CHAPTER 20

# On proximity between spectral elements associated with periodic and almost periodic stationary processes, *by E. N. Cabral*

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**Abstract**. In this contribution, periodic and almost periodic stationary processes are respectively studied in the frequency domain. A relation of proximity is clearly established between one of the spectral tools associated with these processes: the associated random measure. This is a way to consider, for such processes, that the filters resulting from Principal Components Analysis in the frequency domain are close.

**Résumé**. Dans cette contribution, nous nous intéressons à l'étude, dans le domaine des fréquences, des processus stationnaires périodique et quasi-périodique. Une relation de proximité est établie de façon claire entre l'un des outils spectraux associés à ces processus: les mesures aléatoires associées. C'est pour cette raison que l'on pourrait, au moyen d'analyses en composantes principales, dans le domaine des fréquences, de ces processus, considérer que les filtres issus de ces analyses sont voisins.

## 1. Introduction

Given a stationary process $(X_t)_{t\in\mathbb{R}}$ and $\eta \in ]0, 2\|X_0\|[$, we say that $(X_t)_{t\in\mathbb{R}}$ is $(T, \eta)$-almost periodic when $\|X_t - X_{t+nT}\| \leq \eta$, for any pairs $(t, n)$ of $\mathbb{R} \times \mathbb{Z}$. There are many physical phenomena that occur with a certain period of time. But these repetitions, which can be considered as independent, are not perfectly periodic. Since, in reality, there is no process exactly periodic, it may be interesting to study almost periodicity of such random phenomena. However, in literature, apart from the works on the spectrum and structure of $K$-periodically correlated fields (see, e.g. Dehay (1994) or Dehay *et al.* (2014)), no study has been done about periodic stationary continuous random function in the frequency domain. Hence, it is interesting to investigate this issue since there are many areas of application where such phenomena meet; these areas, among others, include the signal theory (acoustic or vibratory signal processing, modulation of speech, ...), meteorology (analysis of the data given by equations), medicine (analyzes of electroencephalogram, electrocardiogram), etc.

Thus, given a $(T, \eta)$-almost periodic stationary continuous random function, it is legitimate to think that its usually associated spectral elements (random measure, projector valued spectral measure and unitary operator) have a proximity relationship with those of a periodic stationary continuous random function. We establish such a result only for their associated random measures; this reinforces the idea of the one-to-one correspondence which exists between a stationary random function and its associated random measure (see, e.g. Azencott and Dacunha-Castelle (1984) or Boudou and Romain (2002)), i.e. a stationary random function can be considered as the Fourier transform of a random measure. In other words, we show, on the one hand, that the random measure associated with a $(2\pi, \eta)$-almost periodic stationary process is almost concentrated on $\mathbb{Z}$ (which is the spectrum of a $2\pi$-periodic stationary process).

On the other hand, with a counter-example, we show that, about the associated (projector valued) spectral measures and unit operators, there is no necessarily proximity relation. This is the goal of the section 4, after presenting, in section 3, the spectral specificities of a $2\pi$-periodic stationary process and recalling in section 2, some tools and notations necessary for the understanding of this text. And before concluding in section 6, we show in section 5, the proximity between the filters resulting from the principal components analyzes of periodic and almost periodic stationary processes, respectively. In this paper, we use the spectral tools classically associated with a stationary random function defined and developed in Boudou and Romain (2002), for example.

## 2. Notation and recalls

The aim of this paragraph is to precise notation which will be used in this text and recall some mathematical tools which are necessary for the understanding.

When $H$ is a $\mathbb{C}$-Hilbert space (which, from a perspective of multidimensional approach, will be of $L_p^2 := L_{\mathbb{C}^p}^2(\Omega, \mathcal{A}, \mathbb{P})$ type, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space), the set of the orthogonal projectors is denoted $\mathcal{P}(H)$. By $G$, we will designate a locally compact Abelian group, which dual $\widehat{G}$ is of countable basis. For example, $G$ can be $\mathbb{Z}^k$, $\mathbb{R}^k$ or $\Pi^k$, where $\Pi = [-\pi, \pi[$.

**2.1. Random measure and stochastic integral.** A *random measure (r.m.)* $Z$ defined on $\mathcal{E}$, $\sigma$-field of parts of a set $E$, taking values in $H$, is a *vector measure* such that $< Z(A), Z(B) >_H = 0$, for any pairs $(A, B)$ of disjoint elements of $\mathcal{E}$.

It is then easy to verify that the mapping $\mu_Z : A \in \mathcal{E} \mapsto \|Z(A)\|_H^2 \in \mathbb{R}^+$ is a bounded measure. When $G$ is $\mathbb{R}^k$ or $\Pi^k$ and if $\mu_Z$ is absolutely continuous with respect to the Lebesgue measure, then the derivative of $\mu_Z$ with respect to the Lebesgue measure is called *spectral density* (commonly used in signal theory).

The *stochastic integral*, relatively to the r.m. $Z$, can be defined as the *unique isometry* from $L_{\mathbb{C}}^2(E, \mathcal{E}, \mu_Z)$ onto $H_Z := \overline{span}\{Z(A); A \in \mathcal{E}\}$ which, with $\mathbf{1}_A$, associates $Z(A)$, this for any $A$ of $\mathcal{E}$.

The image of an element $\varphi$ of $L^2(E, \mathcal{E}, \mu_Z)$ by this isometry is denoted $\int \varphi dZ$, and called *integral of $\varphi$ relatively to the r.m.* $Z$. Let $(F, \mathcal{F})$ be a second measurable space and $f$ a measurable mapping defined on $E$ onto $F$. The mapping $f(Z) : A \in \mathcal{F} \mapsto Z(f^{-1}(A)) \in H$ is a r.m. called *r.m. image of $Z$ by $f$* and we easily verify that

    **(i)** $\mu_{f(Z)}$ is the measure image of $\mu_Z$ by $f$;

    **(ii)** when $\varphi$ is an element of $L^2_{\mathbb{C}}(F, \mathcal{F}, \mu_{f(Z)})$, then $\varphi \circ f \in L^2_{\mathbb{C}}(E, \mathcal{E}, \mu_Z)$ and $\int \varphi df(Z) = \int \varphi \circ f dZ$.

**Multidimensional approach.** When $X$ is an element of $L^2_p$, the mapping $y \in L^2_{\mathbb{C}}(\mathbb{P}) \mapsto \mathbb{E}(yX) \in \mathbb{C}^p$, denoted by $\widetilde{X}$, is a *Hilbert-Schmidt operator*. Its adjoint $\widetilde{X}^*$ is the mapping $u \in \mathbb{C}^p \mapsto <u, X>_{\mathbb{C}^p} \in L^2_{\mathbb{C}}(\mathbb{P})$.

A *$p$-random measure ($p$-r.m.)* $Z$ is a vector measure defined on $\mathcal{B}_{\widehat{G}}$, the Borel $\sigma$-field of $\widehat{G}$, with values in $L^2_p$, such that holds the stronger condition $\widetilde{Z(A)} \circ \widetilde{Z(B)}^* = 0$, which implies $<Z(A), Z(B)>_{L^2_p} = 0$, for each pair $(A, B)$ of disjoint elements of $\mathcal{B}_{\widehat{G}}$.

Let $\sigma_2(p, q)$ (resp. $\sigma_2(p)$) be the $\mathbb{C}$-Hilbert space of all Hilbert-Schmidt operators mapping $\mathbb{C}^p$ into $\mathbb{C}^q$ (resp. $\mathbb{C}^p$) with the inner product defined by $<., .>_2 : (L, K) \mapsto trL \circ K^*$, and $M_Z$ be the vector-valued measure in $\sigma_2(p)$ defined by

$$(2.1) \qquad M_Z(A) = \widetilde{Z(A)} \circ \widetilde{Z(A)}^* = \mathbb{E}[Z(A) \otimes Z(A)],$$

for any $A$ of $\mathcal{B}_{\widehat{G}}$.

For any $\sigma$-finite measure $\nu$ defined on $(\widehat{G}, \mathcal{B}_{\widehat{G}})$ which dominates $M_Z$, the Radon-Nikodym derivative $\frac{dM_Z}{d\nu}$ of $M_Z$ with respect to $\nu$ admits (see Rosenberg (1974) p.174) a measurable Schmidt decomposition

$$\frac{dM_Z}{d\nu}(\cdot) = \sum_{j=1}^{p} \mu_j(\cdot) a_j(\cdot) \otimes a_j(\cdot).$$

More precisely, there exists two families of measurable applications, $(\mu_j(\gamma))_{j=1,\cdots,p}$ and $(a_j(\gamma))_{j=1,\cdots,p}$, defined on $\widehat{G}$ with values in $\mathbb{R}^+$ and $\mathbb{C}^p$, respectively. Furthermore, for each $\gamma$ of $\widehat{G}$, $\sum_{j=1}^{j=p} \mu_j(\gamma)a_j(\gamma) \otimes a_j(\gamma)$ is a Schmidt decomposition of $\frac{dM_Z}{d\nu}(\gamma)$ which belongs to $\sigma_2(p)$.

### 2.2. Stationary continuous random function.

A *stationary continuous random function (c.r.f.)* $(X_g)_{g \in G}$, defined on $G$ and with values in $H$ is a family of elements of $H$, such that

**(i)** the mapping $g \in G \mapsto X_g \in H$ is continuous;

**(ii)** $< X_{g_1}, X_{g_2} >_H = < X_{g_1-g_2}, X_0 >_H$, for any pairs $(g_1, g_2)$ of elements of $G$.

We can then show that *there exists one, and only one, r.m. $Z$, called r.m. associated with the stationary c.r.f. $(X_g)_{g \in G}$ defined on $\mathcal{B}_{\widehat{G}}$, and with values in $H$, such that $H_Z = \overline{span}\{X_g; g \in G\}$ and, for any $g$ of $G$, $X_g = \int (\cdot, g)_{\widehat{G}G} dZ(\cdot)$,* and where $(\cdot, \cdot)_{\widehat{G}G}$ denotes the duality bracket.

- When $G = \mathbb{Z}$, we speak about *stationary series*, its dual $\widehat{G}$ is identifiable to $\Pi$ which is a group for $\lambda_1 \oplus \lambda_2 = \lambda_1 + \lambda_2 - \left[\frac{\lambda_1+\lambda_2+\pi}{2\pi}\right]$ (where $[x]$ designates the integer part of $x$). The r.m. associated with the stationary series $(X_n)_{n \in \mathbb{Z}}$ is defined on $\mathcal{B}_\Pi$, the Borel $\sigma$-field of $\Pi$, such that $X_n = \int e^{i\lambda n} dZ(\lambda)$, for any $n \in \mathbb{Z}$.

- When $G = \mathbb{Z}^k$, we also speak about *stationary series*. The r.m. associated with the stationary series $(X_{n_1,n_2,\ldots,n_k})_{(n_1,n_2,\ldots,n_k) \in \mathbb{Z}^k}$ is defined on $\mathcal{B}_{\Pi^k} := \mathcal{B}_\Pi \otimes \cdots \otimes \mathcal{B}_\Pi$, and, for any $(n_1, n_2, \ldots, n_k)$ of $\mathbb{Z}^k$, we have $X_{n_1,n_2,\ldots,n_k} = \int e^{i(\lambda_1 n_1 + \lambda_2 n_2 + \cdots + \lambda_k n_k)} dZ(\lambda_1, \lambda_2, \ldots, \lambda_k)$.

- When $G = \mathbb{R}$, we speak about *stationary (continuous time) process*, its dual $\widehat{G}$ is identifiable to $\mathbb{R}$ and the r.m. associated with the stationary process $(X_t)_{t \in \mathbb{R}}$ is defined on $\mathcal{B}_\mathbb{R}$, the Borel $\sigma$-field of $\mathbb{R}$, such that $X_t = \int e^{i\lambda t} dZ(\lambda)$, for any $t \in \mathbb{R}$.

Let us now recall the correlated stationarity.

*Two stationary c.r.f.* $(X_g)_{g \in G}$ *et* $(Y_g)_{g \in G}$, *taking values in* $H$, *are stationarily correlated when* $< X_{g_1}, Y_{g_2} >_H = < X_{g_1 - g_2}, Y_0 >_H$, *for any pairs* $(g_1, g_2)$ *of elements of* $G$.

The correlated stationarity can be expressed in the frequency domain.

Indeed, *two stationary c.r.f.* $(X_g)_{g \in G}$ *and* $(Y_g)_{g \in G}$ *are stationarily correlated if and only if the associated r.m.* $Z$ *and* $Z'$, *respectively, are also stationarily correlated, i.e. such that* $< Z(A), Z'(B) >_H = 0$, *for any pairs* $(A, B)$ *of disjoint elements of* $\mathcal{B}_{\widehat{G}}$.

REMARK 16. *When* $H$ *is of* $L_p^2$ *type, we say about* $p$-*stationary c.r.f, in which case it is stationary. But a c.r.f. can be stationary without being* $p$-*stationary (this is the case, for example, when the spectral components are not stationarily correlated).*

Let us now examine the measures which take values in $\mathcal{P}(H)$.

### 2.3. Spectral measure and unitary operator. A *(projector valued) spectral measure (s.m.)* on $\mathcal{B}_{\widehat{G}}$ for $H$ is an application $\varepsilon$ from $\mathcal{B}_{\widehat{G}}$ in $\mathcal{P}(H)$ such that

**(i)** $\varepsilon(A \cup B) = \varepsilon(A) + \varepsilon(B)$, for any pair $(A, B)$ of disjoint elements of $\mathcal{B}_{\widehat{G}}$;

**(ii)** for any sequence $(A_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{B}_{\widehat{G}}$ which decreasingly converges to $\emptyset$, $\lim_n \varepsilon(A_n) X = 0$, for any $X$ of $H$.

**(iii)** $\varepsilon(\widehat{G}) = id_H$.

Then it is clear that, for any $X$ of $H$, the mapping $Z_\varepsilon^X$ defined hereafter is a r.m.: $Z_\varepsilon^X : A \in \mathcal{B}_{\widehat{G}} \mapsto \varepsilon(A) X \in H$.

From a family of r.m., we can define a s.m.. Indeed, if $\{Z^X ; X \in H\}$ is a family of r.m., defined on $\mathcal{B}_{\widehat{G}}$ and taking values in $H$, pairwise stationarily correlated and such that $Z^X(\widehat{G}) = X$, for any $X$ of $H$, then

- the mapping $\varepsilon(A) : X \in H \mapsto Z^X(A) \in H$ is an orthogonal projector, for any $A$ of $\mathcal{B}_{\widehat{G}}$;

- the mapping $\varepsilon : A \in \mathcal{B}_{\widehat{G}} \mapsto \varepsilon(A) \in \mathcal{P}(H)$ is a s.m. on $\mathcal{B}_{\widehat{G}}$ for $H$.

For any r.m. $Z$ defined on $\mathcal{B}_{\widehat{G}}$ with values in $H$, we can match one and only one s.m. $\varepsilon$ on $\mathcal{B}_{\widehat{G}}$ for $H_Z$, such that $\varepsilon(A)(\int \varphi dZ) = \int \mathbf{1}_A \varphi dZ$, for any pairs $(A, \varphi)$ of $\mathcal{B}_{\widehat{G}} \times L^2_{\mathbb{C}}(E, \mathcal{E}, \mu_Z)$. This s.m. $\varepsilon$ is called *s.m. associated* with $Z$, and we have $\varepsilon(A)(Z(\widehat{G})) = Z(A)$.

If $G'$ is a second locally compact abelian group which dual $\widehat{G}'$ is of countable basis, and if $f$ is a mapping from $\widehat{G}$ into $\widehat{G}'$, measurable with respect to the $\sigma$-fields $\mathcal{B}_{\widehat{G}}$ and $\mathcal{B}_{\widehat{G}'}$, we call *image of $\varepsilon$ by $f$* the mapping $f(\varepsilon) : A' \in \mathcal{B}_{\widehat{G}} \mapsto \varepsilon(f^{-1}(A')) \in \mathcal{P}(H)$; it is a s.m. on $\mathcal{B}_{\widehat{G}'}$ for $H$.

If $U$ is a unitary operator of $H$, *there exists a s.m. $\varepsilon$, and only one, called s.m. associated with the unitary operator $U$, on $\mathcal{B}_\Pi$ for $H$, such that $UX = \int e^{i \cdot 1} dZ^X_\varepsilon$, for any $X$ of $H$*. Conversely, *if $\varepsilon$ is a s.m. on $\mathcal{B}_\Pi$ for $H$, the mapping $X \in H \mapsto \int e^{i \cdot 1} dZ^X_\varepsilon \in H$ is a unitary operator of associated s.m. $\varepsilon$.*

For any $g$ of $G$, let us denote by $f_g$ the mapping which, with $\gamma$ of $\widehat{G}$, associates the unique element $f_g(\gamma)$ of $\Pi$ such that $e^{i f_g(\gamma)} = (\gamma, g)_{\widehat{G}G}$; $f_g$ is a continuous homomorphism, so measurable.

We call then *group of the unitary operators deduced from $\varepsilon$, s.m. on $\mathcal{B}_{\widehat{G}}$ for $H$, the family $\{U_g; g \in G\}$ of the unitary operators of $H$, where, for any $g$ of $G$, $U_g$ is the unitary operator of associated s.m. $f_g(\varepsilon)$, for any $g$ of $G$.*

If $U$ is a unitary operator of associated s.m. $\varepsilon$, then $\{U^n; n \in \mathbb{Z}\}$ *is the group of the unitary operators deduced from $\varepsilon$, s.m. on $\mathcal{B}_\Pi$ for $H$.*

If $U$ is a unitary operator of associated s.m. $\varepsilon$, we show that
$$\lim_p \sum_{k=0}^{p-1} e^{i(-\pi + k\frac{2\pi}{p})} \varepsilon([-\pi + k\frac{2\pi}{p}, -\pi + (k+1)\frac{2\pi}{p}[) X = UX, \text{ for any } X \text{ of } H.$$ The mapping $\sum_{k=0}^{p-1} e^{i(-\pi + k\frac{2\pi}{p})} \varepsilon([-\pi + k\frac{2\pi}{p}, -\pi + (k+1)\frac{2\pi}{p}[)$ is, for any integer $p \geqslant 2$, a unitary operator for $H$.

Finally, *when $\{U_g; g \in G\}$ is the group of the unitary operators deduced from $\varepsilon$, s.m. on $\mathcal{B}_{\widehat{G}}$ for $H$, $(U_g X)_{g \in G}$ is a stationary c.r.f. of associated s.m. $Z^X_\varepsilon$, this for any $X$ of $H$.*

For all these above questions, we can refer to Boudou (2007), Boudou and Romain (2002), Boudou and Romain (2011).

**2.4. Principal Component Analysis in the frequency domain.** By $L^2_{pq}(M_Z)$ (resp. $L^2_p(M_Z)$) we denote (see, for more details, Boudou and Dauxois (1994)) the space of functions whose values are operators belonging to $\sigma_2(p,q)$ (resp. $\sigma_2(p)$. This space can be regarded as a quotient space of a linear subspace of $[\sigma_2(p,q)]^{\widehat{G}}$ (resp. $[\sigma_2(p)]^{\widehat{G}}$), by a relation of equivalence defined from $M_Z$. Each element $\varphi$, i.e. a coset in $L^2_{pq}(M_Z)$, contains a $\mathcal{B}_{\widehat{G}}$-measurable function which is also denoted by $\varphi$. The space $L^2_{pq}(M_Z)$ associated with the inner product

$$(2.2) \qquad < \varphi, \psi > \ = \ \int tr[\varphi(\lambda) \circ (\frac{dM_Z}{d\nu}(\lambda)) \circ \psi^*(\lambda)]d\nu(\lambda)$$

is a Hilbert space, where $\nu$ is a $\sigma$-finite measure which dominates $M_Z$.

Let $(X_g)_{g \in G}$ be a $p$-stationary c.r.f., $Z$ its associated $p$-r.m.. The image of $(X_g)_{g \in G}$ by any filter (whose transfer function is) $\varphi$ belonging to $L^2_{pq}(M_Z)$, is the $q$-stationary c.r.f. $(Y_g)_{g \in G} = (\int (\gamma, g)_{\widehat{G}G} \varphi(\gamma)dZ(\gamma))_{g \in G}$ of associated $q$-r.m. $Z_\varphi : A \in \mathcal{B} \mapsto \int \mathbf{1}_A \varphi dZ \in L^2_q$ which is stationarily correlated with $Z$. If $\psi$ belongs to $L^2_{ql}(M_{Z_\varphi})$, then $\psi(\cdot)\varphi(\cdot)$ belongs to $L^2_{pl}(M_Z)$ and $\int \psi dZ_\varphi = \int \psi(\cdot)\varphi(\cdot)dZ$.

Now given a $p$-stationary c.r.f. $(X_g)_{g \in G}$ of associated $p$-r.m. $Z$, we want to summarize it by a $q$-stationary c.r.f. $(X'_g)_{g \in G}$ (with $q < p$), of associated $q$-r.m. $Z'$, which is stationarily correlated with $(X_g)_{g \in G}$, that is $\widetilde{X_g} \circ \widetilde{Y_{g'}}^* = \widetilde{X_{g-g'}} \circ \widetilde{Y_0}^*$, for any $(g, g') \in G \times G$, or equivalently, $\widetilde{Z(A)} \circ \widetilde{Z'(B)}^* = 0$, for all pairs $(A, B)$ of disjoint elements of $\mathcal{B}_{\widehat{G}}$. The stationarity properties permit us to affirm that any $p$-dimensional filter $(W_g)_{g \in G}$ of $(X'_g)_{g \in G}$ is such that $\|X_g - W_g\|_{L^2_p} = \|X_0 - W_0\|_{L^2_p}$, for all $g$ in $G$. So we will measure the quality of the $q$-dimensional summary $(X'_g)_{g \in G}$ of $(X_g)_{g \in G}$ by $\inf\{\|X_0 - \int \varphi dZ'\|_{L^2_p}; \varphi \in L^2_{qp}(M_{Z'})\}$. Of course, between all the possible summaries, we will choose the best one from this point of view. So this leads to the following

DEFINITION 7. *The Principal Component Analysis (PCA) of order $q$ of the $p$-stationary c.r.f. $(X_g)_{g \in G}$, of associated $p$-r.m. $Z$, is a $q$-stationary c.r.f. $(X'_g)_{g \in G}$, of associated $q$-r.m. $Z'$, stationarily correlated with $(X_g)_{g \in G}$, and an element $\varphi$ of $L^2_{qp}(M_{Z'})$ such that the norm $\|X_0 - \int \varphi dZ'\|_{L^2_p}$ is minimum.*

If we see that $X_0 = \int I_p dZ$, where $I_p := id_{\mathbb{C}^p}$, we can remark that this analysis is equivalent to the PCA of order $q$ of $Z$ (see Boudou and Dauxois (1994),

Boudou and Viguier-Pla (2006), Boudou *et al.* (2010)). Let $\sum_{j=1}^{p} \mu_j(\cdot) a_j(\cdot) \otimes a_j(\cdot)$ be a measurable Schmidt decomposition of $M_Z'$, the derivative of $M_Z$ with respect to $\nu$ and let $\{f_1, \cdots, f_q\}$ be the canonical basis of $\mathbb{C}^q$. According to Boudou and Dauxois (1994), we can give the

PROPOSITION 8. *A solution of the PCA of order $q$ of the $p$-r.m. $Z$ is given by*
$$Z' := Z_\alpha \text{ and } \varphi(\cdot) := \alpha^*(\cdot) = \sum_{j=1}^{q} f_j \otimes a_j(\cdot), \text{ where } \alpha \text{ is given by } \alpha(\cdot) := \sum_{j=1}^{q} a_j(\cdot) \otimes f_j.$$

## 3. Spectral elements associated with periodic stationary process

We consider in this section, a $p$-stationary process $(X_t)_{t \in \mathbb{R}} \subset L_p^2$ which is $T$-periodic (where $T > 0$), i.e. $\|X_t - X_{t+nT}\|_{L_p^2} = 0$, for any pairs $(t, n)$ of $\mathbb{R} \times \mathbb{Z}$. For more simplicity in the computations, we set $T = 2\pi$. And all other periods $T > 0$ can be treated by an homothetic transformation, that is *when $(X_t)_{t \in \mathbb{R}}$ is $2\pi$-periodic, the process $(Y_t)_{t \in \mathbb{R}}$ defined by $Y_t = X_{\frac{t2\pi}{T}}$, for any $t$ of $\mathbb{R}$, is $T$-periodic.*

Let $Z$ be the associated $p$-r.m. of $(X_t)_{t \in \mathbb{R}}$ and $M_Z$ the vector-valued measure defined in (2.1) and which is dominated by its trace $\mu_Z$. Denote by $S := \left(\frac{dM_Z}{d\mu_Z}\right)^{1/2}$, the square root of the derivative of $M_Z$ with respect to $\mu_Z$. Since $X_{2\pi} = X_0$ in $L_p^2$, we have

$$
\begin{aligned}
\|X_{2\pi} - X_0\|^2 &= \int \|e^{i \cdot 2\pi} S - S\|^2 d\mu_Z &&= \int |e^{i \cdot 2\pi} - 1|^2 \|S\|^2 d\mu_Z \\
&= \int |e^{i \cdot 2\pi} - 1|^2 tr(\frac{dM_Z}{d\mu_Z}) d\mu_Z &&= \int |e^{i \cdot 2\pi} - 1|^2 d\mu_Z \\
&= \|e^{i \cdot 2\pi} - 1\|^2_{L^2(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_Z)} &&= 0
\end{aligned}
$$

So $e^{i \cdot 2\pi} = 1$ $\mu_Z$-almost everywhere and we can give the

LEMMA 35. *For any element $B$ of $\mathcal{B}_\mathbb{R}$, one has $Z(B) = Z(B \cap \mathbb{Z})$.*

**Proof**. If we remark that
$$\{\lambda \in \mathbb{R} : e^{i\lambda 2\pi} = 1\} = \{\lambda \in \mathbb{R} : \lambda 2\pi = k2\pi, k \in \mathbb{Z}\} = \mathbb{Z},$$
we have $\mu_Z(\mathbb{R} \setminus \mathbb{Z}) = 0$. Let $B$ in $\mathcal{B}_\mathbb{R}$, we have $B = (B \cap \mathbb{Z}) \cup (B \cap (\mathbb{R} \setminus \mathbb{Z}))$ and $Z(B) = Z(B \cap \mathbb{Z}) + Z(B \cap (\mathbb{R} \setminus \mathbb{Z}))$. The result of lemma comes since $\|Z(B \cap (\mathbb{R} \setminus \mathbb{Z}))\|^2 = \mu_Z(B \cap (\mathbb{R} \setminus \mathbb{Z})) \leq \mu_Z(\mathbb{R} \setminus \mathbb{Z})$ and then $Z(B \cap (\mathbb{R} \setminus \mathbb{Z})) = 0$. $\square$

If we denote $Z_k := Z(\{k\})$, for any $k$ of $\mathbb{Z}$, we can establish the next result. Let us note that the summability evoked below is in the sense, for example, of Choquet (1964).

PROPOSITION 9. *For any element $B$ of $\mathcal{B}_\mathbb{R}$, the family $\{\delta_k(B)Z_k; k \in \mathbb{Z}\}$ of elements of $L_p^2$ is summable of sum $Z(B)$, i.e.*

(3.1)
$$Z(B) = \sum_{k \in \mathbb{Z}} \delta_k(B) Z_k$$

*where $\delta_k$ is the Dirac measure defined on $\mathcal{B}_\mathbb{R}$ and concentrated in $k$.*

To prove this proposition, let's first state the following

LEMMA 36. *Let $\{x_k; k \in \mathbb{Z}\}$ be a family of pairwise orthogonal elements of a $\mathbb{C}$-Hilbert space $H$, such that the sequence $\left( \sum_{k=-n}^{n} x_k \right)_{n \in \mathbb{N}^*}$ is a Cauchy sequence. Then, the family $\{x_k; k \in \mathbb{Z}\}$ is summable.*

**Proof**. Let $P_f$ be the set of finite parts of $\mathbb{Z}$ and let us set any positive element $\epsilon$ of $\mathbb{R}$ and denote by $x$ the limit of the Cauchy sequence $\left( \sum_{k=-n}^{n} x_k \right)_{n \in \mathbb{N}^*}$. Then, $\|x\|$ is the limit of the sequence $\left( \| \sum_{k=-n}^{n} x_k \| \right)_{n \in \mathbb{N}^*}$ and $\|x\|^2$ the limit of the sequence $\left( \| \sum_{k=-n}^{n} x_k \|^2 \right)_{n \in \mathbb{N}^*}$. Recalling the pairwise orthogonality, one has $\|x\|^2 = \lim_n \sum_{k=-n}^{n} \|x_k\|^2$. Since $\left( \sum_{k=-n}^{n} \|x_k\|^2 \right)_{n \in \mathbb{N}^*}$ is a Cauchy sequence, we deduce that $\{\|x_k\|^2; k \in \mathbb{Z}\}$ is a summable family and therefore satisfies Cauchy's criterion; there exists then an element $J$ of $P_f$ such that, for any $K$ of $P_f$ which verify $K \cap J = \emptyset$, one has $\sum_{k \in K} \|x_k\|^2 < \epsilon^2$. Let $K$ be any element of $P_f$ such that $K \cap J = \emptyset$, then $\| \sum_{k=-n}^{n} x_k \|^2 = \sum_{k \in K} \|x_k\|^2 < \epsilon^2$. The family $\{x_k; k \in \mathbb{Z}\}$ satisfies Cauchy's criterion, it is therefore summable. $\square$

**Proof of Proposition 9**. Let us set $A_n = \mathbb{Z} \setminus \{k \in \mathbb{Z}; |k| \leq n\}$, for any $n$ of $\mathbb{N}^*$. One has $\underset{n \in \mathbb{N}^*}{\cap} A_n = \emptyset$ and $(A_n)_{n \in \mathbb{N}^*}$ is a sequence of elements of $\mathcal{B}_\mathbb{R}$ which decreasingly converges to $\emptyset$. Hence $\lim_n Z(A_n) = 0$ in $L_p^2$ and, furthermore, $\mathbb{Z} = A_n \cup \{k \in \mathbb{Z}; |k| \leq n\}$. Let $B \in \mathcal{B}_\mathbb{R}$, one has $Z(\mathbb{Z} \cap B) = Z(A_n \cap B) + Z(B \cap \{k \in \mathbb{Z}; |k| \leq n\})$.

And according to lemma $35$, $Z(B) = Z(A_n \cap B) + \sum\limits_{k=-n}^{n} Z(B \cap \{k\})$. Since $\lim\limits_{n} Z(A_n \cap B) = 0$, we have $Z(B) = \lim\limits_{n} \sum\limits_{k=-n}^{n} Z(B \cap \{k\})$. On the one hand, $Z(B \cap \{k\}) = \delta_k(B)Z_k$ and $Z(B) = \lim\limits_{n} \sum\limits_{k=-n}^{n} \delta_k(B)Z_k$. On the other hand, $(\sum\limits_{k=-n}^{n} \delta_k(B)Z_k)_{n \in \mathbb{N}^*}$ is a Cauchy sequence which limit is $Z(B)$ and, considering the pairwise orthogonality of $\delta_k(B)Z_k$, we then have, according to lemma $36$, that the family $\{\delta_k(B)Z_k; k \in \mathbb{Z}\}$ is summable of sum $Z(B)$. $\square$

We can, furthermore, show the

PROPOSITION 10. *Let $(X_t)_{t \in \mathbb{R}}$ be a $p$-stationary c.r.f. which is $2\pi$-periodic, and of associated $p$-r.m. $Z$. For any $\varphi$ of $L^2_{pq}(M_Z)$, the family $\{\varphi(k) \circ Z_k; k \in \mathbb{Z}\}$ of elements of $L^2_q$ is summable of sum $\int \varphi dZ$, i.e. $\int \varphi dZ = \sum\limits_{k \in \mathbb{Z}} \varphi(k) \circ Z_k$. In particular, for any $t$ of $\mathbb{R}$, the family $\{e^{ikt} Z_k; k \in \mathbb{Z}\}$ is summable of sum $X_t$.*

**Proof**. Let us consider the $p$-r.m. $Z_\varphi : A \in \mathcal{B}_\mathbb{R} \mapsto \int \mathbf{1}_A \varphi dZ \in L^2_p$. Since $\mu_Z(\mathbb{R} \setminus \mathbb{Z}) = 0$, one has

$$\|Z_\varphi(\mathbb{R} \setminus \mathbb{Z})\|^2 = \| \int \mathbf{1}_{\mathbb{R} \setminus \mathbb{Z}} \varphi dZ \|^2 = \int \mathbf{1}_{\mathbb{R} \setminus \mathbb{Z}} |\varphi|^2 d\mu_Z = 0.$$

Therefore, $Z_\varphi(\mathbb{R} \setminus \mathbb{Z}) = 0$ and

$$(3.2) \qquad \int \varphi dZ = Z_\varphi(\mathbb{R}) = Z_\varphi(\mathbb{Z}).$$

Since $\{\{k\}; k \in \mathbb{Z}\}$ is a countable family of elements of $\mathcal{B}_\mathbb{R}$ which are pairwise disjoint, then $\{Z_\varphi(\{k\}); k \in \mathbb{Z}\}$ is a summable family of sum

$$(3.3) \qquad Z_\varphi(\cup_{k \in \mathbb{Z}} \{k\}) = Z_\varphi(\mathbb{Z}).$$

For any $k$ of $\mathbb{Z}$, one has

$$Z_\varphi(\{k\}) = \int \mathbf{1}_{\{k\}} \varphi dZ = \int \mathbf{1}_{\{k\}} \varphi(k) dZ = \varphi(k) \circ \int \mathbf{1}_{\{k\}} dZ = \varphi(k) \circ Z_k,$$

which, reported in equation ($3.3$), ensures the summability of the family $\{\varphi(k) \circ Z_k; k \in \mathbb{Z}\}$ of sum $Z_\varphi(\mathbb{Z})$. And, according to equation ($3.2$), the family $\{\varphi(k) \circ Z_k; k \in \mathbb{Z}\}$ is summable of sum $\int \varphi dZ$.
In particular, it comes that the family $\{e^{ikt} Z_k; k \in \mathbb{Z}\}$ is summable of sum

$\int e^{i.t} dZ = X_t$. $\square$

Denoting $M_k := \widetilde{Z_k} \circ \widetilde{Z_k}^* = \mathbb{E}[Z_k \otimes Z_k]$ and $\mu_k := tr M_k$, for any $k$ of $\mathbb{Z}$, we can establish the next three results.

PROPOSITION 11. *For any $B$ of $\mathcal{B}_\mathbb{R}$, the family $\{\delta_k(B)M_k; k \in \mathbb{Z}\}$ of elements of $\sigma_2(p)$ is absolutely summable of sum $M_Z(B) := \sum_{k \in \mathbb{Z}} \delta_k(B)M_k$.*

**Proof**. Let $B$ be an element of $\mathcal{B}_\mathbb{R}$, according to proposition 9, one has $Z(B) = \lim_n \sum_{k=-n}^{n} \delta_k(B)Z_k$, and therefore $\|Z(B)\|^2 = \lim_n \sum_{k=-n}^{n} \|\delta_k(B)Z_k\|^2$. Since $\|\delta_k(B)M_k\| \leq \delta_k(B)\|Z_k\|^2$, the family $\{\delta_k(B)M_k; k \in \mathbb{Z}\}$ of elements of $\sigma_2(p)$ is absolutely summable of sum $\lim_n \sum_{k=-n}^{n} \delta_k(B)M_k$.

Otherwise, one has

$$
\begin{aligned}
M_Z(B) &= \widetilde{Z(B)} \circ \widetilde{Z(B)}^* = (\lim_n \sum_{k=-n}^{n} \delta_k(B)\widetilde{Z_k}) \circ \widetilde{Z(B)}^* \\
&= \lim_n \sum_{k=-n}^{n} \delta_k(B)\widetilde{Z_k} \circ \widetilde{Z(B)}^* \\
&= (\lim_n \sum_{k=-n}^{n} \delta_k(B)\widetilde{Z_k}) \circ (\lim_m \sum_{i=-m}^{m} \delta_k(B)\widetilde{Z_i}^*) \\
&= \lim_n \sum_{k=-n}^{n} \delta_k(B)\widetilde{Z_k} \circ \widetilde{Z_k}^* = \lim_n \sum_{k=-n}^{n} \delta_k(B)M_k. \square
\end{aligned}
$$

PROPOSITION 12. *For any $B$ of $\mathcal{B}_\mathbb{R}$, the family $\{\delta_k(B)\mu_k; k \in \mathbb{Z}\}$ of positive elements of $\mathbb{R}$ is summable of sum $\mu_Z(B) := \sum_{k \in \mathbb{Z}} \delta_k(B)\mu_k$.*

**Proof**. Let $B$ be an element of $\mathcal{B}_\mathbb{R}$. According to proposition 11, it is clear that

$$
\mu_Z(B) = tr M_Z(B) = tr(\lim_n \sum_{k=-n}^{n} \delta_k(B)M_k) = \lim_n \sum_{k=-n}^{n} \delta_k(B) tr M_k.
$$

$\square$

Let $M'$ denote the density of $M_Z$ with respect to $\mu_Z$. We then have

PROPOSITION 13. *For any $k$ of $\mathbb{Z}$, $\mu_k M'(k) = \widetilde{Z_k} \circ \widetilde{Z_k}^*$.*

**Proof**. Let $N$ denote the measurable mapping defined on $\mathbb{R}$ with values in $\sigma_2(p)$ by

$$
N : \begin{cases}
k \in \mathbb{Z} \ \mapsto \ \mu_k^{-1}\widetilde{Z}_k \circ \widetilde{Z}_k^*, & \text{if } \mu_k \neq 0 \\[2mm]
k \in \mathbb{Z} \ \mapsto \ M'(k), & \text{if } \mu_k = 0 \\[2mm]
x \notin \mathbb{Z} \ \mapsto \ 0.
\end{cases}
$$

On the one hand, for any $x$ of $\mathbb{Z}$, one has $N(x) = M'(x)$ for three reasons:
- firstly, if $x \in \mathbb{Z}$, then it is of the type $x = k$ and
$\widetilde{Z}_k \circ \widetilde{Z}_k^* = M_Z(k) = \int \mathbf{1}_{\{k\}}(.)M'(.)d\mu_Z = \int \mathbf{1}_{\{k\}}(.)M'(k)d\mu_Z = M'(k)\mu_k$;

- secondly, if $\mu_k \neq 0$, then $N(k) = \mu_k^{-1}\widetilde{Z}_k \circ \widetilde{Z}_k^* = M'(k)$;

- and thirdly, if $\mu_k = 0$, then $N(k) = M'(k)$.

On the other hand, since $\mathbb{Z} \subset (N = M')$ or $(N \neq M') \subset (\mathbb{R} \setminus \mathbb{Z})$, one has $N = M'$ $\mu_Z$-almost everywhere and $\mu_Z(\mathbb{R} \setminus \mathbb{Z}) = 0$.

Therefore, we can choose as a derivative of $M_Z$ with respect to $\mu_Z$ the map $M'$ which associates $\mu_k^{-1}\widetilde{Z}_k \circ \widetilde{Z}_k^*$ to $k \in \mathbb{Z}$, when $\mu_k \neq 0$, and associates $0$ to $x \notin \mathbb{Z}$. For any $k$ of $\mathbb{Z}$:

- either $\mu_k = 0$ and therefore $Z_k = 0$ and $\mu_k M'(k) = \widetilde{Z}_k \circ \widetilde{Z_k}^*$;

- or $\mu_k \neq 0$ and therefore $\mu_k M'(k) = \widetilde{Z}_k \circ \widetilde{Z}_k^*$. $\square$

REMARK 17. *From proposition 10, it follows a technique of truncation of the process spectral density, which consists in approximating $X_t$ only by the elements $Z_k$ of maximum norm. Indeed, we can say that the family $\{\|Z_k\|^2; k \in \mathbb{Z}\}$ of positive elements of $\mathbb{R}$ is summable of sum $\mu_Z(\mathbb{R}) = \mu_Z(\mathbb{Z})$ and therefore $\lim_k \|Z_k\| = 0$. That is, from a certain rank $N_0$, we can express $X_t$ only by $Z_k$, with $|k| \leq N_0$, which have maximum norm and therefore $X_t \backsimeq \sum_{|k| \leq N_0} e^{ikt}Z_k$.*

Hence comes the

PROPOSITION 14. *The sequence* $(X_t^n)_{n \in \mathbb{N}}$ *of elements of* $L_p^2$, *defined by* $X_t^n = \sum\limits_{|k| \leq n} e^{ikt} Z_k$, *converges uniformly in* $t \in \mathbb{R}$ *to* $X_t$.

**Proof**. Since $\{\|Z_k\|^2, k \in \mathbb{Z}\}$ is summable of sum $\mu_Z(\mathbb{R}) = \mu_Z(\mathbb{Z})$, we have $\lim\limits_n [\mu_Z(\mathbb{Z}) - \sum\limits_{|k| \leq n} \|Z_k\|^2] = 0$.

Let $\epsilon$ be any positive element of $\mathbb{R}$, then there exits $N_0(\epsilon) \in \mathbb{N}$ such that, for any $n > N_0(\epsilon)$, one has $\sum\limits_{|k| > n} \|Z_k\|^2 < \epsilon^2$.

Let $t \in \mathbb{R}$ and $n > N_0(\epsilon)$, one has:

$$
\begin{aligned}
\|X_t^n - X_t\|^2 &= \|\sum_{|k| \leq n} e^{itk} Z_k - \sum_{l \in \mathbb{Z}} e^{itl} Z_l\|^2 = \|\sum_{|k| > n} e^{itk} Z_k\|^2 \\
&= \sum_{|k| > n} \|Z_k\|^2 < \epsilon^2.
\end{aligned}
$$

Hence comes the uniform convergence in $t$. $\square$

## 4. Almost periodic stationary process

DEFINITION 8. *We say that a $p$-stationary process $(X_t)_{t \in \mathbb{R}}$ is $(T, \eta)$-almost periodic, where $T > 0$ and $\eta \in ]0, 2\|X_0\|[$, if $\|X_t - X_{t+nT}\|_{L_p^2} \leq \eta$, for any pairs $(t, n) \in \mathbb{R} \times \mathbb{Z}$.*

Let us consider a $(T, \eta)$-almost periodic $p$-stationary process $(X_t)_{t \in \mathbb{R}}$ of associated $p$-r.m. $Z$. Our goal, in this section, is at first to find the $p$-r.m. $\mathfrak{Z}$ associated with the $p$-stationary series $(X_{nT})_{n \in \mathbb{Z}}$. Secondly, we study the $p$-r.m. $\mathfrak{Z}$ which may lead to interesting consequences about the behavior of the spectral tools associated with $(X_t)_{t \in \mathbb{R}}$. Recall that the series $(X_{nT})_{n \in \mathbb{Z}}$ is $p$-stationary in the sense that

$$
\begin{aligned}
\widetilde{X}_{nT} \circ \widetilde{X}_{mT}^* &= \mathbb{E}(X_{nT} \otimes X_{mT}) = \mathbb{E}(X_{nT-mT} \otimes X_0) \\
&= \mathbb{E}(X_{(n-m)T} \otimes X_0) = \widetilde{X}_{(n-m)T} \circ \widetilde{X}_0^*.
\end{aligned}
$$

for any pair of elements $n$ and $m$ of $\mathbb{Z}$. Thus there exists one, and only one, $p$-r.m. $\mathfrak{Z}$ associated with $(X_{nT})_{n \in \mathbb{Z}}$ and defined on $\mathcal{B}_\Pi$ with values in $L_p^2$. Let $f$ be the mapping defined by

$$
(4.1) \qquad f \; : \; \begin{cases} \mathbb{R} & \to & \Pi \\[2mm] \lambda & \mapsto & \lambda T - 2\pi[\frac{\lambda T + \pi}{2\pi}] \end{cases}
$$

It is a continuous mapping, since being the transpose of the continuous homomorphism $n \in \mathbb{Z} \mapsto nT \in \mathbb{R}$ (see Boudou (2007)); hence, $f$ is measurable and we have

$$
(4.2) \qquad f^{-1}(A) = \bigcup_{k \in \mathbb{Z}} (k\frac{2\pi}{T} + \frac{1}{T}A), \quad \text{for any Borel set } A \text{ of } \Pi.
$$

In particular, $f^{-1}(\{0\}) = \frac{2\pi}{T}\mathbb{Z}$. According to section 2.1, it is clear that $f(Z)$ is a $p$-r.m., image of $Z$ by $f$, and we have the

PROPOSITION 15. *If $(X_t)_{t \in \mathbb{R}}$ is a $p$-stationary process of associated $p$-r.m. $Z$, then the series $(X_{nT})_{n \in \mathbb{Z}}$ is $p$-stationary of associated $p$-r.m. $\mathfrak{Z} := f(Z)$, where $f$ is given by (4.1).*

**Proof**. Let $(Y_n)_{n \in \mathbb{Z}}$ be the $p$-stationary series associated with the $p$-r.m. $f(Z)$. We then have, for any $n$ of $\mathbb{Z}$ and where $I_p := id_{\mathbb{C}^p}$, $Y_n = \int \overline{e^{i \cdot n} I_p} df(Z) = \int \overline{e^{if(\cdot)n} I_p} dZ$. A representative of $e^{if(\cdot)n} I_p$ is the mapping defined by $t \in \mathbb{R} \mapsto e^{i(tTn - 2\pi n[\frac{tT+\pi}{2\pi}])} I_p \in \sigma_2(p)$, i.e. the mapping $t \in \mathbb{R} \mapsto e^{itTn} I_p \in \sigma_2(p)$. So $\overline{e^{if(\cdot)n} I_p} = \overline{e^{i \cdot Tn} I_p}$ in $L_p^2(M_Z)$ and for any $n$ of $\mathbb{Z}$, we have $Y_n = \int \overline{e^{if(\cdot)n} I_p} dZ = \int \overline{e^{i \cdot Tn} I_p} dZ = X_{nT}$. $\square$

The $(T, \eta)$-almost periodic $p$-stationary process $(X_t)_{t \in \mathbb{R}}$ is such that, in particular, one has

$$
(4.3) \qquad \|X_0 - X_{nT}\| \le \eta, \qquad \forall n \in \mathbb{Z}.
$$

We then can state the

PROPOSITION 16. *With the same notations, we have*

$$
(4.4) \qquad \|\mathfrak{Z}(\{0\}) - X_0\| \le \eta \quad ;
$$
$$
(4.5) \qquad \mu_{\mathfrak{Z}}(\Pi \setminus \{0\}) \le \eta^2 \quad ;
$$
$$
(4.6) \qquad \mu_Z(\mathbb{R} \setminus \mathbb{Z}) \le \eta^2 \quad .
$$

**Proof**. According to Von Neumann's ergodic theorem (see, e.g. Krengel (1985) in chapter $1$ p.$4$ or Riesz and Nagy (1968) in chapter $X$ p.$397$), we can deduce that $\mathfrak{Z}(\{\lambda\}) = \lim_n \frac{1}{n} \sum_{k=1}^{n} e^{-i\lambda k} X_{2\pi k}$, for any $\lambda \in \Pi$. In particular, $\mathfrak{Z}(\{0\}) = \lim_n \frac{1}{n} \sum_{k=1}^{n} X_{2\pi k}$. And so

$$
\begin{aligned}
\|\mathfrak{Z}(\{0\}) - X_0\| &= \|\lim_n \frac{1}{n} \sum_{k=1}^{n} X_{2\pi k} - X_0\| = \|\lim_n \frac{1}{n} \sum_{k=1}^{n} (X_{2\pi k} - X_0)\| \\
&= \lim_n \frac{1}{n} \|\sum_{k=1}^{n} (X_{2\pi k} - X_0)\| \leq \lim_n \frac{1}{n} \sum_{k=1}^{n} \|X_{2\pi k} - X_0\| \\
&\leq \eta.
\end{aligned}
$$

Hence comes the inequality (4.4). To prove (4.5), one has

$$
\begin{aligned}
\mu_{\mathfrak{Z}}(\Pi \setminus \{0\}) &= \|\mathfrak{Z}(\Pi \setminus \{0\})\|^2 = \|\mathfrak{Z}(\Pi) - \mathfrak{Z}(\{0\})\|^2 = \|X_0 - \mathfrak{Z}(\{0\})\|^2 \\
&\leq \eta^2.
\end{aligned}
$$

By relations (4.5) and (4.1), with $T = 2\pi$, we get easily the inequality (4.6). $\square$

These consequences are very useful since they allow us to study the proximity or not between the spectral tools associated with, respectively, $2\pi$-periodic and $(2\pi, \eta)$-almost periodic stationary processes. Indeed, the $p$-r.m. defined by $Z' : A \in \mathcal{B}_{\mathbb{R}} \mapsto Z(A \cap \mathbb{Z}) \in L^2_p$ is associated with the $p$-stationary process $(\int_{\mathbb{R}} e^{i.t} dZ')_{t\in\mathbb{R}} = (\int_{\mathbb{R}} \mathbf{1}_{\mathbb{Z}} e^{i.t} dZ)_{t\in\mathbb{R}}$, which is $2\pi$-periodic.

### 4.1. A non-proximity between unitary operators.
The proximity between $2\pi$-periodic and $(2\pi, \eta)$-almost periodic $p$-stationary processes does not also imply that of the associated unit operators.

Indeed, let's study the following counter-example. Let $F$ be a closed subspace of $L^2_p$ and $P$ (resp. $P^\perp$) be the projector on $F$ (resp. $F^\perp$, the orthogonal of $F$). Consider the operators $U = P + e^{i\lambda} P^\perp$ and $I = P + P^\perp$. It is clear that $U$ and $I$ are unitary in $L^2_p$ and we verify that $(U^n X)_{n\in\mathbb{Z}}$ and $(I^n X)_{n\in\mathbb{Z}}$ are stationary series, for any $X$ in $L^2_p$. We have $\|U - I\| = \|e^{i\lambda} P^\perp - P^\perp\| = \|e^{i\lambda} - 1\| \|P^\perp\|$. Since $\|P^\perp\| = 1$, we then have $\|U - I\| = |e^{i\lambda} - 1|$ and $\|X_n - X_0\| = 2|\sin \frac{n\lambda}{2}| \|P^\perp X\|$.

- If $|\lambda|$ is small and $\|P^\perp X\|$ is large, then $U$ is close to $I$. For any $n$ in $\mathbb{Z}$ such that $n\lambda \simeq \pi$, we have $\|X_n - X_0\| \simeq 2\|P^\perp X\|$, which is large.

- If $\lambda$ is large and $\|P^\perp X\|$ is small, then $\|U - I\| = |e^{i\lambda} - 1| \simeq 2$, but $\|X_n - X_0\| \leq 2\|P^\perp X\|$, so $X_n$ is close to $X_0$.

This non-proximity of unit operators, for fairly close stationary processes, may surprise.

**4.2. A proximity between associated $p$-r.m.** Let $A$ be an element of $\mathcal{B}_\mathbb{R}$. With the same notations, we have
$Z(A) = Z(A \cap \mathbb{Z}) + Z(A \cap (\mathbb{R} \setminus \mathbb{Z})) = Z'(A) + Z(A \cap (\mathbb{R} \setminus \mathbb{Z}))$ and so

$$
\begin{aligned}
\|Z(A) - Z'(A)\|^2 &= \|Z(A \cap (\mathbb{R} \setminus \mathbb{Z}))\|^2 = \mu_Z(A \cap (\mathbb{R} \setminus \mathbb{Z})) \\
&\leq \mu_Z(\mathbb{R} \setminus \mathbb{Z}) = \|Z(\mathbb{R} \setminus \mathbb{Z})\|^2 \leq \eta^2.
\end{aligned}
$$

Hence the following proposition comes.

PROPOSITION 17. *If $(X_t)_{t\in\mathbb{R}}$ is a $p$-stationary process of associated $p$-r.m. $Z$, such that $\|X_0 - X_{2\pi n}\| \leq \eta$, for any element $n$ of $\mathbb{Z}$, then*

**(i)** $\|X_t - X_{t+2\pi n}\| \leq \eta$, *for any pairs $(n,t)$ of $\mathbb{Z} \times \mathbb{R}$;*

**(ii)** *the image $(X_t')_{t\in\mathbb{R}}$ of $(X_t)_{t\in\mathbb{R}}$ by the filter $\mathbf{1}_\mathbb{Z} I_{C^p}$ is $2\pi$-periodic and its associated $p$-r.m. is $Z' : A \in \mathcal{B}_\mathbb{R} \mapsto Z(A \cap \mathbb{Z}) \in L_p^2$ ;*

**(iii)** $\|Z(A) - Z'(A)\|^2 \leq \eta^2$, *for any element $A$ of $\mathcal{B}_\mathbb{R}$.*

From the last inequality above, it can be affirmed that the proximity relation between $2\pi$-periodic and $(2\pi, \eta)$-almost periodic $p$-stationary processes, ensures that of associated $p$-r.m..

REMARK 18. *The results proposed in this paper are valid in a $\mathbb{C}$-Hilbert space $H$ even though if we have approached the particular case where $H$ is of $L_p^2$ type and the r.m. $Z$ is a $p$-r.m..*

## 5. Periodic and almost periodic PCA in the frequency domain

In this section, we show that the filters resulting from PCA, in the frequency domain, of $2\pi$-periodic and $(2\pi, \eta)$-almost periodic stationary processes respectively, are close. This result is obtained from a proximity

property between the associated r.m., thus making it possible to perform the comparison of periodic and almost periodic PCA.

**5.1. A proximity property.** Let $Z$ be a $p$-r.m. (which is a r.m.) defined on $\mathcal{B}_{\widehat{G}}$ with values in $L_p^2$ and verifying: *there exists an element $O$ of $\mathcal{B}_{\widehat{G}}$ such that $\|Z(O)\|_{L_p^2} \leq \eta$, where $0 < \eta < 2\|Z(\widehat{G})\|_{L_p^2}$.*
Consider $Z_1$ and $Z_2$ the $p$-r.m. defined by

$$
\begin{aligned}
Z_1 &: \quad A \in \mathcal{B}_{\widehat{G}} \mapsto Z(A \cap (\widehat{G} \setminus O)) \in L_p^2 \\
Z_2 &: \quad A \in \mathcal{B}_{\widehat{G}} \mapsto Z(A \cap O) \in L_p^2.
\end{aligned}
$$

Since $(\widehat{G} \setminus O) \cup O = \widehat{G}$, we can easily remark that $Z(A) = Z_1(A) + Z_2(A)$, for any $A$ of $\mathcal{B}_{\widehat{G}}$. Furthermore, recalling the definition of r.m., the $p$-r.m. $Z_1$ and $Z_2$ are stationarily correlated, i.e. for any pairs $(A, B)$ of elements of $\mathcal{B}_{\widehat{G}}$, since $A \cap (\widehat{G} \setminus O)$ and $B \cap O$ are disjoint, we have

$$
\widetilde{Z_1(A)} \circ \widetilde{Z_2(B)}^* = Z(A \cap (\widehat{G} \setminus O)) \circ \widetilde{Z(B \cap O)}^* = 0
$$

Hence comes the next proposition.

PROPOSITION 18. *For any $A$ of $\mathcal{B}_{\widehat{G}}$, we have*

(5.1) $$\mu_Z(A) = \mu_{Z_1}(A) + \mu_{Z_2}(A);$$
(5.2) $$M_Z(A) = M_{Z_1}(A) + M_{Z_2}(A).$$

From relation (5.1), we can affirm that the measure $\mu_Z$ dominates $M_{Z_1}$ and $M_{Z_2}$. And if we denote by $M_1'$ (resp. $M_2'$) the derivative of $M_{Z_1}$ (resp. $M_{Z_2}$) with respect to $\mu_Z$, then $M' := M_1' + M_2'$ is the derivative of $M_Z$ with respect to $\mu_Z$. Indeed, for any $A$ of $\mathcal{B}_{\widehat{G}}$, we have

$$
\begin{aligned}
\int \mathbf{1}_A M' dZ &= \int \mathbf{1}_A (M_1' + M_2') dZ = \int \mathbf{1}_A M_1' dZ + \int \mathbf{1}_A M_2' dZ \\
&= M_{Z_1}(A) + M_{Z_2}(A) = M_Z(A).
\end{aligned}
$$

Let $\sum_{j=1}^{p} \lambda_j(\cdot) a_j(\cdot) \otimes a_j(\cdot)$ (resp. $\sum_{j=1}^{p} \alpha_j(\cdot) b_j(\cdot) \otimes b_j(\cdot)$) be a measurable Schmidt decomposition of $M_1'$ (resp. $M_2'$). We pose, for any $j$ in $\{1, 2, \cdots, p\}$, by $\gamma_j := \mathbf{1}_{\widehat{G} \setminus O}(\cdot) \lambda_j + \mathbf{1}_O(\cdot) \alpha_j$ and $c_j := \mathbf{1}_{\widehat{G} \setminus O}(\cdot) a_j + \mathbf{1}_O(\cdot) b_j$. Then $\sum_{j=1}^{p} \gamma_j(\cdot) c_j(\cdot) \otimes c_j(\cdot)$ is a measurable Schmidt decomposition of $M'$. So, if we denote by $\{f_1, \ldots, f_q\}$ the canonical basis of $\mathbb{C}^q$, a solution of the PCA of order $q$ of $Z$ (resp. $Z_1$

and $Z_2$) is given by the $q$-r.m.

$$Z_\varphi = \int \varphi dZ \text{ (resp. } Z_{\varphi_1} = \int \varphi_1 dZ_1 \text{ and } Z_{\varphi_2} = \int \varphi_2 dZ_2\text{),}$$

where $\varphi = \sum\limits_{j=1}^{q} c_j(\cdot) \otimes f_j$ (resp. $\varphi_1 = \sum\limits_{j=1}^{q} a_j(\cdot) \otimes f_j$ and $\varphi_2 = \sum\limits_{j=1}^{q} b_j(\cdot) \otimes f_j$). We have $\mathbf{1}_{\widehat{G}\backslash O}\varphi = \varphi_1$ and $\mathbf{1}_O\varphi = \varphi_2$ and the relation between the $q$-r.m. $Z_\varphi$, $Z_{\varphi_1}$ and $Z_{\varphi_2}$ is given, for any $A$ in $\mathcal{B}_{\widehat{G}}$, by

$$
\begin{aligned}
Z_\varphi(A) &= \int \mathbf{1}_A \varphi dZ = \int \mathbf{1}_A \mathbf{1}_{\widehat{G}\backslash O}\varphi dZ + \int \mathbf{1}_A \mathbf{1}_O \varphi dZ \\
&= \int \mathbf{1}_A \mathbf{1}_{\widehat{G}\backslash O}\varphi_1 dZ + \int \mathbf{1}_A \mathbf{1}_O \varphi_2 dZ = \int \mathbf{1}_A \varphi_1 dZ_1 + \int \mathbf{1}_A \varphi_2 dZ_2 \\
&= Z_{\varphi_1}(A) + Z_{\varphi_2}(A).
\end{aligned}
$$

Furthermore, we can affirm the

PROPOSITION 19. *For any $A$ of $\mathcal{B}_{\widehat{G}}$, we have $\|Z_\varphi(A) - Z_{\varphi_1}(A)\|^2 \leq q\eta^2$.*

**Proof**. Indeed, let $A$ be an element of $\mathcal{B}_{\widehat{G}}$. We have

$$
\begin{aligned}
\|Z_\varphi(A) - Z_{\varphi_1}(A)\|^2 &= \|Z_{\varphi_2}(A)\|^2 = \| \int \mathbf{1}_A \varphi_2 dZ_2 \|^2 \\
&= \| \int \mathbf{1}_A \mathbf{1}_O \varphi_2 dZ \|^2 = \| \int \mathbf{1}_{A\cap O} \varphi_2 dZ \|^2 \\
&= \int \mathbf{1}_{A\cap O} \|\varphi_2\|^2 d\mu_Z = q\mu_Z(A \cap O) \\
&\leq q\mu_Z(O) \leq q\eta^2.
\end{aligned}
$$

$\square$

From proposition 19, we can say that the PCA of order $q$ of $Z$ and $Z_1$, respectively, are close. This proximity relation is also reflected in the PCA, in the frequency domain, of $p$-stationary processes associated with $Z$ and $Z_1$ which, as we shall see in the next section, are respectively $(2\pi, \eta)$-almost periodic and $2\pi$-periodic.

### 5.2. Applications to stationary series and stationary process.

5.2.1. *When $G = \mathbb{Z}$*. We consider, in this section, a $p$-stationary series $(X_n)_{n\in\mathbb{Z}} \subset L_p^2$ of associated $p$-r.m. $Z$, which is $(r, \eta)$-almost periodic, where $r \in \mathbb{N}^*$ and $0 < \eta < 2\|X_0\|_{L_p^2}$, i.e. $\|X_n - X_{n+mr}\|_{L_p^2} \leq \eta$, for any pairs $(n, m)$ of elements of $\mathbb{Z}$.

Recall that when $G = \mathbb{Z}$, the dual $\widehat{G}$ of $G$ is identifiable to $\Pi = [-\pi, \pi[$ which is a group for the law $\lambda_1 \oplus \lambda_2 = \lambda_1 + \lambda_2 - 2\pi[\frac{\lambda_1 + \lambda_2 + \pi}{2\pi}]$ (where $[x]$ denotes the integer part of $x$). The $\sigma$-field $\mathcal{B}_{\widehat{G}}$ is the trace of $\mathcal{B}_{\mathbb{R}}$ on $\Pi$, denoted by $\mathcal{B}_\Pi$. The $p$-r.m. associated with a $p$-stationary series $(X_n)_{n \in \mathbb{Z}}$ is such that $X_n = \int e^{i \cdot n} dZ$, for any $n$ of $\mathbb{Z}$. Then we can affirm that

PROPOSITION 20. *The $p$-dimensional series $(X_{nr})_{n \in \mathbb{Z}}$ is $p$-stationary with associated $p$-r.m $\mathfrak{Z} := f(Z)$, which is the image of $Z$ by the measurable homomorphism $f : \lambda \in \Pi \mapsto r\lambda - 2\pi[\frac{r\lambda + \pi}{2\pi}] \in \Pi$.*

REMARK 19. *We have $f^{-1}(\{0\}) = \{\lambda_k; k = 0, 1, \ldots, r - 1\}$, where $\lambda_k$ is given by $\lambda_k := k\frac{2\pi}{r} - 2\pi[\frac{k}{r} + \frac{1}{2}]$.*

Furthermore, we can state the

PROPOSITION 21. *The $p$-r.m. $Z$ is almost concentrated on the set $\{\lambda_k; k = 0, 1, \ldots, r - 1\}$, i.e. $\mu_Z(\Pi \setminus \{\lambda_k; k = 0, 1, \ldots, r - 1\}) \le \eta^2$.*

Let $Z'$ be the $p$-r.m. defined by $Z' : A \in \mathcal{B}_\Pi \mapsto \sum_{k=0}^{r-1} \delta_{\lambda_k}(A) Z(\{\lambda_k\}) \in L_p^2$. This $p$-r.m. is associated with the $p$-stationary series $(X'_n)_{n \in \mathbb{Z}}$ defined, for any $n$ in $\mathbb{Z}$, by $X'_n = \sum_{k=0}^{r-1} e^{i\lambda_k n} Z(\{\lambda_k\})$, which is $r$-periodic. And, for any $A$ of $\mathcal{B}_\Pi$, we have $\|Z(A) - Z'(A)\| \le \eta$.

Since perform the PCA of order $q$ of a $p$-stationary series is equivalent to perform the PCA of the associated $p$-r.m. and since

$$\|Z(\Pi \setminus \{\lambda_k; k = 0, 1, \ldots, r - 1\})\| \le \eta,$$

it is possible, according to the proximity property of section 5.1, to approach the PCA of order $q$ of the $p$-stationary series $(X_n)_{n \in \mathbb{Z}}$ by that of the $r$-periodic $p$-stationary series $(X'_n)_{n \in \mathbb{Z}}$.

5.2.2. *When $G = \mathbb{R}$.* The dual $\widehat{G}$ is identifiable to $\mathbb{R}$ and the $p$-r.m. $Z$ associated with a $p$-stationary c.r.f. $(X_t)_{t \in \mathbb{R}}$ is defined on the Borel $\sigma$-field $\mathcal{B}_{\mathbb{R}}$ of $\mathbb{R}$; it is such that $X_t = \int e^{i \cdot t} dZ$, for any $t$ of $\mathbb{R}$.

Let $(X_t)_{t \in \mathbb{R}}$ be a $p$-stationary process of associated $p$-r.m. $Z$ and such that $\|X_t - X_{t+n2\pi}\|_{L_p^2} \le \eta$, for any pairs $(n, t)$ in $\mathbb{Z} \times \mathbb{R}$ and where $0 < \eta < 2\|X_0\|_{L_p^2}$.

It is clear that:

(a) the series $(X_{n2\pi})_{n\in\mathbb{Z}}$ is $p$-stationary with associated $p$-r.m. $\mathfrak{Z}$ given by $\mathfrak{Z} := f(Z)$, image of $Z$ by the measurable mapping

$$f : \lambda \in \mathbb{R} \mapsto 2\pi\lambda - 2\pi\left[\frac{2\pi\lambda + \pi}{2\pi}\right] \in \Pi, \text{ with } f^{-1}(\{0\}) = \mathbb{Z};$$

(b) the $p$-r.m. $Z$ is concentrated on $\mathbb{Z}$, i.e. $\mu_Z(\mathbb{R} \setminus \mathbb{Z}) \leq \eta^2$.

Let $Z'$ be the $p$-r.m. defined by $Z' : A \in \mathcal{B}_{\mathbb{R}} \mapsto Z(A \cap \mathbb{Z}) \in L_p^2$. It is associated with the $p$-stationary c.r.f. $(\int e^{i\cdot t}dZ')_{t\in\mathbb{R}} = (\int e^{i\cdot t}\mathbf{1}_{\mathbb{Z}}dZ)_{t\in\mathbb{R}}$, which is $2\pi$-periodic. And we verify easily that $\|Z(A) - Z'(A)\| \leq \eta$, for any $A$ in $\mathcal{B}_{\mathbb{R}}$. Hence, it is possible to approach the PCA of order $q$, in the frequency domain, of $(X_t)_{t\in\mathbb{R}}$ by that of $(\int e^{i\cdot t}\mathbf{1}_{\mathbb{Z}}dZ)_{t\in\mathbb{R}}$, since $\|Z(\mathbb{R} \setminus \mathbb{Z})\| \leq \eta$.

## 6. Conclusion

The spectral analysis of periodic and almost periodic stationary processes has shown that:

- on the one hand, the existence of a proximity property, which intensity related to $\eta$ is controlled, between the respective associated r.m.; this only reinforces the one-to-one correspondence relation between a stationary process and its associated r.m.

- on the other hand, each of the other spectral elements usually associated with both processes (projector valued spectral measures and unit operators) do not check such a property and may even have completely "non-controllable" behaviors;

- it would also be possible to approach the PCA, in the frequency domain, of a almost periodic stationary process by that of its periodic version.

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