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CHAPTER 15

The $\partial\bar{\partial}$-problem for extendable currents defined on a half space of $\mathbb{C}^n$, *by M. Eramane Bodian, W. Ndiaye and S. Sambou*

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Abstract. We solve the $\partial\bar{\partial}$-problem for extendable currents defined on a half space of $\mathbb{C}^n$. $\Diamond$

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1. Introduction

In this paper, we solve the $\partial\bar{\partial}$-problem for extendable currents defined on $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \text{Im}(z_n) < 0\}$ that is an example of unbounded pseudo-convex domain as well as its complement. The de Rham cohomology group of the boundary $H^j(b\Omega)$ is trivial for $1 \leqslant j \leqslant 2n-1$. In this context, we first solve the equation $dS = T$ where T is an extendable current defined on Ω and we have :

THEOREM 40. *The de Rham cohomology group for extendable currents*

$$\check{H}^j(\Omega) = 0 \quad \textit{for} \quad 1 \leq j \leq 2n-1.$$

◊

The domain Ω is fat i.e $\overset{\circ}{\bar{\Omega}} = \Omega$ therefore according to Martineau (1996) the extendable currents defined on Ω are topological dual of differential forms with compact support on $\bar{\Omega}$. for that we are led to solve the equation $df = g$ where f et g are differential forms with compact support on Ω and go to the extendable currents by duality. The first particularity lies on the resolution with prescribed support by the operator d because if we solve with compact support in $\mathbb{C}^n$, then we can not as in Bodian et al. (2017b) correct by the solution with compact support. We use the results of Brinkschulte (2004) and Seeley (2002) to get a solution with compact support and then as the concave case, we use the same techniques to correct the solutions because the space of differential forms with compact support on $\bar{\Omega}$ is not a Frechet space but rather an inductive limit of Frechet spaces.

The second particularity compared to Bodian et al. (2016) and Bodian et al. (2017b) lies to resolution of the $\bar{\partial}$ with prescribed support because Ω being the unbounded Levi flat domain, we can not use the techniques of Sambou (2002a).Then we use the results of Brinkschulte (2004) to solve with prescribed support the equation $\bar{\partial}S = T$ in the unbounded domain Ω in order to establish:

THEOREM 41. *Let $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \mathit{Im}(z_n) < 0\} \subset \mathbb{C}^n$ be a domain, then for all extendable (p,q)-current T defined on Ω and d-closed, there is S an extendable $(p-1,q-1)$-current defined on Ω such that $\partial\bar{\partial}S = T$ for $1 \leqslant p, q \leqslant n-1$.*

1.1. Notations.

we note by $\check{D}^p_X(\Omega)$ the space of p-currents defined on Ω and extendable in X, $D^p(\bar{\Omega})$ the space of smooth differential p-forms defined in X with compact support in $\bar{\Omega}$. If X is a complex manifold of dimension n, then we note by $\check{D}_X^{p,q}(\Omega)$ the space of extendable (p,q)-currents defined in Ω and $D^{p,q}(\bar{\Omega})$ the space of differential (p,q)-forms with compact support in $\bar{\Omega}$. We note by $\check{H}^p(\Omega)$ the de Rham cohomology group for extendable currents defined in Ω , $\check{H}^{p,q}(\Omega)$ the Dolbeault cohomology group for extendable currents defined in Ω. $H_\infty^p(X)$ is the cohomology group of de Rham for smooth differential p-forms defined in X, $H_c^p(X)$ is the de Rham cohomology group for smooth differential p-forms defined in X with compact support in $\bar{\Omega}$ and finaly $\Lambda^p(\Omega)$ the space of smooth differential p-forms in Ω.

2. Resolution of the equation $dS = T$ for a half space Ω of $\mathbb{R}^{n+1}$

we consider

$$\Omega = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} < 0\} \subset \mathbb{R}^{n+1}$$

a convex domain, its boundary $b\Omega = \mathbb{R}^n \times \{0\} \simeq \mathbb{R}^n$ and the interior of its complement

$$\mathfrak{C} = \mathbb{R}^{n+1} \backslash \bar{\Omega} = \mathbb{R}^n \times \{x_{n+1} > 0\}.$$

Ω is convex and unbounded and so is its complement $\mathfrak{C}$. So we have $H^j(\Omega) = 0$ and $H^j(b\Omega) = 0$ for $j \geqslant 1$. Then the principal result of this part is :

Theorem 40. *The de Rham cohomology group for extendable currents*

$$\check{H}^j(\Omega) = 0 \quad pour \quad 1 \leq j \leq n.$$

For giving the proof we need the following lemma :

LEMMA 28.

$$D^p(\bar{\Omega}) \cap \ker d = d(D^{p-1}(\bar{\Omega}))$$

for $1 \leq p \leq n$.

Proof. Let $f \in D^p(\bar{\Omega}) \cap \ker d$, then there is Ω' a ball of center z_0 and radius R such that for $f \in D^p(\Omega') \cap \ker d$, $0 < p \leqslant n$, there is $g \in D^{p-1}(\Omega')$ with $dg = f$. This implies that $dg_{|B} = 0$ where $B = \Omega' \cap (\mathbb{R}^{n+1} \backslash \Omega)$. If $p = 1$, then g is a constant with compact support so $g = 0$ in B.

If $1 < p \leqslant n$, then $g_{|B}$ is a differential $(p-1)$-form d-closed then it exists a differential smooth $(p-2)$-form h in $\bar{B}$ such that $dh = g_{|B}$. Let $\tilde{h}$ a smooth extension with compact support of h in Ω' (we can use the extension operator of Seeley Seeley (2002)), $u = g - d\tilde{h}$ is a smooth differential $(p-1)$-form in $\mathbb{R}^{n+1} \backslash \Omega$ with compact support in $\bar{\Omega}$ and $du = f$. □

Proof (Theorem 40). . According to Martineau Martineau (1996), since $\overset{\circ}{\bar{\Omega}}= \Omega$, the currents defined in Ω and extendable in $\mathbb{R}^{n+1}$ are the elements of $(D^p(\bar{\Omega}))'$ topological dual of smooth differential p-forms in $\mathbb{R}^{n+1}$ with compact support in $\bar{\Omega}$. However $\bar{\Omega}$ being unbounded, $D^p(\bar{\Omega})$ is an inductive limit of Fréchet spaces.

we consider a compact $K \subset \bar{\Omega}$ of $\mathbb{R}^{n+1}$ and $D^p(K)$ the space of p-forms in $\mathbb{R}^{n+1}$ with compact support in K. We set

$$\begin{array}{rcl} L_T^K : d(D^p(\Omega) \cap D^p(K) \cap \ker d) & \longrightarrow & \mathbb{C} \\ \bar{\partial}\varphi & \longmapsto & \langle T, \varphi\rangle \end{array}$$

a continuous linear application, and then L_T^K extend as an continuous linear operator :
$\tilde{L}_T^K : D^{p+1}(\bar{\Omega}) \cap D^{p+1}(K) \longrightarrow \mathbb{C}$. It is an extendable current and $d\tilde{L}_T^K = (-1)^{n-p+1}T$ on $\overset{\circ}{K}$.

We consider a family $(K_n)_{n\in\mathbb{N}}$ of compacts set of $\bar{\Omega}$ then we can find in K_n, a current S_n extendable such that $dS_n = T$ in $\overset{\circ}{K}_n$ with $K_n \Subset \overset{\circ}{K}_{n+1}$.
$S_{n+1} - S_n$ is d-closed and $S_{n+1} - S_n = dv_n$ in $\overset{\circ}{K}_{n+1}$.

Let χ be a smooth function on $\mathbb{R}^{n+1}$ with compact support in $\overset{\circ}{K}_{n+1}$ such that $\chi = 1$ in a neighborhood of K_n contained in K_{n+1} and

$$S_{n+1} - d(\chi v_n) = S_n + d(1-\chi)v_n \quad \text{on} \quad \overset{\circ}{K}_n \,.$$

Let us put $U_{n+1} = S_{n+1} - d(\chi v_n)$ and $U_n = S_n + d(1-\chi)v_n$.

We have $dU_{n+1} = dU_n = T$ in $\overset{\circ}{K}_n$ and $U_{n+1} = U_n$ in K_n. We set

$$S = \lim_n U_{n+1}.$$

This is an extendable current in Ω and verifies $dS = T$.

3. Resolution of the $\partial\bar{\partial}$ for extendable currents in a half space of type $\{\text{Im}\,(z_n) < 0\} \subset \mathbb{C}^n$

We give the following fundamental result of $\bar{\partial}$-problem with prescribed support:

THEOREM 42. *Let Ω be a domain and $f \in D^{p,q}(\bar{\Omega}) \cap \ker \bar{\partial}$. Then it exists $g \in D^{p,q-1}(\bar{\Omega})$ such that $\bar{\partial} g = f$; $1 \leqslant q \leqslant n-1$.*

Proof. This is a consequence of the result of a resolution of the $\bar{\partial}$ with prescribed support (Theorem 4.2 in Brinkschulte (2004)). If the support of f is compact in Ω, then we choose pseudo-convex domain Ω' in Ω which contains the support of f. According to Theorem 4.2 in Brinkschulte (2004), there is $g \in D^{p,q-1}(\bar{\Omega}')$ such that $\bar{\partial} g = f$.

If now $\text{supp}(f) \cap b\Omega \neq \varnothing$, since f has compact support and $b\Omega$ is Levi flat, we can find $K \subset \bar{\Omega}$ a compact with pseudo-convex interior and smooth boundary which contains the support of f. According to Theorem 4.2 in Brinkschulte (2004), it exists h a differential $(p, q-1)$-form with support in K such that $dh = f$. We extend h by 0 in $\mathbb{C}^n \backslash K$ and we have the desired solution. So for all $f \in D^{p,q}(\bar{\Omega}) \cap \ker \bar{\partial}$, it exists $g \in D^{p,q-1}(\bar{\Omega})$ such that $\bar{\partial} g = f$. □

By classical duality (refer theorem 40), we have the following result:

THEOREM 43.
Let $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid Im(z_n) < 0\}$ and T be an extendable current of bi-degree (p, q) $\bar{\partial}$-closed in Ω. Then there is an extendable current S defined in Ω such that $\bar{\partial} S = T$ for $1 \leqslant p \leqslant n$ and $1 \leqslant q \leqslant n-1$.

We are going to establish the following result.

Theorem 41. *Let $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid Im(z_n) < 0\} \subset \mathbb{C}^n$ be a domain, then for all extendable (p, q)-current defined in Ω and d-closed, it exists S a extendable $(p-1, q-1)$-current defined in Ω such that $\partial\bar{\partial} S = T$ with $1 \leqslant p, q \leqslant n-1$.*

Proof.
Let T a (p, q)-current, $1 \leqslant p \leqslant n-1$ and $1 \leqslant q \leqslant n-1$, d-closed defined in Ω and extendable in $\mathbb{C}^n$ with $1 \leqslant p + q \leqslant 2n - 2$. Since the theorem 40 assures us that $\check{H}^{p+q}(\Omega) = 0$, it exists a extendable current μ defined in Ω such that $d\mu = T$. μ is an extendable $(p+q-1)$-current, it breaks down into $(p-1, q)$-current μ_1 and into $(p, q-1)$-current μ_2. We have

$$d\mu = d(\mu_1 + \mu_2) = d\mu_1 + d\mu_2 = T.$$

Since $d = \partial + \bar{\partial}$, we have for bi-degree reasons , $\partial\mu_2 = 0$ and $\bar{\partial}\mu_1 = 0$. We get by theorem 43 $\mu_1 = \partial u_1$ and $\mu_2 = \bar{\partial} u_2$ where u_1 and u_2 are extendable currents defined in Ω. So we have :

$$\begin{aligned} T &= \partial\mu_2 + \bar{\partial}\mu_1 \\ &= \partial\bar{\partial} u_2 + \bar{\partial}\partial u_1 \\ &= \partial\bar{\partial}(u_2 - u_1) \end{aligned}$$

We set $S = u_2 - u_1$, then S is an extendable $(p-1, q-1)$-current defined in Ω such that $\partial\bar{\partial} S = T$. □

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