ESSAY III. CONSTRUCTION OF STATIONARY STRONG-MARKOV TRANSITION PROBABILITIES

Let X_t be a continuous parameter stochastic process on $(\Omega, \mathcal{F}, P)$ with values in a metrizable Lusin space $(E, \mathcal{E})$ (i.e., $\mathcal{E}$ is the Borel σ-field of a Borel set E in a compact metric space $\bar{E}$). In order just to state the property of X_t that it be a "time-homogeneous Markov process", it is necessary to introduce some form of conditional probability function to serve as transition function. From an axiomatic standpoint it is of course desirable to assume as little as possible about this function. An interesting and difficult problem is then to deduce from such assumptions the existence of a complete Markov transition probability $p(t,x,B)$ for (P, X_t) which satisfies the Chapman-Kolmogorov identities

$$p(s+t,x,B) = \int p(s,x,dy)p(t,y,B) , \tag{1.1}$$

thus giving rise to a family $(P^x,\ x \in E)$ of Markovian probabilities for which

$$P^x(X_{s+t} \in B \mid \sigma(X_\tau,\ \tau \le s)) = P^{X_s}\{X_t \in B\} . \tag{1.2}$$

The analogous time-inhomogeneous problem (of obtaining a $p(s,x;s+t,B)$) was treated by J. Karush (1961), and considerably later the present problem was taken up by J. Walsh [9]. It seems, however, that for the homogeneous case the solution remained complicated and conceptually difficult.[1]

Since the publication of these two works, a new tool has appeared on the scene which has an obvious bearing on the problem, namely, the "prediction process" of [5] and [8]. Accordingly, the present essay aims to show what can be done by using this method. But it is not simply a question of applying a new device. Our view is that the prediction process is fundamental to the problem, and the hypotheses which are needed to apply it give a basic understanding of the nature of the difficulties. A suggested way of viewing the entire matter is as follows. The prediction process is in some sense the best approximation to X_t by a process which does have a

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[1] The hypotheses of Theorem 3 of [9] are ultimately consequences of ours (Corollary 1.9 below).

stationary strong-Markov transition function. The problem is thus to formulate the conditions under which the prediction process becomes identifiable with X_t itself.

Two immediate requirements are that the paths of X_t be sufficiently regular, and that their probability space be sufficiently tractable, so that the assumed conditional probabilities may be identified P-almost surely for each t with the regular conditional probabilities which constitute the prediction process. We will make the following initial assumption (to be relaxed in Theorem 1.12).

ASSUMPTION 1.1. Let $(\Omega, \theta_t, \mathcal{F}_t^o)$ denote the space of right-continuous E-valued paths $w(t)$, $t \geq 0$, with left limits for $t > 0$, and the usual translation operators and generated σ-fields. We assume the canonical representation $X_t(w) = w(t)$.

We now introduce the two basic definitions with which we will be concerned.

DEFINITION 1.2. Let $Q(x,S)$, $X \in E$, $S \in \mathcal{F}^o (= \bigvee_t \mathcal{F}_t^o)$, be a probability kernel, i.e. a probability in S for each x and $\mathcal{E}$-measurable in x for each S. A probability P on $\mathcal{F}^o$ is called homogeneous Markov relative to Q and $\mathcal{F}_{t+}^o$ $(= \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^o)$, $t \geq 0$, if for each t and $S \in \mathcal{F}^o$

$$(1.3) \qquad P(\theta_t^{-1} S \,|\, \mathcal{F}_{t+}^o) = Q(X_t, S) \quad P\text{-a.s.}$$

DEFINITION 1.3. The Chapman-Kolmogorov identities for $Q(x,S)$ are

$$(1.4) \qquad Q(x, \theta_{s+t}^{-1}(S)) = \int Q(x, \{X_s \in dy\})\, Q(y, \theta_t^{-1} S)$$

$0 \leq s,t$; $x \in E$, $S \in \mathcal{F}^o$.

REMARKS. Since regular conditional probabilities exist over $\mathcal{F}^o$, the assumption of a Q as in Definition 1.2 is equivalent to assuming only a marginal conditional probability kernel $Q_s(x,B)$, $B \in \mathcal{E}$, for each $s \geq 0$. In fact, it is enough to have Q_s for rational s, since then $\int Q_{s_1}(X_\tau, dy) Q_{s_2}(y,B) = Q_{s_1+s_2}(X_\tau, B)$ except on a P-null set for each τ. We can then use this identity, along with the fact that regular conditional probabilities assign probability one to the r.c.l.l. paths, to construct a Q satisfying (1.3). In fact, the measures generated by Q_s on the space of E-valued functions of rational $s \geq 0$ must reduce, when X_τ is substituted as initial value, to the restriction to rational s of any regular conditional probability on the r.c.l.l. paths given $\mathcal{F}_{\tau+}^o$. Hence they extend to measures on the r.c.l.l. paths, P-a.s. for every τ. The set of restrictions to rational $s \geq 0$ of r.c.l.l. paths is a Borel set in the countable product space, so the condition that this set have probability 1 gives a Borel set of initial values. Outside this set, we may take $Q(x,S) = I_S(w(0))$.

The most that follows from (1.3), however, is that (1.4) holds for all $\tau \geq 0$ and $S \in F^o$ except for x in a set $E(\tau,s)$ with $P\{X_\tau \in E(\tau,s)\} = 0$. In short, one can eliminate the dependence on t and S since the paths are right-continuous and F^o is countably generated. But we do not see how to eliminate dependence on s, much less on τ, without further assumptions.

Secondly, the reason for conditioning on F^o_{t+} in Definition 1.1 is in one sense trivial: We could have used F^o_{t-} and X_{t-} instead, but it is less convenient. However, the distinction between F^o_t and F^o_{t-} is "unobservable" for the prediction process (see, for example, the Remark following Theorem 1.9 of Essay I). So it is unrealistic to condition on F^o_t except when it is shown (as following Theorem 1.12 below) that this is equivalent to F^o_{t+} . The point here is that the prediction process is automatically a strong-Markov process relative to F^o_{t+} . Thus our method dictates that the same will be true of X_t .

The problem is now to identify conditions (presumably verifiable in practice) under which, given a Q satisfying (1.3), there exists a $Q^*(x,S)$ satisfying both (1.3) and (1.4). To this end we first state the relevant properties of the prediction process of X_t, as obtained in [5], and [8], and Essay 1. Let $(H,\mathcal{H})$ be the set of probability measures $z(S)$ on F^o and the σ-field generated by $\{z : z(S) \leq a\}$, $S \in F^o$, $a \in R$. Further, for each $t \geq 0$ let F^z_{t+} be the σ-field generated by F^o_{t+} and all z-null sets. Then the z-prediction process $Z^z_t = Z^z_t(S,w)$, for each $z \in H$, is an F^z_{t+}-optional process with space $(H,\mathcal{H})$ such that for each optional time $T < \infty$, and all $S \in F^o$,

$$P^z(\theta_T^{-1} S | F^z_{T+}) = Z^z_T(S) \ , \quad z\text{-a.s.} \tag{1.5}$$

(where P^z is another notation for z itself). The process Z^z_t is unique up to z-equivalence.

REMARK. In [5] the spaces Ω and H were "larger" than the present ones. But since Ω is here a Lusin space it is easy to see that the probabilities $Z(t)$ of [5] must already equal one on the Borel image of this Ω in the space Ω' of [5], for all t z-a.s. Hence we can assume the present $(H,\mathcal{H})$.

The second essential feature of the processes Z^z_t concerns their behavior as z varies. From Theorem 1.15 of Essay I we have:

THEOREM 1.4. There is a jointly Borel transition function $q(t,y,A)$ on $(H,\mathcal{H})$ such that for each z the process Z^z_t (with the probability z

itself) is a homogeneous strong-Markov process relative to F^z_{t+}, with transition function $q(t,y,A)$. In particular, q satisfies the Chapman-Kolmogorov identities (1.1).

An advantage of restricting to a space of right-continuous paths is that one can be quite explicit about the connection of Z^z_t and X_t. Indeed we have a simple functional dependence.

THEOREM 1.5. There is an H/E-measurable function φ such that for all $z \in H$

$$P^z\{X_t = \varphi(Z^z_t) \quad \text{for all} \quad t \geq 0\} = 1 .$$

PROOF. It is convenient to introduce the set of non-branching points of Z

$$\begin{aligned} H_0 &= \{z \in H\colon \ P^z\{Z_0{}^z = z\} = 1\} \\ &= \{z \in H\colon \ q(0,z,\{z\}) = 1\} . \end{aligned}$$

We have $H_0 \in H$, and by Proposition 2 of Meyer [8] for all $z \in H$ we have $P^z\{Z^z_t \in H_0, \ t \geq 0\} = 1$ (in fact, the distributions of $Z^{(\cdot)}_{(\cdot)}$ on H_0 are those of a right process on H_0 with transition function q). For $z \in H_0$ and $B \in E$, since

$$\begin{aligned} Z^z_0\{X_0 \in B\} &= P^z(X_0 \in B \mid F^z_{0+}) \\ &= P^z\{X_0 \in B\} \qquad , z \ \text{a.s.}, \end{aligned}$$

we must have $P^z\{X_0 = \varphi(z)\} = 1$ for some function $\varphi(z)$ on H_0. Since $z(S)$ is H-measurable for $S \in F^o$ and X_0 is F^o-measurable, we see that $E^z f(X_0)$ is H-measurable for $f \in b(E)$ (the bounded E-measurable functions). Then we have $\{z\colon \varphi(z) \in B\} = \{z\colon E^z I_B(X_0) = 1\} \in H$, so φ is H-measurable on H_0. We set $\varphi(z) = x_0$ on $H - H_0$ for some fixed $x_0 \in E$. Now for any F^z_{t+}-stopping time $T < \infty$ we have for $B \in E$

$$\begin{aligned} I_B(X_T) &= P^z \ (X_T \in B \mid F^z_{T+}) \\ &= P^{Z^z_T} \ (X_0 \in B) \\ &= I_B(\varphi(Z^z_T)) , \qquad\qquad z\text{-a.s.} \end{aligned}$$

It follows easily that $X_T = \varphi(Z^z_T)$, z-a.s. Then, since both X_t and $\varphi(Z^z_t)$ are F^z_{t+}-optional processes, the optional section theorem of [1, IV, 84] finishes the proof.

Before proceeding, let us review our notations. P without superscript refers to the original process on Ω, and at the same time we have $P \in H$. P^z and E^z are simply z and its expectation, for $z \in H$, but we do not write P^P. Z^z is the prediction process of z; in particular, Z^P is that of P. We will need to use $Q(x,S)$ in three distinct senses: first, as a probability kernel; second, as a mapping $Q : E \to H$ defined by $Q(x) = Q(x,(\cdot))$; and third as a set mapping $Q\{x \in S\} = \{Q(x) : x \in S\}$.

The essential requirement for using the processes Z^z to construct a transition function for X_t is that the mapping $Q: E \to H$ defined by the given kernel $Q(x,S)$ should have a range $Q(E)$ sufficiently large that $P\{Z_t^P \in Q(E),\ t \geq 0\} = 1$. The most natural way to insure this is to introduce:

ASSUMPTION 1.6. Q is continuous for the given topology on E and some topology on H such that

i) $\mathcal{H}$ is the σ-field generated by the open sets, and

ii) Z_t^P is P-a.s. right continuous in t.

There are usually many different topologies generating $\mathcal{H}$ and making Z_t^P a.s.-right-continuous. Perhaps the most natural one is the weak*-topology with respect to the topology of weak convergence on Ω, discussed below. We postpone further discussion of Assumption 1.6 until the construction of the transition function $Q^*(x,S)$ is complete.

LEMMA 1.7. Under Assumption 1.6 there is a $K_0 \subset H_0$, $K_0 \in \mathcal{H}$, such that $Q\varphi$ = identity on K_0 and $P\{Z_t^P \in K_0,\ t \geq 0\} = 1$.

PROOF. By (1.3) we have for each $t \geq 0$, $P\{Q(X_t) = Z_t^P\} = 1$, hence $P\{Q(x_r) = Z_r^P$ for all rational $r \geq 0\} = 1$. By right-continuity of X_t and Z_t^P it follows that $P\{Q(X_t) = Z_t^P,\ t \geq 0\} = 1$. Next, let $S(x)$ denote $\{w : w(0) = x\}$. By Theorem 1.5 we have $P\{X_t = \varphi Q(X_t),\ t \geq 0\} = 1$, and since we have $Z_t^P \in H_0$ this implies $P\{Q(X_t, S(X_t)) = 1,\ t \geq 0\} = 1$. We set $K_0 = Q\{x:\ Q(x,S(x)) = 1\} \cap H_0$. Since $\{x:\ Q(x,S(x)) = 1\} \cap \{x:\ Q(x) \in H_0\} \in \mathcal{E}$, and on the above intersection we have $\varphi\, Qx = x$, then Q is one-to-one on this set, whose image under Q is K_0. It follows by [1, III, 21] that $K_0 \in \mathcal{H}$. We have $Q\,\varphi\, Qx = Qx$, hence $Q\varphi$ is the identity on K_0 and the proof is complete.

REMARKS. We did not quite have to require that $Q(x)$ be continuous, but only that it be measurable and that its graph be closed in $E \times H$. Furthermore under the not unreasonable conditions that $Q(x,S(x)) = 1$ for all x and that $Q(x, \cdot \mid \mathcal{F}_{0+}^{o}) = Q(x,\cdot)$ for all x (where the conditioning is on $Q(x)$) we have $K_0 = Q(E)$.

We now use the set K_0 of Lemma 1.7 to construct a state space for the prediction process on which it can be identified with X_t.

LEMMA 1.8. There is a $K_1 \subset K_0$, $K_1 \in H$, such that $P\{Z_0^P \in K_1\} = 1$ and for all $z \in K_1$,

$$P^z\{Z_t^z \in K_1, \quad t \geq 0\} = 1 .$$

REMARK. In the terminology of Essay I, Definition 2.1, 3), K_1 is a Borel packet of Z_t.

PROOF. (In part like Theorem 2.4 a) of [6]). We begin by setting

$$K = \{z \in H_0: \; P^z\{Z_t^z \in K_0, \quad t > 0\} = 1\} .$$

Then in the terminology of [3, Section 12] for $\alpha > 0$ we have $K = \{z \in H_0: \; P^\alpha_{H_0-K_0} 1(z) = 0\}$ where $P^\alpha_{H_0-K_0} 1$ is α-excessive for the transition function q. Since q is Borel and the prediction process is a right-process on H_0 (see Remark III. e. of Meyer [8]) K is a nearly Borel set for the prediction process. It follows that for $z \in K$, $I_K(Z_t^z)$ is P^z-indistinguishable from a well-measurable (optional) process of F_t^z, and so the section theorem implies that $P^z\{Z_t^z \in K, \; t \geq 0\} = 1$. We have, by Lemma 1.7, $P\{Z_0^P \in K\} = 1$, hence $P\{Z_0^P \in K \cap K_0\} = 1$. Also, for $z \in K \cap K_0$ we have by definition of K that $P^z\{Z_t^z \in K \cap K_0, \; t \geq 0\} = 1$, so we may consider $K \cap K_0$ as state space for the prediction process, and by Lemma 1.7 $Q\,\varphi$ = identity on $K \cap K_0$.

It remains to show that $K \cap K_0$ may even be replaced by a Borel subset K_1. We use an argument due to P. A. Meyer [7] (see also the end of [9]). Since $K \cap K_0$ is nearly Borel, it has a Borel subset K_2 such that $P\{Z_t^P \in K_2, \; t \geq 0\} = 1$. Let K_2' denote the nearly Borel set

$$K_2' = \{z \in K_2: \; P^z\{Z_t^z \in K_2, \quad t > 0\} = 1\} .$$

As before, we have

i) $P\{Z_0^P \in K_2'\} = 1$ and

ii) $P^z\{Z_t^z \in K_2', \; t \geq 0\} = 1$ for $z \in K_2'$.

Similarly, we define by induction a sequence $K_2' \supset K_3 \supset K_3' \ldots \supset K_n \supset K_n' \ldots$, where K_n is Borel, and K_n' is nearly Borel and satisfies i) and ii). Now let $K_1 = \bigcap_{n \geq 2} K_n$. Then K_1 is Borel, and obviously satisfies i). But for $z \in K_1$ we have $P^z\{Z_t^z \in K_n', \; t \geq 0\} = 1$ for every n.

Since $K_1 = \bigcap_{n\geq 2} K'_n$, K'_1 also satisfies ii) and the proof is complete.

We can now prove the main theorem.

THEOREM 1.9. Under Assumptions 1.1 and 1.6, given $Q(x,S)$ and P as in Definition 1.2 there exists a $Q^*(x,S)$ for the same P which satisfies the identities (1.4).

PROOF. We have $Q\varphi$ = identity on K_1, and $P\{X_t \in \varphi(K_1),\ t \geq 0\} = 1$. By [1, III, 21], $\varphi(K_1) \in \mathcal{E}$. Now we define

$$Q^*(x,S) = \begin{cases} Q(x,S) & \text{if } x \in \varphi(K_1) \\ I_S(w_x) & \text{if } x \notin \varphi(K_1) \end{cases},$$

where $w_x(t) = x$ for all $t \geq 0$. Obviously Q^* is a probability kernel and satisfies (1.3) for P, and (1.4) for $x \notin \varphi(K_1)$. Finally, for $x \in \varphi(K_1)$, $0 \leq t_1 \leq \cdots \leq t_n$, and $B_1,\dots,B_n \in \mathcal{E}$, by (1.5) and Theorem 1.4 we have

$$Q^*(x, \bigcap_{k=1}^n X_{t_k} \in B_k)$$

$$= P^{Q(x)}(\bigcap_{k=1}^n \{\varphi(Z_{t_k}^{Q(z)}) \in B_k \cap \varphi(K_1)\})$$

$$= \int_{Q(B_n)\cap K_1} \cdots \int_{Q(B_1)\cap K_1} q(t_1, Q(x), dz_1) \dots q(t_n - t_{n-1}, z_{n-1}, dz_n)$$

$$= \int_{B_n \cap \varphi(K_1)} \dots \int_{B_1\cap\varphi(K_1)} Q^*(x, X_{t_1} \in dy_1) \dots Q^*(y_{n-1}, X_{t_n - t_{n-1}} \in dy_n)$$

where we used the fact that Q^{-1} is an isomorphism of $\mathcal{H}|K_1$ onto $\mathcal{E}|\varphi(K_1)$ for the last equality (again by [1, III]). In the last term we may omit the $\varphi(K_1)$'s just as for the first equality. Choosing $B_1 = E$, $t_1 = s$, and $t_2 - t_1 = t$, this establishes (1.4) for $S = \bigcap_{k=2}^n \{X_{t_k - t_2} \in B_k\}$. The general case follows immediately by the familiar uniqueness of the extension.

COROLLARY 1.9. For every initial distribution μ, we have the strong Markov property:

$$P^\mu(\theta_T^{-1} S \mid \mathcal{F}_{T+}^\mu) = Q^*(X_T, S), \quad P^\mu\text{-a.s.}$$

where $P^\mu(S) = \int Q^*(x,S)\mu(dx)$, and T is any finite stopping time of the completed σ-fields $\mathcal{F}_{T+}^\mu$.

REMARK. It follows that $\mathcal{F}_{T+}^\mu = \mathcal{F}_T^\mu$.

PROOF. For μ concentrated on $\varphi(K_1)$ this follows from the analogous property of $Z_t^{P^\mu}$, by writing P^μ-a.s. $X_\cdot = \varphi(Z_\cdot^{P^\mu})$ as in the former proof. The part of μ outside of $\varphi(K_1)$ causes no difficulty since, for every T, we have $\{X_0 \notin \varphi(K_1)\} \in \mathcal{F}_{T+}^\mu$.

We turn to a discussion of Assumption 1.6, which of course is the main question mark in the theory. The essential fact in identifying such a topology is

THEOREM 1.10. Let f be bounded and $\mathcal{F}^o$-measurable $(f \in b(\mathcal{F}^o))$. If $f \circ \theta_t$ is right-continuous (resp. with left limits) in t for all $w \in \Omega$, then for every $z \in H$

$$P^x\{E^{Z_t^z} f \text{ is right-continuous (resp. with left limits)}\} = 1 .$$

PROOF. This follows immediately from two known results:

a) $E^{Z_t^z} f$ is the $\mathcal{F}_t^z$-optional projection of $f \circ \theta_t$ [1, III, Theorem 2], and

b) The $\mathcal{F}_t^z$-optional projection of a right-continuous bounded process (resp. with left limits) is itself right-continuous (resp. with left limits) z-a.s. [7, Appendice 2].

Therefore, we have immediately

COROLLARY 1.10. Let $\{f_n \in b(\mathcal{F}^o),\ 1 \le n\}$ satisfy the two conditions

a) for each w and n, $f_n \circ \theta_t$ is right-continuous in $t \ge 0$, and

b) the monotone linear bounded closure of $\{f_n\}$ is $b(\mathcal{F}^o)$.

Then the topology on H generated by the functions $E^z f_n$, $1 \le n$ satisfies i) and ii) of Assumption 1.6.

PROOF. Only i) needs comment. But since each $E^z f_n$ is measurable with respect to the σ-field generated by the open sets, so is $E^z f$ for f if the closure $b(\mathcal{F}^o)$, as required.

There are many possibilities for such f_n . Perhaps the most obvious is to take $f_n = g_m(X_r)$ where r runs over the non-negative rationals and g_m runs over a uniformly dense set of continuous functions on a compact metric space $\bar{E}$ containing E as a Borel subset. Then the condition that Q satisfy Assumption 1.6 becomes the <u>Feller property</u> $E^{Q(x)} g_m(X_r) \in C(E)$ for rational r .

A weaker type of requirement, but one which still involves the given topology of E, utilizes all finite products

$$(1.6) \qquad f_n = \prod_{i=1}^{k} \int_{r_i}^{\infty} e^{-t} g_{m_i}(X_t)\, dt ,$$

for $0 \le r_i$ rational and the g_m's as above. Here the topology generated on Ω by the f_n is just the weak topology of the sojourn measures $\mu(t,A)$ defined by $\mu(t,A) = \int_0^t I_A(X_s)ds$. Indeed, we have $\int_0^t g_m(X_s)\,ds = \int_E g_m(x)\mu(t,dx)$. Hence, convergence of these integrals for all m is just weak convergence of $\mu(t,\cdot)$. On the other hand, this convergence for all t and m is easily seen to be equivalent to that generated by the f_n. This topology is metrizable, for example, with metric $d(w_1,w_2) = \sum_m w^{-n}|f_n(w_1) - f_n(w_2)|$, whence Ω is embedded as a Borel subset of its compactification, which is the space $\bar{\Omega}$ of (equivalence classes of) measurable functions with values in the closure of E (for this argument, see Essay 1, Theorem 1.2, where an analogous but weaker topology is treated).

Accordingly, we can consider on H the weak-*topology generated by this topology on $\bar{\Omega}$, by setting $h(\bar{\Omega} - \Omega) = 0$ for $h \in H$. Again, continuity of $Q(x)$ for this topology on its range can be expressed in more familiar terms.

THEOREM 1.11. Continuity $Q(x)$ for the weak-*topology generated by $E^z f_n$ for the f_n of (1.6) is equivalent to the continuity on E, for all $\lambda > 0$ and continuous g on $\bar{E}$, of

$$E^{Q(x)} \int_0^\infty e^{-\lambda t} g(X_t)dt \,. \tag{1.7}$$

REMARK. Let $R_\lambda g(x)$ denote (1.7). Then the last continuity is just the Ray property of $R_\lambda g(x)$, except that we are not assuming the resolvent equation. The proof below is not self-contained, but in the present context it does not seem to merit that degree of emphasis.

PROOF. We rely on the construction of [5], where the coordinate functions h_n are the present g_n. By the argument just given, convergence in the space Ω' of [5] induces on Ω the topology of weak convergence of sojourn time distributions.

Consequently, the topology of H in [5] reduces to the same weak-* topology as above. The assertion of our theorem now follows from the proof of Theorem 3.1.1 of [5] in two steps. First, we observe that the proof of $R_\lambda\colon\ C(E) \longrightarrow C(E_Q)$ needs no change, where E_Q is E with the Q-induced topology. This is simply the observation that each $R_\lambda g_n(x)$ is continuous on E_Q since each $\int_0^\infty e^{-\lambda t} g_n(X_t)dt$ is continuous on Ω. Consequently, if $E^{Q(x)} f_n$ are continuous on E then $R_\lambda\colon\ C(E) \longrightarrow C(E)$. Second, we note that the proof of Lemma 3.1.1 of [5] does not use the resolvent equation or the compactness of E. Accordingly it applies unchanged, and we obtain that if $R_\lambda\colon\ C(E) \longrightarrow C(E)$ holds, then

$$\int_0^{t_k} g_{n_k}(X_s)ds,\ 1 \le k \le n,\ \ t_k \ge 0,$$

have joint distributions for Q^x which are weakly continuous in x (for any choice of n, n_k, and t_k). This easily implies continuity of $E^{Q(x)} f_n$ for the f_n of (1.6) so the proof is complete.

As seen above, both the Feller property and the Ray property are essentially special cases of Assumption 1.6. It is thus of interest to note that (at least formally) the later is much more general than either of these. According to Corollary 1.10, if $g_k \in b(\mathcal{E})$ is any sequence such that the monotone linear bounded closure of $\{g_k\}$ is all of $b(\mathcal{E})$ then the topology on H generated by $E^z f_n$ for all $f_n = \int_r^\infty e^{-s} g_k(X_s)ds$, $0 \le r$ rational, $1 \le k$, will satisfy the requirements i) and ii) of Assumption 1.6. Hence one need only find a $Q(x)$ continuous in such a topology to obtain the conclusions of Theorem and Corollary 1.9. Moreover, since the g_k involve only the σ-field $\mathcal{E}$ (and not the topology of E), one is now free to change the topology of E provided that X_t may still be assumed to have right-continuous paths with left limits. Therefore, rather than starting with Assumption 1.1, we could just as well assume such a continuity of $E^{Q(X_t)} f_n$. This leads to the following statement.

THEOREM 1.12. Let $(\Omega, \theta_t, \mathcal{F}_t^o)$ be the space of Lebesgue measurable $(E,\mathcal{E})$-valued paths $x_t(w) = w(t)$, $t \ge 0$, with the σ-fields $\mathcal{F}_t^o$ augmented to include $\sigma(\int_0^s f(X_\tau)\,d\tau,\ s < t,\ f \in b(\mathcal{E}))$. Suppose given P on $\mathcal{F}^o$ and a probability kernel $Q(x,S)$ satisfying (1.3). Let $g_k \in b(\mathcal{E})$ be any sequence having monotone linear bounded closure $b(\mathcal{E})$, and let f_n be an enumeration of the random variables $\int_r^\infty e^{-s} g_k(X_s)\,ds$, $0 \le r$ rational, $1 \le k$. Suppose that the family $h_n(x) = E^{Q(x)} f_n$ generates the σ-field $\mathcal{E}$, and that the processes $h_n(X_t)$ are P^*-a.s. right-continuous with left limits, where P^* is P-outer-measure. Then the conclusions of Theorem and Corollary 1.9 hold when $(\Omega, \theta_t, \mathcal{F}_t^o)$ is replaced by the space of right-continuous paths with left limits in the topology on E generated by the $h_n(x)$, and when P is transferred to this space.

FINAL REMARKS. Such a P on $\mathcal{F}^o$ is induced through completion by any progressively measurable process. For $0 \le g_k$ the processes $e^{-t} h_n(X_t)$ are measurable supermartingales with respect to $\mathcal{F}_t^o$ and P, as seen by a familiar computation. Hence the martingale convergence theorems can be used to aid in checking the right-continuity with left limits. The question is simply whether, by making a standard modification of X_t, the martingale right-limits along rational t can be evaluated by substitution of X_t in h_n. It is important to note that this is always possible if we permit the standard modification to take values in H instead of just in E (regarded as a Borel subset of H through identification with its image by the mapping Q). Thus by (1.3) the limits along rational t may be evaluated a.s. at each t by substitution of X_t. Letting Z_t^P denote the general prediction process on $\bar{\Omega}$ (see Section 1 of Essay I) we may assume without loss of generality that for each r in a countable dense set $P\{X_r = \varphi(Z_r^P)\} = 1$. Then if we replace X_t by Z_t^P whenever this evaluation fails, and then replace Z_t^P by $\varphi(Z_t^P)$ whenever $Z_t^P \in Q(E)$, we get a standard modification of X_t with values in $E \cup (\bar{H}-Q(E))$ which satisfies the conclusions of Theorem and Corollary 1.9.

It is also of interest to note that for Theorem 1.12 one need only assume (1.3) relative to F_t^o. Then the familiar "Hunt's Lemma" argument shows that the $h_n(X_t)$ are in any case conditional expectations relative to F_{t+}^o, and therefore $Q(X_t,S)$ satisfies (1.3) relative to F_{t+}^o. The analytical question of giving conditions on a semigroup P_t under which, for any corresponding Markov process, F_t^o and F_{t+}^o are equivalent, is dealt with at length in Englebert (1978). Here it has been implicitly assumed (see the second remark after Definition 1.3).

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