BAYESIAN ROBUSTNESS AND STABILITY

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The role of stability of a Bayes decision problem in quantitative Bayesian robustness is analyzed. An important consequence of stability is the differentiability of the optimal Bayes decision for a smooth decision problem. The applications of the derivative to global and local sensitivity analysis are discussed.

1. Introduction. Statistical inferences are based on, in addition to the observations, some prior assumptions about the underlying situation. In the Bayesian decision theoretic framework, these assumptions take the form of specification of the basic inputs of the decision problem: the **loss**, the **likelihood**, and the **prior distribution** of the relevant unknowns (parameter). These specifications are not supposed to be exactly true – quite often they are mathematically convenient rationalizations of somewhat imprecise knowledge of the underlying situation. Such rationalizations are often justified by appealing to a vague notion of "stability" principle or continuity.

Kadane and Chuang (1978) introduced two precise, well-formulated concepts of stability for Bayes decision problems to address their qualitative "robustness' and gave sufficient conditions (also, see Chuang, 1984) for stability in some special cases. The most general results in this direction are due to Salinetti (1994). In particular, Salinetti gives a complete characterization of Strong Stability I (see section 3) for a general decision problem. The second concept, Strong Stability II, is treated in Kadane and Srinivasan (1994).

This paper is motivational in nature and its main focus is the role of Strong Stability in quantitative robustness analysis with respect to perturbation in the prior distribution. The examples given show that it is reasonably easy to verify Strong Stability in smooth problems with differentiable loss functions. The main results of the paper are that Strong Stability leads to the Gâteaux differentiability of the optimal Bayes decision for smooth problems and the derivatives can be used to carry out both global and local sensitivity analyses.

There is an extensive literature on sensitivity analysis of Bayes decision problems with respect to the prior distribution. Early work in this direction is due to Edwards, Lindeman and Savage (1963). Also, over the last decade, there has been considerable activity (Berger,1994) in the area of

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Bayesian global sensitivity analysis. More recently, the focus has shifted to (Trusczynska, 1990, Dey and Bimriwal, 1990, Srinivasan and Trusczynska, 1992, Ruggeri and Wasserman, 1993, Gustafson, Srinivasan and Wasserman, 1994,and, Srinivasan and Wasserman, 1994) local sensitivity analysis and, particularly, applications of the local sensitivity measures to diagnostics.

2. Preliminaries. The formulation and discussion of stability requires a complete specification of a decision problem. Towards this, suppose the parameter space $\Theta \subset \boldsymbol{R}$ is a complete separable metric subspace of the real numbers $\boldsymbol{R}$, the decision space $D \subset \boldsymbol{R}$ is open and the loss function $L(\theta, \delta)$: $\Theta \times D \to \boldsymbol{R}^+$ is continuous in θ for each $\delta \in D$ and finite in d for each $\theta \in \Theta$. Let $l(\theta|x)$ denote a bounded continuous (in θ) likelihood function satisfying

(1) $\quad l(\theta|x) > 0$ for all $\theta \in \Theta$;

(2) $\quad L(\theta, \delta)l(\theta|x)$ is bounded in θ for every $\delta \in D$ and x.

Here x is the value of the observable which is taken to be fixed through the discussion.

A few comments are in order about the conditions on $l(\theta|x)$ and $L(\theta, \delta)$. While (1) does not lead to any loss of generality, the continuity of $L(\theta, \delta)l(\theta|x)$ in θ imposes some restrictions. Invariably, (2) is satisfied in statistical contexts. Also, the likelihoods of commonly used statistical models for observables are continuous in their parameters. The continuity of $L(\theta, \delta)$ in θ, however, is not always met in all statistical decision problems. While the subsequent discussion can certainly be extended to address these general situations, we have chosen to make this assumption to avoid the additional technical details and, thus, not be distracted from the motivational goals of this article.

Denoting the class of all prior probability distributions on Θ by $\mathcal{P}$ and its generic elements by P or Q, the posterior distribution of P is given by

$$dP^x(\theta) = l(\theta|x)dP(\theta)/m_p(x) \tag{3}$$

where $m_p(x) \int_\Theta l(\theta|x)dp(\theta)$.

The concept of stability involves convergence of prior probabilities and, unless otherwise stated, this is taken to be weak convergence. The notation "$P_n \Rightarrow P_0$" will stand for P_n converges to P_0 weakly.

As indicated earlier, an important consequence of stability is the differentiability of the optimal Bayes decision with respect to the underlying prior distribution. The concept of differentiability that is used in this article is the following Gâteaux differentiability. Let $\tilde{\mathcal{P}}$ be the linear space spanned by $\mathcal{P}$ and, $T : \tilde{\mathcal{P}} \to \boldsymbol{R}$ be a functional.

DEFINITION: The functional T is said to be Gâteaux differentiable at $P_0 \in \mathcal{P}$ if there exists a continuous linear function $T'_{P_0}(\cdot) : \tilde{\mathcal{P}} \to R$ such that every $h \in \tilde{\mathcal{P}}$,

$$\lim_{t \to 0} \frac{T(P_0 + th) - T(P_0)}{t} = T'_{P_0}(h)\,. \tag{4}$$

Typically, in Bayesian analysis, the class of prior distributions of interest is a "small" subset Γ of $\mathcal{P}$. In such cases, by Gâteaux differentiability of T we mean (4) holds for all $h \in \tilde{\Gamma}$, the linear space of Γ.

3. Stability. The concept of stability was first introduced in the statistical context by Kadane and Chuang (1978) as a framework to address the qualitative robustness of a Bayes decision problem. Roughly, stability implies that "small" changes in the inputs of the decision problem lead to a small difference in the optimal risk. The focus of this paper is robustness with respect to changes in the prior distribution and, therefore, the following definition of stability is tailored to this situation.

DEFINITION I: The decision problem (L, l, P_0) is Strongly Stable I (SSI) if for every sequence $P_n \Rightarrow P_0$,

$$\lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} \left[\int L\,(\theta, \delta_0(\epsilon))\, dP_n^x(\theta) - \inf_{\delta \in D} \int L(\theta, \delta) dP_n^x(\theta) \right] = 0 \tag{5}$$

for every $\delta_0(\epsilon)$ such that

$$\int L\,(\theta, \delta_0(\epsilon))\, dP_0^x(\theta) \le \inf_{\delta \in D} \int L(\theta, \delta) dP_0^x(\theta) + \epsilon\,. \tag{6}$$

See Kadane and Chuang (1978) for the motivation of the above definition. They also introduced a more stringent definition of stability. Though the main focus of this paper is Strong Stability I, this is stated for the sake of completeness.

DEFINITION II: The decision problem (L, l, P_0) is Strongly Stable II (SSII) for all sequences $P_n \Rightarrow P_0$, $Q_n \Rightarrow P_0$,

$$\lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} \left[\int L\,(\theta, \delta_{Q_n}(\epsilon))\, dP_n^x(\theta) - \inf_{\delta \in D} \int L(\theta, \delta) dP_n^x(\theta) \right] = 0 \tag{7}$$

for every $\delta_{Q_n}(\epsilon)$ satisfying

$$\int L\,(\theta, \delta_{Q_n}(\epsilon))\, dQ_n^x(\theta) \le \inf_{\delta \in D} \int L(\theta, \delta) dQ_n^x(\theta) + \epsilon\,. \tag{8}$$

It is easy to see, by setting $Q_n \equiv P_0$, that SSII implies SSI for a decision problem. The converse, however, is not true in general. Chuang (1984), and Kadane and Srinivasan (1994) address the converse problem and give a variety of sufficient conditions for a decision problem to satisfy SSII. The notion of SSII has several interesting applications. The appropriate framework for the robustness analysis of computer intensive Bayesian decision theory based on Gibbs sampling or Monte Carlo techniques turns out to be SSII. Another interesting consequence of SSII is sharp estimates of oscillations of the optimal decision with respect to variations in the prior or other inputs of the decision problem. These estimates in turn can be used to address the rates of convergence problems in theoretical as well as computational studies.

Reverting to Strong Stability (i.e., SSI), it follows from the boundedness and continuity of $l(\theta|x)$ and $L(\theta,\delta)l(\theta|x)$ in θ that (5) is equivalent to

$$(9) \quad \limsup_{n\to\infty} \left[\int L(\theta,\delta_0)l(\theta|x)dP_n(\theta) - \inf_{\delta\in D} \int L(\theta,\delta)l(\theta|x)dP_n^x(\theta) \right] = 0$$

where δ_0 is an optimal decision of (L,l,P_0). Hence (9) can be used to obtain necessary and sufficient conditions for Strong Stability. The most general result, giving necessary and sufficient condition, is due to Salinetti (1994). The rest of this paper is centered around this result and the appropriate version in the framework of this paper is as follows.

THEOREM 3.1 (SALINETTI) *Suppose $L(\theta,\delta)$ is jointly lower semi-continuous in θ and δ. Then $\{(L,l,P_0)\}$ is Strongly Stable (i.e., SSI) if, and only if, for every sequence $P_n \Rightarrow P_0$ and every $\epsilon > 0$ the sequence $\{(L,l,P_n)\}$ has a bounded sequence $\{\delta_{P_n}(\epsilon)\}$ of ϵ-optimal decisions.*

The joint lower semi-continuity of $L(\theta,\delta)$ typically holds in most statistical contexts. However, the existence of bounded ϵ-optimal solutions depends very much on the loss function and the class of prior distributions under consideration and may turn out to be difficult to verify. Kadane and Srinivasan (1994) give general sufficient conditions for the existence of a bounded sequence of ϵ-optimal decisions.

In many practical applications, however, the decision problems are reasonably smooth in the sense the loss $L(\theta,t)$ is differentiable in t and the class of possible prior distributions may possess rich properties. It is often easy to verify, as the examples below show, Strong Stability for such smooth problems. The following elementary result, based on Taylor expansion, plays a crucial role in these examples.

PROPOSITION 3.2

Suppose $L(\theta, t)$ is differentiable in t with derivative $L'(\theta, t)$. Then for any sequence $P_n \Rightarrow Po$, $\{\delta_{P_n}\}$ is bounded if there exists a compact set $K \subset \Theta$ such that

$$(i) \quad \sup_{\theta \in K} \frac{\mid L'(\theta, t) \mid}{L(\theta, t)} = O\left(\frac{1}{|t|}\right) \text{ for all large } t$$

and

$$(ii) \liminf_{n \to \infty} \inf_{t \in D} \int_K \mid L'(\theta, t) \mid \ell(\theta \mid x) dP_n(\theta) > 0.$$

PROOF: Assume the conditions and suppose $\mid \delta_{P_n} \mid \to \infty$. Then, by condition (i), there exists a constant $c > 0$ such that for all sufficiently large n

$$\begin{aligned} \int_K \mid L'(\theta, \delta_{P_n}) \mid \ell(\theta \mid x) dP_n(\theta) &\leq \mid \frac{C}{\delta_{P_n}} \mid \int L(\theta, \delta_{P_n}) \ell(\theta \mid x) dP_n(\theta) \\ &\leq \frac{C}{\mid \delta_{P_n} \mid} \int L(\theta, \delta_0) \ell(\theta \mid x) dP_n(\theta). \\ &\leq \frac{MC}{\mid \delta_{P_n} \mid} \end{aligned}$$

where $M = \sup_{\theta \in \Theta} L(\theta, \delta_o) \ell(\theta \mid x)$. Therefore

$$\mid \delta_{P_n} \mid \leq \frac{MC}{\liminf_{n \to \infty} \inf_{t \in D} \int_K \mid L'(\theta, t) \mid \ell(\theta \mid x) dP_n(\theta)} < \infty$$

for all sufficiently large n. This contradiction completes the proof. □

The condition (i) on the loss function is satisfied by all the commonly used loss functions. It is a tail condition which prevents the loss from approaching zero as $\mid t \mid \to \infty$ for θ in a compact set.

This proposition, as the following examples illustrate, leads to an easily verifiable sufficient condition for Strong Stability in "smooth" neighborhoods of prior distributions popular in robustness analysis. The reader is referred to Berger (1994) for a discussion of these. In the examples that follow, $L(\theta, t)$ is jointly lower semi-continuous, differentiable in $t \in D$, and satisfies the condition (i) of the proposition.

EXAMPLE 1 (ϵ-CONTAMINATION CLASS) Let

$$\mathcal{P}_0 = \left\{ P_Q^\epsilon \equiv P_0 + \epsilon(Q - P_0) : \ 0 \le \epsilon < 1, \ Q \in \tilde{\boldsymbol{Q}} \right\}$$

where $\tilde{\boldsymbol{Q}}$ is a class of probability distributions on θ and introduce the mode of convergence $P_Q^\epsilon \to P_0$ if $\epsilon \to 0$. Clearly, this implies $P_Q^\epsilon \Rightarrow P_0$ as $\epsilon \to 0$. Under this notion of convergence,

$$\liminf_{\epsilon \to 0} \inf_{t \in D} \int_K |L'(\theta,t)|\, l(\theta|x) dP_Q^\epsilon(\theta) > \frac{1}{2} \inf_{t \in D} \int_K |L'(\theta,t)|\, l(\theta|x) dP_0 .$$

Therefore, (L, l, P_0) is Strongly Stable for an ϵ-contamination neighborhood if

$$\inf_{t \in D} \int_K |L'(\theta,t)|\, l(\theta|x) dP_0(\theta) > 0 . \tag{10}$$

Another class of prior distributions extensively used in the robustness literature is the Density Ratio Class, first introduced by DeRobertis and Hartigan (1981). The following addresses Strong Stability in this case.

EXAMPLE 2 (DENSITY RATIO CLASS) Let U and V be two finite positive measures on Θ satisfying $V \le U$ i.e., for every borel set $B \subset \Theta$, $V(B) \le U(B)$. The density ratio class of probability distributions Γ_{DR}, is given by

$$\Gamma_{DR} = \left\{ P_\mu \equiv \frac{\mu}{\mu(\Theta)} : \ V \le \mu \le U \right\} . \tag{11}$$

Equip Γ_{DR} with weak convergence and suppose $P_0 \in \Gamma$. Let $\{P_{\mu_n}\}$ be a sequence from Γ such that $P_{\mu_n} \Rightarrow P_0$. Then for every n,

$$\int_K |L'(\theta,t)|\, l(\theta|x) dP_{\mu_n}(\theta) \ge \frac{\int_K |L'(\theta,t)|\, l(\theta|x) dV(\theta)}{U(\Theta)}$$

and hence

$$\liminf_{n \to \infty} \inf_{t \in D} \int_K |L'(\theta,t)|\, l(\theta|x) dP_{\mu_n}(\theta) \ge \inf_{t \in D} \frac{\int_K |L'(\theta,t)|\, l(\theta|x) dV(\theta)}{U(\Theta)} .$$

Therefore, a sufficient condition for Strong Stability of (L, l, P_0) under Density Ratio neighborhoods is

$$\inf_{t \in D} \int_K |L'(\theta,t)|\, l(\theta|x) dV(\theta) > 0 . \tag{12}$$

The next example treats the class of unimodal prior distributions with specified quantiles discussed in O'Hagan and Berger (1988).

EXAMPLE 3 (UNIMODAL QUANTILE CLASS) Let the parameter space $\Theta = \boldsymbol{R}$ be partitioned into m intervals $I_1, I_2, \ldots, I_m$ where $I_j = [a_{j-1}, a_j]$, $1 \le j \le m$. Assume, without loss of generality, $a_0 < a_1 < \cdots < a_m$. Also, let $\gamma = (\gamma_1, \ldots, \gamma_m)$ be an element of m simplex (i.e., $\gamma_i \ge 0$, $\sum_1^m \gamma_i = 1$) such that the constants $q_i = \frac{\gamma_i}{(a_i - a_{i-1})}$, $i = 1, 2, \ldots, m$, satisfy the condition

$$q_1 \le q_2 \le \cdots q_{k-1} \le q_k \ge q_{k+1} \ge \cdots \ge q_m \tag{13}$$

for some k. Assume $q_2 > 0$ and $q_{m-1} > 0$.

The unimodal quantile class $\Gamma_{U,\gamma}$ consists of all absolutely continuous unimodal prior distributions with a_i as $\left(\sum_1^i \gamma_j\right)$ 100 percentile, i.e.,

$$\Gamma_{U,\gamma} = \left\{ p(\theta) : \int_{a_{i-1}}^{a_i} p(\theta) d\theta = \gamma_i,\ 1 \le i \le m \text{ and } p \text{ is unimodal} \right\}.$$

This class is non-empty due to the condition (14). Moreover, every $p \in \Gamma_{U,\gamma}$ has (see O'Hagan and Berger, 1988) the following properties:

(i) p is increasing in every interval I_j, $j \le k-1$;

(ii) p is decreasing in every interval I_j, $j \ge k+1$;

(iii) $p(a_{j-1}) \ge q_{j-1}$ for $j \le k-1$;

(iv) $p(a_j) \ge q_{j+1}$ for $j \ge k+1$;

(v) p is unimodal in I_k and $p(a_{k-1}) \ge q_{k-1}$, $p(a_k) \ge q_{k+1}$.

Equip $\Gamma_{U,\gamma}$ with weak convergence. Suppose $P_0 \in \Gamma_{U,\gamma}$ and $P_n \Rightarrow P_0$ where $P_n \in \Gamma_{U,\gamma}$. Letting p_n denote the density of P_n, the above properties imply

$$\begin{aligned}\int_K |L'(\theta,t)|\, l(\theta|x) p_n(\theta) d\theta \ &\ge \sum_{j=1}^{k-1} q_{j-1} \int_{K \cap I_j} |L'(\theta,t)|\, l(\theta|x) d\theta \\ &\quad + \sum_{j=k+1}^{m} q_{j+1} \int_{K \cap I_j} |L'(\theta,t)|\, l(\theta|x) d\theta \\ &\quad + \min(q_{k-1}, q_{k+1}) \int_{K \cap I_k} |L'(\theta,t)|\, l(\theta|x) d\theta .\end{aligned}$$

Therefore, setting $q^* = \min(q_2, q_3, \ldots, q_{m-1})$,

$$\inf_t \int_K |L'(\theta,t)|\, l(\theta|x) p_n(\theta) d\theta \ge q^* \inf_t \int_{(a_1, a_{m-1}) \cap K} |L'(\theta,t)|\, l(\theta|x) d\theta .$$

Hence, (L, l, P_0) is Strongly Stable for the unimodal quantile class if

$$\inf_t \int_{K \cap (a_1, a_m)} |L'(\theta, t)| \, l(\theta|x) d\theta > 0 \, .$$

The last example considers Gustafson's classes (P. Gustafson, 1994) of prior distributions. These classes possess many interesting properties and are well suited for local sensitivity analysis.

EXAMPLE 4 (GUSTAFSON'S CLASSES) Suppose P_0 has density p_0 with respect to the Lebesgue measure. Gustafson's linear class is given by

$$G_L^q(p_0; \alpha) = \left\{ \frac{p_0 + u}{1 + \int u(\theta) d\theta} : \ u \geq 0, \ \int_\Theta \left(\frac{u}{p_0} \right)^q dP_0 \leq \alpha^q \right\}$$

where $1 \leq q < \infty$. Clearly $P_0 \in G_L^q(P_0; \alpha)$. Suppose $P_n \in G_L^q(P_0; \alpha)$ and is given by the density $\frac{p_0 + u_n}{1 + \int u_n(\theta) d\theta}$. The mode for convergence introduced by Gustafson for this class is $P_n \to P_0$ if $\int \left(\frac{u_n}{p_0} \right)^q dP_0 \to 0$ as $n \to \infty$. This is a fairly strong notion of convergence and, in particular, implies $P_n \Rightarrow P_0$. Suppose $P_n \to P_0$. Then, by Jensen's inequality, for all large n

$$\begin{aligned} \inf_{t \in D} \int |L'(\theta, t)| \, l(\theta|x) dP_n(\theta) &\geq \inf_{t \in D} \int |L'(\theta, t)| \, l(\theta|x) \frac{p_0(\theta)}{1 + \int u_n(\theta) d\theta} d\theta \\ &\geq \frac{1}{2} \inf_{t \in D} \int_K |L'(\theta, t)| \, l(\theta|x) p_0(\theta) d\theta \end{aligned}$$

Hence,

$$\inf_{t \in D} \int_K |L'(\theta, t)| \, l(\theta|x) p_0(\theta) d\theta > 0 \tag{14}$$

implies (L, l, P_0) is Strongly Stable for the class $G_L^q(p_0; \alpha)$.

Gustafson also introduced a nonlinear class as a model for prior uncertainty. It can be shown that Strong Stability follows for the nonlinear class if (14) holds.

4. Differentiability of Bayes Decisions. A remarkable consequence of Strong Stability is the Gâteaux differentiability of the optimal decision with respect to the prior distribution when the loss is reasonably smooth. These derivatives have applications in global and local sensitivity analyses of the underlying decision problem.

In this section we establish the differentiability of the optimal Bayes procedures for a special class of loss functions and sketch the application to global sensitivity analysis. The general results in this direction are more technically involved and will appear elsewhere.

Suppose $\Theta = D = \boldsymbol{R}$ and $L(\theta, \delta) \equiv L(\theta - \delta)$ is a strictly convex twice continuously differentiable loss function. In addition, assume the following conditions hold.

(A1) For each δ, $L'(\theta - \delta)l(\theta|x)$ and $L''(\theta - \delta)l(\theta|x)$ are bounded in θ. Here L' and L'' are, respectively, the first and second derivatives of L.

(A2) The level sets of L are bounded.

The next two propositions establish the smoothness of the optimal decisions δ_{P_n} of (L, l, P_n) for a sequence of priors P_n converging weakly to P_0.

PROPOSITION 4.1 Suppose $P_n \Rightarrow P_0$ as $n \to \infty$ and $\{\delta_{P_n}\}$ are bounded. Then $(\delta_{P_n} - \delta_{P_0}) \to 0$ as $n \to \infty$ and $(\delta_{P_n} - \delta_{P_0}) = O(P_n - P_0)$.

PROOF: Let B be a compact convex set containing $\{\delta_{P_n}\}$ and δ_0. Since $L''(\theta - \delta)$ is continuous, it is uniformly continuous on any compact set and, hence, the family of functions $\{L''(\cdot - t) : t \in B\}$ are equi-continuous at every $\theta \in \Theta$. Therefore, by a result of Rao on uniformity classes (Bhattacharya and Rao, 1975, Corollary 2.7),

$$\lim_{n\to\infty} \sup_{\delta \in B} \left| \int L''(\theta - \delta)l(\theta|x)dP_n(\theta) - \int L''(\theta - \delta)l(\theta|x)dP_n(\theta) \right| = 0\,. \tag{15}$$

Now, using the Taylor expansion,

$$\begin{aligned} &\int L'(\theta - \delta_{P_0})l(\theta|x)dP_n(\theta) \\ &= (\delta_{P_0} - \delta_{P_n}) \int L''(\theta - \delta_n^*)l(\theta|x)dP_n(\theta) \\ &= (\delta_{P_0} - \delta_{P_n}) \left[\int L''(\theta - \delta_n^*)l(\theta|x)dP_0(\theta) + \int L''(\theta - \delta_n^*)l(\theta|x)d(P_n - P_0)(\theta) \right]. \end{aligned} \tag{16}$$

Take limit as $n \to \infty$ and note that the left side of (16) converges by (A1) to $\int L'(\theta - \delta_{P_0})l(\theta|x)dP_0(\theta) = 0$. Also, by (15), the second term on the right side converges to zero. Combining these facts, $\delta_{P_n} \to \delta_{P_0}$ follows from the assumption $L(\theta - \delta)$ is strictly convex and, hence, $L''(\theta - \delta)$ is everywhere positive continuous function.

Finally, $(\delta_{P_n} - \delta_{P_0}) = O(P_n - P_0)$ follows from (16) since

$$\begin{aligned} (\delta_{P_n} - \delta_{P_0}) &= O\left(\int L'(\theta - \delta_{P_0})l(\theta|x)dP_n(\theta) \right) \\ &= O\left(\int L'(\theta - \delta_{P_0})l(\theta|x)d(P_n - P_0)(\theta) \right) \\ &= O(P_n - P_0) \end{aligned}$$

by the assumption (A1) and the strict convexity of L.

PROPOSITION 4.2 Suppose (L, l, P_0) is Strongly Stable. Then for any sequence $P_n \Rightarrow P_0$, the sequence $\{\delta_{P_n}\}$ is bounded.

PROOF: Let $\rho_n(\delta)$ and $\rho_0(\delta)$ denote the integrated risks with respect to P_n and P_0, respectively. By assumption (A2) the strictly convex function $\rho_0(\delta)$ has bounded level sets and, therefore, it follows (see Rockafeller, 1967, Theorem 27.2) that for $\epsilon > 0$ there exists $\eta > 0$ such that $|\delta_\eta - \delta_{P_0}| < \epsilon$ for every η-minimal solution δ_η of ρ_0 i.e., $\rho_0(\delta_\eta) \le \rho_0(\delta_{P_0}) + \eta \Rightarrow |\delta_\eta - \delta_{P_0}| < \epsilon$.

Under the conditions of this section, it is known (Chuang, 1984, Kadane and Srinivasan, 1994) that (L, l, P_0) is Strongly Stable (i.e., SSI) if, and only if, it is Strongly Stable II. Therefore, by setting $Q_n = P_n$ and $P_n = P_0$ in the definition of SSII (see Section 3), $\rho_0(\delta_{P_n}) \to \rho_0(\delta_{P_0})$ as $n \to \infty$. Hence, for all sufficiently large n $\rho_0(\delta_{P_n}) \le \rho(\delta_{P_0}) + \eta$ and, consequently, $|\delta_{P_0} - \delta_{P_0}| < \epsilon$. This concludes the proof.

A version of the preceding proposition could have been established directly from Salinetti's theorem, guaranteeing the existence of a bounded subsequence of $\{\delta_{P_n}\}$. Such a result is sufficient for the purposes of this paper. We chose the above version to indicate the usefulness of Strong Stability II. Another point worth noting is, since ϵ is arbitrary, the convergence of δ_{P_n} to δ_{P_0} follows from this proposition.

The main result of this section is the following theorem which establishes the differentiability of δ_{P_0} and gives a representation for its derivative.

THEOREM 4.3 *Suppose (L, l, P_0) is Strongly Stable. Then δ_{P_0} is Gâteaux differentiable and the derivative in the direction of $Q \in \mathcal{P}$ is*

$$\delta'_{P_0}(Q) = \frac{-2 \int L'(\theta - \delta_{P_0}) l(\theta|x) dQ(\theta)}{\int L''(\theta - \delta_{P_0}) l(\theta|x) dP_0(\theta)}.$$

PROOF: Let $\{P_n\}$ be a sequence of priors, converging to P_0 (i.e., $P_n \Rightarrow P_0$), of the form $P_n \equiv (1 - \epsilon_n)P_0 + \epsilon_n Q$ where Q is some probability distribution on Θ and $\epsilon_n \downarrow 0$ as $n \to \infty$. Then, by Taylor series and Propositions 4.1 and 4.2,

$$\int L(\theta - \delta_{P_n}) l(\theta|x) dP_n(\theta) - \int L(\theta - \delta_{P_0}) l(\theta|x) dP_n(\theta)$$
$$= (\delta_{P_n} - \delta_{P_0}) \int L'(\theta - \delta_{P_0}) l(\theta|x) dP_n + \tfrac{(\delta_{P_n} - \delta_{P_0})^2}{2} \int L''(\theta - \delta_{P_0}) l(\theta|x) dP_n + R_n$$

where R_n, the remainder term, is $O\left((\delta_{P_n} - \delta_{P_0})^2\right)$. Therefore, by Strong Stability of (L, l, P_0), the left-side goes to zero as $n \to \infty$ and it is of the order $O\left((\delta_{P_n} - \delta_{P_0})^2\right)$ because $O(P_n - P_0) = O(\delta_{P_n} - \delta_{P_0})$, and $\int L'(\theta -$

$\delta_{P_0})l(\theta|x)dP_0(\theta) = 0$. Divide by $(\delta_{P_n} - \delta_{P_0})$ to conclude

$$\lim_{n\to\infty}\left[\int L'(\theta-\delta_{P_0})l(\theta|x)dP_n(\theta) + \frac{(\delta_{P_n}-\delta_{P_0})}{2}\int L''(\theta-\delta_{P_0})l(\theta|x)dP_n(\theta)\right] = 0\,.$$

Now substitute the representation $P_n = (1-\epsilon_n)P_0 + \epsilon_n Q$, appeal to the fact $\int L'(\theta-\delta_{P_0})l(\theta|x)dP_0 = 0$ and the assumption $\int L''(\theta-\delta_{P_0})l(\theta|x)$ is bounded continuous to get

$$\lim_{n\to\infty}\left[\epsilon_n\frac{2\int L'(\theta-\delta_{P_0})l(\theta|x)dQ(\theta)}{\int L''(\theta-\delta_{P_0})l(\theta|x)dP_0(\theta)} + (\delta_{P_n}-\delta_{P_0})\right] = 0\,,$$

i.e.,

$$\delta'_{P_0}(Q) = \lim_{n\to\infty}\frac{\delta_{P_n}-\delta_{P_0}}{\epsilon_n} = \frac{2\int L'(\theta-\delta_{P_0})l(\theta|x)dQ(\theta)}{\int L''(\theta-\delta_{P_0})l(\theta|x)dQ(\theta)}\,.$$

This establishes the differentiability as well as the representation of the derivative.

As indicated earlier, the derivative can be used in global robustness analysis of the optimal decisions. The global analysis involves computation of the range of a relevant posterior quantity as the prior varies in a class of prior distributions, say Γ, and making judgments about the robustness of the posterior quantity. The derivative of a posterior quantity can be used to derive an algorithm called "Linearizing Algorithm" to compute its range. The algorithm is essentially an infinite dimensional linear optimization.

The following elementary fact plays a crucial role in this development. Let Γ be a class of prior distributions and $T : \Gamma \to R$ be a functional. Assume T has extrema in Γ with $\bar{T} = T(\bar{P}) = \max_{P\in\Gamma} T(P)$, and $\underline{T} = T(\underline{P}) = \min_{P\in\Gamma} T(P)$.

PROPOSITION 4.4 Suppose $T(P)$ is Gâteaux differentiable everywhere in Γ and, for any $P_1, P_2 \in \Gamma$, $0 < \alpha < 1$,

$$(*)\quad \min(T(P_0), T(P_2)) \le T(\alpha P_1 + (1-\alpha)P_2) \le \max(T(P_1), T(P_2))\,.$$

Then

$$\inf_{P\in\Gamma} T'_{\underline{P}}(P - \underline{P}) = \inf_{P\in\Gamma} T'_{\underline{P}}(P) = 0$$

and

$$\sup_{P\in\Gamma} T'_{\bar{P}}(P - \bar{P}) = \sup_{P\in\Gamma} T'_{\bar{P}}(P) = 0\,.$$

PROOF: See Srinivasan and Truszczynska (1992).

If Γ is convex, the condition $(*)$ in the above proposition can be dispensed with and the conclusion holds without any additional conditions.

Suppose now the problem (L, l, P) is Strongly Stable at every $P \in \Gamma$ and the interest is in the range of δ_P as P varies over Γ. Let $\bar{\delta}$ and $\underline{\delta}$ be the maximum and minimum of δ_P, respectively.

THEOREM 4.5 *Assume either the condition (*) holds for δ_P or Γ is convex. Suppose (L, l, P) is Strongly Stable for every $P \in \Gamma$. Then*

(i) $\bar{\delta}$ is given by the solution of

$$\inf_{Q \in \Gamma} \int L'(\theta, t) l(\theta|x) dQ(\theta) = 0$$

and

(ii) $\underline{\delta}$ is given by the solution of

$$\sup_{Q \in \Gamma} \int L'(\theta, t) l(\theta|x) dQ(\theta) = 0$$

PROOF: By Strong Stability, δ_P is differentiable at every $P \in \Gamma$. The theorem follows by appealing to Proposition 4.4.

Thus the computation of $\bar{\delta}$ ($\underline{\delta}$) is reduced to minimization (maximization) of a linear function and solving for "t" in the minimum. Since $L(\theta - t)$ is strictly convex, $L'(\cdot)$ has exactly one sign change and this makes the search for the solution fairly easy.

In the context of global analysis the authors (Kadane and Srinivasan, 1994b) have argued for examining the range of the optimal risk. The authors have preliminary results in this direction and are currently examining their computational feasibility.

5. Local Robustness. Stability, in addition to providing qualitative results regarding the "robustness" of a decision problem, offers tools for quantitative assessment of sensitivity. The purpose of this section is to briefly sketch a program, currently pursued by the authors, that may lead to a formal theory of quantitative Bayesian robustness.

Suppose the (L, l, P) is Strongly Stable at $P_0 \in \Gamma$, a class of prior distributions and the optimal decision is differentiable. Since the derivative quantifies the local sensitivity, a natural local index of decision robustness is

$$I(P_0; \Gamma) = \sup_{Q \in \Gamma} \left| \delta'_{P_0}(Q) \right| .$$

This is analogous to the "Influence Function" in the frequentist robustness theory. If the index $I(P_0; \Gamma)$ is not too "large", the decision δ_{P_0} may be considered "robust" for the class Γ. This interpretation immediately suggests the problem of finding decisions with smaller index of robustness if $I(P_0; \Gamma)$ is too "large". To formulate a well-posed problem, one has to eliminate from consideration "constant" decisions with zero derivative which are, in a sense, trivial procedures. A way to avoid these trivial decisions is to focus attention only on optimal decisions. Define

$\mathcal{D} = \{\delta_{P,L}$: $\delta_{P,L}$ is the optimal decision of (L, l, P) for some $P \in \mathcal{P}$ and $L\}$.

Restricting attention to the decisions in the class $\mathcal{D}$, the problem can be posed as follows:

"Given $0 < \alpha < 1$ and $\epsilon > 0$, find a $\delta_1 \in \mathcal{D}$ such that

(17) $I(\delta_1; \Gamma) \leq \alpha I(\delta_{P_0}; \Gamma)$;

and

(18) δ_1 is an ϵ-optimal decision of (L, l, P_0)."

The problem as posed above may not have, in general, a solution if α and ϵ are too small. However, as the following example illustrates, it has a solution for moderate values of α and ϵ if $I(P_0; \Gamma)$ is "large".

EXAMPLE 5.1 Let $L(\theta - \delta) = (\theta - \delta)^2$, $l(\theta|x) = e^{-\frac{1}{4}(\theta-x)^2}$ and P_0 have density $p_0(\theta) = e^{-\frac{1}{4}\theta^2}$. The optimal decision of (L, l, P_0) is $\delta_{P_0}(x) \equiv \frac{x}{2}$ and its derivative in the direction of Q is

$$\delta'_{P_0}(Q) = \frac{-\int \left(\theta - \frac{x}{2}\right) e^{-\frac{1}{4}(\theta-x)^2} dQ(\theta)}{\int e^{-\frac{1}{4}(\theta-x)^2} e^{-\frac{1}{4}\theta^2} d\theta} .$$

Consider now the loss

$$L_\gamma(\theta - t) = (1-\gamma)(\theta - t)^2 + \gamma|\theta - t|, \quad 0 \leq \gamma \leq 1 .$$

For any absolutely continuous prior P, the optimal decision $\delta_{\gamma,P}$ of (L_γ, l, P) is given by the solution of

$$-2(1-\gamma)\int(\theta - t)l(\theta|x)dP(\theta) + \frac{\gamma}{2}\left[\int_{-\infty}^{t} l(\theta|x)dP(\theta) - \int_{t}^{\infty} l(\theta|x)dP(\theta)\right] = 0$$

and, therefore, $\delta_{\gamma,P_0} = \frac{x}{2} = \delta_{P_0}$. Consequently, δ_{γ,P_0} is (L, l, P_0) optimal. Moreover, δ_{γ,P_0} is differentiable (Srinivasan and Truszczynska,1992) and its derivative has the representation

$$\delta'_{\gamma,P_0}(Q) = C(\gamma)\delta'_{P_0}(Q) + (1 - C(\gamma))\,\delta'_{M,P_0}(Q)$$

with

$$C(\gamma) = \frac{(1-\gamma)\int l(\theta|x)dP_0(\theta)}{\left[(1-\gamma)\int l(\theta|x)dP(\theta) + \frac{\gamma}{2} l\left(\frac{x}{2}\right) p_0\left(\frac{x}{2}\right)\right]}$$

and

$$\delta'_{M,P_0}(Q) = \frac{\frac{1}{2}\left[\int_{-\infty}^{x/2} l(\theta|x)dQ - \int_{x/2}^{\infty} l(\theta|x)dQ\right]}{l\left(\frac{x}{2}\right) p_0\left(\frac{x}{2}\right)} .$$

Hence, for any Q,

$$\left|\delta'_{\gamma,P_0}(Q)\right| \leq C(\gamma)\left|\delta'_{P_0}(Q)\right| + (1 - C(\gamma))\left|\delta'_{M,P_0}(Q)\right|$$

and

$$I(\delta_{\gamma,P_0};\Gamma) \leq C(\gamma) I(\delta_{P_0};\Gamma) + (1 - C(\gamma)) I(\delta_{M,P_0};\Gamma) .$$

This implies,

(19) $$I(\delta_{\gamma,P_0};\Gamma) < I(\delta_{P_0};\Gamma)$$

if

(20) $$I(\delta_{M,P_0};\Gamma) < I(\delta_{P_0};\Gamma) .$$

Now suppose Γ is the class of all unimodal densities with mode at zero. Then (20) always holds and δ_{γ,P_0} is more robust compared to δ_{P_0}. If, on the other hand, Γ is the class of all symmetric unimodal densities, then for small values of $|x|$, the two indices in (20) are close, and as $|x|$ gets larger, δ_{M,P_0} becomes more robust relative to δ_{P_0}. Thus, by appropriately choosing γ, it is possible to construct a more robust procedure, relative to δ_{P_0}, with risk close to that of δ_{P_0} when (L, l, P_0) holds.

Another important issue in the context of the above problem is the existence of the most Γ-robust decision among all decisions in $\mathcal{D}$ (or a smooth subclass of $\mathcal{D}$) satisfying (18). The following example indicates the possibility of finding such a most robust decision. The example is based on a result due to Trusczynska (1990).

EXAMPLE 5.2 Let $l(\theta|x)$ be a likelihood with maximum likelihood estimator $\hat{\theta}$ and P_0 be an absolutely continuous prior with positive density. Consider the loss $L_{A_0}(\theta,\delta) = (I_{A_0}(\theta) - \delta)^2$. The optimal decision of (L_{A_0}, l, P_0) is $\delta_{P_0,A_0} = \int I_{A_0}(\theta)dP_0^x(\theta) \equiv \gamma_0$. For any probability Q on Θ, the derivative of δ_{P_0,A_0} in the direction of Q is

$$\delta'_{P_0,A_0}(Q) = \frac{\int (I_{A_0} - \gamma_0) l(\theta|x) dQ(\theta)}{m_{P_0}(x)} .$$

and its index of robustness over the class of all priors is

$$I(\delta_{P_0,A_0}) = \sup_Q \left|\delta'_{P_0,A_0}(Q)\right| .$$

Suppose we want to find the most robust decision, say δ^*, among all decisions $\delta \in \mathcal{D}$ satisfying

$$\int L_{A_0}(\theta,\delta^*)dP_0^x(\theta) = \int L_{A_0}(\theta,\delta_{P_0,A_0})dP_0^x(\theta)$$

i.e.,

$$I(\delta^*) = \inf_{\delta \in D_{\gamma_0}} I(\delta)$$

where $\mathcal{D}_{\gamma_0} = \{\delta_{P_0,A}\colon \delta_{P_0,A}$ is (L_A, l, P_0)-optimal and $\delta_{P_0,A} = \gamma_0\}$.
It is proved in Trusczynska (1990) that δ^* exists and is given by the likelihood region

$$A^* = \left\{\theta : \frac{l(\theta|x)}{l(\hat{\theta}|x)} \geq C\right\}$$

where C is such that $\int I_{A^*}(\theta) dP_0^x(\theta) = \gamma_0$. Thus, the most robust decision on the basis of the "local" index I is $\delta^* = \delta_{P_0,A^*}$. Interestingly, δ_{P_0,A^*} is also the globally "robust" (Wasserman, 1988) in the sense that it has the smallest range among all decisions in $\mathcal{D}_{\gamma_0}$.

The authors have developed general results based on the ideas of the preceding examples and they will appear, hopefully, in the near future.

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Bayesian Robustness and Stability

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For a Bayesian Decision Problem (BDP), Strong Stability (SS), in its different types and versions as introduced by Kadane and Chuang (1978) in the statistical, context, is a relevant formalization of qualitative robustness which analyses the stability of the problem through the behavior of the posterior expected loss.

In the more specific situation considered in the paper, in presence of uncertainty in the elicitation of the prior P_0, with fixed loss L and likelihood ℓ, the decision problem (L, ℓ, P_0) is *strongly stable* if for any sequence $\{P_n\}$ of probability measures weakly converging to $P_0, P_n \to P_0$, we have

$$\limsup_{n\to\infty}[\rho(P_n, \delta_{P_0}) - \inf_{\delta\in D} \rho(P_n, \delta)] = 0,$$

where δ_{P_0} is the Bayes decision of (L, ℓ, P_0) and $p(P, \delta)$ denotes the posterior expected loss of a decision δ of the problem (L, ℓ, P).

This amounts to say that for every $\epsilon > 0$, δ_{P_0}, optimal for (L, ℓ, P_0), is ϵ-optimal for all but finitely many (L, ℓ, P_n). Since, under reasonable conditions we also have $\rho(P_n, \delta_{P_0}) \to \rho(P_0, \delta_{P_0}$, SS is also equivalent to

$$\left|\inf_{\delta\in D} \rho(P_0, \delta) - \inf_{\delta\in D} \rho(P_n, \delta)\right| < \epsilon,$$

which explicitly connects the stability to the continuity of the posterior expected loss of the problem.

Along these lines the paper sensibly enlarges the set of sufficient conditions for SS of a BDP, provides "applicable" sufficient conditions and, based on them, verifies SS for the most interesting neighborhood classes of priors considered in Bayesian robustness.

Starting from SS as qualitative robustness, the paper proceeds then towards the quantitative analysis of robustness and delineates interesting developments along which constructing a formal, local and global, theory of robustness for BDP's.

In this direction the key result is the Gateaux differentiability at P_0 of the Bayesian functional $P \to \delta_P$ mapping the prior P into the Bayesian decision δ_P for (L, ℓ, P). The result is derived from SS when the loss function is reasonably smooth (Theorem 4.3). A "natural" measure of local sensitivity with respect to a class Γ containing P_0 is then proposed, based on the differential $\delta'_{P_0}(\cdot)$, in the form $\sup_{Q\in\Gamma} \mid \delta'_{P_0}(Q) \mid$.

As first comment, technical in a sense, it can be of some interest to evidentiate that SS (with all its topological involvements) can be an unnecessary requirement to get a directional derivative. In fact, in the conditions of Theorem 4.3, without appealing to SS, but simply referring to the structure of the problem $\inf_{\delta \in D} \rho(P_c, \delta) \to \inf_{\delta \in D} \rho(P_c, \delta)$, one can get

$$\inf_{\delta \in D} \rho(P_c, \delta) \to \inf_{\delta \in D} \rho(P_0, \delta) \text{ and } \delta_{P_c} \to \delta_{P_0}$$

where $P_c = (1 - \epsilon)P_0 + \epsilon Q$. These relations and the regularity assumptions on the loss L and its first and second derivatives give the Gateaux differential of $P \to_P$ at P_0.

However it has to be added that the type of argument used in the proof of the theorem and based on SS, could actually lead to a stronger differential, consistent with the metric chosen in the space of the priors to formalize the uncertainty about P_0.

The second comment concerns the measure $\sup_{Q \in \Gamma} \mid \delta'_{P_0}(Q) \mid$ proposed for the local sensitivity and based on the variations of the Bayesian decision. In fact, coherently with the decision scheme where decisions are compared and chosen for their losses, and in line with the corresponding notion of SS as qualitative robustness and based on the behavior of the posterior expected loss, it could be more appropriate to measure the local sensitivity of the BDP through the differential of a Bayesian functional related to the posterior expected loss.

In accordance with SS and in view of the observations at the beginning of this discussion, natural candidates could be the differential $\rho'_{P_0}(\cdot)$ of the Bayesian function $P \to \inf_{\delta \in D} \rho(P, \delta)$ and $\sup_{Q \in \Gamma} \mid \rho'_{P_0}(Q) \mid$ as measure of local sensitivity.

In fact it can be shown that, under rather mild conditions, certainly in the conditions of the paper, the functional $Prightarrow \inf_{\delta \in D} \rho(P, \delta)$ is differentiable and the differential is

$$\rho'_{P_0}(Q) = \frac{m(Q)}{m(P_0)}[\rho(Q, \delta_{P_0}) - \rho(P_0, \delta_{P_0})]$$

where $m(P) = f\ell dP$ (Conigliani and Salinetti, 1995). Rewritten as linear functional in the form

$$\rho'_{P_0} = \int \left[\frac{L(\theta, \delta_{P_0})}{m(P_0)} - \frac{\rho(P_0, \delta_{P_0})}{m(P_0)}\right] \ell(\delta) Q(d\theta),$$

computing the measure of local sensitivity $\sup_{Q \in \Gamma} \mid \rho'_{P_0}(Q) \mid$ substantially reduces to the maximization of a linear functional over Γ.

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REJOINDER

J. Kadane and C. Srinivasan

We thank Gabriella Salinetti for her careful reading of our paper, and for her thoughtful comments. Taking each point in turn,

1. We agree that strong stability is redundant for Theorem 4.3 . However, Theorem 4.3 remains true if (A1) and (A2) are relaxed but strong stability is maintained. Getting the conditions exactly right requires a longer and more technical treatment than what we were able to do here.

2. We also agree that robustness with respect to the expected loss is a very important criterion for Bayesian robustness in a decision-theoretic context. In fact, we have argued this ourselves in Kadane and Srinivasan (1994b).

3. We agree with the result cited from Conigliani and Salinetti (1995). This result can also be found in Truszczynska (1990) and in Srinivasan and Truszczynska (1992), where it is Lemma 4.2.

We believe that the study of strong stability promises considerable possibilities for further progress in understanding Bayesian robustness.