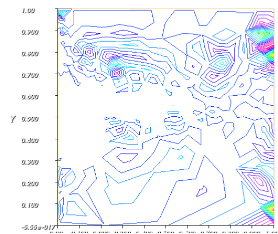
$(e): \ u(x,y); \ R = 1.$



$(f): \ p(x,y); \ R = 1.$

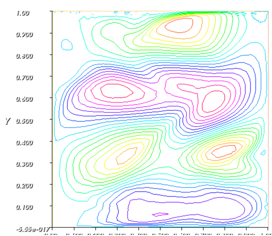


$(g): \ u(x,y); \ R = 200.$

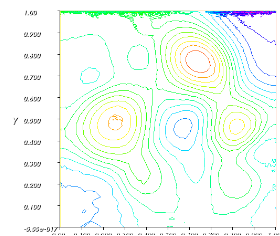


$(h): \ p(x,y); \ R = 200.$

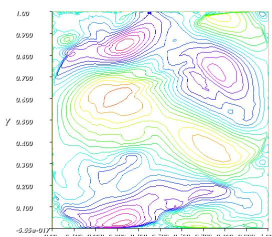
Laminar flows



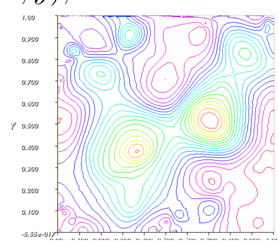
$(i): \ u(x,y); \ Re = 10^4.$



$(j): \ p(x,y); \ Re = 10^4.$



$(k): \ u(x,y); \ Re = 10^6.$



$(l): \ p(x,y); \ Re = 10^6.$

Turbulent flows

4. Some concluding remarks and extensions

The adapted projection method allows us to simulate Navier-Stokes systems for all types flows (laminar, viscous and turbulent).

In order to obtain more accurate schemes, other forms of the projection method can be derived. At least three types of higher order projection schemes have been proposed. Namely, schemes via improved intermediate velocity boundary conditions (Kim and Moin (1985)), schemes via pressure-correction (Van Kan (1986), Bell *et al.* (1989), Gresho and Chan (1990)) and schemes via improved pressure boundary condition (Orszag *et al.* (1986), Karniadakis *et al.* (1991)); see also Shen (1992).

In forthcoming works, we will intend to study obstacles position problems in dynamic fluids of Navier Stokes types by coupling the adapted projection method and topological optimizations tools.

Bibliography

Bell J., Colella P. and Glaz H. (1989). A second-order projection method for the incompressible Navier-Stokes equations. *J. Comput. Phys.* **85**, 257-283.

Brezzi F. and Fortin M. (1991). *Mixed and Hybrid Finite Element Methods*. Springer, New York.

Constantin P. and Foias C. (1988). *Navier-Stokes Equations*, University of Chicago Press.

Chorin A.J. (1967). Numerical study of thermal-convection. *J. Comput. Phys.* **2**, 12-26.

Chorin A.J. (1967). The numerical solution of the Navier-Stokes equations for an incompressible fluid. *Bull. Amer Math. Soc.* **73**, 928-931.

Chorin A.J. (1968). Numerical solution of the Navier-Stokes equations. *Math. Comp.* **22**, 745-762.

Chorin A.J. (1968). Numerical solution of incompressible flow problems. *Stud. Numer, Stud. Numer Anal.* **2**, 64-71.

Girault V. and Raviart P.A. (1986). *Finite Elements Methods for Navier-Stokes Equations*. Series in Computational Mathematics **5** (Springer, New York).

Gresho P.M. and Chan S. (1990). On the theory of semi-implicit projection methods for viscous incompressible flow and its implementation via a finite element method that also introduces a nearly consistent mass matrix. Part 2. Implementation. *Internat. J. Numer Methods Fluids.* **11**, 621-659.

Gresho P.M. and Sani R.L. (1996). *Incompressible Flow and the Finite Element Method, I: Advection Diffitsion and Isothermal Laminar Flow*. book in preparation.

Gunzburger M. (1989). *Finite Element Methods for Viscous Incompressible Flows: A Guide to Theory, Practice, and Algorithms*. Academic Press.

Karniadakis G.E., Israeli M. and Orszag S.A. (1991). High-order splitting methods for the incompressible Navier-Stokes equations. *J. Comput. Phys.* **97**, 414-443.

Kim J. and Moin P. (1985). Application of a fractional-step method to incompressible Navier-Stokes equations. *J. Comput. Phys.* **59**, 308-323.

Ladyzhenskaya O. (1963). *The Mathematical Theory of Viscous Incompressible Flow*, revised English edition (translated from the Russian by Richard A. Silverman). Gordon & Breach, New York.

Lions J. L. (1969) *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris.

Lions P. L. (1996). *Mathematical topics in fluid mechanics, Vol. 1 : Incompressible models*. Oxford Lecture Series in Mathematics and its applications, Vol. 3, Oxford University Press, New York.

Marchuk G.I. (1990). Splitting and alternating direction methods. In *P.G. Ciarlet and J.L. Lions, eds., Handbook of Numerical Analysis* **I**, (North-Holland, Amsterdam) 197-462.

Marion M. and Temain R. (1998). Navier-Stokes Equations: Theory and Approximation. In *P.G. Ciarlet and J.L. Lions, eds., Handbook of Numerical Analysis* **VI**.

Orszag S.A., Israeli M. and Deville M.O. (1986). Boundary conditions for incompressible flows. *J. Sci. Comput.* **1**, 75-111.

Pironneau O. (1988). *Médthode des Elements Finis pour les Fluides*. Collection Recherches en Mathbmatiques Appliques 7 (Masson, Paris).

Shen J. (1992). On error estimates of projection methods for Navier-Stokes equations: first-order schemes.*SIAM J. Numer. Anal.* **29** (1), 57-77.

Shen J. (1992). On error estimates of some higher order projection and penalty projection methods for Navier-Stokes equations. *Numer. Math.* **62**, 49-73.

Shen J. (1994). Remarks on the pressure error estimates for the projection methods. *Numer. Math.* **17**, 513-520.

Tatar L. (2006). *An Introduction to the Navier-Stokes equations and Oceanography*. Lecture Notes of the Unione Mathematica Italiana. Springer.

Temam R. (1966). Sur l'approximation de la solution des équations de Navier-Stokes. *C. R. Acad. Sci*,Paris, Série A **262**. 219-221.

Temam R. (1979). *Navier Stokes Equations, Studies in Mathematics and its Applications*. North-Holland Publishing Compagny, Amsterdam, New-York, Oxford.

Temam R. (1966). Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires, *C. R. Acad. Sci*,Paris, Série A, **263**, 241-244, 265-267, 459-462.

Temam R. (1968). Sur la stabilité et la convergence de la méthode des pas fractionnaires. *Ann. Mat. Pura Appl.* Serie 4, **LXXIV**, 191-380.

Temam R. (1968). Une méthode d'approximation de la solution des equations de Navier-Stokes. *Bull. Soc. Math. France* **98**, 115-152.

Temam R. (1969). Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires, *Arch. Ration. Mech. Anal.* **33**, 377-385.

Temam R. (1969). Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires (I). *Arch. Ration. Mech. Anal.* **33**, 377-385.

Temam R. (1984). *Navier-Stokes Equations, 3rd revised edition.* North-Holland, Amsterdam, 1984.

Temam R. (1991). Remark on the pressure boundary condition for the projection method. *Theoret. Comput. Fluid Dynamics*. **3**, 181-184.

Thomasset F. (1981). *Implementation of Finite Elements Methods for Navier-Stokes Equations*, Series in Computational Physics, Springer, New York.

Van Kan J. (1986). A second-order accurate pressure-correction scheme for viscous incompressible flow. *SIAM J. Sci. Statist. Comput.***7**, 870-891.

Yanenko N.N. (1971). *The Method of Fractional Steps for Multidimensional Problems in Mathematical Physics (in Russian). French translation Theorie des Pas Fractionnaires*,(Armand Colin, Paris, 1968). English translation, The Method of Fractional Steps Springer, New York.