COMPARISONS FOR MAINTENANCE POLICIES INVOLVING COMPLETE AND MINIMAL REPAIR

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Maintenance policies are compared under various types of aging. Formerly, a standard assumption was that when a component or system failed, it was replaced by a new one. Preventive maintenance usually took the form of replacement according to an age or block policy. Under the assumption that a component can be minimally repaired, new results involving block replacement policies can be obtained. An analog of age replacement, called repair replacement, is also discussed and compared with other policies.

1. Introduction. The study of operating characteristics of maintenance policies in reliability has a long history. For a survey of the very early developments see Barlow and Proschan (1965). In this article we shall review one aspect of this area, the comparison of maintenance policies.

A maintenance policy involves repairing or replacing a system or component when it fails. This cycle is continued indefinitely. We shall not consider the time taken to repair the component in this paper. An assumption equivalent to not considering these repair times, which we shall make, is that repairs or replacements are instantaneous. Rather than waiting for components to fail, intervention is possible in the sense that replacements may be planned. That is, working components can be replaced (or overhauled, but we shall consider only replacements for planned intervention in this paper) before failure. Two standard forms of intervention are block and age policies. A block policy is said to be in effect if components are replaced on a fixed schedule determined a priori and not depending on unplanned failures which may occur. Unplanned failures are handled as if there were no block policy. An age policy mandates replacement on a fixed schedule starting at time zero and continuing until an unplanned failure occurs at which time a new schedule starts.

If components are used which wear out it would seem that intervention of the age or block type would result in fewer unplanned failures. It has been shown

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that under certain types of wearout that this is in fact the case. Also comparisons between age and block policies have been made in order to determine which of these two policies is preferable under various types of wearout. Most of these results have been obtained under the assumption that unplanned replacements are complete (i.e., replacement is with new components). Equivalent to saying that replacements are complete is that components are repaired to a good as new state. This type of assumption is unrealistic in the situation where components can be repaired so they are only as good as they were immediately prior to failure. This is called the "bad as old" or "minimal repair" assumption and although the concept is not a new one, recent research has shown that many results holding in the good as new case also hold in the bad as old situation. We shall discuss some of these results for the comparison of maintenance policies.

In Section 3 we give a review of the comparison of policies where failed components are completely repaired. The comparison of policies where components are minimally repaired and block replacements are undertaken is discussed in Section 4. Section 5 is similar to Section 4 where instead of block replacement, a generalization of age replacement which we call repair replacement is considered. Section 6 considers comparisons for the totality of all planned replacements and unplanned repairs. An appendix gives mathematical details for the type of stochastic comparisons considered.

2. Preliminaries. We use a counting process $\{N(t),\ t \geq 0\}$ to model the number of failures of a system or a component, where the times between failure are given by $\{X_n,\ n \geq 1\}$. If the failed system or component is repaired and if there is no block or age policy, N will count only these unplanned repairs. If the repairs are complete then $\{X_n,\ n \geq 1\}$ will consist of independent failure times. If, in addition, the components used for replacement are alike, the process N can be assumed to be a renewal process. The assumption that the repairs are minimal is modeled by assuming that the process is nonhomogeneous Poisson and in this case we use the notation $\{N_m(t),\ t \geq 0\}$. For background concerning minimal repair see Asher and Feingold (1984).

For a block replacement policy where replacements are planned at times T, $2T$, $3T, \ldots$ we use the notation $N^B(T,t)$ to indicate the number of unplanned complete repairs up to time t. We designate the process by $N^B(T)$. The process of all unplanned repairs and planned replacements is given by $R^B(T)$. If the underlying repair process is minimal we write $N_m^B(T)$ and $R_m^B(T)$ for the corresponding block replacement processes. Formerly R processes were called removal processes (see Barlow and Proschan (1981), p. 181). We shall call these *renovation processes.*

For an age replacement policy where the replacements occur every T units after a complete repair we let $N^A(T,t)$ be the number of unplanned repairs up to time t and $R^A(T,t)$ be the total number of unplanned repairs and planned replacements up to time t where $N^A(T)$ and $R^A(T)$ designate whole processes. If we want to emphasize the distribution of the underlying process, N or N_m, we will write N_F or $N_{m,F}$ where F is the distribution time until the first event of the process.

The basic results we will discuss will involve counting processes N_1 and N_2 related to two different maintenance policies. Some of the earliest results (see Barlow and Proschan, 1965) were of the type

$$E[N_1(t)] \geq E[N_2(t)] \quad \text{for all } t. \tag{1}$$

Other results involve limiting relationships which can be obtained from the above by letting $t \to \infty$. Another standard result is of the form

$$P\{N_1(t) \geq n\} \geq P\{N_2(t) \geq n\} \quad \text{for } n = 1, 2, \ldots, \text{ all } t, \tag{2}$$

which is equivalent to assuming that $E[f(N_1(t))] \geq E[f(N_2(t))]$ holds for all increasing functions f and all t. In this case we write

$$N_1(t) \overset{\text{st}}{\geq} N_2(t) \quad \text{for all } t.$$

We call this type of result a marginal stochastic comparison since it only involves the one dimensional marginals of the processes N_1 and N_2. A more general type of result would be

$$(N_1(t_1), N_1(t_2)) \overset{\text{st}}{\geq} (N_2(t_1), N_2(t_2)) \quad \text{for all } t_1 < t_2, \tag{3}$$

which means

$$E[f(N_1(t_1), N_1(t_2))] \geq E[f(N_2(t_1), N_2(t_2))] \quad \text{for all } t_1 < t_2$$

and all f increasing in both variables.

The stochastic comparisons mentioned above are subsumed under a more general stochastic comparison denoted by

$$N_1 \overset{\text{st}}{\geq} N_2. \tag{4}$$

This stochastic comparison is defined in the Appendix following (A.3). The definition in essence gives that the measure induced by N_1 on the space of counting processes stochastically dominates the measure induced by N_2. It is shown that this latter comparison is equivalent to the stochastic ordering of all finite dimensional distributions. Consequently this implies the two dimensional version given by (3).

3. Comparison of Policies with Complete Repair. Various results have been given which compare $N(t)$, $N^A(T,t)$, $N^B(T,t)$, $R^A(T,t)$ and $R^B(T,t)$. One elementary result is that without any assumptions on the lifetime

$$R^A(T,t) \overset{\text{st}}{\geq} R^B(T,t) \quad \text{for all } t. \tag{5}$$

This is Theorem 4.1 (in different notation) of Barlow and Proschan (1965, p. 67). A second more interesting, but still intuitive, result is that if the lifetimes are IFR, then

$$N(t) \overset{\text{st}}{\geq} N^A(T,t) \overset{\text{st}}{\geq} N^B(T,t) \quad \text{for all } t. \tag{6}$$

This is also proven in Barlow and Proschan (1965, Theorem 4.4, p. 69). As an immediate consequence, using the second stochastic inequality and asymptotic results, these authors conclude (Corollary 4.5, p. 71) that if F is an IFR lifetime, then

$$E[N(t)] \leq \frac{tF(t)}{\int_0^t \bar{F}(x)dx}. \tag{7}$$

In Marshall and Proschan (1972) various improvements of these results were obtained where the weaker property of NBU is assumed rather than IFR. These results appear in Barlow and Proschan (1981) and we mention the most important of these:

$$N(t) \overset{\text{st}}{\geq} N^A(T,t) \quad \text{for all } t,T \quad \text{if and only if} \quad F \text{ is NBU;} \tag{8}$$

$$N(t) \overset{\text{st}}{\geq} N^B(T,t) \quad \text{for all } t,T \quad \text{if and only if} \quad F \text{ is NBU;} \tag{9}$$

$$N^B(T,t) \overset{\text{st}}{\leq} N^B(kT,t) \quad \text{for all } t,T,k \quad \text{if and only if} \quad F \text{ is NBU;} \tag{10}$$

$$N^A(T,t) \overset{\text{st}}{\leq} N^A(kT,t) \quad \text{for all } t,T,k \quad \text{if and only if} \quad F \text{ is NBU;} \tag{11}$$

$$N^A(T_1,t) \overset{\text{st}}{\leq} N^A(T_2,t) \quad \text{for all } T_1 \leq T_2 \text{ for all } t \text{ if and only if } F \text{ is IFR} \tag{12}$$

where $kT = < kT, 2kT, 3kT, \cdots >$ and k is a positive integer. These are all proven in Chapter 6, Section 4 of Barlow and Proschan (1981).

One other result which is of the above type, but is given in a disguised form as Theorem 3.2 of Chapter 6 of Barlow and Proschan (1981) is that if F is the distribution function of X_1 from the nonhomogeneous Poisson process N_m and also the distribution function associated with the renewal process N then

$$F \text{ is NBU implies } N_m(t) \overset{\text{st}}{\geq} N(t) \quad \text{for all } t. \tag{13}$$

This was also proven by Blumenthal, Greenwood, and Herbach (1976).

4. Block Replacement Policies. Block, Langberg, and Savits (1990) have generalized the previous results in several ways. First, they considered more general policies Z. By a policy Z we mean that replacements occur at times $z_1 < z_2 < z_3 \cdots$ and we use the notation $Z =< z_1, z_2, \cdots >$. In the special case that the z's occur every T units (i.e., $z_i = iT$) we write $Z = T$. Consequently, we use the notation $N^B(Z)(N_m^B(Z))$ to designate the process giving the number of unplanned replacements in a complete (minimal) repair block process. We continue to use the term block here in the sense that the replacement schedule is made a priori (in a block) even though the z_i need not be equal. Similarly $R^B(Z)$ and $R_m^B(Z)$ will designate the renovation process. These authors obtained comparisons of the whole processes of type (4) given in Section 2. Also, these authors considered the comparison of minimal repair policies involving block replacement. A basic result used to obtain some of these generalizations involves comparison of minimal repair processes with stochastically ordered lifetimes. That is, let F and G be the distribution functions of the lifetimes associated with the nonhomogeneous Poisson processes $\{N_{m,F}(t),\, t \geq 0\}$ and $\{N_{m,G}(t),\, t \geq 0\}$. The result is

$$N_{m,F} \overset{\text{st}}{\geq} N_{m,G} \quad \text{if and only if} \quad F \overset{\text{st}}{\leq} G \quad (\text{i.e., } F(t) \geq G(t) \quad \text{for all } t).$$

See, for example, Theorem 3.1.1(a) of the paper cited above. The following comparisons are also obtained:

$$N_m^B(Z) \overset{\text{st}}{\geq} N^B(Z) \quad \text{for all } Z \text{ if } F \text{ is NBU;} \tag{14}$$

$$N_m \overset{\text{st}}{\geq} N_m^B(Z) \quad \text{for all } Z \text{ if and only if } F \text{ is NBU;} \tag{15}$$

$$N_m \overset{\text{st}}{\geq} N^B(Z) \quad \text{for all } Z \text{ if and only if } F \text{ is NBU.} \tag{16}$$

Notice that (14) above generalizes the result (13) and gives that for any block replacement policy Z, a minimal repair policy for unplanned failures produces stochastically more failures than a complete repair process. The result (15) gives a result which generalizes (9) and the result (16) is a hybrid of (14) and (15).

In Theorem 4.1 of Block, Langberg, and Savits (1990), several other results which generalize (5) are also given.

5. Repair Replacement Policies. Block, Langberg, and Savits (1989) consider repair replacement policies. In these policies items are either minimally or completely repaired at unplanned failures or they are replaced if they survive a certain fixed time from the last repair without suffering an unplanned failure. If at failure only complete repairs are allowed, then the repair replacement policy reduces to an age replacement policy. Consequently, the concept of a repair replacement policy is a more general type of replacement policy than an age replacement policy. We consider repair replacement policies for two reasons. First, upon repair it may be that a replacement is scheduled within a short period of time; however, when the repair was made it may be clear that the component was in good shape and did not require an immediate replacement. Consequently, the replacement should be deferred. One way of deferring it is to start a new replacement schedule from the time of the repair. Secondly, as we will see in (18), a repair replacement policy has fewer unscheduled repairs than a minimal repair policy under IFR lifetimes. This is often a desirable outcome if cost is not a consideration.

As before we let N and N_m denote the processes with only complete and only minimal repair respectively with no intervention. We now define the repair replacement processes with scheduled planned replacement determined by $Z = < z_k >$. Planned replacements occur at times $z_1, z_1 + z_2, \dots$ until an unplanned repair occurs. Assume this occurs between times $\Sigma_{i=1}^{n_1-1} z_i$ and $\Sigma_{i=1}^{n_1} z_i$. The planned replacement schedule is then restarted from the time of the unplanned repair and the schedule of planned replacements is given by $z_{n_1+1}, z_{n_1+1} + z_{n_1+2}, \dots$ units of time *after the unplanned repair.* This process continues. Notice that Z here is different than in the previous section in that the z_i here give times between planned replacements. See Block, Langberg, and Savits (1989) for more details.

If only complete repairs are allowed, i.e., the N process is used, and $z_k = T$ for all k we have that at each unplanned complete repair, the schedule of planned replacement times *from that repair* is $T, 2T, 3T, \dots$. This yields the usual age replacement policy.

If the repairs are minimal, i.e., the process N_m is used, the general repair replacement policy results.

If only complete repairs are permitted we let $N^A(Z)$ be the process counting the number of unplanned (complete) repairs of the repair replacement policy (i.e., the age replacement policy). For $Z = T$, $N^A(Z)$ reduces to the process $N^A(T)$ discussed in (6).

For minimal repairs, we use $N_m^R(Z)$ for the process counting the number of unplanned (minimal) repairs of the repair replacement.

We are now able to state some results of Block, Langberg, and Savits (1989). If we let F be the underlying distribution of the renewal process N we have

$$(17) \qquad N \overset{\text{st}}{\geq} N^A(Z) \quad \text{for all } Z \text{ if and only if } F \text{ is NBU.}$$

This is given in Theorem 3.2(b) of the aforementioned paper. This result extends the first inequality of (6) in two directions. First, instead of T, a general time schedule Z is used and, second, the above result is a comparison of the type (4).

A second result, given in Theorem 3.2(d) of Block, Langberg, and Savits (1989), is that

$$(18) \qquad N_m \overset{\text{st}}{\geq} N_m^R(Z) \quad \text{for all } Z \text{ if } F \text{ is IFR.}$$

This is a companion result to (15) and gives a generalization of (17) from complete to minimal repairs. Various other comparisons can be obtained among the processes N_m, $N^A(Z)$, and $N_m^R(Z)$. An example which is an analog to (14) for block policies is that

$$N_m^R(Z) \overset{\text{st}}{\geq} N^A(Z) \quad \text{for all } Z \text{ if } F \text{ is NBU.} \tag{19}$$

This is given in Theorem 3.2(c) of Block, Langberg, and Savits (1989).

Finally, the second inequality of (6) holds for processes as well as marginally and is given by Theorem 5.2 of Block, Langberg, and Savits (1989).

6. Comparisons for Renovations. We now consider the total number of renovations, i.e., the total number of planned replacements and unplanned repairs for both the block and repair replacement policies. We recall that $R^B(Z)$ and $R_m^B(Z)$ are the total renovations for the renewal process and minimal repair process with block policy Z. Similarly for the renewal process we use $R^A(Z)$ and $R_m^R(Z)$.

The following results have been shown in Block, Langberg, and Savits (1990):

$$N \overset{\text{st}}{\leq} R^B(Z) \quad \text{for all } Z; \tag{20}$$

$$R^B(Z) \overset{\text{st}}{\leq} R_m^B(Z) \quad \text{for all } Z \text{ if } F \text{ is NBU;} \tag{21}$$

$$N_m \overset{\text{st}}{\leq} R^B(Z) \quad \text{for all } Z \text{ if } F \text{ is NWU.} \tag{22}$$

Result (22) says that if a lifetime undergoes beneficial aging, there are more renovations (i.e., removals) for a renewal process with block replacement, then there are repairs for a minimal repair process.

For repair replacement policies Block, Langberg, and Savits (1989) have shown, among other results, that

$$N \overset{\text{st}}{\leq} R^A(Z) \quad \text{for all } Z, \tag{23}$$

$$R^A(Z) \overset{\text{st}}{\leq} R_m^R(Z) \quad \text{for all } Z \text{ if } F \text{ is NBU} \tag{24}$$

and

$$N_m \overset{\text{st}}{\leq} R^A(Z) \quad \text{for all } Z \text{ if } F \text{ is NWU.} \tag{25}$$

Appendix. In order to define the stochastic ordering of two processes N_1 and N_2 in (4), it would be enough to restrict ourselves to the class of all counting processes by which we mean the class of stochastic processes whose sample paths are nonnegative right-continuous step functions, starting at 0, only increasing by jumps of size one, and endowed with the Skorohod topology. We denote the set of all such sample paths by $S([0,\infty))$. However, it is convenient to enlarge our viewpoint somewhat. The framework which we shall follow is that delineated in Kamae, Krengel, and O'Brien (1977).

Let E be a Polish space (i.e., a complete separable metric space) equipped with a closed partial ordering $\leq$. A partial ordering $\leq$ is said to be closed if its graph $\{(x,y) : x \leq y\}$ is a closed subset of $E \times E$ in the product topology. The Borel σ-algebra on E is denoted by $\mathcal{E}$.

The principal examples considered in this paper are listed below. First, however, we state two useful facts.

(A.1) A countable product of partially ordered Polish spaces is also a partially ordered Polish space under the product topology and the coordinatewise partial ordering.

(A.2) A closed subset of a partially ordered Polish space is itself a partially ordered Polish space with the induced topology and partial ordering.

Examples:

(1) If $E_i = [0,\infty)$ with the usual topology and ordering, then we denote the partially ordered Polish space $\prod_{i=1}^{\infty} E_i$ by $\mathbf{R}_+^{\infty}$.

(2) Let I be the interval $[a,b]$ and $D(I)$ the set of all functions $x : I \to R$ which are right-continuous with left-hand limits. Then it is well-known that $D(I)$ equipped with the Skorohod metric and pointwise partial ordering (i.e., $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in I$) is a partially ordered Polish space. Sometimes we also want to consider the case $I = [a,b)$ with b possibly infinite. In this case, the metric can be modified so that $D(I)$ is again a partially ordered Polish space (see Stone (1963)).

(3) For $D(I)$ as above, let $S(I)$ denote the subset of all nondecreasing step-functions $s : I \to \{0,1,2,\ldots\}$ having only jumps of size one and satisfying $s(a) = 0$. It is not hard to show that $S(I)$ is closed and hence itself a partially ordered Polish space.

We now consider the notion of stochastic order on a partially ordered Polish space E. A Borel set $U \in \mathcal{E}$ is said to be an upper set if $x \in U$ and $x \leq y$ implies $y \in U$. If λ_1 and λ_2 are two probability measures on $(E,\mathcal{E})$, we say that λ_1 is stochastically smaller than λ_2, written as $\lambda_1 \overset{\text{st}}{\leq} \lambda_2$, if $\lambda_1(U) \leq \lambda_2(U)$ for all upper sets U. This is equivalent to the condition that $\int f d\lambda_1 \leq \int f d\lambda_2$ for all nonnegative nondecreasing Borel measurable functions f. (A function $f : E \to \mathbf{R}$ is said to be nondecreasing if $x \leq y$ implies that $f(x) \leq f(y)$).

Another preservation property which is useful is the following:

(A.3) If $\phi : E \to F$ is a nondecreasing measurable mapping between two partially ordered Polish spaces and λ_1, λ_2 are two probability measures on $(E,\mathcal{E})$ with $\lambda_1 \overset{\text{st}}{\leq} \lambda_2$, then the induced measures $\mu_i = \lambda_i \circ \phi^{-1}$ on $(F,\mathcal{F})$ satisfy $\mu_1 \overset{\text{st}}{\leq} \mu_2$.

Let E be a partially ordered Polish space. By an E-valued stochastic process X we mean a measurable mapping from a probability space $(\Omega, \mathcal{F}, P)$ into $(E, \mathcal{E})$. We denote the induced probability measure $P \circ X^{-1}$ on $(E, \mathcal{E})$ by λ_X. If X and Y are two E-valued stochastic processes, we say that X is stochastically smaller than Y, denoted by $X \overset{\text{st}}{\leq} Y$, if $\lambda_X \overset{\text{st}}{\leq} \lambda_Y$. We say that X and Y are stochastically equivalent if $\lambda_X \overset{\text{st}}{\leq} \lambda_Y$ and $\lambda_Y \overset{\text{st}}{\leq} \lambda_X$; i.e., $\lambda_X = \lambda_Y$.

Several of the results in Block, Langberg, and Savits (1990) are a consequence of the following theorem. Let E_i, $i = 1, 2, \ldots$ be a sequence of partially ordered Polish spaces and set $E = \prod_{i=1}^{\infty} E_i$. We define the projection map $\pi_i : E \to E_i$ as usual. If X is an E-valued stochastic process, denote the i^{th} coordinate of X by X_i, i.e., $X_i = \pi_i X$. For $n = 1$, define the probability measure p_1 on E_1 by $p_1(A) = P(X_1 \in A)$. For $n \geq 2$, let $p_n(A|x_1, \ldots, x_{n-1})$ be a regular conditional probability of $P(X_n \in A|X_1 = x_1, \ldots, X_{n-1} = x_{n-1})$. Such exists because $E_1 \times \cdots \times E_{n-1}$ is Polish. (See, e.g., Breiman (1968)). We shall call the collection $< p_n >$ a system of transition probabilities (for X). Note that such a system completely determines the induced probability measure λ_X on $E = \prod_{i=1}^{\infty} E_i$.

Now suppose X and Y are two $E = \prod_{i=1}^{\infty} E_i$-valued stochastic processes with corresponding systems $< p_n >$ and $< q_n >$ of transition probabilities. We then write $< p_n > \overset{\text{st}}{\leq} < q_n >$ if

(i) $p_1 \overset{\text{st}}{\leq} q_1$ and

(ii) $p_n(\cdot|x_1, \ldots, x_{n-1}) \overset{\text{st}}{\leq} q_n(\cdot|y_1, \ldots, y_{n-1})$

whenever $x_i \leq y_i$, $i = 1, \ldots, n-1$ and $n = 2, 3, \ldots$.

We are now ready to state the following result. It is essentially a reformulation of Theorem 2 in Kamae, Krengel, and O'Brien (1977).

THEOREM A.4. *Let $E = \prod_{i=1}^{\infty} E_i$ be the product of partially ordered Polish spaces and X, Y two E-valued stochastic processes with corresponding systems $< p_n >$ and $< q_n >$ of transition probabilities. If $< p_n > \overset{st}{\leq} < q_n >$, then $X \overset{st}{\leq} Y$.*

COROLLARY 1. *Let $\{K(t); t \geq 0\}$ and $\{L(t); t \geq 0\}$ be two counting processes with corresponding interarrival times $< X_n >$ and $< Y_n >$. Assume that*

(i) $X_1 \overset{st}{\leq} Y_1$

and

(ii) $(X_n|X_1 = x_1, \ldots, X_{n-1} = x_{n-1}) \overset{st}{\leq} (Y_n|Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1})$

for $n \geq 2$ and whenever $x_1 \leq y_1, \ldots, x_{n-1} \leq y_{n-1}$. Then $K \overset{st}{\geq} L$.

PROOF. Let $E_i = [0,\infty)$. Then $\mathbf{X} =< X_1, X_2, \ldots >$ is an $E = \prod_{i=1}^{\infty} E_i$-valued stochastic process with a system $< p_n >$ of transition probability given by

$$p_n(A|x_1,\ldots,x_{n-1}) = P\{X_n \in A | X_1 = x_1,\ldots,X_{n-1} = x_{n-1}\}.$$

Similarly $\mathbf{Y} =< Y_1, Y_2, \ldots >$ is an E-valued stochastic process with system $< q_n >$ where

$$q_n(A|y_1,\ldots,y_{n-1}) = P\{Y_n \in A | Y_1 = y_1,\ldots,Y_{n-1} = y_{n-1}\}.$$

By assumptions (i) and (ii), we have $< p_n > \overset{\text{st}}{\leq} < q_n >$. Hence $\mathbf{X} \overset{\text{st}}{\leq} \mathbf{Y}$. Since $K = \Phi(< \mathbf{X}_i >)$ and $L = \Phi(< \mathbf{Y}_i >)$ with Φ a nonincreasing function on E, we deduce that $K \overset{\text{st}}{\geq} L$ by (A.3). Here Φ is defined as the mapping $\Phi : \mathbf{R}_+^{\infty} \to S([0,\infty))$ given by

$$\Phi(< x_i >)(t) = \sum_{i=1}^{\infty} I_{[0,t]}(x_1 + \cdots + x_i).$$

For the next result we need to introduce another mapping. Let $E_i = S([0, w_i])$ for $i = 1, 2, \ldots$ and set $w = \sum_{i=1}^{\infty} w_i (0 < w \leq \infty)$. We define the mapping

$$\Psi : \prod_{i=1}^{\infty} E_i \to E = S([0,w))$$

by

$$\Psi(< s_i >)(t) = \begin{cases} s_1(t) & \text{if } 0 \leq t < w_1, \\ \sum_{i=1}^{k-1} s_i(w_i) + s_k(t - \sum_{i=1}^{k-1} w_i) & \text{if } \sum_{i=1}^{k-1} w_i \leq t < \sum_{i=1}^{k} w_i. \end{cases}$$

COROLLARY 2. *Let $K_i = \{K_i(t);\ 0 \leq t \leq w_i\}$ and $L_i = \{L_i(t);\ 0 \leq t \leq w_i\}$ be counting processes such that $K_i \overset{st}{\leq} L_i$ for each $i = 1, 2, \ldots$. Suppose that each sequence $< K_i >$ and $< L_i >$ is independent. Then if $K = \Psi(< K_i >)$ and $L = \Psi(< L_i >)$ we have $K \overset{st}{\leq} L$.*

PROOF. Let $E_i = S([0, w_i])$ and set $X_i = K_i$ and $Y_i = L_i$. Then $\mathbf{X} = < X_1, X_2, \ldots >$ and $\mathbf{Y} =< Y_1, Y_2, \ldots >$ are $E = \prod_{i=1}^{\infty} E_i$-valued stochastic processes. By independence, the corresponding system of transition probabilities are given by

$$p_1(A) = P\{X_1 \in A\} = P\{K_1 \in A\},$$

$$q_1(A) = P\{Y_1 \in A\} = P\{L_1 \in A\},$$

and, for $n \geq 2$,

$$p_n(A|x_1,\ldots,x_{n-1}) = P\{X_n \in A|X_1 = x_1,\ldots,X_{n-1} = x_{n-1}\} = P\{K_n \in A\},$$

$$q_n(A|y_1,\ldots,y_{n-1}) = P\{Y_n \in A|Y_1 = y_1,\ldots,Y_{n-1} = y_{n-1}\} = P\{L_n \in A\}.$$

Thus $< p_n > \overset{\text{st}}{\leq} < q_n >$ since $K_n \overset{\text{st}}{\leq} L_n$ for all n by assumption. Consequently, $\mathbf{X} \overset{\text{st}}{\leq} \mathbf{Y}$. Since Ψ is a nondecreasing function on E, the result follows from (A.3).

Finally, in Section 2 following (4), we stated that the notion of stochastic ordering for counting processes could be equivalently expressed in terms of their finite dimensional distributions. We now give a proof of this obvious result.

Let X be a $D(I)$-valued stochastic process and $T = \{t_1,\ldots,t_n\} \subset I$. We assume without loss of generality that $t_1 < \cdots < t_n$. The induced probability measure $P\{(X(t_1),\ldots,X(t_n)) \in A\}$ on $(\mathbf{R}^n, \mathcal{B}^n)$ is called a *finite dimensional distribution of* X and is denoted by $\lambda_X^T(A)$.

The next lemma will be used in the proof of Theorem A.6 and may be of some independent interest.

LEMMA A.5. *Let $\lambda \overset{st}{\leq} \mu$ on a partially ordered Polish space E and suppose F is a closed subset of E. Then if $\lambda(F) = \mu(F) = 1$, it follows that $\lambda|F \overset{st}{\leq} \mu|F$.*

PROOF. Let U be an upper set in F and $\epsilon > 0$. Then there exists a compact set $K \subset U$ such that $\lambda(K) \geq \lambda(U) - \epsilon$. If $V = \{y \in E : y \geq x \text{ for some } x \in K\}$, then V is a closed upper set in E which contains K (see, e.g., Nachbin (1965)). Hence $\lambda(V) \leq \mu(V)$ and consequently $\lambda(V \cap F) = \lambda(V) \leq \mu(V) = \mu(V \cap F)$. But $V \cap F \subset U$ and so

$$\lambda(U) - \epsilon \leq \lambda(K) \leq \lambda(V) \leq \mu(V \cap F) \leq \mu(U).$$

THEOREM A.6. *Let X and Y be two $S(I)$-valued stochastic processes. Then $X \overset{st}{\leq} Y$ if and only if $\lambda_X^T \overset{st}{\leq} \lambda_Y^T$ for all finite sets $T \subset I$.*

PROOF. The necessity is clear since the mapping $\pi^T : S(I) \to \mathbf{R}^n$ given by $\pi^T(s) = (s(t_1),\ldots,s(t_n))$ is nondecreasing. Thus the result follows from (A.3).

Now we suppose that $\lambda_X^T \overset{\text{st}}{\leq} \lambda_Y^T$ for all finite sets $T \subset I$. Since $S(I) \subset D(I)$ we may consider X and Y as stochastic processes on $D(I)$. According to the proof of Theorem 4 in Kamae, Krengel, and O'Brien (1977) we can assert that $X \overset{\text{st}}{\leq} Y$ as $D(I)$-valued stochastic processes. The conclusion now follows from Lemma A.5.

REMARK A.7. Although Theorem 4 of Kamae, Krengel, and O'Brien (1977) is stated in terms of stochastic ordering between conditional probabilities, it is easy to reformulate it in terms of stochastic ordering between finite dimensional distributions. Note the equivalent version of Theorem A.6 for the space $D(I)$ is a consequence of their Theorem 4. In fact, if their Lemma 1 was adapted to the space $S(I)$, then Theorem A.6 would be a direct consequence of a suitably revised version of their Theorem 4.

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