ROBUSTNESS OF MANN-WHITNEY-WILCOXON TEST FOR SCALE TO DEPENDENCE IN THE VARIABLES

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The asymptotic efficiency of the Mann-Whitney-Wilcoxon (MWW) test for scale relative to the likelihood ratio test for equality of exponential scale parameters is evaluated. This efficiency is studied when the underlying variables have a bivariate exponential distribution of the form due to Morgenstern (1956), Gumbel (1960), Marshall and Olkin (1967), Downton (1970), Cowan (1987), and Sarkar (1987).

1. Introduction. Serfling (1968) studied the use of the Wilcoxon test statistic when there is some dependence among the X's and among the Y's. Hollander, Pledger, and Lin (1974) showed that the two-sample Wilcoxon test is asymptotically conservative when the X's and Y's having a bivariate distribution which is positively quadrant dependent. Govindarajulu (1975) studied the sensitivity of the Mann-Whitney-Wilcoxon (MWW) test for location alternatives when X and Y are dependent having an unknown bivariate distribution with continuous marginals. In the present paper we study the sensitivity of MWW test for scale alternatives when X and Y are dependent. In particular, we evaluate the Pitman efficiency of the MWW test relative to the likelihood ratio test for scale alternatives when (X, Y) has a bivariate exponential distribution. Several bivariate exponential distributions are available in the literature. See, for instance, Basu (1986) and Sarkar (1987) for a survey of these forms. Here we select a few of the bivariate exponential forms and evaluate the Pitman efficiencies of the MWW test.

2. An Asymptotically Distribution-free Test. Let $X[Y]$ be distributed as $F[G]$ where F and G are continuous. We wish to test the null hypothesis

$$H_0 : F(x) = G(x) \text{ for all } x$$

against the alternative

$$H_1 : F(x) \geq G(x) \text{ with strict inequality for some } x.$$

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Let (X_i, Y_i), $i = 1, 2, \ldots, n$ denote a random sample of size n drawn from $H(x, y)$. Let $H_n(x, y)$, $F_n(x)$ and $G_n(y)$ respectively denote the empirical distribution functions (e.d.f.'s) based on the samples $(X_i, Y_i)(i = 1, \ldots, n), (X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$. Let $Z_{ij} = 1$ or 0 according as $X_i \leq Y_j$ or $X_i > Y_j$ respectively for $1 \leq i, j \leq n$.

Define

$$U = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} Z_{ij} = \int_{-\infty}^{\infty} F_n(x) dG_n(x).$$

Then we have the following result of Govindarajulu (1975), which was independently obtained by Hollander, Pledger, and Lin (1974).

RESULT 1. With the above notation, for all continuous F and G we have

$$\lim_{n \to \infty} P\{n^{\frac{1}{2}}(U - p)/\sigma \leq z\} = \Phi(z), \text{ for all } z,$$

where

$$p = \int F dG. \tag{1}$$

$$\begin{aligned} \sigma^2 = 2 \iint_{x<y} F(x)[1 & - F(y)] dG(x) dG(y) \\ & + 2 \iint_{x<y} G(x)[1 - G(y)] dF(x) dF(y) \\ & - 2 \iint_{-\infty}^{\infty} [H(x, y) - F(x)G(y)] dG(x) dF(y), \end{aligned} \tag{2}$$

and Φ denotes the standard normal distribution function.

PROOF. See Theorem 2.1 of Govindarajulu (1975).

One can rewrite σ^2 as

$$\sigma^2 = \int F^2 dG + \int G^2 dF - 2 \int \int H(x, y) dF(y) dG(x) - (1 - 2p)^2. \tag{3}$$

In order to test H_0 against H_1, we reject H_0 when U exceeds some $k_\alpha(\frac{1}{2} < k_\alpha < 1)$ where k_α is determined by α. Now since $F \geq G, E[F(Y) - G(Y)] = 0$ if and only if $F(Y) = G(Y)$ with probability one. Thus the test is consistent against H_1. To see this clearly, for large n we have $k_\alpha = (\frac{1}{2}) + \sigma(H_0) z_\alpha n^{-\frac{1}{2}}$ and the power of the test is $\Phi\left[\left\{(p - \frac{1}{2}) n^{\frac{1}{2}} - \sigma(H_0) z_\alpha\right\} / \sigma(H_1)\right]$ which tends to one as $n \to \infty$ since $p > \frac{1}{2}$ where $\sigma(H_0)$ and $\sigma(H_1)$ respectively denote the values of σ under H_0 and H_1. Also note that $\sigma^2(H_1) > \sigma^2(H_0) - (1 - 2p)^2$. Since σ^2 under H_0 is not free of $H(x, y)$, the test is not distribution-free. A consistent estimator of σ^2 under H_0 is given by

$$\hat{\sigma}^2 = \frac{2}{3} - 2n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} H_n(Y_i, X_j). \tag{4}$$

Thus, an asymptotically distribution-free test of H_0 against H_1 is obtained by using $\hat{\sigma}^2$ in the place of σ^2. Thus

$$k_\alpha = \frac{1}{2} - \hat{\sigma} n^{-\frac{1}{2}} \Phi^{-1}(\alpha) \text{ for large } n.$$

Certain Remarks.

(i) Let $U^* = [n(n-1)]^{-1} \sum\sum_{i \neq j} Z_{ij}$. Consider

$$U - U^* = \{-1/n^2(n-1)\} \sum_{i \neq j}\sum Z_{ij} + n^{-2} \sum_{i=1}^{n} Z_{ii}.$$

Thus

$$n^{\frac{1}{2}} |U - U^*| \leq 2/n^{\frac{1}{2}} \to 0 \text{ as } n \to \infty.$$

Hence, U^* is asymptotically equivalent to U.

(ii) Suppose that n_N is random which is independent of (X_i, Y_i), $i = 1, 2, \ldots$, and there exists a positive integer N such that n_N/N converges to λ $(0 < \lambda < \infty)$ in probability. Then

$$P[n_N^{\frac{1}{2}}(U - p)/\sigma \leq z] \to \Phi(z) \text{ as } N \to \infty.$$

(This result is useful for handling the censored samples case.)

(iii) If one wishes to test H_0 against $H_2 : F \leq G$, one should interchange the roles of X and Y in the test procedure for H_0 against H_1.

3. Parametric Competitor. In this section we assume independent exponential marginals for the distributions of X and Y and derive the likelihood ratio test procedure for testing the null hypothesis of equality of the scale parameters. Since the scale parameters of the exponential marginals are the means of the distributions, it is not inappropriate to use the MWW test for testing the equality of the scale parameters.

Let X have the distribution $F(x) = 1 - \exp(-\lambda_1 x)$, $x > 0$ and Y have the distribution function $G(x) = 1 - \exp(-\lambda_2 x)$, $x > 0$. We wish to test $H_0 : \lambda_1 = \lambda_2$ against the alternative $H_1 : \lambda_2 \neq \lambda_1$. If $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ denote random samples from F and G respectively, one can easily show that the likelihood ratio criterion is given by Λ (since $\hat{\lambda}_1 = 1/\bar{X}, \hat{\lambda}_2 = 1/\bar{Y}$ and $\hat{\lambda} = 2/(\bar{X}+\bar{Y})$ when $\lambda_1 = \lambda_2 = \lambda$) where

$$\frac{1}{4}\Lambda^{\frac{1}{n}} = \bar{X}\bar{Y}/(\bar{X}+\bar{Y})^2,\ \bar{X} = n^{-1}\sum_1^n X_i,\ \bar{Y} = n^{-1}\sum_1^n Y_i. \tag{5}$$

Suppose we reject H_0 when

$$T = n\bar{X}\bar{Y}/(\bar{X}+\bar{Y})^2 < k_\alpha \tag{6}$$

where k_α is determined by the level of significance α. Notice that $\lambda_1\bar{X}$ and $\lambda_2\bar{Y}$ tend to unity in probability as n becomes large. If Z and W denote independent standard exponential random variables and $\bar{Z}$ and $\bar{W}$ denote sample means based on random samples of size n each, then by Slutsky's theorem, T has the same asymptotic distribution as T' where

$$T' = \lambda_1\lambda_2(\lambda_1+\lambda_2)^{-2}n\bar{Z}\bar{W}. \tag{7}$$

Next we compute the Pitman efficacy of T' (and hence that of T) assuming that Z and W have a bivariate exponential distribution with standard exponential marginals (that is, having scale parameters equal to unity) and correlation coefficient ρ^*. Note that for large n, $(\sqrt{n}\bar{Z}, \sqrt{n}\bar{W})$ is bivariate normal with means $(\sqrt{n}, \sqrt{n})$, unit variances and covariance $= E(ZW) - 1 = \text{corr}(Z, W) = \rho^*$, since

$$\text{Cov}(\sqrt{n}\bar{Z}, \sqrt{n}\bar{W}) = E(n\bar{Z}\bar{W}) - n = E(ZW) + n - 1 - n.$$

We need the following lemma.

LEMMA 3.1. *If (V_1, V_2) is bivariate normal with mean (0,0), unit variances and correlation ρ^*, then*

$$\begin{aligned} &(i)\quad E(V_1^2V_2) = E(V_1V_2^2) = 0, \text{ and} \\ &(ii)\quad E(V_1^2V_2^2) = 1 + 2\rho^{*2}. \end{aligned} \tag{8}$$

PROOF. (i) follows from the fact that all moments of odd orders are equal to zero and

$$\begin{aligned} E(V_1^2V_2^2) = E\{V_1^2E(V_2^2|V_1)\} &= E[V_1^2\{(1-\rho^{*2}) + \rho^{*2}V_1^2\}] \\ &= (1-\rho^{*2}) + 3\rho^{*2} = 1 + 2\rho^{*2}. \end{aligned}$$

For computing the Pitman efficacy of the test procedure based on T, we need to evaluate

$$E(n\bar{Z}\bar{W}|H_0) = n \operatorname{Cov}(\bar{Z},\bar{W}|H_0) + n = \rho^* + n, \tag{9}$$

$$E(n\bar{Z}\bar{W}|H_1) = n \operatorname{Cov}(\bar{Z},\bar{W}|H_1) + \frac{n}{\lambda_2} = (\rho^* + n)/\lambda_2, \text{ when } \lambda_1 = 1. \tag{10}$$

Also

$$\begin{aligned} E(n^2\bar{Z}^2\bar{W}^2|H_0) &= E\{n^2(\bar{Z}-1)^2(\bar{W}-1)^2\} + 4n^2E(\bar{Z}-1)(\bar{W}-1) + \\ &\quad 2n^2E(\bar{Z}-1)^2 + n^2 \\ &= (1 + 2\rho^{*^2}) + 4n\rho^* + 2n + n^2. \end{aligned} \tag{11}$$

Hence

$$\begin{aligned} \operatorname{Var}(n\bar{Z}\bar{W}|H_0) &= 1 + 2\rho^{*^2} + 4n\rho^* + 2n + n^2 - (p^* + n)^2 = \\ &\quad 1 + \rho^{*^2} + 2n\rho^* + 2n. \end{aligned} \tag{12}$$

Now letting $\theta = \lambda_2/\lambda_1$, with $\lambda_1 = 1$ and assuming that $\theta = 1 \pm \xi/n^{\frac{1}{2}}$, we obtain the Pitman efficacy of T to be

$$\begin{aligned} e(T) \stackrel{\text{def}}{=} \lim_{n\to\infty} &\frac{[E(T'|H_1) - E(T'|H_0)]^2}{\xi^2 \operatorname{var}(T'|H_0)} \\ &= \lim_{n\to\infty} \frac{E^2(n\bar{Z}\bar{W})(\frac{1-\lambda_2}{\lambda_2})^2}{\xi^2\operatorname{var}(n\bar{Z}\bar{W})} \\ &= \lim_{n\to\infty} \frac{(\rho^*+n)^2(\xi^2)}{n\xi^2(1 \pm \xi/n^{\frac{1}{2}})^2(1+\rho^{*^2}+2\rho^*n+2n)} \\ &= \frac{1}{2(1+\rho^*)}. \end{aligned} \tag{13}$$

4. Asymptotic Efficiency with Respect to Scale Alternatives. Govindarajulu (1975) studied the asymptotic efficiency of U relative to Student's t-test against location alternatives. Here, we will evaluate the asymptotic efficiency of U relative to the T-test against scale alternatives, especially when (X, Y) has a bivariate exponential distribution of the form due to Morgenstern (1956), Gumbel (1960), Marshall and Olkin (1967), Downton (1970), Cowan (1987), and Sarker (1987). Let us assume that $G(x) = F(\theta x)$. Then we can rewrite the null and alternative hypotheses as

$$H_0 : \theta = 1 \text{ versus } H_1 : \theta < 1.$$

Furthermore, the efficacy of the U-test is

$$e(U) = (\int yf^2(y)dy)^2/A \tag{14}$$

where

$$\begin{aligned} A &= (2/3) - 2I(\rho) \\ I(\rho) &= \int_0^1 \int_0^1 H(F^{-1}(u), F^{-1}(v))du\, dv. \end{aligned} \tag{15}$$

Hence, the asymptotic efficiency of U-test relative to T-test is given by

$$e(U,T) = 2(1+\rho^*)(\int yf^2(y)dy)^2/A. \tag{16}$$

Next we will evaluate (16) when $H(x,y)$ is a bivariate exponential distribution having standard exponential distribution for the marginals. Then, computations yield

$$\int_0^\infty yf^2(y)dy = 1/4. \tag{17}$$

Then

$$e(U,F) = (1+\rho^*)/8A. \tag{18}$$

The Bivariate Exponential Distribution of Morgenstern (1956).

The joint density (in the standard form) is given by

$$h(x,y) = e^{-x-y}[1+\rho(2e^{-x}-1)(2e^{-y}-1)], \;\; x,y>0, \;\; -1<\rho<1.$$

Hence

$$H(x,y) = (1-e^{-x})(1-e^{-y})(1+\rho e^{-x-y})$$

and the correlation between X and $Y = \rho^* = \rho/4$. Hence

$$H(F^{-1}(u), F^{-1}(v)) = uv\{1+\rho(1-u)(1-v)\}.$$

Thus

$$\begin{aligned} I(\rho) &= \int_0^1 \int_0^1 H(F^{-1}(u), F^{-1}(v))du\, dv \\ &= (1/4) + (\rho/36); \\ A(\rho) &= (2/3) - 2I(\rho) = (3-\rho)/18. \end{aligned}$$

So

$$e(U,t) = 9(4+\rho)/16(3-\rho). \tag{19}$$

Table 4.1. Values of $e(U,T)$

ρ	-1	-0.5	0	0.5	1
$e(U,T)$	0.42	0.56	0.75	1.01	1.41

Bivariate Exponential Distribution of Gumbel (1960).
The joint density (in standard form) is

$$h(x,y) = e^{-x-y-\rho xy}\{(1+\rho x)(1+\rho y) - \rho\},\ \ x,y > 0,\ \rho > 0.$$

Then

$$H(x,y) = 1 - e^{-x} - e^{-y} + \exp(-x - y - \rho xy),\ \ x,y > 0,\ 0 < \rho < 1.$$

Correlation between X and $Y = \rho^* = \frac{e^{1/\rho}}{\rho}E_1(\rho^{-1}) - 1$, where $E_1(x) = \int_x^\infty (e^{-u}/u)du$ stands for the exponential integral. Hence

$$\begin{aligned} I(\rho) &= \int_0^1\int_0^1 [1-(1-u)-(1-v)+(1-u)(1-v)e^{-\rho\ln(1-y)\ln(1-v)}]du\, dv \\ &= \int_0^1\int_0^1 [1-s-t+ste^{-\rho\ln s\ln t}]ds\, dt \\ &= \int_0^1\int_0^1 ste^{-\rho\ln s\ln t}ds\, dt. \end{aligned}$$

Let

$$a(t) = \int_0^1 se^{-\rho\ln s\ln t}ds = \int_0^1 s^{(1-\rho\ln t)}ds = (2-\rho\ln t)^{-1}.$$

Hence

$$\begin{aligned} I(\rho) &= \int_0^1 \{t/(2-\rho\ln t)\}dt \\ &= e^{4/\rho}\int_2^\infty \frac{e^{-2v/\rho}}{\rho v}dv \\ &= \frac{e^{4/\rho}}{\rho}\int_{4/\rho}^\infty \frac{e^{-w}}{w}dw = \frac{e^{4/\rho}}{\rho}E_1(4/\rho). \end{aligned} \tag{20}$$

So,

$$e(U,T) = (1+\rho^*)/4[(4/3) - (4/\rho)e^{4/\rho}E_1(4/\rho)]. \tag{21}$$

Computations yield the following table.

Table 4.2. Values of $e(U,T)$

ρ	0	0.1	0.2	0.25	0.3	0.4	0.5	0.75	0.8	0.9
$1+\rho^*$	1	0.92	0.85	0.82	0.80	0.876	0.72	0.65	0.64	0.62
$e(U,T)$	0.75	0.64	0.56	0.53	0.50	0.45	0.42	0.34	0.33	0.31

The Bivariate Exponential Distribution of Marshall and Olkin (1967). In standard form, we have

$$P(X > x, Y > y) = \exp[-x - y - \rho \max(x,y)], \quad \rho, x, y > 0$$

where

$$EX = EY = (1+\rho)^{-1}$$

and correlation between X and $Y = \rho^* = \rho/(2+\rho)$. Thus

$$\begin{aligned} H(x,y) &= P(X \le x) + P(Y \le y) - 1 + P(X > x, Y > y) \\ &= 1 - e^{-x(1+\rho)} - e^{-y(1+\rho)} + e^{-x-y-\rho\max(x,y)}. \end{aligned} \tag{22}$$

Hence

$$\begin{aligned} I(\rho) &= \int_0^1 \int_0^1 [u + v - 1 + \exp\{\frac{\ln(1-u)+\ln(1-v)}{(1+\rho)} - \frac{\rho}{(1+\rho)} \\ &\quad \times \max(-\ln(1-u), -\ln(1-v))\}] du\, dv \\ &= 2 \iint_{0<v<u<1} (1-u)(1-v)^{(1+\rho)^{-1}} dv\, du = (1+\rho)/(4+3\rho). \end{aligned}$$

So,

$$A = (2/3) - 2I(\rho) = 2/3(4+3\rho). \tag{23}$$

Consequently,

$$e(U,T) = 3(1+\rho)(4+3\rho)/8(2+\rho). \tag{24}$$

Table 4.3. Values of $e(U,T)$

ρ	0	0.5	1.0	1.5	2.0	2.5
$e(U,T)$	0.75	1.24	1.75	2.28	2.81	3.35

Bivariate Exponential Distribution of Downton (1970). The joint density (in standard form) is given by

$$h(x,y) = (1-\rho)^{-1}\exp\left\{-\frac{(x+y)}{1-\rho}\right\} I_0\left(\frac{2\sqrt{\rho xy}}{1-\rho}\right), \quad x,y,\rho > 0,$$

where I_0 denotes the modified Bessel function of order zero. Thus

$$H(x,y) = (1-\rho)^{-1}\int_0^x\int_0^y e^{-(u+v)/(1-\rho)} I_0\left(\frac{2\sqrt{\rho uv}}{1-\rho}\right) du\, dv.$$

Using the expansion $I_0(z) = \sum_{k=0}^{\infty}(\frac{1}{4}z^2)^k/(k!)^2$ we have

$$H(x,y) = \sum_{k=0}^{\infty}(1-\rho)\rho^k\left(\int_0^{x/(1-\rho)}\frac{u^k e^{-u}}{k!}du\right)\left(\int_0^{y/(1-\rho)}\frac{v^k e^{-v}}{k!}dv\right).$$

Hence

$$\begin{aligned} I(\rho) = \int_0^{\infty}\int_0^{\infty} H(x,y)e^{-(x+y)}dx\, dy \quad &= \quad \sum_{k=0}^{\infty}(1-\rho)\rho^k \\ &\quad \times\left[\int_{x=0}^{\infty} e^{-x}\left(\int_0^{x/(1-\rho)}\frac{u^k e^{-u}}{k!}du\right)dx\right]^2. \end{aligned}$$

Consider

$$\begin{aligned} a(\rho) &= \int_0^{\infty} e^{-x}\left(\int_0^{x/(1-\rho)}\frac{u^k e^{-u}}{k!}du\right) \\ &= (1-\rho)\int_0^{\infty} e^{-(1-\rho)s}\left(\int_0^s\frac{u^k e^{-u}}{k!}du\right)ds \\ &= (1-\rho)\int_0^{\infty}\frac{u^k e^{-u}}{k!}\left(\int_u^{\infty} e^{-(1-\rho)s}ds\right)du \\ &= \int_0^{\infty} e^{-(2-\rho)u}\frac{u^k}{k!}du = (2-\rho)^{-(k+1)}. \end{aligned}$$

Substituting this in the expression for $I(\rho)$ we obtain

$$I(\rho) = \sum_{k=0}^{\infty}(1-\rho)\rho^k(2-\rho)^{-2(k+1)} = (1-\rho)/\{(2-\rho)^2-\rho\}.$$

Consequently,

$$A(\rho) = \frac{2}{3} - 2I(\rho) = \frac{2(1-\rho)^2}{3\{(2-\rho)^2-\rho\}}.$$

Straightforward computations yield

$$\int_0^\infty \int_0^\infty xyh(x,y)dx\,dy = (1-\rho)^3 \sum_{k=0}^{\infty}(k+1)^2\rho^k.$$

Next, writing $(k+1)^2 = (k+1)(k+2) - (k+1)$ and summing the right hand side series we obtain

$$E(ZW) = 1 + \rho.$$

Hence $\rho^* = \rho$ and consequently

$$\begin{aligned} e(U,T) = (1+\rho)/8A &= 3(1+\rho)\{(2-\rho)^2 - \rho\}/16(1-\rho)^2 = \\ &\quad 3(1+\rho)(4-\rho)/16(1-\rho). \end{aligned} \tag{25}$$

Table 4.4. Giving the Values of $e(U,T)$

ρ	0	0.25	0.5	0.75	0.9	1
$e(U,T)$	0.75	1.17	1.97	4.27	11.04	∞

The Bivariate Exponential Distribution of Cowan (1987).
Since

$$P[X > x, Y > y] = P[X \le x, Y \le y] - P[X \le x] - P[Y \le y] + 1,$$

one can write the general bivariate distribution function as

$$\begin{aligned} H(x,y) &= 1 - e^{-\lambda_1 x} - e^{-\lambda_2 y} + \\ &\quad \exp\left[-\frac{1}{2}\{\lambda_1 x + \lambda_2 y + (\lambda_1^2 x^2 + \lambda_2^2 y^2 - 2\lambda_1\lambda_2 xy\cos a)^{\frac{1}{2}}\}\right], \\ &\quad x, y > 0,\ 0 < a < \pi. \end{aligned}$$

Using the standard bivariate distribution

$$\begin{aligned} I(a) &= \int_0^\infty \int_0^\infty H(x,y)dF(x)dF(y) \\ &= \int_0^\infty \int_0^\infty [1 - e^{-x} - e^{-y} + e^{-\frac{1}{2}\{(x+y)+(x^2+y^2-2xy\cos a)^{\frac{1}{2}}\}}]e^{-x-y}dx\,dy \\ &= \int_0^\infty \int_0^\infty e^{-\frac{3}{2}(x+y)-\frac{1}{2}(x^2+y^2-2xy\cos a)^{\frac{1}{2}}}dx\,dy \\ &= \int_0^{\pi/2} \frac{4d\theta}{9(\cos\theta+\sin\theta)^2+1-\sin 2\theta\cos a+6(\cos\theta+\sin\theta)(1-\sin 2\theta\cos a)^{\frac{1}{2}}} \end{aligned}$$

after using a polar coordinate transformation. Hence

$$
\begin{aligned}
I(a) &= 4\int_0^{\pi/2} \frac{d\theta}{10 + (9 - \cos a)\sin 2\theta + 6(\cos\theta + \sin\theta)(1 - \sin 2\theta \cos a)^{\frac{1}{2}}} \\
I(a) &= 2\int_0^{\pi/2} \frac{d\theta}{5 + (5 - \eta)\sin 2\theta + 3(\cos\theta + \sin\theta)(1 + \sin 2\theta - 2\eta \sin 2\theta)^{\frac{1}{2}}}.
\end{aligned}
$$

where $\eta = (1 + \cos a)/2$. One can compute (starting with the double integral)

$$I(\pi) = 1/4, \text{ and } I(0) = 1/3.$$

$$
\begin{aligned}
A &= \frac{2}{3} - 2I(a) \\
2A &= 4(\frac{1}{3} - I(a)) \\
e(U,T) &= (1 + \rho^*)/8A.
\end{aligned}
$$

Also, Cowan (1987) gives

$$
\begin{aligned}
\mathrm{Corr}(X,Y) &= 1 \quad \text{if } a = 0 \\
&= -1 + \frac{4}{1+\cos a}\left[1 - \frac{1-\cos a}{1+\cos a}\log(\frac{2}{1-\cos a})\right], \quad \text{for } 0 < a < \pi, \\
&= 0 \quad \text{if } a = \pi.
\end{aligned}
$$

Computations yield Table 4.5 giving values of $I(a)$, the correlation between X and Y and $e(U,T)$.

Table 4.5. Values of $I(a)$, the Correlations between X and Y and $e(U,T)$

a	0	30°	60°	90°	120°	105°	180°
Corr(X,Y)	1	.728	.434	.227	.096	.023	0
$I(a)$	.333	.318	.295	.276	.262	.253	.250
$e(U,T)$	∞	6.88	2.36	1.33	0.95	0.79	0.75

Bivariate Exponential Distribution of Sarkar (1987).

Sarkar (1987) obtains an absolutely continuous bivariate exponential distribution given by (for $\lambda_1, \lambda_2 > 0$ and $\lambda_{12} \geq 0$)

$$
\begin{aligned}
P[X \geq x, Y \geq y] &= \exp\{-(\lambda_2+\lambda_{12})y\}\{1-[B(\lambda_1 y)]^{-\gamma}[B(\lambda_1 x)]^{1+\gamma}\} \text{ if } 0 < x \leq y \\
&= \exp\{-(\lambda_1+\lambda_{12})x\}\{1-[B(\lambda_2 x)]^{-\gamma}[B(\lambda_2 y)]^{1+\gamma}\} \text{ if } x \geq y > 0,
\end{aligned}
$$

where $\gamma = \lambda_{12}/(\lambda_1+\lambda_2), B(z) = 1-\exp(-z)$ for $z>0$. Notice that X and Y are independent if $\lambda_{12}=0$. Considering the standard form, that is, when $\lambda_1=\lambda_2=1$ and $\lambda_{12}=\rho$ we have $E(Z)=E(\lambda_1 X)=(1+\rho)^{-1}$, $EW=E(\lambda_2 Y)=(1+\rho)^{-1}$, Var $Z=$ Var $W=(1+\rho)^{-2}$, and correlation between Z and W is

$$\operatorname{Corr}(Z,W)=\rho^*=\frac{\rho}{2+\rho}\left[1+2\sum_{j=1}^{\infty}\frac{j!}{2+\rho+2j}\prod_{k=1}^{j}(2+\rho+k)^{-1}\right].$$

Since

$$P[Z\le z, W\le w]=1-P(Z\ge z)-P(W\ge w)+P(Z\ge z, W\ge w).$$

We obtain

$$\begin{aligned}
I(\rho) &= \int_0^1\int_0^1 H(F^{-1}(u),F^{-1}(v))du\,dv\\
&= 1-\int_0^1(1-u)du-\int_0^1(1-v)dv\\
&\quad+\iint_{u\le v}(1-v)^{(1+\rho)}\{1-v^{-\rho/2}u^{1+(\rho/2)}\}du\,dv\\
&\quad+\iint_{u>v}(1-u)^{(1+\rho)}\{1-u^{-\rho/2}v^{1+(\rho/2)}\}du\,dv\\
&= 2\iint_{0<u\le v<1}(1-v)^{(1+\rho)}\{1-v^{-\rho/2}u^{1+(\rho/2)}\}du\,dv\\
&= 2\left[\int_0^1(1-v)^{(1+\rho)}v\,dv-(2+\rho/2)^{-1}\int_0^1 v^2(1-v)^{1+(\rho/2)}dv\right]\\
&= 2\left[\frac{1}{(3+\rho)(2+\rho)}-(2+\rho/2)^{-1}\frac{2}{(4+\rho/2)(3+\rho/2)(2+\rho/2)}\right]\\
&= 2\left[\frac{1}{(3+\rho)(2+\rho)}-\frac{32}{(8+\rho)(6+\rho)(4+\rho)^2}\right].
\end{aligned}$$

Hence

$$A(\rho)=2\left[\frac{1}{3}-\frac{2}{(3+\rho)(2+\rho)}+\frac{64}{(8+\rho)(6+\rho)(4+\rho)^2}\right].$$

Then one can easily evaluate

$$e(U,T)=(1+\rho^*)/8A.$$

Table 4.6. Values of ρ^*, $A(\rho)$ and $e(U,T)$ for some selected values of ρ

ρ	0	0.5	1.0	1.5	2.0	2.5	3.0
$A(\rho)$	0.167	0.324	0.414	0.472	0.511	0.539	0.560
ρ^*	0	0.274	0.418	0.507	0.555	0.617	0.653
$e(U,T)$	0.75	0.49	0.43	0.40	0.38	0.38	0.37

Concluding Remarks. Mann-Whitney-Wilcoxon test is more robust to positive dependence in the X, Y variables while testing for scale alternative with certain bivariate exponential distributions for (X, Y). This is true for the bivariate exponential forms due to Morgenstern (1956), Gumbel (1960), Downton (1970), Cowan (1987), and Marshall and Olkin (1967). However, surprisingly for the bivariate form due to Sarkar (1987), the MWW test is sensitive to positive dependence between the variables X and Y.

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