EMPIRICAL BAYES STOCK MARKET PORTFOLIOS

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We consider sequential investments in a stock market with the goal of performing as well as if we knew the empirical distribution of future market performance. In particular, we wish to outperform the best stock.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \geq 0$ denote a market vector for one investment period, where x_i is the number of units returned from an investment of 1 unit in the i-th stock. A portfolio $\mathbf{b} = (b_1, b_2, \ldots, b_m)$, $b_i \geq 0$, $\Sigma b_i = 1$, is the proportion of the current capital invested in each of the m stocks. Thus $S = \mathbf{b}^t\mathbf{x} = \Sigma b_i x_i$ is the factor by which the capital is increased in one investment period using portfolio $\mathbf{b}$.

If portfolio $\mathbf{b}$ is used for n investment periods, readjusting stock holdings as necessary, then the stock sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ results in captial S_n at time n given by

$$S_n = \prod_{i=1}^{n} \mathbf{b}^t\mathbf{x}_i = e^{n(\frac{1}{n}\sum_{i=1}^{n} \ln \mathbf{b}^t\mathbf{x}_i)} .$$

Define the expected log return $W(\mathbf{b},F)$ for portfolio $\mathbf{b}$ against stock distribution F, by

$$W(\mathbf{b},F) = E_F \ln \mathbf{b}^t\mathbf{X} = \int \ln \mathbf{b}^t \mathbf{x} dF(\mathbf{x}),$$

and let

$$W^*(F) = \max_{\mathbf{b}} W(\mathbf{b},F).$$

We observe that

$$S_n = \prod_{i=1}^{n} \mathbf{b}^t\mathbf{x}_i = e^{nW(\mathbf{b},F_n)} \leq e^{nW^*(F_n)},$$

where F_n is the empirical c.d.f. of $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n$.

Suppose the stock vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n$ have no underlying distribution. However, we shall constrain the sequence to take values in some finite set $\mathbf{X}$. Our bounds will depend on the cardinality of this set.

THEOREM. There exists a sequence of portfolios $\mathbf{b}_k$, where $\mathbf{b}_k$ depends only on the past $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_{k-1}$ and the set $\mathbf{X}$, such that the cumulative log return satisfies

$$\frac{1}{n}\log S_n = \frac{1}{n}\sum_{k=1}^{n} \ln \mathbf{b}_k^t \mathbf{x}_k \geq W^*(F_n) - \frac{c}{\sqrt{n}},$$

for all $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n \in \mathbf{X}$ and for all n, where the constant c depends only on the range $\mathbf{X}$.

Thus one can perform asymptotically as well on sequential investments as if one knew F_n ahead of time. The proof appears in Cover and Gluss (1986).

REFERENCE

Cover, T.M., and Gluss, D.H. (1986). Empirical Bayes Stock Market Portfolios, to appear in Adv. Appl. Math.