DYNAMIC CONSTRUCTION AND SIMULATION OF RANDOM VECTORS

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In this paper described is a novel method of generation of nonnegative random variables $T_1, \ldots, T_n$ which may be dependent and which have an absolutely continuous joint distribution. In this method first $\min(T_1, \ldots, T_n)$ is generated and then one of the indices $1, \ldots, n$ (j_1, say) is chosen and T_{j_1}, is determined. Once $T_{j_1}, \ldots, T_{j_k}$ have been determined, then $\min_{j \epsilon \{1,\ldots,n\}-\{j_1,\ldots,j_k\}}(T_j)$ is generated and one of the remaining indices (j_{k+1}, say) is chosen and $T_{j_{k+1}}$ is determined. The novel method has a clear intuitive meaning, mainly for applications in reliability theory. The new method is applied to obtain stochastic comparisons of two absolutely continuous random vectors consisting of nonnegative random variables. Also, the use of the new method is illustrated in obtaining some multivariate aging properties and positive dependence properties of vectors of random lifetimes.

1. Introduction. Consider an absolutely continuous nonnegative random variable T with distribution function F, survival function $\bar{F} = 1 - F$ and hazard function $\Lambda = -\log \bar{F}$. The random variable T can be thought of as a lifetime of a device. The hazard rate (or the instantaneous failure rate of the device) at time t is defined as

$$\lambda(t) = \frac{f(t)}{P(T \geq t)} = \frac{f(t)}{\bar{F}(t)} = \frac{d}{dt}\Lambda(t),\ t \geq 0,$$

where $f = \frac{d}{dt}F$ is the density function of T. It is well known (and easy to verify) that T is stochastically equal to (that is, has the same distribution as) the time of the first epoch of a nonhomogeneous Poisson process on $[0, \infty)$ with intensity function λ. Thus, in order to generate a random variable $\hat{T}$ which has the same

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distribution as T, one can generate a nonhomogeneous Poisson process with intensity function λ and let $\hat{T}$ be the time of the first epoch in that process. This can be done by generating a standard exponential random variable S and define $\hat{T}$ by

$$\hat{T} = \inf\{t : \Lambda(t) > S\} = \Lambda^{-1}(S);$$

see, e.g., Lewis and Shedler (1979), Ross (1985), and Shanthikumar (1986).

The purpose of this paper is to give a multivariate analog of this univariate result. This is done in Section 2 where we introduce a method (called the *dynamic construction*) which, for every nonnegative absolutely continuous random vector $\mathbf{T} = (T_1, \ldots, T_n)$, constructs a random vector $\hat{\mathbf{T}} = (\hat{T}_1, \ldots, \hat{T}_n)$ of times of first epoch of some nonhomogeneous Poisson processes such that $\hat{\mathbf{T}} \overset{\text{st}}{=} \mathbf{T}$ (in this paper $\overset{\text{st}}{=}$ denotes stochastic equality).

A random variable X is said to be *stochastically smaller* than a random variable Y (denoted $X \overset{\text{st}}{\leq} Y$) if, for every t, $P\{X > t\} \leq P\{Y > t\}$. A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be stochastically smaller than a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ [denoted $\mathbf{X} \overset{\text{st}}{\leq} \mathbf{Y}$] if $g(\mathbf{X}) \overset{\text{st}}{\leq} g(\mathbf{Y})$ for every increasing Borel measurable real function g. A function g is called increasing if $g(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ whenever $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$. [In this paper 'increasing' and 'decreasing' are not used in the strict sense. For vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ we denote $\mathbf{x} \leq \mathbf{y}$ to mean $x_i \leq y_i$, $i, \ldots, n$.] It is well known that $\mathbf{X} \overset{\text{st}}{\leq} \mathbf{Y}$ if and only if

$$Eg(\mathbf{X}) \leq Eg(\mathbf{Y}) \tag{1}$$

for every increasing Borel-measurable real function g for which the expectations exist. Another condition which is equivalent to $\mathbf{X} \overset{\text{st}}{\leq} \mathbf{Y}$, is

$$P\{\mathbf{X} \epsilon U\} \leq P\{\mathbf{Y} \epsilon U\}$$

for every Borel set U which has an increasing indicator function [such sets are called increasing (or upper) sets].

If X and Y are nonnegative absolute continuous random variables with hazard rate functions μ and η, respectively, then it is well known (and easy to verify) that

$$[\mu(t) \geq \eta(t),\ t \geq 0] \Longrightarrow X \overset{\text{st}}{\leq} Y. \tag{2}$$

It follows that if $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ are vectors of independent nonnegative absolutely continuous random variables such that the hazard rate functions of X_i and Y_i are μ_i and η_i, respectively, $i = 1, \ldots, n$, then

$$[\mu_i(t) \geq \eta_i(t),\ t \geq 0,\ i = 1, \ldots, n] \Longrightarrow \mathbf{X} \overset{\text{st}}{\leq} \mathbf{Y}. \tag{3}$$

It is not hard to show that the converse of (2), and hence also of (3), is false.

In this paper we extend (3) to random vectors $\mathbf{X}$ and $\mathbf{Y}$ which may have dependent components. This is done in Section 3 where random vectors $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ are constructed simultaneously (on a common probability space), using the dynamic construction, such that $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ satisfy: $\hat{\mathbf{X}} \overset{\text{st}}{=} \mathbf{X}$, $\hat{\mathbf{Y}} \overset{\text{st}}{=} \mathbf{Y}$ and $P\{\hat{\mathbf{X}} \leq \hat{\mathbf{Y}}\} = 1$. The results are similar to those of Shaked and Shanthikumar (1987a), but the proofs are different.

The rest of the paper (Sections 4 and 5) consists of applications of the results of Section 3. In Section 4 we identify some conditions on the hazard rates of a random vector $\mathbf{T}$ (the hazard rates are defined in Section 2) which imply that $\mathbf{T}$ satisfies the MIHR$|\ \mathcal{F}_t$ property of Arjas (1981a). In Section 5 we show that another set of conditions on the hazard rates imply that $\mathbf{T}$ has the WBF (weakened by failures) property of Arjas and Norros (1984). In particular, such random variables are associated in the sense of Esary, Proschan, and Walkup (1967).

In this paper, for $t_* \geq 0$, we will consider nonhomogeneous Poisson processes on $[t_*, \infty)$ with intensity function λ defined on $[t_*, \infty)$. By this phrase we mean nonhomogeneous Poisson processes which start counting at time t_*, or, equivalently, nonhomogeneous Poisson processes with intensity 0 on $[0, t_*)$ and intensity λ on $[t_*, \infty)$.

2. The Dynamic Construction. For the purpose of generating a random vector $\hat{\mathbf{T}}$, such that $\hat{\mathbf{T}} \overset{\text{st}}{=} \mathbf{T}$ for some given $\mathbf{T}$, two alternative constructions have been used. They are the *standard construction* (see, e.g., Arjas and Lehtonen (1978)) and the *total hazard construction* (see, e.g., Norros (1986) and Shaked and Shanthikumar (1986a, 1987b)). If $\mathbf{T}$ is n-dimensional, then each of these constructions requires n uniform random variables in order to generate $\hat{\mathbf{T}}$. The dynamic construction, described below, requires more than n uniform random variables but it has an intuitive meaning which has theoretical and practical advantages, especially in reliability theory.

In the dynamic construction, the random variables $T_1, \ldots, T_n$, are thought of as the lifetimes of n components numbered $1, 2, \ldots, n$. The dynamic point of view can be described as follows: Let $t \geq 0$ be zero or an observed time of failure of one of the components. Assume that at that time t, it is known which components are still alive and the failure times of the components which fail before or at time t. Given this information, the dynamic construction considers then the conditional distribution of the time to next failure, t' say ($t' > t$), and the conditional probability that this next failure is of a particular component of those still alive at time t. The time t' is a new starting point at which the conditional distribution of the following failure and the identity of the next failed component are considered. This is done inductively until all n components have failed.

Given that $t \geq 0$ is a failure time of a component and that components $i_1, \ldots, i_k$ are still alive then [if $t = 0$ then $k = n$ and $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$], the next failure time is $\min(T_{i_1}, \ldots, T_{i_k})$ and, in the dynamic construction below, this time is described (as in the univariate case, see Section 1) as the time of the first epoch of a nonhomogeneous Poisson process, starting at time t, with intensity function depending on the observed 'history' up to time t. Once $\min(T_{i_1}, \ldots, T_{i_k})$ is observed, at time t', say, the identity of the failed component (which must be one of $i_1, \ldots, i_k$) is chosen according to the conditional distribution [on $\{i_1, \ldots, i_k\}$] of the component identities given in the history up to time $t'-$ and that a failure has occurred at time t'.

For example, in the bivariate case there are two components, 1 and 2, which start to live at time 0. First consider the distribution of $\min(T_1, T_2)$ with hazard rate function λ, say, on $[0, \infty)$. It is the distribution of the time of the first epoch in a nonhomogeneous Poisson process with intensity function λ. Next, given that $\min(T_1, T_2) = t$, say, choose an i from $\{1, 2\}$ according to the probability $p_t(i)$, $i = 1, 2$, where $p_t(i)$ is the conditional probability [given $\min(T_1, T_2) = t$] that component i fails at time t [that is, that in fact $\min(T_1, T_2) = T_i$]. Note that $p_t(1) + p_t(2) = 1$ for all t. This way, one of the T_i's (the smallest of the two) is stochastically represented as the time of the first epoch of a nonhomogeneous Poisson process. Finally, given that $\min(T_1, T_2) = t$ and that $\min(T_1, T_2) = T_i$ for some $i \epsilon \{1, 2\}$, let λ_{3-i} (which may depend on t) be the (conditional) hazard rate function of the surviving component on $[t, \infty)$. The random variable T_{3-i} can also be stochastically represented as the time of the first epoch of a nonhomogeneous Poisson process (with intensity function λ_{3-i}). It is not hard to see that the functions $\lambda(\cdot)$, $p_\cdot(1)$, $p_\cdot(2)$, $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ determine the distribution F of (T_1, T_2) [see, e.g., Cox (1972) or Shaked and Shanthikumar (1986b)]. In the bivariate case these functions (and similar ones in the case $n > 2$) will be the building blocks of the dynamic construction described below.

In general, let $\mathbf{T} = (T_1, \ldots, T_n)$ be a nonnegative absolutely continuous random vector to be thought of as a vector of lifetimes of n components. For $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, let $\mathbf{t}_I$ denote $(t_{i_1}, \ldots, t_{i_k})$. The complement of I will be denoted by $\bar{I} = \{1, \ldots, n\} - I$ and if $\bar{I} = \{j_1, \ldots, j_{n-k}\}$ then $\mathbf{t}_{\bar{I}} = (t_{j_1} \ldots, t_{j_{n-k}})$. Let $\mathbf{e} = (1, \ldots, 1)$. The length of $\mathbf{e}$ will vary from one formula to another, but it will always be possible to determine it from the expression in which $\mathbf{e}$ appears.

We will often consider the conditional distribution of $\mathbf{T}_I$ given that $\mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}$ and that $\mathbf{T}_I \geq t\mathbf{e}$ for some $\mathbf{t}_{\bar{I}} \geq 0\mathbf{e}$ and $t \geq \bigvee_{i \epsilon \bar{I}} t_i \equiv \max\{t_i : i \epsilon \bar{I}\}$. Then, for $i \epsilon I$, the conditional density of T_i, at time t, given the above information, will be called the *conditional hazard rate* of T_i (or the *conditional instantaneous failure rate* of component i) at time t. It will be denoted by $\lambda_i(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}},\ \mathbf{T}_I \geq t\mathbf{e})$. Formally, for $i \epsilon I$,

$$
\begin{aligned}
&\lambda_i(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}},\ \mathbf{T}_I \geq t\mathbf{e}) \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\{t \leq T_i \leq t + \Delta t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}},\ \mathbf{T}_I \geq t\mathbf{e}\}.
\end{aligned} \tag{4}
$$

The absolute continuity of $\mathbf{T}$ ensures that this limit exists. To save space we sometimes suppress the condition $\mathbf{T}_I \geq t\mathbf{e}$ and just write $\lambda_i(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}},\ \ \cdot)$ but the reader should keep in mind that '$\cdot$' means $\mathbf{T}_I \geq t\mathbf{e}$ with t being the same as the first argument of λ_i. The function $\lambda_i(\cdot \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \ \cdot)$ will be of interest to us only on the (random) interval $\left(\max_{j\epsilon\bar{I}} T_j, \ \min_{i\epsilon I} T_i\right)$, however, to avoid a discussion of such random hazard rate functions [such a discussion can be found in Arjas (1981b)] we do not emphasize this point here. Note however, that $\lambda(\cdot \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot)$ of (4) is well defined for almost every $t \geq \bigvee_{j\epsilon\bar{I}} t_j$.

The absolute continuity implies that, with probability one, no two failures can occur at the same time. Thus, for $t \geq \bigvee_{i\epsilon\bar{I}} t_i$,

$$\lambda_I(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \ \mathbf{T}_I \geq t\mathbf{e}) \equiv \sum_{i\epsilon I} \lambda_i(t \mid \mathbf{T}_{\bar{I}} \geq \mathbf{t}_{\bar{I}}, \ \mathbf{T}_I \geq t\mathbf{e}) \tag{5}$$

is the conditional hazard rate of $\min_{i\epsilon I}(T_i)$ at time t. For $i\epsilon I$, denote

$$\begin{aligned} & p_t(i \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \ \mathbf{T}_I \geq t\mathbf{e}) \\ & \quad \equiv \frac{\lambda_i(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \ \mathbf{T}_I \geq t\mathbf{e})}{\lambda_I(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \ \mathbf{T}_I \geq t\mathbf{e})} \\ & \quad = P\left\{\min_{j\epsilon I} T_j = T_i \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \ \mathbf{T}_I \geq t\mathbf{e}, \ \min_{j\epsilon I}(T_j) = t\right\}. \end{aligned} \tag{6}$$

In the sequel we will also suppress sometimes the condition $\mathbf{T}_I \geq t\mathbf{e}$ in (5) and (6) and just write $\lambda_I(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot)$ and $p_t(i \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot)$. Note that for $t \geq \bigvee_{j\epsilon\bar{I}} t_j$, $\sum_{i\epsilon I} p_t(i \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot) = 1$ and that

$$\lambda_i(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot) = p_t(i \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot)\lambda_I(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot). \tag{7}$$

For a nonnegative random vector $(T_1, \ldots, T_n)$ the *dynamic construction* of $(\hat{T}_1, \ldots, \hat{T}_n)$, such that $(\hat{T}_1, \ldots, \hat{T}_n) \overset{\text{st}}{=} (T_1, \ldots, T_n)$, consists of the following n steps:

Step 1: Consider n independent nonhomogeneous Poisson processes on $[0, \infty)$ indexed by $i\epsilon\{1, \ldots, n\}$ with intensity function $\lambda_i(t \mid \mathbf{T} \geq t\mathbf{e})$, $t \geq 0$, $i = 1, \ldots, n$. If Process j_1 yields the first epoch (out of all the n processes) then let the time of this epoch be $\hat{T}_{j_1}$.

Step 2: Given that Step 1 resulted in $\hat{T}_{j_1} = t_{j_1}$, consider $n - 1$ independent nonhomogeneous Poisson processes on $[t_{j_1}, \infty)$ indexed by $i\epsilon I \equiv \{1, \ldots, n\} - \{j_1\}$. For $i\epsilon I$ let the intensity function of Process i be $\lambda_i(t \mid T_{j_1} = t_{j_1}, \mathbf{T}_I \geq t\mathbf{e})$, $t \geq t_{j_1}$. If Process $j_2(\epsilon I)$ yields the first epoch (out of all the $n - 1$ processes) then let the time of this epoch be $\hat{T}_{j_2}$.

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Step $k + 1$: Given that Steps $1, \ldots k$ resulted in $\hat{T}_{j_1} = t_{j_1}, \ldots, \hat{T}_{j_k} = t_{j_k}$, let $I = \{1, \ldots, n\} - \{j_1, \ldots, j_k\}$ and consider $n - k$ independent nonhomogeneous

Poisson processes on $[\bigvee_{j\in\bar{I}} t_j, \infty)$ indexed by $i\in I$. For $i\in I$ let the intensity function of Process i be $\lambda_i(\cdot \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot)$ on $[\bigvee_{j\in\bar{I}} t_j, \infty)$. If Process $j_{k+1}(\in I)$ yields the first epoch (out of the $n-k$ processes) then let $\hat{T}_{j_{k+1}}$ be the time of this epoch.

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Step n: Given that Steps $1,\ldots,n-1$ resulted in $\hat{T}_{j_1} = t_{j_1},\ldots,\hat{T}_{j_{n-1}} = t_{j_{n-1}}$, let $I = \{j_n\} \equiv \{1,\ldots,n\} - \{j_1,\ldots,j_{n-1}\}$ and consider a nonhomogeneous Poisson process on $[\bigvee_{i\in\bar{I}} t_i, \infty)$ with intensity functions $\lambda_{j_n}(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}},\ T_{j_n} \geq t)$. Let the time of the first epoch of this process be $\hat{T}_{j_n}$.

The verification that $(\hat{T}_1,\ldots,\hat{T}_n) \stackrel{\text{st}}{=} (T_1,\ldots,T_n)$ is straightforward. It follows from the following representation of the joint density $f(t_1,\ldots,t_n)$ of $(T_1,\ldots,T_n)$: For $0 \leq t_1 \leq \cdots \leq t_n$,

$$
\begin{aligned}
f(t_1,\ldots,t_n) &= \lambda_1(t_1 \mid \mathbf{T} \geq t_1\mathbf{e})\exp\{-\Lambda_{\{1,\ldots,n\}}(t_1)\} \\
&\times\{\prod_{i=2}^{n}[\lambda_i(t_i \mid \mathbf{T}_{\{1,\ldots,i-1\}} = \mathbf{t}_{\{1,\ldots,i-1\}},\ \cdot) \\
&\times \exp\{-(\Lambda_{\{i,\ldots,n\}}(t_i \mid \mathbf{T}_{\{1,\ldots,i-1\}} = \mathbf{t}_{\{1,\ldots,i-1\}},\ \mathbf{T}_{\{i,\ldots,n\}} \geq t_{i-1}\mathbf{e}))\}]\},
\end{aligned}
$$

and similar expressions are valid when $0 \leq t_{\pi(1)} \leq \cdots \leq t_{\pi(n)}$ for any permutation π of $(1,\ldots,n)$ [see, e.g., Cox (1972) or Lemma 1.1 of Shaked and Shanthikumar (1986b)]. Here $\Lambda_{\{1,\ldots,n\}}(t) \equiv \int_0^t \lambda_{\{1,\ldots,n\}}(u \mid \mathbf{T} \geq u\mathbf{e})du$, for $t \geq 0$, and $\Lambda_I(t \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}},\ \mathbf{T}_I \geq (\bigvee_{i\in\bar{I}} t_i)\mathbf{e}) \equiv \int_{\bigvee_{i\in\bar{I}} t_i}^{t} \lambda_I(u \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}},\ \mathbf{T}_I \geq u\mathbf{e})du$ for $I \subset \{1,\ldots,n\}$ and $t \geq \bigvee_{i\in\bar{I}} t_i$.

For some purposes the following *modified dynamic construction* is more useful than the dynamic construction described above:

Step 1: Consider a nonhomogeneous Poisson process on $[0,\infty)$ with intensity function $\lambda_{\{1,\ldots,n\}}(t \mid \mathbf{T} \geq t\mathbf{e})$, $t \geq 0$. At the time s, say, of the first epoch, choose an index i with probability $P\{\text{the index } i \text{ is chosen}\} = p_s(i \mid \mathbf{T} \geq s\mathbf{e})$, $i = 1,\ldots,n$. If the chosen i is j_1 then let $\hat{T}_{j_1} = s$.

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Step k + 1: Given that Steps $1,\ldots,k$ resulted in $\hat{T}_{j_1} = t_{j_1},\ldots,\hat{T}_{j_k} = t_{j_k}$ let $I = \{1,\ldots,n\} - \{j_1,\ldots,j_k\}$. Consider a nonhomogeneous Poisson process on $(\bigvee_{i\in\bar{I}} t_i, \infty)$ with intensity function $\lambda_I(\cdot \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot)$. At the time s, say, of the first epoch, choose an index from I with probability $p_s(i \mid \mathbf{T}_{\bar{I}} = \mathbf{t}_{\bar{I}}, \cdot)$, $i\in I$. If the chosen index i is j_{k+1} then let $\hat{T}_{j_{k+1}} = s$.

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Step n: The last step is the same as in the dynamic construction.

3. Stochastic Ordering Via Conditions on the Hazard Rates. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two nonnegative absolutely continuous random vectors. For any set $I \subset \{1, \ldots, n\}$ and fixed $\mathbf{t}_{\bar{I}} \geq 0\mathbf{e}$, $t \geq \bigvee_{j \epsilon \bar{I}} t_j$ and $i \epsilon I$, let the conditional hazard rates of X_i and Y_i be defined (as in (4)) by

$$\begin{aligned} &\mu_i(t \mid \mathbf{X}_{\bar{I}} = t_{\bar{I}},\ \mathbf{X}_I \geq t\mathbf{e}) = \mu_i(t \mid \mathbf{X}_{\bar{I}} = t_{\bar{I}}, \cdot) \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\{t \leq X_i \leq t + \Delta t \mid \mathbf{X}_{\bar{I}} = t_{\bar{I}},\ \mathbf{X}_I \geq t\mathbf{e}\} \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\eta_i(t \mid \mathbf{Y}_{\bar{I}} = t_{\bar{I}},\ \mathbf{Y}_I \geq t\mathbf{e}) = \eta_i(t \mid \mathbf{Y}_{\bar{I}} = t_{\bar{I}}, \cdot) \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\{t \leq Y_i \leq t + \Delta t \mid \mathbf{Y}_{\bar{I}} = t_{\bar{I}},\ \mathbf{Y}_I \geq t\mathbf{e}\}. \end{aligned} \tag{9}$$

In this section we find sufficient conditions on the μ_i's and η_i's which imply that $\mathbf{X} \overset{\text{st}}{\leq} \mathbf{Y}$. The following main result of this section is a multivariate analog of (2). Here, and in the remainder of the paper, for $I \subset \{1, \ldots, n\}$ and $t_{\bar{I}} \geq 0\mathbf{e}$, we denote $M(\mathbf{t}_I) = \bigvee_{i \epsilon I} t_i$ where $\bigvee_{i \epsilon I} t_i = 0$ if $I = \emptyset$. Similarly, for (not necessarily disjoint) sets $I, J, K, \cdots \subset \{1, \ldots, n\}$ and $\mathbf{t}_I \geq 0\mathbf{e}$, $\mathbf{t}_J \geq 0\mathbf{e}$, $\mathbf{t}_K \geq 0\mathbf{e}, \ldots$, we denote $M(\mathbf{t}_I,\ \mathbf{t}_J,\ \mathbf{t}_K, \ldots) = (\bigvee_{i \epsilon I} t_i) \bigvee \left(\bigvee_{j \epsilon J} t_j\right) \bigvee (\bigvee_{k \epsilon K} t_k) \bigvee \cdots$.

THEOREM 3.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *have conditional hazard rates as in (8) and (9). If for all disjoint sets* $I, J \subset \{1, \ldots, n\}$ *such that* $\overline{I \cup J} \neq \emptyset$ *and for all fixed* $\mathbf{t}_J \geq 0\mathbf{e}$, *the following holds:*

$$\begin{aligned} &\mu_k(M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u \mid \mathbf{X}_I = \mathbf{t}_I,\ \mathbf{X}_J = \mathbf{t}_J,\ X_{\overline{I \cup J}} \geq (M(\mathbf{t}_J,\ \tilde{\mathbf{t}}_I) + u)\mathbf{e}) \\ &\geq \eta_k(M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u \mid \mathbf{Y}_I = \tilde{\mathbf{t}}_I,\ \mathbf{Y}_{\bar{I}} \geq (M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u)\mathbf{e}) \end{aligned} \tag{10}$$

whenever $\mathbf{t}_I \leq \tilde{\mathbf{t}}_I$, $u \geq 0$ *and* $k \epsilon \overline{I \cup J}$ *(J may be the empty set), then* $\mathbf{X} \overset{st}{\leq} \mathbf{Y}$.

REMARK 3.2. Roughly speaking, the events on which the failure rates are conditioned in (10) are two histories of the same length $M(\mathbf{t}_J, \tilde{\mathbf{t}}_I)$. The history on the left hand side of (10) has more failures than the history on the right hand side, and, for components which failed in both histories, the failure times in the former are earlier than in the latter. Condition (10) says that whenever the histories of $\mathbf{X}$ and $\mathbf{Y}$ can thus be compared, then the failure rate, under the law of $\mathbf{X}$, of each surviving component in the history of $\mathbf{X}$, is larger than the failure rate of the same component under the law of $\mathbf{Y}$.

REMARK 3.3. If $\mathbf{X}$ and $\mathbf{Y}$ are vectors of independent random variables with hazard rate functions satisfying the condition in (3) then (10) holds. Thus Theorem 3.1 contains Result (3) as a special case.

The proof of Theorem 3.1 is given in the Appendix.

In Sections 4 and 5 we need a slight generalization of Theorem 3.1. We will need to compare random vectors $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ where $\mathbf{Y}$ is nonnegative and absolutely continuous on $[0, \infty)^n$, but $\mathbf{X}$ may have some components which are identically zero, that is, $\mathbf{X}$ is of the form [relabelling components if necessary] $\mathbf{X} = (0, \ldots, 0, \ X_{\ell+1}, \ldots, X_n)$ for some $\ell \epsilon \{0, 1, \ldots, n-1\}$, where $(X_{\ell+1}, \ldots, X_n)$ is nonnegative and absolutely continuous on $[0, \infty)^{n-\ell}$. We will condition on events of the form $\{\mathbf{X}_I = \mathbf{t}_I, \ \mathbf{X}_{\bar{I}} \geq t\mathbf{e}\}$ $t \geq M(\mathbf{t}_I)$ where $I \supset \{1, \ldots, \ell\}$ and $\mathbf{t}_{\{1,\ldots,\ell\}} = 0\mathbf{e}$. For $i \epsilon \bar{I}$, the conditional hazard rate at time t of X_i, given $\mathbf{X}_I = \mathbf{t}_I$ and $\mathbf{X}_{\bar{I}} \geq t\mathbf{e}$, denoted by $\mu_i(t \mid \mathbf{X}_I = \mathbf{t}_I, \ \mathbf{X}_{\bar{I}} \geq t\mathbf{e})$, $t \geq M(\mathbf{t}_I)$, is well defined as in (8). Condition (10) then is well defined for every $I \supset \{1, \ldots, \ell\}$.

THEOREM 3.4. *Let $\mathbf{X}$ and $\mathbf{Y}$ be as described above. If for all disjoint sets I, $J \subset \{1, \ldots, n\}$ such that $(I \cup J) \supset \{1, \ldots, \ell\}$ and $\overline{I \cup J} \neq \emptyset$ and for all fixed $\mathbf{t}_J$, condition (10) holds whenever $\mathbf{t}_I \leq \tilde{\mathbf{t}}_I$ $[\mathbf{t}_{\{1,\ldots,\ell\}} = 0\mathbf{e}]$, and $k \epsilon \overline{I \cup J}$, then $\mathbf{X} \overset{st}{\leq} \mathbf{Y}$.*

The proof of Theorem 3.4 is similar to the proof of Theorem 3.1. Instead of starting with Step 1 one defines $\hat{X}_1 = \cdots = \hat{X}_\ell \equiv 0$ and then starts the construction (described in the proof of Theorem 3.1) in step $(\ell + 1)$.1. We omit the details.

4. Hazard Rates and the MIHR| $\mathcal{F}_t$ Property. Let $T_1, \ldots, T_n$ be nonnegative random variables to be thought of as lifetimes of components numbered $1, \ldots, n$. Let Z_i be the life indicator of component i, that is,

$$\begin{aligned} Z_i(t) &= 1 \text{ if } t < T_i, \\ &= 0 \text{ if } t \geq T_i. \end{aligned}$$

For $t \geq 0$, let $\mathcal{F}_t$ be the σ-field generated by $\{Z_i(s) : 0 \leq s \leq t, \ i = 1, \ldots, n\}$, that is,

$$\mathcal{F}_t = \sigma(Z_i(s) : 0 \leq s \leq t, \ 1 \leq i \leq n). \tag{11}$$

We will condition on sets in $\mathcal{F}_t$ of the form

$$A_t = \{\mathbf{T}_I = \mathbf{t}_I, \ \mathbf{T}_{\bar{I}} \geq t\mathbf{e}\},$$

where $I \subset \{1, \ldots, n\}$ and $t \geq M(\mathbf{t}_I)$. Thus A_t is an observed history of components $1, \ldots, n$ until time t. It describes which components are still alive at time t and the failure times of the components that are already dead at time t.

Let θ_t denote a shift by t in time and define

$$\theta_t T_i = (T_i - t)^+ = \max(T_i - t,\ 0),\ \ i = 1, \dots, n,\ \ t \geq 0.$$

Denote $\mathbf{T} = (T_1, \dots, T_n)$ and $\theta_t \mathbf{T} = (\theta_t T_1, \dots, \theta_t T_n)$.

Arjas (1981a) considered the class of multivariate increasing hazard rate (MIHR) random vectors described in the following definition.

DEFINITION 4.1. The random vector $(T_1, \dots, T_n)$ is called MIHR relative to $(\mathcal{F}_t)_{t \geq 0}$ [denoted by MIHR$|\ (\mathcal{F}_t)_{t \geq 0}$ or just by MIHR$|\ \mathcal{F}_t$] if for all $t \leq t'$ and all Borel upper sets U in R^n,

$$P\{\theta_t \mathbf{T} \epsilon U \mid \mathcal{F}_t\} \geq P\{\theta_{t'} \mathbf{T} \epsilon U \mid \mathcal{F}_{t'}\} \quad \text{a.s.} \tag{12}$$

For the random vector $\mathbf{T}$ let the conditional hazard rates be defined as in (4).

THEOREM 4.2. *Suppose that the conditional hazard rates of* $\mathbf{T}$ *satisfy:*

(i) For disjoint sets $I,\ J \subset \{1, \dots, n\}$, $J \neq \emptyset$ *and fixed* $\mathbf{t}_{I \cup J} \geq 0\mathbf{e}$, $\tilde{\mathbf{t}}_I \geq 0\mathbf{e}$, $k \epsilon \overline{I \cup J}$,

$$\begin{aligned} &\lambda_k(M(\mathbf{t}_I, \tilde{\mathbf{t}}_I, \mathbf{t}_J) + u \mid \mathbf{T}_I = \mathbf{t}_I,\ \mathbf{T}_J = \mathbf{t}_J, \cdot) \\ &\geq \lambda_k(M(\mathbf{t}_I, \tilde{\mathbf{t}}_I, \mathbf{t}_J) + u \mid \mathbf{T}_I = \tilde{\mathbf{t}}_I, \cdot),\ u \geq 0. \end{aligned} \tag{13}$$

(ii) For disjoint sets $I,\ J \subset \{1, \dots, n\}$ *and fixed* $k \epsilon \overline{I \cup J}$, $t \leq t'$, $\mathbf{t}_I \leq t\mathbf{e}$, $\tilde{\mathbf{t}}_J \geq t\mathbf{e}$ *and* $\mathbf{t}_J \geq t\mathbf{e}$ *such that* $\tilde{\mathbf{t}}_J - t\mathbf{e} \geq \mathbf{t}_J - t'\mathbf{e}$,

$$\begin{aligned} &\lambda_k(t' + M(\tilde{\mathbf{t}}_J) - t + u \mid \mathbf{T}_I = \mathbf{t}_I,\ \mathbf{T}_J = \mathbf{t}_J, \cdot) \\ &\geq \lambda_k(M(\tilde{\mathbf{t}}_J) + u \mid \mathbf{T}_I = \mathbf{t}_I,\ \mathbf{T}_J = \tilde{\mathbf{t}}_J, \cdot),\ \ u \geq 0. \end{aligned} \tag{14}$$

Then $\mathbf{T}$ *is MIHR*$|\ \mathcal{F}_t$.

REMARKS 4.3.

(a) Condition (i) means that the smaller is the working set, the larger are the instantaneous failure rates of the surviving components.

(b) In Condition (ii) two histories are compared. The 'future' of one starts at $M(\tilde{\mathbf{t}}_J) = t + M(\tilde{\mathbf{t}}_J) - t$ and the future of the other starts at $t' + M(\tilde{\mathbf{t}}_J) - t$. In both histories $\mathbf{t}_I$ are identical. The (known) $\mathbf{T}_J$ may be different, however the (known) $\mathbf{T}_J$ in the respective histories satisfy $\theta_t \tilde{\mathbf{t}}_J \geq \theta_{t'} \mathbf{t}_J$. Condition (ii) states that then $\theta_{M(\tilde{\mathbf{t}}_J)} \mathbf{T}_{\overline{I \cup J}}$ (which are the 'future' of one history) have smaller instantaneous failure rates than $\theta_{t' + M(\tilde{\mathbf{t}}_J) - t} \mathbf{T}_{\overline{I \cup J}}$ [which are the 'future' of the other history].

(c) Substituting $t = 0$, $\tilde{\mathbf{t}}_J = \mathbf{t}_J$, $I = \emptyset$ in (14) we see that for every set $J \subset \{1, \ldots, n\}$ and every $t' \geq 0$ and $k \epsilon J$,

$$\begin{aligned} & \lambda_k(M(\mathbf{t}_J) + t' + u \mid \mathbf{T}_J = \mathbf{t}_J, \cdot) \\ (15) \qquad & \geq \lambda_k(M(\mathbf{t}_J) + u \mid \mathbf{T}_J = \mathbf{t}_J, \cdot), \quad u \geq 0. \end{aligned}$$

That is, all the conditional hazard rate functions are increasing between failures. Our condition (ii) requires more than this local IHR property.

(d) In the proof of Theorem 4.2, Condition (i) is used to compare two conditional hazard rates given (two) histories in which different numbers of components are known to have failed already. In contrast, Condition (ii) is used to compare two conditional failure rates given (two) histories in both of which the same components (indexed by $i \epsilon I \cup J$) have already failed and the components indexed by $i \epsilon \overline{I \cup J}$ are still alive.

(e) If $T_1, \ldots, T_n$ are independent, absolutely continuous IHR random variables then it is easily seen that (i) and (ii) hold. Thus Theorem 4.2 agrees with a result of Arjas (1981a) which states that independent IHR random variables are MIHR$|\, \mathcal{F}_t$.

PROOF OF THEOREM 4.2.: Fix t and t' ($t' \geq t \geq 0$). Let

$$(16) \qquad D_t = \{\mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_{\bar{B}} \geq t\mathbf{e}\}$$

$$(17) \qquad E_{t'} = \{\mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_C = \mathbf{t}_C,\ T_{\overline{B \cup C}} \geq t'\mathbf{e}\}$$

where B, $C \subset \{1, \ldots, n\}$ are disjoint and $\mathbf{t}_B \leq t\mathbf{e}$, $t\mathbf{e} < \mathbf{t}_C \leq t'\mathbf{e}$. Denote the cardinality of $\bar{B}$ by $\tilde{n}(\leq n)$. Notice that given D_t [respectively, $E_{t'}$], $\theta_t \mathbf{T}_B = 0\mathbf{e}$ [respectively, $\theta_{t'} \mathbf{T}_B = 0\mathbf{e}$]. Let $\mathbf{V}$ [respectively, $\mathbf{W}$] be an $\tilde{n}$-dimensional random vector distributed according to the conditional distribution of $\theta_t \mathbf{T}_{\bar{B}}$ given D_t [respectively, $\theta_{t'} \mathbf{T}_{\bar{B}}$ given $E_{t'}$]. We will show that

$$(18) \qquad \mathbf{V} \overset{\text{st}}{\geq} \mathbf{W}$$

and then (12) follows.

Notice that when $C \neq \emptyset$ then, with probability one, $\mathbf{W}_C = 0\mathbf{e}$ whereas $\mathbf{V}_C \geq 0\mathbf{e}$. Hence in order to prove (18) we will use Theorem 3.1 when $C = \emptyset$ and Theorem 3.4 when $C \neq \emptyset$.

Given, for some $I \subset \bar{B}$, that $\mathbf{V}_I = \mathbf{s}_I$ and $\mathbf{V}_{\overline{I \cup B}} \geq s\mathbf{e}$ (where $s\mathbf{e} \geq \mathbf{s}_I$), the conditional hazard rate of V_k at s (where $k \epsilon \overline{I \cup B}$) is

$$\begin{aligned} & \eta_k(s \mid \mathbf{V}_I = \mathbf{s}_I,\ \mathbf{V}_{\overline{B \cup I}} \geq s\mathbf{e}) \\ (19) \qquad & = \lambda_k(t + s \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_I = \mathbf{s}_I + t\mathbf{e}, \cdot),\ s \geq M(\mathbf{s}_I), \end{aligned}$$

where λ_k is the conditional hazard rate function of T_k as defined in (4). Similarly, given, for some $i \subset \overline{B \cup C}$, that $\mathbf{W}_I = \mathbf{s}_I$ and $\mathbf{W}_{\overline{I \cup B \cup C}} \geq s\mathbf{e}$ (where $s\mathbf{e} \geq \mathbf{s}_I$), the conditional hazard rate of W_k at s (where $k \epsilon \overline{I \cup B \cup C}$) is

$$\begin{aligned} &\mu_k(s \mid \mathbf{W}_I = \mathbf{s}_I,\ \mathbf{W}_{\overline{B \cup C \cup I}} \geq s\mathbf{e}) \\ &= \lambda_k(t' + s \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_C = \mathbf{t}_C,\ \mathbf{T}_I = \mathbf{s}_I + t'\mathbf{e}, \\ &\mathbf{T}_{\overline{B \cup C \cup I}} \geq (s + t')\mathbf{e}),\ s \geq M(\mathbf{s}_I). \end{aligned} \tag{20}$$

In order to prove (18) we will show that the η_k's and μ_k's defined in (19) and (20) satisfy the $\tilde{n}$-dimensional version of (10), that is, for all disjoint sets I, $J \subset \bar{B}$, such that $\bar{B} \cap (\overline{I \cup J}) \neq \emptyset$ and $(I \cup J) \supset C$,

$$\begin{aligned} &\mu_k(M(\mathbf{t}_J,\ \tilde{\mathbf{t}}_I) + u \mid \mathbf{W}_I = \mathbf{t}_I,\ \mathbf{W}_J = \mathbf{t}_J,\ W_{\overline{B \cup I \cup J}} \geq (M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u)\mathbf{e}) \\ &\geq \eta_k(M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u \mid \mathbf{V}_I = \tilde{\mathbf{t}}_I,\ V_{\overline{B \cup I}} \geq (M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u)\mathbf{e}), \end{aligned} \tag{21}$$

whenever $\mathbf{t}_I \leq \tilde{\mathbf{t}}_I$, $u \geq 0$ and $k \epsilon \overline{I \cup J \cup B}$.

Two cases will be considered.

Case 1: $C = \emptyset$. First we show that (21) holds when $J \neq \emptyset$. Let I and J be as in (21) and let $\mathbf{t}_I \leq \tilde{\mathbf{t}}_I$, $u \geq 0$ and $k \epsilon \overline{I \cup J \cup B}$. Then, from (19) and (20) [here LHS and RHS stand for 'left hand side' and 'right hand side'],

$$\begin{aligned} \text{LHS(21)} &= \lambda_k(M(t'\mathbf{e} + \mathbf{t}_J,\ t'\mathbf{e} + \tilde{\mathbf{t}}_I) + u \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_I = \mathbf{t}_I + t'\mathbf{e}, \\ &\qquad \mathbf{T}_J = \mathbf{t}_J + t'\mathbf{e},\ \cdot), \\ \text{RHS(21)} &= \lambda_k(M(t\mathbf{e} + \mathbf{t}_J,\ t\mathbf{e} + \tilde{\mathbf{t}}_I) + u \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_I = \tilde{\mathbf{t}}_I + t\mathbf{e},\ \cdot). \end{aligned}$$

Hence

$$\begin{aligned} &\text{RHS(21)} \leq \lambda_k(M(t'\mathbf{e} + \mathbf{t}_J, t'\mathbf{e} + \tilde{\mathbf{t}}_I) + u \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_I = \tilde{\mathbf{t}}_I + t\mathbf{e},\ \cdot) \\ &\leq \text{LHS(21)}, \end{aligned} \tag{22}$$

where the first inequality follows from (15) and the second from (13).

Now we show that (21) holds also when $J = \emptyset$ (recall that we still assume $C = \emptyset$). Let $I \subset \bar{B}$,

$$\mathbf{t}_I \leq \tilde{\mathbf{t}}_I, \tag{23}$$

and $k \epsilon \overline{I \cup B}$. Then

$$\begin{aligned} &\text{LHS(21)} = \mu_k(M(\tilde{\mathbf{t}}_I) + u \mid \mathbf{W}_I = \mathbf{t}_I,\ \mathbf{W}_{\overline{I \cup B}} \geq (M(\tilde{\mathbf{t}}_I) + u)\mathbf{e}) \\ &= \lambda_k(M(t'\mathbf{e} + \tilde{\mathbf{t}}_I) + u \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_I = t'\mathbf{e} + \mathbf{t}_I,\ \cdot), \end{aligned} \tag{24}$$

$$
\begin{aligned}
\mathrm{RHS}(21) &= \eta_k(M(\tilde{\mathbf{t}}_I) + u \mid \mathbf{V}_I = \tilde{\mathbf{t}}_I,\ \mathbf{V}_{\overline{I \cup B}} \geq (M(\tilde{\mathbf{t}}_I) + u)\mathbf{e}) \\
&= \lambda_k(M(t\mathbf{e} + \tilde{\mathbf{t}}_I) + u \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_I = t\mathbf{e} + \tilde{\mathbf{t}}_I,\ \cdot), \qquad (25)
\end{aligned}
$$

Now, in (14) plug:

$$
\begin{array}{rl}
B & \text{in place of } I, \\
I & \text{in place of } J, \\
\mathbf{t}_I + t'\mathbf{e} & \text{in place of } \mathbf{t}_J, \\
\tilde{\mathbf{t}}_I + t\mathbf{e} & \text{in place of } \tilde{\mathbf{t}}_J.
\end{array} \qquad (26)
$$

The resulting LHS (14) is equal to (24) and the resulting RHS (14) is equal to (25). By assumption, (14) holds if t, t', $\mathbf{t}_J$ and $\tilde{\mathbf{t}}_J$ *there* (in (14)) satisfy $\tilde{\mathbf{t}}_J - t\mathbf{e} \geq \mathbf{t}_J - t'\mathbf{e}$. This inequality translates (through substitution (26)) to $\mathbf{t}_I \leq \tilde{\mathbf{t}}_I$ *here* (in (24) and (25)); the latter is true by (23). Thus, from (ii) we obtain that LHS (21) [i.e., (24)] $\geq$ RHS (21) [i.e., (25)].

Case 2: $C \neq \emptyset$. Let $I,\ J \subset \bar{B}$ be disjoint sets such that $\bar{B} \cap (\overline{I \cup J}) \neq \emptyset$ and $(I \cup J) \supset C$. In LHS (21) we only have to condition on $\mathbf{W}_{I \cap \bar{C}}$ and $\mathbf{W}_{J \cap \bar{C}}$ because $\mathbf{W}_C = 0$. Let $\tilde{\mathbf{t}}_{I \cap C} \geq 0\mathbf{e}$, $\mathbf{t}_{J \cap \bar{C}} \geq 0\mathbf{e}$, $\tilde{\mathbf{t}}_{I \cap \bar{C}} \geq \mathbf{t}_{I \cap \bar{C}} \geq 0\mathbf{e}$, $u \geq 0$ and $k \epsilon \overline{I \cup J \cup B}$. Then

$$
\begin{aligned}
&\mathrm{LHS}(21) = \lambda_k(M(\mathbf{t}_{J \cap \bar{C}} + t'\mathbf{e},\ \tilde{\mathbf{t}}_{I \cap \bar{C}} + t'\mathbf{e}) + u \mid \\
&\mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_C = \mathbf{t}_C,\ \mathbf{T}_{I \cap \bar{C}} = \mathbf{t}_{I \cap \bar{C}} + t'\mathbf{e}, \\
&\mathbf{T}_{J \cap \bar{C}} = \mathbf{t}_{J \cap \bar{C}} + t'\mathbf{e},\ \cdot), \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{RHS}(21) = \lambda_k(M(\mathbf{t}_{J \cap \bar{C}} + t\mathbf{e},\ \tilde{\mathbf{t}}_{I \cap \bar{C}} \\
&+ t\mathbf{e}) + u \mid \mathbf{T}_B = \mathbf{T}_B,\ \mathbf{T}_I = \tilde{\mathbf{t}}_I + t\mathbf{e},\ \cdot). \qquad (28)
\end{aligned}
$$

If $J \neq \emptyset$ then in LHS (21) [of (27)] it is given that more components have already failed than in the condition given in RHS (21) [of (28)]. So the fact that (27) $\geq$ (28) follows from (15) and (i) as in (22).

If $J = \emptyset$ then in both (27) and (28) it is given that the same number of components have failed (though possibly at different times). For this case the proof of (27) $\geq$ (28) uses (ii) and is similar to (though notationally somewhat more involved than) the proof of (24) $\geq$ (25). We omit the details. ||

REMARK 4.4. Condition (i) of Theorem 4.2 looks simple (see Remark 4.3 (a)) but it is stronger than what is really required. A careful study of the proof of Theorem 4.2 shows that the following condition (which is weaker than (i) and (ii) combined) implies that $\mathbf{T}$ is MIHR$|\ \mathcal{F}_t$:

(iii) For disjoint sets $I,\ J,\ L \subset \{1,\dots,n\}$ and fixed $k\epsilon\overline{I \cup J \cup L}$, $t \le t'$, $\mathbf{t}_I \le t\mathbf{e}$, $\tilde{\mathbf{t}}_J \ge t\mathbf{e}$, $\mathbf{t}_J \ge t\mathbf{e}$ and $\mathbf{t}_L \ge t\mathbf{e}$ such that $\tilde{\mathbf{t}}_J - t\mathbf{e} \ge \mathbf{t}_J - t'\mathbf{e}$ (L may be empty),

$$\begin{aligned} &\lambda_k(t' - t + M(\tilde{\mathbf{t}}_J,\ \mathbf{t}_L) + u \mid \mathbf{T}_I = \mathbf{t}_I,\ \mathbf{T}_J = \mathbf{t}_J,\ \mathbf{T}_L = \mathbf{t}_L,\ \cdot) \\ &\ge \lambda_k(M(\tilde{\mathbf{t}}_J,\ \mathbf{t}_L) + u \mid \mathbf{T}_I = \mathbf{t}_I,\ \mathbf{T}_J = \tilde{\mathbf{t}}_J,\ \cdot),\quad u \ge 0. \end{aligned} \tag{29}$$

To see that indeed (iii) is weaker than (i) and (ii) combined, note that if $L \neq \emptyset$ then (29) follows from (13) and (15). If $L = \emptyset$ then (29) is the same as (14).

5. Hazard Rates and the WBF Property. Let $T_1,\dots,T_n$ be nonnegative random lifetimes as in Section 4. Fix $t > 0$, $B \subset \{1,\dots,n\}$ [such that $\bar{B} \neq \emptyset$], $\ell\epsilon\bar{B}$ and $\mathbf{t}_B \le t\mathbf{e}$. Consider the two histories

$$D = \{\mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_{\bar{B}} \ge t\mathbf{e}\}, \tag{30}$$

$$E = \{\mathbf{T}_B = \mathbf{t}_B,\ T_\ell = t,\ T_{\bar{B}-\{\ell\}} \ge t\mathbf{e}\}. \tag{31}$$

Let $\mathbf{V}$ [respectively, $\mathbf{W}$] be distributed according to the conditional distribution of $\theta_t \mathbf{T}_{\bar{B}}$ given D [respectively, E]. Arjas and Norros (1984) studied random vectors $\mathbf{T} = (T_1,\dots,T_n)$ which have the property given in the following definition.

Definition 5.1. The random vector $\mathbf{T} = (T_1,\dots,T_n)$ is said to be *weakened by failures* (WBF) if for all $t \ge 0$, $B \subset \{1,\dots,n\}$ [such that $\bar{B} \neq \emptyset$], $\ell\epsilon\bar{B}$ and $\mathbf{t}_B \le t\mathbf{e}$, the random vectors $\mathbf{V}$ and $\mathbf{W}$ satisfy

$$\mathbf{V} \overset{\text{st}}{\ge} \mathbf{W}. \tag{32}$$

Theorem 5.2. *Suppose the conditional hazard rates of* $\mathbf{T}$ *satisfy (i) of Theorem 4.2 and*

(iv) For every set $I \subset \{1,\dots,n\}$ and fixed $\mathbf{t}_I$, $\tilde{\mathbf{t}}_I$ [such that $\mathbf{t}_I \le \tilde{\mathbf{t}}_I$] and $k\epsilon\bar{I}$,

$$\lambda_k(M(\tilde{\mathbf{t}}_I) + u \mid \mathbf{T}_I = \mathbf{t}_I,\ \cdot) \ge \lambda_k(M(\tilde{\mathbf{t}}_I) + u \mid \mathbf{T}_I = \tilde{\mathbf{t}}_I,\ \cdot),\quad u \ge 0.$$

Then $\mathbf{T}$ *has the WBF-property.*

Remark 5.3. In Condition (iv) two histories of the same length and with the same number of failures are compared. The history with the earlier failure times yields higher failure rates for the surviving components.

Proof of Theorem 5.2: Suppose the cardinality of $\bar{B}$ is $\tilde{n}(< n)$. Since $W_\ell = 0$ with probability one, whereas $V_\ell \ge 0$, use will be made of Theorem 3.4.

For $I \subset \bar{B}$, given $\mathbf{V}_I = \mathbf{s}_I$ and $V_{\overline{I \cup B}} \ge s\mathbf{e}$ (where $s \ge M(\mathbf{s}_I)$), the conditional hazard rate of V_k at time s, where $k\epsilon\overline{B \cup I}$, is

$$\eta_k(s \mid \mathbf{V}_I = \mathbf{s}_I,\ \mathbf{V}_{\overline{B \cup I}} \geq s\mathbf{e}) = \lambda_k(t+s \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_I = \mathbf{s}_I + t\mathbf{e},\ \cdot), \tag{33}$$

where λ_k is defined as in (4). Similarly, for $I \subset \overline{B \cup \{\ell\}}$, given $\mathbf{W}_I = \mathbf{s}_I$, $W_{\overline{B \cup I \cup \{\ell\}}} \geq s\mathbf{e}$ (where $s \geq M(\mathbf{s}_I)$), the conditional hazard rate of W_k at s, where $k \epsilon \overline{B \cup I \cup \{\ell\}}$ is

$$\begin{aligned}&\mu_k(s \mid \mathbf{W}_I = \mathbf{s}_I,\ \mathbf{W}_{\overline{B \cup I \cup \{\ell\}}} \geq s\mathbf{e})\\ &= \lambda_k(t+s \mid \mathbf{T}_B = \mathbf{t}_B,\ T_\ell = t,\ \mathbf{T}_I = \mathbf{s}_I + t\mathbf{e},\ \cdot).\end{aligned} \tag{34}$$

In order to prove (32) we will show that the η_k's and μ_k's defined in (33) and (34) satisfy the $\tilde{n}$-dimensional version of (10) required in Theorem 3.4. That is, for disjoint sets $I,\ J \subset \bar{B}$, such that $\bar{B} \cap (\overline{I \cup J}) \neq \emptyset$ and $\ell \epsilon I \cup J$,

$$\begin{aligned}&\mu_k(M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u \mid \mathbf{W}_I = \mathbf{t}_I,\ \mathbf{W}_J = \mathbf{t}_J,\ \mathbf{W}_{\overline{B \cup I \cup J}} \geq (M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u)\mathbf{e})\\ &\geq \eta_k(M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u \mid \mathbf{V}_I = \tilde{\mathbf{t}}_I,\ \mathbf{V}_{\overline{B \cup I}} \geq (M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + u)\mathbf{e}),\end{aligned} \tag{35}$$

whenever $\mathbf{t}_I \leq \tilde{\mathbf{t}}_I$, $u \geq 0$, $k \epsilon \overline{I \cup J \cup B}$. Since $\ell \epsilon I \cup J$, the LHS (35) is well defined only if $t_\ell = 0$, $\tilde{t}_\ell \geq 0$.

If $\ell \notin I$ then $J \neq \emptyset$ because $\ell \epsilon J$. Then from (i) it follows that

$$\begin{aligned}&\lambda_k(M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + t + u \mid \mathbf{T}_B, = \mathbf{t}_B,\ \mathbf{T}_I = \mathbf{t}_I + t\mathbf{e},\ \mathbf{T}_J = \mathbf{t}_J + t\mathbf{e}, \cdot)\\ &\geq \lambda_k(M(\mathbf{t}_J, \tilde{\mathbf{t}}_I) + t + u \mid \mathbf{T}_B, = \mathbf{t}_B,\ \mathbf{T}_I = \mathbf{t}_I + t\mathbf{e},\ \cdot).\end{aligned} \tag{36}$$

But in this case $[J \neq \emptyset]$ (36) is equivalent to (35).

If $\ell \epsilon I$ and $J \neq \emptyset$ then, in a similar manner, one can again obtain (35) using (i).

If $J = \emptyset$ (then of course $\ell \epsilon I$) then (35) is equivalent to

$$\begin{aligned}&\lambda_k(M(\tilde{\mathbf{t}}_I) + t + u \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_I = \mathbf{t}_I + t\mathbf{e},\ \cdot)\\ &\geq \lambda_k(M(\tilde{\mathbf{t}}_I) + t + u \mid \mathbf{T}_B = \mathbf{t}_B,\ \mathbf{T}_I = \tilde{\mathbf{t}}_I + t\mathbf{e},\ \cdot).\end{aligned} \tag{37}$$

But (37) follows from (iv).

Thus (35) holds for every choice of $I,\ J \subset \bar{B}$ [such that $\bar{B} \cap (\overline{I \cup J}) \neq \emptyset$ and $\ell \epsilon (I \cup J)$], $\mathbf{t}_I \leq \tilde{\mathbf{t}}_I$, $u \geq 0$ and $k \epsilon \overline{I \cup J \cup B}$ and the stochastic comparison (32) now follows from Theorem 3.4. ||

From Theorem 1 of Arjas and Norros (1984) it follows that if an absolutely continuous $\mathbf{T}$ has the WBF property then $\mathbf{T}$ is a vector of associated random variables (in the sense of Esary, Proschan, and Walkup (1967)). Thus from Theorem 5.2 we obtain:

THEOREM 5.4. *Suppose the conditional hazard rate functions of* $\mathbf{T}$ *satisfy (i) of Theorem 4.2 and (iv) of Theorem 5.2. Then* $\mathbf{T}$ *is a vector of associated random variables.*

REMARK 5.5. In the proof of Theorem 5.2, Condition (i) is applied only with $\mathbf{t}_I \leq \tilde{\mathbf{t}}_I$. Thus the conclusions of Theorems 5.2 and 5.4 are valid under the following condition (which is weaker than (i) and (iv) combined):

(v) For disjoint sets I, $J \subset \{1, \ldots, n\}$ and fixed $\mathbf{t}_I$, $\tilde{\mathbf{t}}_I$, $\mathbf{t}_J$ [such that $\mathbf{t}_I \leq \tilde{\mathbf{t}}_I$] and $k \epsilon \overline{I \cup J}$ (J may be empty),

$$\lambda_k(M(\tilde{\mathbf{t}}_I, \mathbf{t}_J) + u \mid \mathbf{T}_I = \mathbf{t}_I, \ \mathbf{T}_J = \mathbf{t}_J \ \cdot) \\ \geq \lambda_k(M(\tilde{\mathbf{t}}_I, \mathbf{t}_J) + u \mid \mathbf{T}_I = \tilde{\mathbf{t}}_I, \ \cdot), u \geq 0.$$

The results of Sections 4 and 5 can be applied to various stochastic models. We will not give details of the applications here because they are similar to the applications given in Section 6 in Shaked and Shanthikumar (1987a).

Appendix: Proof of Theorem 3.1. Using the dynamic construction described in Section 2 we will construct simultaneously two random vectors $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ on a common probability space such that

$$\hat{\mathbf{X}} \stackrel{\text{st}}{=} \mathbf{X}, \tag{38}$$

$$\hat{\mathbf{Y}} \stackrel{\text{st}}{=} \mathbf{Y}, \tag{39}$$

and

$$\hat{\mathbf{X}} \leq \mathbf{Y} \text{ with probability one.} \tag{40}$$

Then, for every increasing Borel measurable function g,

$$Eg(\mathbf{X}) = Eg(\hat{\mathbf{X}}) \leq Eg(\hat{\mathbf{Y}}) = Eg(\mathbf{Y}),$$

provided the expectations exist, and the desired result then follows from (1).

We describe the construction of $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ according to the steps (Step 1 through Step n) of the construction of $\hat{\mathbf{X}}$.

Step 1: Consider n independent nonhomogeneous Poisson processes on $[0, \infty)$ indexed by $i \epsilon \{1, \ldots, n\}$ with intensity functions $\mu_i(t \mid \mathbf{X} \geq t\mathbf{e})$, $t \geq 0$, $i = 1, \ldots, n$. If Process j_1 yields the first epoch (out of all n processes) at time t_{j_1}, say, then let this time be $\hat{X}_{j_1}$. Also, with probability $\eta_{j_1}(t_{j_1} \mid \mathbf{Y} \geq t_{j_1}\mathbf{e}) / \mu_{j_1}(t_{j_1} \mid \mathbf{X} \geq t_{j_1}\mathbf{e})$ let this time, t_{j_1}, be also $\hat{Y}_{j_1}$ and with probability $1 - \eta_{j_1}(t_{j_1} \mid \mathbf{Y} \geq t_{j_1}\mathbf{e}) / \mu_{j_1}(t_{j_1} \mid \mathbf{X} \geq t_{j_1}\mathbf{e})$ delay the determination of $\hat{Y}_{j_1}$ for a later step. If $\hat{X}_{j_1} = \hat{Y}_{j_1}$ then go to Step 2.2. If $\hat{Y}_{j_1}$ has not yet been determined then go to Step 2.1.

The proof of Theorem 3.1 continues after Remark A.1.

REMARK A.1.

(a) For Step 1 to be sensible it is required that

$$\eta_{j_1}(t_{j_1} \mid \mathbf{Y} \geq t_{j_1}\mathbf{e})/\mu_{j_1}(t_{j_1} \mid \mathbf{X} \geq t_{j_1}\mathbf{e}) \leq 1, \text{ but this follows from (10).}$$

(b) At the conclusion of Step 1, $\hat{X}_{j_1}$ has been determined (and perhaps also $\hat{Y}_{j_1}$) and

$$\hat{X}_{j_1} \leq \hat{Y}_{j_1} \text{ with probability one,} \tag{*.1}$$

because either $\hat{Y}_{j_1} = \hat{X}_{j_1}$ or $\hat{Y}_{j_1}$ is going to be determined at (and then be equal to) some time after t_{j_1}.

(c) The next steps will be indexed by a pair $m.\ell$ ($\ell \leq m$). For $1 \leq \ell < m \leq n$, Step $m.\ell$ produces one of the following:

(α) Generates just the m-th $\hat{X}$ (and the procedure then proceeds to Step $(m+1).\ell$).

(β) Generates just the ℓ-th $\hat{Y}$ (and the procedure then proceeds to Step $m.(\ell+1)$).

(γ) Generates both the m-th $\hat{X}$ and the ℓ-th $\hat{Y}$ (and the procedure then proceeds to Step $(m+1).(\ell+1)$).

For $\ell = m\epsilon\{1,\ldots,n\}$, Step $m.m$ produces either (α) or (γ). For $\ell < m = n+1$, Step $(n+1).\ell$ produces only (β). The procedure ends upon entrance to Step $(n+1).(n+1)$ which is vacuous.

PROOF OF THEOREM 3.1 (continued):

Step 2.1: It is given, for some $j_1\epsilon\{1,\ldots,n\}$ and $t_{j_1} \geq 0$, that $\hat{X}_{j_1} = t_{j_1}$, $\hat{\mathbf{X}}_{\{1,\ldots,n\}-\{j_1\}} \geq t_{j_1}\mathbf{e}$ and $\hat{\mathbf{Y}} \geq t_{j_1}\mathbf{e}$. Consider $(n-1)+1$ independent nonhomogeneous Poisson processes on $[t_{j_1},\infty)$. Let the first $n-1$ processes be called *processes of type 1* and call the last one a *process of type 2.* The $n-1$ type 1 processes are indexed by $i\epsilon I \equiv \{1,\ldots,n\} - \{j_1\}$. For $i\epsilon I$ let the intensity function of the type 1 process i be $\mu_i(t \mid X_{j_1} = t_{j_1}, \mathbf{X}_I \geq t\mathbf{e})$, $t \geq t_{j_1}$. Let the intensity of the type 2 process be $\eta_{j_1}(t \mid \mathbf{Y} \geq t\mathbf{e})$, $t \geq t_{j_1}$. If the type 2 process yields the first epoch (out of all n processes) at time $\tilde{t}_{j_1}$, say, then let the time of this epoch be $\hat{Y}_{j_1}$ and go to Step 2.2. If the type 1 process $j_2(\epsilon I)$ yields the first epoch (out of all n processes) at time t_{j_2}, say, then let the time of this epoch be $\hat{X}_{j_2}$. Also, in this case, with probability $\eta_{j_2}(t_{j_2} \mid \mathbf{Y} \geq t_{j_2}\mathbf{e})/\mu_{j_2}(t_{j_2} \mid X_{j_1} = t_{j_1}, \mathbf{X}_I \geq t_{j_2}\mathbf{e})$ [which is ≤ 1 by (10)] let the time of this epoch, t_{j_2}, be also $\hat{Y}_{j_2}$ and go to Step 3.2. With probability $1 - \eta_{j_2}(t_{j_2} \mid \mathbf{Y} \geq t_{j_2}\mathbf{e})/\mu_{j_2}(t_{j_2} \mid X_{j_1} = t_{j_1}, \mathbf{X}_I \geq t_{j_2}\mathbf{e})$ delay the determination of $\hat{Y}_{j_2}$ for a later step and go to Step 3.1.

Step 2.2: It is given, for some $j_1 \epsilon \{1, \ldots, n\}$ and fixed $t_{j_1} \le \tilde{t}_{j_1}$, that $\hat{X}_{j_1} = t_{j_1}$, $\hat{Y}_{j_1} = \tilde{t}_{j_1}$, $\mathbf{X}_{\{1,\ldots,n\}-\{j_1\}} \ge \tilde{t}_{j_1}\mathbf{e}$ and $\mathbf{Y}_{\{1,\ldots,n\}-\{j_1\}} \ge \tilde{t}_{j_1}\mathbf{e}$. Consider $n-1$ independent Poisson processes on $[\tilde{t}_{j_1}, \infty)$ indexed by $i\epsilon I = \{1, \ldots, n\} - \{j_1\}$. For $i \epsilon I$ let the intensity function of Process i be $\mu_i(t \mid \mathbf{X}_{j_1} = t_{j_1}, \mathbf{X}_I \ge t\mathbf{e})$, $t \ge \tilde{t}_{j_1}$. If Process j_2 yields the first epoch (out of all the $n-1$ processes) at time t_{j_2}, say, then let the time of this epoch be $\hat{X}_{j_2}$. Also with probability $\eta_{j_2}(t_{j_2} \mid Y_{j_1} = \tilde{t}_{j_1}, \mathbf{Y}_I \ge t_{j_2}\mathbf{e})/\mu_{j_2}(t_{j_2} \mid X_{j_1} = t_{j_1}, \mathbf{X}_I \ge t_{j_2}\mathbf{e})$ let the time of this epoch also be $\hat{Y}_{j_2}$ and with probability $1 - \eta_{j_2}(t_{j_2} \mid Y_{j_1} = \tilde{t}_{j_1}, \mathbf{Y}_I \ge t_{j_2}\mathbf{e})/\mu_{j_2}(t_{j_2} \mid X_{j_1} = t_{j_1}, \mathbf{X}_I \ge t_{j_2}\mathbf{e})$ delay the determination of $\hat{Y}_{j_2}$ for a later step. Again, the last sentence is sensible because, by (10), $\eta_{j_2}(t_{j_2} \mid Y_{j_1} = \tilde{t}_{j_1}, \mathbf{Y}_I \ge t_{j_2}\mathbf{e})/\mu_{j_2}(t_{j_2} \mid X_{j_1} = t_{j_1}, \mathbf{X}_I \ge t_{j_2}\mathbf{e}) \le 1$. If $\hat{X}_{j_2} = \hat{Y}_{j_2}$ then go to Step 3.3. If $\hat{Y}_{j_2}$ has not yet been determined then go to Step 3.2.

The proof of Theorem 3.1 continues after Remark A.2.

REMARK A.2. After the conclusion of Step 2 (that is, just after the last step of the form 2.i, for some $i\epsilon\{1,2\}$, has been executed) $\hat{X}_{j_1}$ and $\hat{X}_{j_2}$ (and perhaps also $\hat{Y}_{j_1}$ and $\hat{Y}_{j_2}$) have been determined. In addition to (*.1) we also have

$$(*.2) \qquad \hat{X}_{j_2} \le \hat{Y}_{j_2} \text{ with probability one.}$$

PROOF OF THEOREM 3.1 (continued):

STEP $(m+1).(\ell+1)$ [for $\ell \le m \le n$, $\ell < n$]: It is given, for some $I = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, n\}$, $J = \{j_{\ell+1}, \ldots, j_m\} \subset \{1, \ldots, n\}$ [such that $I \cap J = \emptyset$] and for some fixed $0\mathbf{e} \le \mathbf{t}_I \le \tilde{\mathbf{t}}_I$ and $\mathbf{t}_J \ge 0\mathbf{e}$, that $\mathbf{X}_I = \mathbf{t}_I$, $\mathbf{Y}_I = \tilde{\mathbf{t}}_I$, $\mathbf{X}_J = \mathbf{t}_J$, $\mathbf{X}_{\overline{I \cup J}} \ge (M(\mathbf{t}_J, \tilde{\mathbf{t}}_I))\mathbf{e}$ and $\mathbf{Y}_I \ge (M(\mathbf{t}_J, \tilde{\mathbf{t}}_I))\mathbf{e}$ [note that if $\ell = m$ then $J = \emptyset$ and if $m+1 = n+1$ then $\overline{I \cup J} = \emptyset$]. Consider $(n-m)$ nonhomogeneous Poisson processes of type 1 indexed by $k\epsilon\overline{I \cup J}$ and $(m-\ell)$ nonhomogeneous Poisson processes of type 2 indexed by $j\epsilon J$. All processes are independent and on $[M(\mathbf{t}_J, \tilde{\mathbf{t}}_I), \infty)$. For $k\epsilon\overline{I \cup J}$ let the intensity function of type 1 process k be $\mu_k(t \mid \mathbf{X}_I = \mathbf{t}_I, \mathbf{X}_J = \mathbf{t}_J, \cdot)$. For $j\epsilon J$ let the intensity function of type 2 process j be $\eta_j(t \mid \mathbf{Y}_I = \tilde{\mathbf{t}}_I, \cdot)$. If type 2 process $j_{\ell+1}\epsilon J$ yields the first epoch (out of all $n-\ell = (n-m)+(m-\ell)$ processes) at time $\tilde{t}_{j_{\ell+1}}$, say, then let the time of this epoch be $\hat{Y}_{j_{\ell+1}}$ and go to set $(m+1).(\ell+2)$. If type 1 process $j_{m+1}\epsilon\overline{I \cup J}$ yields the first epoch (out of all $n-\ell$ processes) at time $t_{j_{m+1}}$, say, then let the time of this epoch be $\hat{X}_{j_{m+1}}$. Also, in this case, with probability $\eta_{j_{m+1}}(t_{j_{m+1}} \mid \mathbf{Y}_I = \tilde{\mathbf{t}}_I, \cdot)/\mu_{j_{m+1}}(t_{j_{m+1}} \mid \mathbf{X}_I = \mathbf{t}_I, \mathbf{X}_J = \mathbf{t}_J, \cdot)$ [which is ≤ 1 by (10)] let the time of this epoch, $t_{j_{m+1}}$, be also $Y_{j_{m+1}}$ and go to Step $(m+2).(\ell+2)$. With probability $1 - \eta_{j_{m+1}}(t_{j_{m+1}} \mid \mathbf{Y}_I = \tilde{\mathbf{t}}_I, \cdot)/\mu_{j_{m+1}}(t_{j_{m+1}} \mid \mathbf{X}_I = \mathbf{t}_I, \mathbf{X}_J = \mathbf{t}_J, \cdot)$ delay the determination of $\hat{Y}_{j_{m+1}}$ for a later step and go to Step $(m+2).(\ell+1)$.

If $m \le n-1$, then at the conclusion of Step $m+1$ (that is, just after the last step of the form $(m+1).i$ for some $i\epsilon\{1, \ldots, m+1\}$ has been executed) $\hat{X}_{j_1}, \ldots, \hat{X}_{j_{m+1}}$ (and some of $\hat{Y}_{j_1}, \ldots, \hat{Y}_{j_{m+1}}$) have been determined. In addition to $(*.1), (*.2), \ldots, (*.m)$ we also have

(*.m+1) $\hat{X}_{j_{m+1}} \leq \hat{Y}_{j_{m+1}}$ with probability one.

Executing all the steps in sequence (the last step must be the one before entrance to Step $(n+1).(n+1)$) we obtain $(*.1),\ldots,(*.n)$. From this it follows that (40) holds. Notice that at the conclusion of Step m, the m-th $\hat{X}_j$ has been determined as described in the dynamic construction in Section 2, $m = 1, 2, \ldots, n$. That is, $\mathbf{X}$ and $\hat{\mathbf{X}}$ have the same instantaneous failure rates. Hence, by Lemma 1.1 of Shaked and Shanthikumar (1986b) we have (38). Using well known results about thinning of nonhomogeneous Poisson processes (see e.g. Savits (1988)) it is seen that for each ℓ, just after the last step of the form $i.\ell$, the ℓ-th $\hat{Y}_j$ have been determined as described in the dynamic construction in Section 2. Hence, again by Lemma 1.1 of Shaked and Shanthikumar (1986b), we have (39).

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