A COMPARISON OF BONFERRONI-TYPE AND PRODUCT-TYPE INEQUALITIES IN PRESENCE OF DEPENDENCE

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Let $X_1, \ldots, X_n$ be a sequence of dependent random variables and for $j = 1, \ldots, n$, $A_j = (X_j \varepsilon I_j)$ where I_j's are infinite intervals of the same type, $I_j = (-\infty, a_j)$ or $I_j = (b_j, \infty)$. In this article we compare the performance of the Bonferroni-type and product-type inequalities in approximating the probabilities $P\{\cup_{i=1}^n A_i\}$ or $P\{\cap_{i=1}^n B_i\}$ where B_i is the complementary event of A_i.

The following results are proved. If $X_1, \ldots, X_n$ possess a positive dependence structure (MTP_2 or sub-Markov with respect to a sequence of infinite intervals of the same type) the product-type inequalities dominate the Bonferroni-type inequalities. If, on the other hand, the sequence of random variables is negatively dependent (S-MRR_2 or super-Markov with respect to a sequence of infinite intervals of the same type) the product-type inequalities complement the Bonferroni-type inequalities in approximating the probabilities mentioned above. Three examples are presented to illustrate the results obtained in this paper.

1. Introduction. Let $X_1, \ldots, X_n$ be a sequence of dependent random variables and for $j = 1, 2, \ldots, n$

$$A_j = (X_j \varepsilon I_j), \tag{1}$$

where I_j are infinite intervals of the same type; $I_j = (-\infty, a_j)$ or $I_j = (b_j, \infty)$. We are interested in studying the approximations for

[1]Supported in part by the Research Foundation of the University of Connecticut.

AMS 1980 subject classifications. Primary 60E15; secondary 62E10.

Key words and phrases. Probability inequalities, positive dependence, negative dependence, multivariate distributions.

I would like to thank the editors and the referee for their helpful comments on an earlier version of this paper.

$$P_1 = P\{\cup_{i=1}^n A_i\}, \tag{2}$$

or equivalently,

$$1 - P_1 = P\{\cap_{i=1}^n B_i\}, \tag{3}$$

where $B_i = A_i^c$ is the complementary event of A. These approximations play an important role in many areas of statistics; to list just a few: multiple comparison analysis (Fuchs and Sampson, 1987; Games, 1977; Kenyon, 1986a; Sidak, 1971; and Tong, 1970), simultaneous prediction (Chew, 1968), location and scale shift detection (Bauer and Hackl, 1978, 1980, and 1985; Glaz, 1983; Glaz and Johnson, 1987; and Worsley, 1979), scan statistics (Berman and Eagleson, 1985; Gates and Westcott, 1984; Glaz, 1989; Glaz and Naus, 1983; Naus, 1982; and Samuel-Cahn, 1983), sequential testing (Bauer and Hackl, 1985; Glaz and Johnson, 1986; and Kenyon, 1986b), and outlier detection (Ellenberg, 1976; Galpin and Hawkins, 1981; and Joshi, 1972).

In Section 2 of this article, we briefly outline the up-to-date development in the area of Bonferroni-type inequalities. In Section 3 the product-type inequalities will be introduced along with the necessary dependence concepts. We then compare the Bonferroni-type and product-type inequalities for certain dependence structures for $X_1, \ldots, X_n$. In Section 4 three examples will be presented for the evaluation of Bonferroni-type and product-type inequalities. A brief discussion comparing these two classes of inequalities and evaluating the numerical results from Section 4 will be given in Section 5.

2. Bonferroni-Type Inequalities. The Bonferroni-type inequalities have been used by many authors to obtain bounds for P_1 given in equation (2):

$$S_{1,n} - S_{2,n} \leq P_1 \leq S_{1,n}, \tag{4}$$

where

$$S_{1,n} = \Sigma_{i=1}^n p_i, \quad S_{2,n} = \Sigma_{i=1}^{n-1} \Sigma_{j=i+1}^n p_{i,j} \tag{5}$$

and

$$p_i = P(A_i), \quad p_{i,j} = P(A_i \cap A_j). \tag{6}$$

As these bounds can be quite inaccurate, attempts have been made to improve their performance. Kwerel (1975) has shown that

$$P_1 \geq aS_{1,n} + bS_{2,n}, \tag{7}$$

where $a = 2/k$, $b = -2/k(k-1)$ and $k-2$ is the integer part of $2S_{2,n}/S_{1,n}$. The inequality of (7) is the tightest, given the probabilities (6). The computation of this lower bound, for large n, can be quite tedious and the performance unsatisfactory (Glaz, 1989).

The study of upper bounds for P_1 have received more attention, the reason being that it provides a conservative test or a confidence coefficient in a multiple comparison procedure (see references mentioned in the Introduction). Let $v_1, \ldots, v_n$ be the vertices of the graph G, representing the events $A_1, \ldots, A_n$, respectively. The vertices v_i and v_j are joined by an edge e_{ij} if and only if $A_i \cap A_j \neq \phi$. Hunter (1976) and Worsley (1982) proved that for a subgraph T of G

$$P_1 \leq S_{1,n} - \Sigma_{\{(i,j); e_{ij} \varepsilon T\}}\, p_{i,j}, \tag{8}$$

if and only if T is a tree. An important member of this class of upper bounds is

$$P_1 \leq S_{1,n} - \Sigma_{i=1}^{n}\, p_{i,i+1}, \tag{9}$$

which under certain conditions is the least upper bound in that class. The above statement is valid if the events $A_1, \ldots, A_n$ are exchangeable or are ordered in such a way that for $1 \leq i_1 < i_2 \leq n$, $P(A_{i_1} \cap A_{i_2})$ is maximized for $i_j - i_{j-1} = 1$ (see Worsley, 1982, Examples 3.1 and 3.2).

DEFINITION 2.1. An inequality for P_1 or $1 - P_1$ is of *order k* if it is given in terms of $P\{\cap_{j=1}^{m} A_{i_j}\}$ for $1 \leq m \leq k < n$, and contains the term $P\{\cap_{j=1}^{k} A_{i_j}\}$ for some $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Recently, Hoover (1989) has derived a sequence of Bonferroni-type upper bounds of order k, $1 \leq k \leq n-1$:

$$P_1 \leq P\{\cup_{i=1}^{k} A_i\} + \Sigma_{j=k+1}^{n} P\{A_j \cap [\cap_{\substack{1 \leq i_1 < i_2 < \ldots < i_k < n \\ i_1, i_2, \ldots, i_k \varepsilon S_j}} (\cup_{j=1}^{k} A_{i_j})^c]\}, \tag{10}$$

where S_j is a subset of $\{1, 2, \ldots, j-1\}$ of size $k-1$ and $j \geq k+1$. For $k = 1$ and $k = 2$ the upper bounds in (10) reduce to the Bonferroni upper bound in (4) and the Hunter-Worsley upper bound in (8), respectively. In the case that $A_1, \ldots, A_n$ are naturally ordered in such a way that $P(\cap_{j=1}^{m} A_{i_j})$ is maximized for $i_j - i_{j-1} = 1$, $2 \leq j \leq m$ and $2 \leq m \leq n-1$, the natural ordering with $S_j = \{j-1, j-2, \ldots, j-k\}$ is recommended for the upper bound of order k. In this case (10) reduces to:

$$P_1 \leq S_{1,n} - \Sigma_{i=1}^{n-1}\, p_{i,i+1} - \Sigma_{j=2}^{k-l}\, \Sigma_{i=1}^{n-j}\, p^*_{i,i+1,\ldots,i+j}, \tag{11}$$

where

$$\Sigma_{j=2}^{1}\, d_j \equiv 0 \text{ and for } j \geq 2$$

$$p^*_{i,i+1,\ldots,i+j} = P(A_i \cap A_{i+1}^c \cap \ldots \cap A_{i+j-1}^c \cap A_{i+j}). \tag{12}$$

For $k = 2$ equation (11) reduces to equation (9). If the events $A_1, A_2, \ldots, A_n$ are exchangeable, a further simplification of (10) is obtained:

(13) $$P_1 \leq np_1 - (n-1)p_{1,2} - \Sigma_{j=2}^{k-1} (n-j)p^*_{1,2,\ldots,j+1},$$

where $p^*_{1,2,\ldots,j+1}$ is given by equation (12). In Section 4 three examples will be presented to evaluate these Bonferroni-type inequalities.

3. Product-Type Inequalities. Let $X_1, \ldots, X_n$ be a sequence of dependent random variables and let A_i be the events defined in equation (1). The so-called *product upper bound* for P_1 is given by:

(14) $$P_1 \leq 1 - \Pi_{i=1}^{n} (1-p_i),$$

where p_i is defined in equation (6). This inequality along with the conditions for its validity has been studied by Dunn (1958), Esary, Proschan and Walkup (1967), Jogdeo (1977), Khatri (1967), Sidak (1967, 1968, 1971, and 1973), and Scott (1967).

The following concept of positive dependence introduced by Esary, Proschan, and Walkup (1967) is useful in establishing the inequality (14). $X_1, \ldots, X_n$ are said to be *associated* if for every pair of coordinatewise increasing real valued functions f and g,

$$\mathrm{Cov}[f(\mathbf{X}), g(\mathbf{X})] \geq 0,$$

where $\mathbf{X} = (X_1, \ldots, X_n)$. Esary, Proschan, and Walkup (1967) proved that $X_1, \ldots, X_n$ being associated is a sufficient condition for the validity of (14). It is well-known that the product bound for P_1 is tighter than the Bonferroni upper bound in (4). On the other hand, the upper bound (9) outperforms the product bound (14) (Worsley, 1982, Example 3.1). According to the definition (2.1), the product bound (14) is a first order inequality. The rest of this section is devoted to presenting the product-type inequalities of order k and comparing them with the Bonferroni-type inequalities of corresponding order. In what follows we will assume that the events $A_1, \ldots, A_n$ are naturally ordered in such a way that $P\{\cap_{j=1}^{m} A_{i_j}\}$ is maximized for $i_j - i_{j-1} = 1$, $2 \leq j \leq m$ and $2 \leq m \leq n-1$.

To study the higher order product-type inequalities for P_1, the following concepts of dependence play an important role.

DEFINITION 3.1. (Karlin, 1968). A nonnegative real-valued function of two variables, $f(x, y)$, is *totally positive of order two*, TP_2 (*reverse rule of order two*, RR_2), if

$$f(x_1, y_1)f(x_2, y_2) - f(x_1, y_2)f(x_2, y_1) \geq (\leq) 0$$

for all $x_1 < x_2$ and $y_1 < y_2$.

DEFINITION 3.2. (Karlin and Rinott, 1980a, 1980b). A nonnegative real-valued function of n variables, $f(x_1, \ldots, x_n)$ is *multivariate totally positive of order two*, MTP_2 (*multivariate reverse rule of order two*, MRR_2), if for any pair of arguments x_i and x_j the function f, viewed as a function of x_i and x_j while the rest of the arguments are kept fixed, is TP_2 (RR_2). $f(x_1, \ldots, x_n)$ is said to be strongly MRR_2, S-MRR_2, if for any set of PF_2 functions $\{\phi_j\}$ (a function ϕ is PF_2 if and only if $\phi(x-y)$ is TP_2 in the variables $-\infty < x,\, y < \infty$), the marginals

$$g(x_{i_1}, \ldots, x_{i_k}) = \int \cdots \int f(x_1, \ldots, x_n)\, \Pi_{m=1}^{n-k}\, \phi(x_{j_m}) dx_{j_1} \ldots dx_{j_{n-k}}$$

are MRR_2 in the variables $(x_{i_1}, \ldots, x_{i_k})$, where the set $\{1, \ldots, n\} = \{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\}$. A sequence of random variables, $X_1, \ldots, X_n$, is said to be MTP_2 (S-MRR_2) if its joint density is MTP_2 (S-MRR_2).

The class of random variables with MTP_2 or S-MRR_2 densities is quite rich. For a listing of these densities, see Karlin and Rinott (1980a, 1980b). Barlow and Proschan (1975) defined the TP_2 in pairs property for $(X_1, \ldots, X_n)$. If the support of its distribution function is a product space, then TP_2 in pairs is equivalent to MTP_2.

We introduce the following concept of dependence that is closely related to the higher order product-type bounds.

DEFINITION 3.3. A sequence of random variables $X_1, \ldots, X_n$ is said to be *sub-Markov* (*super-Markov*) with respect to a sequence of intervals $I_1, \ldots, I_n$ if for any $1 \le i < k \le n$

$$P\{X_k \varepsilon I_k \mid \cap_{j=1}^{k-1} (X_j \varepsilon I_j)\} \ge (\le) P\{X_k \varepsilon I_k \mid \cap_{j=i}^{k-1} (X_j \varepsilon I_j)\}.$$

In Glaz and Johnson (1984, Theorems 2.3 and 2.8) it is proved that if the joint density of $X_1, \ldots, X_n$ is MTP_2 (S-MRR_2), then $X_1, \ldots, X_n$ is sub-Markov (super-Markov) with respect to the intervals $I_j = (-\infty, a_j)$ or $I_j = (b_j, \infty)$, $j = 1, \ldots, n$. Moreover, if $X_1, \ldots, X_n$ are MTP_2 (S-MRR_2), we construct a decreasing (increasing) sequence of upper (lower) bounds for P_1:

$$\gamma_{k,n} = 1 - P\{\cap_{j=1}^{k} A_j^c\} \Pi_{m=k+1}^{n} P(A_m^c \mid \cap_{j=m-k+1}^{m-1} A_j^c), \tag{15}$$

where $1 \le k \le n-1$ and A_j^c is the complementary event of $A_j = (X_j \varepsilon I_j)$, $j = 1, \ldots, n$. Note that if $k = 1$, then $\gamma_{1,n}$ is the product bound given by the inequality (14) (in the positive dependence case). For $k \ge 1$, $\gamma_{k,n}$ is the kth order product-type bound for P_1. We now proceed to compare the product-type bounds with the Bonferroni-type bounds. For $k \ge 2$ let

$$\delta_{k,n} = S_{1,n} - \Sigma_{i=1}^{n-1}\, p_{i,i+1} - \Sigma_{j=2}^{k-1}\, \Sigma_{i=1}^{n-j}\, p^*_{i,i+1,\ldots,i+j} \tag{16}$$

denote the kth order Bonferroni-type bound where $S_{1,n}$, $p_{i,i+1}$, and $p^*_{i,i+1,\ldots,i+j}$ are given by equations (5), (6), and (12), respectively. The following result is true:

THEOREM 3.1. *Let $X_1, \ldots, X_n$ be a sequence of dependent random variables and $A_1, \ldots, A_n$ be the events defined in equation (1). Assume that $0 < P_1 < 1$. Then for $k \ge 2$*

$$\gamma_{k,n} \leq \delta_{k,n}, \tag{17}$$

where $\gamma_{k,n}$ and $\delta_{k,n}$ are given by equations (15) and (16), respectively. Moreover, if $n > k$ and for some $1 \leq m \leq k$ and $1 \leq j \leq n-k$

$$P\{A_n \cap (\cap_{i=1}^{k-1} A_{n-i}^c)\} > 0, \;\; P\{A_j \mid \cap_{m=j+1}^{j+k} A_i^c)\} > 0 \tag{18}$$

then the inequality in (17) is sharp.

PROOF. We prove this result by induction on n, the number of the events A_j. For $n = k$,

$$\gamma_{k,k} = 1 - P\{\cap_{j=1}^k A_j^c\} = P\{\cup_{j=1}^k A_j\} = \delta_{k,k}.$$

Assume the conclusion of the theorem is true for $n-1$ events with a weak inequality in (17), and show that it holds for n events. Write for $k \geq 3$

$$\gamma_{k,n} = \gamma_{k,n-1} + (1 - \gamma_{k,n-1})P(A_n \mid \cap_{j=n-k+1}^{n-1} A_j^c).$$

Then by the induction hypothesis, it follows that

$$\begin{aligned}
\gamma_{k,n} &\leq \delta_{k,n-1} + (1 - \gamma_{k,n-1})P(A_n \mid \cap_{j=n-k+1}^{n-1} A_j^c) \\
&= \delta_{k,n} - (\delta_{k,n} - \delta_{k,n-1}) + (1 - \gamma_{k,n-1})P(A_n \mid \cap_{j=n-k+1}^{n-1} A_j^c) \\
&= \delta_{k,n} - \{P(A_n) - P(A_{n-1} \cap A_n) \\
&\quad -\Sigma_{j=2}^{k-1} P[A_{n-j} \cap (\cap_{i=n-j+1}^{n-1} A_i^c) \cap A_n]\} \\
&\quad +(1 - \gamma_{k,n-1})P\{A_n \cap [\cap_{j=1}^{k-1} A_{n-j}^c]\}/P[\cap_{j=1}^{k-1} A_{n-j}^c] \\
&= \delta_{k,n} - P\{A_n \cap [\cap_{j=1}^{k-1} A_{n-j}^c]\} \\
&\quad +(1 - \gamma_{k,n-1})P\{A_n \cap [\cap_{j=1}^{k-1} A_{n-j}^c]\}/P[\cap_{j=1}^{k-1} A_{n-j}^c].
\end{aligned}$$

Since

$$(1 - \gamma_{k,n-1})/P[\cap_{j=1}^{k-1} A_{n-j}^c] = \Pi_{j=1}^{n-k} P\{A_j^c \mid \cap_{m=j+1}^{j+k-1} A_m^c\},$$

we get that

$$\gamma_{k,n} \leq \delta_{k,n} - P\{A_n \cap [\cap_{j=1}^{k-1} A_{n-j}^c]\}\{1 - \Pi_{j=1}^{n-k} P(A_j^c \mid \cap_{m=j+1}^{j+k-1} A_m^c)\}. \tag{19}$$

As the second term on the right-hand side of the inequality (19) is nonnegative, we obtain the inequality (17). It follows from the inequality (19) that if the conditions (18) hold, then the inequality in (17) is sharp. This concludes the proof of Theorem 3.1 for $k \geq 3$. For $k = 2$ the proof is similar, with equation (9) being used instead of (11). $\|$

The following two results are a direct consequence of Theorem 3.1 and Glaz and Johnson (1984, Theorem 2.3 and Theorem 2.8, respectively).

COROLLARY 3.2. *If $X_1, \ldots, X_n$ are MTP_2, then for $k \geq 1$*

$$P_1 \leq \gamma_k \leq \delta_k.$$

Moreover, γ_k and δ_k are nonincreasing sequences of k.

COROLLARY 3.3. *If $X_1, \ldots, X_n$ are S-MRR_2, then for $k \geq 1$*

$$\gamma_k \leq P_1 \leq \delta_k.$$

Moreover, the sequences γ_k and δ_k are nondecreasing and nonincreasing, respectively, in k.

REMARK. The condition of $X_1, \ldots, X_n$ being MTP_2 (S-MRR_2) in Corollary 3.2 (Corollary 3.3) can be relaxed to $X_1, \ldots, X_n$ being sub-Markov (super-Markov) with respect to the intervals $I_1^c, \ldots, I_n^c$.

In Section 4 we present three examples to evaluate the performance of the product-type inequalities.

4. Examples. To illustrate the inequalities discussed in Sections 2 and 3 and to compare their performance, we present three examples. A brief discussion will follow in Section 5.

4.1. Boundary Crossing Probabilities. Let $Z_1, \ldots, Z_n, \ldots$ be independent random variables from a normal distribution with mean 0 and variance 1 and $S_j = \Sigma_{i=1}^{j} Z_i$. Denote by

$$\tau = \inf\{j \geq 1; \mid S_j \mid > c_j\},$$

the first time that the sequence of partial sums cross a symmetric boundary given by the constants c_j. We are interested in approximations for

$$P(\tau > n) = P\{\cap_{j=1}^{n} (\mid S_j \mid \leq c_j)\},$$

$n = 1, 2, \ldots$. Based on these approximations, one can evaluate approximations for $E(\tau)$ and $\mathrm{Var}(\tau)$, the expected time and the variance of the time for the first crossing of the boundary, respectively. It follows from Glaz and Johnson (1986, Theorem 2.1) and Karlin and Rinott (1982) that $\mid S_1 \mid, \ldots, \mid S_n \mid$ is MTP_2. Hence, Corollary 3.2 implies that for $k \geq 1$

$$P(\tau > n) \geq \gamma_{k,n}^* \geq \delta_{k,n}^*,$$

where

$$\gamma^*_{k,n} = 1 - \gamma_{k,n} \text{ and } \delta^*_{k,n} = 1 - \delta_{k,n}$$

are given in equations (15) and (16), respectively. Here, we will evalute the bounds in the case of a triangular boundary

$$c_j = a - bj,\ a > 0,\ b > 0,$$

that has been introduced by Anderson (1960) in the context of sequential tests of hypotheses. For a more elaborate discussion of this subject, the reader is referred to Glaz and Johnson (1986).

In Table 4.1 we compare the Bonferroni-type and product-type bounds of order $k \leq 3$ with the simulated values for $P(\tau > n)$. The triangular boundary in this example is given by $c_j = 7.5 - .2j$. The simulated values for $P(\tau > n)$ are denoted by $\hat{P}(\tau > n)$ and have been estimated from a simulation with 10,000 trials using IMSL (1975).

Table 4.1

Approximations for $P(\tau > n)$, $c_j = 7.5 - .2j$

n	$\delta^*_{1,n}$	$\gamma^*_{1,n}$	$\delta^*_{2,n}$	$\gamma^*_{2,n}$	$\delta^*_{3,n}$	$\gamma^*_{3,n}$	$\hat{P}(\tau > n)$
5	.9955	.9955	.9961	.9961	.9961	.9961	.9963
10	.7878	.8038	.8945	.8953	.8996	.8998	.9025
15	-	.3029	.6384	.6555	.6717	.6789	.6939
20	-	.0325	.2949	.3824	.3811	.4229	.4534
25	-	.0006	-	.1657	.0755	.2030	.2333
30	-	.0000	-	.0413	-	.0573	.0683
35	-	.0000	-	.0015	-	.0022	.0028

NOTE: The - in the table corresponds to values less than 0.

4.2. Moving Window Detection Probabilities. Let $Z_1, \ldots, Z_n, \ldots$ be independent observations from a normal distribution with mean 0 and variance one unit. For fixed $m \geq 2$, define

$$S_{j,m} = \Sigma_{i=j}^{j+m-1} Z_i,\ \ j \geq 1,$$

and

$$\tau_m = \inf\{j \geq 1;\ S_{j,m} > a\} + m - 1.$$

Then τ_m is the first time that the process of moving sums of length m crosses the straight line boundary specified by the constant a, $a > 0$. Applications to quality control are discussed in Bauer and Hackl (1980) and Lai (1974), who employ the first-order product bound $\gamma^*_{1,n}$ to approximate $P(\tau_m > n)$.

Note that in this example the sequence of moving sums, $\{S_{m,j}\}_{j=m}^{n}$, is associated but not MTP_2. Hence we cannot argue that $\gamma^*_{k,n}$ is a lower bound for $P(\tau_m > n)$. One can show (Glaz and Johnson, 1988) that

$$\lim_{n\to\infty} P(\tau_m > n \mid \tau_m > n-1) = \alpha,$$

where $0 < \alpha < 1$, and use this asymptotic stationarity property of $P(\tau_m > n \mid \tau_m > n-1)$ to justify the use of $\gamma^*_{k,n}$ as an approximation for $P(\tau_m > n)$. The quantity $\delta^*_{k,n}$ is still a lower bound for $P(\tau_m > n)$ and from Theorem 3.1 we have that $\gamma^*_{k,n} \geq \delta^*_{k,n}$.

In Table 4.2, for specified values of m, a, and n, we present the kth order Bonferroni-type bounds and product-type approximations, $k \leq 3$, and compare them with the simulated values $\hat{P}(\tau_m > n)$. $P(\tau_m > n)$ have been estimated from a simulation with 10,000 trials using IMSL (1975).

Table 4.2

Approximations for $P(\tau_{10} > n)$, $a = 2.0$

n	$\delta^*_{1,n}$	$\gamma^*_{1,n}$	$\delta^*_{2,n}$	$\gamma^*_{2,n}$	$\delta^*_{3,n}$	$\gamma^*_{3,n}$	$\hat{P}(\tau_{10} > n)$
15	-	.1595	.4436	.4866	.4976	.5148	.5278
20	-	.0346	.1508	.3216	.2723	.3650	.3866
25	-	.0075	-	.2125	.0470	.2588	.2785
30	-	.0016	-	.1404	-	.1835	.2002
35	-	.0004	-	.0928	-	.1300	.1443
40	-	.0000	-	.0613	-	.0922	.1039
45	-	.0000	-	.0405	-	.0654	.0762
50	-	.0000	-	.0268	-	.0463	.0572
60	-	.0000	-	.0112	-	.0233	.0236
70	-	.0000	-	.0051	-	.0117	.0159
80	-	.0000	-	.0022	-	.0059	.0072
90	-	.0000	-	.0010	-	.0029	.0036
100	-	.0000	-	.0004	-	.0015	.0019

NOTE: The - corresponds to values less than 0.

4.3. Multinomial Distribution. Let $\mathbf{X} = (X_1, \ldots, X_m)$ be a multinomial random variable with parameters $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{n} = (n_1, \ldots, n_m)$, where $\Sigma_{i=1}^m p_i = 1$ and $\Sigma_{i=1}^m n_i = N$. It follows from Karlin and Rinott (1980b) that $\mathbf{X}$ is S-MRR_2. We are interested in approximations for $P(X_i \leq a_i;\ i = 1, \ldots, m)$ or

$P(X_i \geq b_i;\ i = 1, \ldots, m)$. We will assume that $p_1 = p_2 = \ldots = p_m = p$, in which case $X_1, \ldots, X_m$ are exchangeable. It follows from Corollary 3.3 that

$$\delta^*_{k,m} \leq P(X_i \leq a_i;\ i = 1, \ldots, m) \leq \gamma^*_{k,m} \tag{20}$$

and

$$\delta^*_{k,m} \leq P(X_i \geq b_i;\ i = 1, \ldots, m) \leq \gamma^*_{k,m}, \tag{21}$$

where $\gamma^*_{k,m} = 1 - \gamma_{k,m}$ and $\delta^*_{k,m} = 1 - \gamma_{k,m}$. Mallows (1968) has proved that $\gamma^*_{1,m}$ is an upper bound for the above probabilities.

An important special case of the approximations given in (20) and (21) is when $a_1 = a_2 = \ldots = a_m = a$ and $b_1 = b_2 = \ldots = b_m = b$. In these cases we obtain approximations for the distribution of

$$X_{(m)} = \max(X_1, \ldots, X_m) \text{ and } X_{(1)} = \min(X_1, \ldots, X_m),$$

respectively. We illustrate the performance of these approximations in the following example. Consider a roulette with $m = 38$ numbers. We would like to test the null hypothesis that $p_1 = p_2 = \ldots = p_{38} = 1/38$. Consider the test that rejects the null hypothesis for large values of $X_{(m)}$.

In Table 4.3 we present bounds for the P-values of this test when $N = 100$. The P-values are given by $P(X_{(38)} \geq n)$, where n is the largest observed cell count.

Table 4.3

Bounds for the P-Values for the Test of Equal Cell Probabilities

n	5	6	7	8	9	10	11
$\gamma_{3,38}$	.9944	.8562	.4758	.1744	.0496	.0121	.0026
$\delta_{3,38}$	> 1	> 1	.6200	.1894	.0507	.0121	.0026

5. Discussion. The Bonferroni-type and product-type inequalities, presented in Sections 2 and 3, have the same degree of complexity. In fact, one can show that both types of the kth order inequalities for $P\{\cup_{j=1}^{n}(X_j \varepsilon I_j)\}$ can be expressed in terms of $P\{\cap_{j=1}^{i}(X_j \varepsilon I_j)\}$, for $1 \leq i \leq k$.

If $X_1, \ldots, X_n$ possesses a positive dependence structure (MTP_2 or sub-Markov with respect to I_j's), the product-type inequalities dominate the Bonferroni-type inequalities (Corollary 3.2). In this case, Table 4.1 of Example 4.1 illustrates the amount of improvement achieved by the kth order product-type inequality over the kth order Bonferroni-type inequality for $k = 1, 2, 3$. The order of the inequality plays an important role in improving the approximations. Example 4.2 supports the use of product-type inequalities as approximations in cases when

$X_1, \ldots, X_n$ are positively dependent but does not necessarily satisfy the conditions of Corollary 2.2. In this situation, the Bonferroni-type inequalities along with the simulations provide a tool for evaluating the accuracy of the product-type approximations. Numerical results in Table 4.2 indicate that $\gamma_{3,n}$ can serve as a respectable approximation for the tail probabilities $P(\tau_m > n)$.

If $X_1, \ldots, X_n$ have a negative dependence structure (S-MRR$_2$ or super-Markov with respect to I_j's), the product-type inequalities complement the Bonferroni-type inequalities in approximating P_1 and $1 - P_1$, given by equation (2) and (3), respectively. This result is quite useful, as there are no tight lower (upper) bounds available for P_1 $(1 - P_1)$. In Example 4.3 both types of inequalities are utilized to approximate the P-value of the test for equal cell probabilities in a multinomial experiment. The numerical results in Table 4.3 indicate that these inequalities can provide us with quite accuracte approximations.

In conclusion, we would like to point out that the product-type bounds have the advantage of always having a value in the interval [0,1], while the Bonferroni-type bounds could have values outside the unit interval (see Tables 4.1–4.3).

REMARKS. Recently, Block, Costigan, and Sampson (1988) developed an optimized version of the second-order product-type inequality under conditions of positive dependence. As part of their work, they show that the second-order product-type inequality developed in Glaz and Johnson (1984) is superior to the corresponding second-order Bonferroni-type inequality, and both are based on the same spanning tree. Their proof of the result is analytical in nature. Hoover (1988) independently used a similar approach to the one used in this paper to derive the proof of Theorem 3.1.

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