A COLLECTION OF PAPERS IN MATHEMATICS AND RELATED SCIENCES,
A FESTSCHRIFT IN HONOUR OF THE LATE GALAYE DIA
EDITORS : PROFS HAMET SEYDI, GANE SAMB LO, ABOUBAKARY DIAKHABY
SPAS-EDS (SAINT-LOUIS, CALGARY, 2018), WWW.STATPAS.ORG/SPASEDS/
IN EUCLID (WWW.PROJECTEUCLID.ORG)
DOI : 10.16929/SBS/2018.100

CHAPTER 16

The Riemann extension of an affine Osserman connection on 3-dimensional manifold, *by A.S. Diallo*

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Abstract. The Riemannian extension of torsion free affine manifolds (M, ∇) is an important method to produce pseudo-Riemannian manifolds. It is know that, if the manifold (M, ∇) is a torsion-free affine two-dimensional manifold with skew symmetric tensor Ricci, then (M, ∇) is affine Osserman manifold . In higher dimensions the skew symmetric of the tensor Ricci is a necessary but not sufficient condition for a affine connection to be Osserman. In this paper we construct affine Osserman connection with Ricci flat but not flat and example of Osserman pseudo-Riemannian metric of signature $(3, 3)$ is exhibited.

Keywords. Affine connection; Jacobi operator; Osserman manifold; Riemann extension . **AMS 2010 Mathematics Subject Classification**. 53B05; 53B20; 53C30.

Cite the chapter as :
Diallo, A.S. (2018). The Riemann extension of an affine Osserman connection on 3-dimensional manifold. In *A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia* (Editors : Seydi H., Lo G.S. and Diakhaby A.). Spas Editions, Euclid Series Book, pp. 283 –293. Doi : 10.16929/sbs/2018.100-03-04

1. Introduction

García-Rio *et al.* (1999) introduced the notion of *affine Osserman connection*. The concept of affine Osserman connection originated from the effort to build up examples of pseudo-Riemannian Osserman manifolds (see Diallo (2011), Diallo and Hassirou (2011),García-Rio *et al.* (2002)) via the construction called the *Riemannian extension*. This construction assigns to every m-dimensional manifold M with a torsion-free affine connection ∇ a pseudo-Riemannian metric $\bar{g}$ of signature (m, m) on the cotangent bundle T^*M. The authors in García-Rio *et al.* (1999) pay attention to dimension $m = 2$. They prove that a 2-dimensional manifold with a connection ∇ is affine Osserman if and only if the Ricci tensor of ∇ is skew-symmetric on M. Recently, the author in Diallo (2011) gave an explicit form of affine Osserman connection on 2-dimensional manifolds. For dimension $m = 3$, to make a description is an interesting problem. Partial results was published in Diallo and Hassirou (2011, 2012).

Our paper is organized as follows. Section 1 introduces this topics. In section 2 we recall some basics definitions and results about affine Osserman connections. In section 3, we study the Osserman condition on a particular affine connection (cf. Proposition 3). Section 4, we will exhibit a non flat pseudo-Riemannian Osserman metric of signature $(3, 3)$ (cf. Proposition 4).

2. Preliminaries

In this section, we give the necessary tools needed to reach our goal. We start by giving the definition of affine connections and we reproduce some basic definitions and results about affine Osserman connections taken from the book García-Rio *et al.* (2002). We recall the definition of the Riemannian extension follows the book Brozos-Vázquez *et al.* (2009).

2.1. Affine connection s. Let M be a 3-dimensional and ∇ a smooth affine connection. We choose a fixed coordinate domain $\mathcal{U}(u_1, u_2, u_3) \subset M$. In $\mathcal{U}$, the connection is given by

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k,$$

where we denote $\partial_i = (\frac{\partial}{\partial_{u_i}})$ and the functions $\Gamma_{ij}^k(i, j, k = 1, 2, 3)$ are called the *Christoffel symbols* for the affine connection relative to the local coordinate system. We define a few tensors fields associated to a given affine connection ∇. The *torsion tensor field* T^∇, which is of type $(1,2)$, is defined by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

The components of the torsion tensor T^∇ in local coordinates are

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

If the torsion tensor of a given affine connection ∇ is 0, we say that ∇ is torsion-free.

The *curvature tensor field* $\mathcal{R}^\nabla$, which is of type $(1,3)$, is defined by

$$\mathcal{R}^\nabla(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The components in local coordinates are

$$\mathcal{R}^\nabla(\partial_k, \partial_l)\partial_j = \sum_i R_{jkl}^i \partial_i.$$

We shall assume that ∇ is torsion-free. If $\mathcal{R}^\nabla = 0$ on M, we say that ∇ is *flat affine connection*. It is known that ∇ is flat if and only if around point there exist a local coordinates system such that $\Gamma_{ij}^k = 0$ for all i, j and k.

We define the *Ricci tensor* Ric^∇, of type $(0,2)$ by

$$Ric^\nabla(Y,Z) = \text{trace}\{X \mapsto \mathcal{R}^\nabla(X,Y)Z\}.$$

The components in local coordinates are given by

$$Ric^\nabla(\partial_j, \partial_k) = \sum_i R_{kij}^i.$$

It is known in Riemannian geometry that the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, that is, $Ric(Y,Z) = Ric(Z,Y)$. But this property is not true for an arbitrary affine connection with torsion-free.

2.2. Affine Osserman manifold s. Let (M, ∇) be a m-dimensional affine manifold, i.e., ∇ is a torsion free connection on the tangent bundle of a smooth manifold M of dimension m. Let $\mathcal{R}^\nabla(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ be the associated curvature operator. We define the *affine Jacobi operator* $J_{\mathcal{R}^\nabla}(X) : T_pM \longrightarrow T_pM$ with respect to a vector $X \in T_pM$ by

$$J_{\mathcal{R}^\nabla}(X)Y := \mathcal{R}^\nabla(Y,X)X.$$

We will write $\mathcal{R}^\nabla$ and $J_{\mathcal{R}^\nabla}$ when it is necessary to distinguish the role of the connection.

Definition 4. *García-Rio et al. (2002) Let (M, ∇) be a m-dimensional affine manifold. Then (M, ∇) is called affine Osserman at $p \in M$ if the characteristic polynomial of $J_{\mathcal{R}^\nabla}(X)$ is independent of $X \in T_pM$. Also (M, ∇) is called affine Osserman if (M, ∇) is affine Osserman at each $p \in M$.*

Theorem 44. *García-Rio et al. (2002) Let (M, ∇) be a m-dimensional affine manifold. Then (M, ∇) is called affine Osserman at $p \in M$ if and only if the characteristic polynomial of $J_{\mathcal{R}^\nabla}(X)$ is $P_\lambda[J_{\mathcal{R}^\nabla}(X)] = \lambda^m$ for every $X \in T_pM$.*

Corollary 14. *(M, ∇) is affine Osserman if the affine Jacobi operator s are nilpotent, i.e., 0 is the only eigenvalue of $J_{\mathcal{R}^\nabla}(\cdot)$ on the tangent bundle TM.*

Corollary 15. *If (M, ∇) is affine Osserman at $p \in M$ then the Ricci tensor is skew-symmetric at $p \in M$.*

Affine Osserman connections are of interest not only in affine geometry, but also in the study of pseudo-Riemannian Osserman metrics since they provide some nice examples without Riemannian analogue by means of the Riemannian extensions. Here it is worth to emphasize that some recent modifications of the usual Riemann extensions allowed some new applications Calviño *et al.* (2009, 2012)

2.3. Riemannian extension construction. Let $N := T^*M$ be the cotangent bundle of an m-dimensional manifold and let $\pi : T^*M \to M$ be the natural projection. A point ξ of the cotangent bundle is represented by an ordered pair (ω, p), where $p = \pi(\xi)$ is a point on M and ω is a 1-form on T_pM. If $u = (u_1, \cdots, u_m)$ are local coordinates on M, let $u' = (u_{1'}, \cdots, u_{m'})$ be the associated dual coordinates on the fiber where we expand a 1-form ω as $\omega = u_{i'}du_i$ $(i = 1, \cdots, m; i' = i+m)$; we shall adopt the Einstein convention and sum over repeated indices henceforth.

For each vector field $X = X^i\partial_i$ on M, the evaluation map $\iota X(p,\omega) = \omega(X_p)$ defines on function on N which, in local coordinates is given by

$$\iota X(u_i, u_{i'}) = u_{i'}X^i.$$

Vector fields on N are characterized by their action on function ιX; the complete lift X^C of a vector field X on M to N is characterized by the identity

$$X^C(\iota Z) = \iota[X, Z], \quad \text{for all} \quad Z \in \mathcal{C}^\infty(TM).$$

Moreover, since a $(0,s)$-tensor field on M is characterized by its evaluation on complete lifts of vectors fields on M, for each tensor field T of type $(1,1)$ on M, we define a 1-form ιT on N which is characterized by the identity

$$\iota T(X^C) = \iota(TX).$$

Let ∇ be a torsion free affine connection on M. The *Riemannian extension* $\bar{g}$ is the pseudo-Riemannian metric on N of neutral signature (m,m) characterized by the identity

$$\bar{g}(X^C, Y^C) = -\iota(\nabla_X Y + \nabla_Y X).$$

If x and y are cotangent vectors, let $x \circ y := \frac{1}{2}(x \otimes y + y \otimes x)$. Expand

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k$$

to define the Christoffel symbols Γ of ∇. One then has:

$$\bar{g} = 2du_i \circ du_{i'} - 2u_{k'}\Gamma_{ij}^k du_i \circ duj.$$

Riemannian extension were originally defined by Patterson and Walker Patterson and Walker (1952) and further investigated in relating pseudo-Riemannian properties of N with the affine structure of the base manifold (M,∇). Moreover, Riemannian extension were also considered in García-Rio *et al.* (1999) in relation to Osserman manifolds. We have the following result:

THEOREM 45. *(García-Rio et al. (1999)) Let $(T^*M, \bar{g})$ be the cotangent bundle of an affine manifold (M,∇) equipped with the Riemannian extension of the torsion free connection ∇. Then $(T^*M, \bar{g})$ is a pseudo-Riemannian globally Osserman manifold if and only if (M,∇) is an affine Osserman manifold* .

3. Affine Osserman connections on 3-dimensional manifolds

Let M a 3-dimensional manifold and ∇ a smooth torsion-free connection. We choose a fixed coordinates domain $\mathcal{U}(u_1, u_2, u_3) \subset M$.

PROPOSITION 3. *Let M be a 3-dimensional manifold with torsion free connection given by*

$$\left\{ \begin{array}{rcl} \nabla_{\partial_1}\partial_1 & = & f_1(u_1,u_2,u_3)\partial_2; \\ \nabla_{\partial_2}\partial_2 & = & f_2(u_1,u_2,u_3)\partial_2; \\ \nabla_{\partial_3}\partial_3 & = & f_3(u_1,u_2,u_3)\partial_2. \end{array} \right. \tag{3.1}$$

Then (M, ∇) is affine Osserman if and only if the Christoffel symbols of the connection (3.1) satisfy:

$$f_2(u_1,u_1,u_3) = f(u_2), \quad \partial_2 f_1 + f_1 f(u_2) = 0 \quad \textit{and} \quad \partial_2 f_3 + f(u_2) f_3 = 0.$$

Proof. We denote the functions $f_1(u_1,u_2,u_3)$, $f_2(u_1,u_2,u_3)$, $f_3(u_1,u_2,u_3)$ by f_1, f_2, f_3 respectively, if there is no risk of confusion. The Ricci tensor of the connection (3.1) expressed in the coordinates (u_1,u_2,u_3) takes the form

$$Ric^{\nabla}(\partial_1,\partial_1) = \partial_2 f_1 + f_1 f_2; \quad Ric^{\nabla}(\partial_1,\partial_2) = -\partial_1 f_2; \tag{3.2}$$

$$Ric^{\nabla}(\partial_3,\partial_2) = -\partial_3 f_2; \quad Ric^{\nabla}(\partial_3,\partial_3) = \partial_2 f_3 + f_2 f_3. \tag{3.3}$$

It is know that the Ricci tensor of any affine Osserman is skew-symmetric, it follows from the expression (3.2) that we have the following necessary condition for the connection (3.1) to be Osserman

$$\partial_1 f_2 = 0, \quad \partial_3 f_2 = 0, \quad \partial_2 f_1 + f_1 f_2 = 0 \quad \text{and} \quad \partial_2 f_3 + f_2 f_3 = 0; \tag{3.4}$$

which implies that the connection is indeed Ricci flat, but not flat. Now, a calculation of the Jacobi operators shows that for each vector $X = \sum_{i=1}^{3} \alpha_i \partial_i$, the associated Jacobi operator is given by

$$(J_{\mathcal{R}^\nabla}(X)) = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}, \tag{3.5}$$

with

$$a_1 = \alpha_3(-\alpha_1\partial_3 f_1 + \alpha_3\partial_1 f_3) \quad \text{and} \quad a_2 = \alpha_1(\alpha_1\partial_3 f_1 - \alpha_3\partial_1 f_3).$$

It follows from the matrix associated to $J_{\mathcal{R}^\nabla}(X)$, that its characteristic polynomial as written as follows:

$$P_\lambda[J_{\mathcal{R}^\nabla}(X)] = \lambda^3.$$

It follows that a connection given by (3.1) is affine Osserman if and only if the Christoffel symbols given by the functions f_1, f_2 and f_3 satisfy:

$$f_2(u_1, u_1, u_3) = f(u_2), \quad \partial_2 f_1 + f_1 f(u_2) = 0 \quad \text{and} \quad \partial_2 f_3 + f(u_2) f_3 = 0. \ \square$$

EXAMPLE 1. *The following connection on $\mathbb{R}^3$ defined by*

$$\nabla_{\partial_1}\partial_1 = u_1u_3\partial_2, \quad \nabla_{\partial_2}\partial_2 = 0, \quad \nabla_{\partial_3}\partial_3 = (u_1 + u_3)\partial_2 \tag{3.6}$$

is a nonflat affine Osserman connection.

COROLLARY 16. *The connection given (3.1) is affine Osserman flat if and only if*

$$\partial_3 f_1(u_1, u_2, u_3) = 0 \quad \textit{and} \quad \partial_1 f_3(u_1, u_2, u_3) = 0.$$

4. Nonflat pseudo-Riemannian Osserman metric of signature $(3, 3)$

In this section we will construct a pseudo-Riemannian Osserman metric of signature $(3, 3)$. Let (M, ∇) be a 3-dimensional affine manifold. Let (u_1, u_2, u_3) be the local coordinates on M. We expand $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$ for $i, j, k = 1, 2, 3$ to define the Christoffel symbols of ∇. We expand a 1-form ω as $\omega = u_4 du_1 + u_5 du_2 + u_6 du_3 \in T^*M$ where (u_4, u_5, u_6) are the dual fiber coordinates. The Riemannian extension is the pseudo-Riemannian metric $\bar{g}$ on the cotangent bundle T^*M of neutral signature $(3, 3)$ expressed by

$$\bar{g} = \begin{pmatrix} -2u_{k'}\Gamma_{ij}^k & \delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix},$$

with respect to $\{\partial_1, \cdots, \partial_6\}(i, j = 1, 2, 3, i' = i + 3)$, where Γ_{ij}^k are the Christoffel symbols of the connection ∇ with respect to the coordinates (u_i) on M.

The Riemannian extenxion $\bar{g}$ on $\mathbb{R}^6$ of the connection (3.6) has the form

$$\bar{g} = \begin{pmatrix} -2u_5u_1u_3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2u_5(u_1 + u_3) & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then the Christoffel symbols of $\bar{g}$,

$$\bar{\Gamma}_{ij}^k = \frac{1}{2}\bar{g}^{kl}\{\partial_j\bar{g}_{il} + \partial_i\bar{g}_{jl} - \partial_l\bar{g}_{ij}\}$$

are given by

$$\bar{\Gamma}_{11}^2 = u_1u_3, \quad \bar{\Gamma}_{11}^4 = -u_3u_5, \quad \bar{\Gamma}_{11}^6 = u_1u_5 \quad \bar{\Gamma}_{13}^4 = -u_1u_5, \quad \bar{\Gamma}_{13}^6 = -u_5,$$
$$\bar{\Gamma}_{15}^4 = -u_1u_3, \quad \bar{\Gamma}_{33}^2 = u_1 + u_3, \quad \bar{\Gamma}_{33}^4 = u_5, \quad \bar{\Gamma}_{33}^6 = -u_5, \quad \bar{\Gamma}_{35}^6 = -(u_1 + u_3);$$

and the others are zero. The non-vanishing covariant derivatives of $\bar{g}$ are given by

$$\begin{array}{rcl}\bar{\nabla}_{\partial_1}\partial_1 & = & u_1u_3\partial_2 - u_3u_5\partial_4 + u_1u_5\partial_6, \quad \bar{\nabla}_{\partial_1}\partial_3 = -u_1u_5\partial_4 - u_5\partial_6, \\ \bar{\nabla}_{\partial_1}\partial_5 & = & -u_1u_3\partial_4, \bar{\nabla}_{\partial_3}\partial_3 = (u_1+u_3)\partial_2 + u_5\partial_4 - u_5\partial_6, \\ \bar{\nabla}_{\partial_3}\partial_5 & = & -(u_1+u_3)\partial_6.\end{array}$$

The non-vanishing components of the curvature tensor of $(\mathbb{R}^6, \bar{g})$ are given by

$$\begin{array}{rcl}R(\partial_1,\partial_3)\partial_1 & = & -u_1\partial_2; \quad R(\partial_1,\partial_3)\partial_3 = \partial_2; \quad R(\partial_1,\partial_3)\partial_5 = u_1\partial_4 - \partial_6; \\ R(\partial_1,\partial_5)\partial_1 & = & -u_1\partial_6; \quad R(\partial_1,\partial_5)\partial_3 = u_1\partial_4; \quad R(\partial_3,\partial_5)\partial_1 = \partial_6; \\ R(\partial_3,\partial_5)\partial_3 & = & -\partial_4.\end{array}$$

Now, If $X = \sum_{i=1}^{6} \alpha_i\partial_i$ is a vector field on $\mathbb{R}^6$, then the matrix associated to the Jacobi operator $J_{\mathcal{R}}(X) = \mathcal{R}(\cdot, X)X$ is given by

$$(J_{\mathcal{R}}(X)) = \begin{pmatrix} A & 0 \\ B & A^t \end{pmatrix},$$

where A is the 3×3 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1-u_1 & 0 & u_1 - 1 \\ 0 & 0 & 0 \end{pmatrix};$$

and B is the 3×3 matrix given by

$$B = \begin{pmatrix} 2u_1 & 0 & -u_1 \\ 0 & 0 & 0 \\ -1 - u_1 & 0 & 1 \end{pmatrix}.$$

Then we have the following

PROPOSITION 4. $(\mathbb{R}^6, \bar{g})$ *is a pseudo-Riemannian Osserman with metric of signature* $(+,+,+,-,-,-)$*. Moreover, the characteristic polynomial of the Jacobi operator s is* $P_\lambda(J_{\mathcal{R}}(X)) = \lambda^6$*.*

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