A NOTE ON ESTIMATORS OF GRADUAL CHANGES

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In the present paper we focus on the estimators in location models with gradual changes described by power α that can be known or unknown. Least squares type estimators of the parameters are studied. It appears that the limit behavior (both the rate of consistency and limit distribution) of the estimators of the change point in location models depends on the type of gradual changes.

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1 Introduction and main results

Consider the location model with a change after an unknown time point m_n:

$$Y_i = \mu + \delta_n\Big(\frac{i-m_n}{n}\Big)_+^{\alpha_0} + e_i,\ i=1,\dots,n, \tag{1.1}$$

where $a_+ = \max(0,a)$, $\mu, \delta_n \neq 0$ and $m_n (< n)$ and $\alpha_0 \in [0,1]$ are unknown parameters. We assume that

$$\begin{aligned} &e_1,\dots,e_n \text{ are independent identically distributed random variables}\\ &Ee_i = 0, \quad 0<\sigma^2<\infty, \quad E|e_i|^{2+\Delta}<\infty \text{ with some } \Delta>0 \end{aligned} \tag{1.2}$$

and

$$m_n = [\gamma n] \text{ with some } \gamma \in (0,1), \tag{1.3}$$

where $[a]$ denotes the integer part of a. Concerning the slope parameter δ_n, we assume that, as $n \to \infty$,

$$|\delta_n| \to 0, \quad \frac{\sqrt{n}|\delta_n|}{\sqrt{\log\log n}} \to \infty, \tag{1.4}$$

which covers local alternatives ($\delta_n \to 0$), and if $\alpha_0 \neq 0$ also fixed alternatives ($\delta_n = \delta \neq 0$).

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We study the least squares type estimators $\widehat{\mu}_n, \widehat{\delta}_n, \widehat{\alpha}_n, \widehat{m}_n$ of the parameters $\mu, \delta_n, \alpha_0, m_n$. These are defined as the minimizers over $\mu \in R$, $\delta \in R$, $\alpha \in [0,1]$ and $k = 1, ..., n$ of

$$\sum_{i=1}^{n} \Big(Y_i - \mu - \delta\Big(\frac{i-k}{n}\Big)_+^{\alpha}\Big)^2.$$

Straightforward (but tedious) calculations give that

$$\widehat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n}\Big(Y_i - \widehat{\delta}_n \frac{1}{n}\Big(\frac{i-\widehat{m}_n}{n}\Big)_+^{\widehat{\alpha}_n}\Big) \tag{1.5}$$

$$\widehat{\delta}_n = \frac{\sum_{i=1}^{n} x_{i,\widehat{m}_n}(\widehat{\alpha}_n) Y_i}{\sum_{i=1}^{n} x_{i,\widehat{m}_n}^2(\widehat{\alpha}_n)}. \tag{1.6}$$

while $\widehat{\alpha}_n$ and $\widehat{m}_n$ are the maximizers over $\alpha \in [0,1]$ and $k = 1, ..., n-1$ of

$$\frac{|\sum_{i=1}^{n} x_{ik}(\alpha) Y_i|}{\big(\sum_{i=1}^{n}(x_{ik}(\alpha))^2\big)^{1/2}}, \tag{1.7}$$

where

$$x_{ik}(\alpha) = \Big(\frac{i-k}{n}\Big)_+^{\alpha} - \frac{1}{n}\sum_{j=1}^{n}\Big(\frac{j-k}{n}\Big)_+^{\alpha}, \quad i = 1, ..., n.$$

If there are more solutions we take the pair $\widehat{m}_n, \widehat{\alpha}_n$ with the smallest first component.

The parameter m_n is the change point and it is the parameter of main interest. The parameter α_0 characterizes the type of change (abrupt - $\alpha_0 = 0$ or gradual $\alpha_0 > 0$) and it is usually also of interest, whereas μ, δ_n and σ^2 are nuisance parameter.

The case when $\alpha_0 \in [0,1]$ is known has been studied in the past, e.g. Horváth and Csörgő (1997), Antoch and Hušková (1998) for survey results in case $\alpha_0 = 0$ and Hušková (1999) for $\alpha_0 \in (0,1]$. However, in reality α is usually unknown.

Related test procedures with $\alpha_0 = 0$ or $\alpha_0 = 1$ are studied in a number of papers, see e.g. Hinkley (1971), Lombard (1987), Jarušková (1998a,b), Csörgő and Horváth (1997), Siegmund and Zhang (1994).

We will study here the limit behavior of the estimators $\widehat{m}_n$ and $\widehat{\alpha}_n$ and compare it with the limit behavior of the corresponding estimators if only one of these parameters is unknown.

We use the notation, for $\gamma \in (0,1)$ and $\alpha \in (1/2, 1]$ (or for $[1,1]$ in the case of a_{11}),

$$A(\alpha,\gamma) = \Big(a_{ij}(\alpha,\gamma)\Big)_{i,j=1,2}, \tag{1.8}$$

$$a_{11}(\alpha,\gamma) = \Big(\int_0^1 (x-\gamma)_+^{2\alpha}\log^2(x-\gamma)_+dx$$

$$(1.9) \qquad - \Big(\int_0^1 (y-\gamma)_+^{\alpha}\log(y-\gamma)_+dy\Big)^2\Big)^2 +$$

$$- \frac{\Big(\int_0^1 (x-\gamma)_+^{2\alpha}\log(x-\gamma)_+dx - \int_0^1 (y-\gamma)_+^{\alpha}\log(y-\gamma)_+dy\int_0^1 (z-\gamma)_+^{\alpha}dz\Big)^2}{\int_0^1 (x-\gamma)_+^{2\alpha}dx - (\int_0^1 (y-\gamma)_+^{\alpha}dy)^2},$$

$$a_{22}(\alpha,\gamma) = \alpha^2\Big(\int_0^1 (x-\gamma)_+^{2\alpha-2}dx - \Big(\int_0^1 ((y-\gamma)_+^{\alpha-1}dy)^2\Big)$$

$$(1.10) \qquad - \alpha^2\frac{\Big(\int_0^1 (x-\gamma)_+^{2\alpha-1}dx - \int_0^1 ((y-\gamma)_+^{\alpha-1}dy\int_0^1 ((z-\gamma)_+^{\alpha}dz\Big)^2}{\int_0^1 (x-\gamma)_+^{2\alpha}dx - (\int_0^1 (y-\gamma)_+^{\alpha}dy)^2},$$

$$a_{12}(\alpha,\gamma) = a_{21}(\alpha,\gamma) = \alpha\Big(\int_0^1 (x-\gamma)_+^{2\alpha-1}\log(x-\gamma)_+dx$$

$$(1.11) \qquad - \Big(\int_0^1 (y-\gamma)_+^{\alpha-1}dy\Big)\Big(\int_0^1 (z-\gamma)_+^{\alpha}\log(z-\gamma)_+dz\Big)\Big)$$

$$- \alpha\Big(\int_0^1 (x-\gamma)_+^{2\alpha-1}dx - \Big(\int_0^1 ((y-\gamma)_+^{\alpha-1}dy\Big)\Big(\int_0^1 (z-\gamma)_+^{\alpha}dz\Big)\Big)$$

$$\times \frac{\int_0^1 (x-\gamma)_+^{2\alpha}\log(x-\gamma)_+dx - \int_0^1 (y-\gamma)_+^{\alpha}dy\int_0^1 (z-\gamma)_+^{\alpha}\log(z-\gamma)_+dz}{\int_0^1 (x-\gamma)_+^{2\alpha}\log(x-\gamma)_+dx - (\int_0^1 (y-\gamma)_+^{\alpha}dy)^2},$$

The integrals can be easily calculated, however, the resulting expressions are neither simpler nor more transparent, e.g. $a_{11}(0,\gamma) = \gamma(1-\gamma)(\log(1-\gamma)-1)^2 + 1 - \gamma - \log(1-\gamma) + (1-\gamma)^{-1}$.

In the following, $a^{ji}(\alpha,\gamma)$ denote the elements of the inverse matrix $A^{-1}(\alpha,\gamma)$.

We formulate the main results in three theorems that cover the cases. In Theorem 1.1 we consider $\alpha_0 \in [0,1/2)$, in Theorem 1.2 we consider $\alpha_0 = 1/2$ and finally in Theorem 1.3 the case $\alpha_0 \in (1/2,1]$.

Theorem 1.1 *($\alpha_0 \in [0,1/2)$) Let (1.1)-(1.4) be satisfied. If $\alpha_0 \in [0,1/2)$, then, as $n \to \infty$,*

$$(1.12) \qquad \left(\delta_n^2 n\right)^{1/(2\alpha_0+1)}\frac{\widehat{m}_n - m_n}{n\sigma^2} \xrightarrow{\mathcal{D}} V_{\alpha_0},$$

$$(1.13) \qquad \delta_n\sqrt{n}(\widehat{\alpha}_n - \alpha_0) \xrightarrow{\mathcal{D}} N(0,\sigma^2 a_{11}^{-1}(\alpha_0,\gamma)),$$

where

$$(1.14) \qquad V_\alpha = \arg\max\{W_\alpha(t) - \int_{-\infty}^{\infty} ((x+t)_+^{\alpha} - x_+^{\alpha})^2dx/2;\ t \in R\},$$

with $\{W_\alpha(t); t \in R\}$ being a Gaussian process with zero mean and covariance structure, for $s, t \in R$,

$$\text{cov}(W_\alpha(t), W_\alpha(s)) = \int_{-\infty}^{\infty} ((x+s)_+^\alpha - x_+^\alpha)((x+t)_+^\alpha - x_+^\alpha)dx, \tag{1.15}$$

If $\alpha_0 = 0$ then, as $n \to \infty$,

$$\delta_n^2 n \frac{\widehat{m}_n - m_n}{n\sigma^2} \xrightarrow{\mathcal{D}} V_0,$$

$$\delta_n \sqrt{n}(\widehat{\alpha}_n - \alpha_0) \xrightarrow{\mathcal{D}} \max(Y, 0),$$

where Y has distribution $N(0, \sigma^2 a_{11}^{-1}(0, \gamma))$. The estimators $\widehat{\alpha}_n$ and $\widehat{m}_n$ are asymptotically independent.

Theorem 1.2 ($\alpha_0 = 1/2$) *Let (1.1)–(1.4) be satisfied. If$\alpha_0 = 1/2$, then, as $n \to \infty$, (1.13) holds true and*

$$\delta_n \sqrt{n} \frac{\sqrt{\log(n - m_n)}}{2} \frac{\widehat{m}_n - m_n}{n} \xrightarrow{\mathcal{D}} N(0, \sigma^2). \tag{1.16}$$

The estimators $\widehat{\alpha}_n$ and $\widehat{m}_n$ are asymtotically independent.

Theorem 1.3 ($\alpha_0 \in (1/2, 1]$) *Let (1.1)–(1.4) be satisfied. If $\alpha_0 \in (1/2, 1)$, then, as $n \to \infty$,*

$$\delta_n \sqrt{n} \begin{pmatrix} \widehat{\alpha}_n - \alpha_0 \\ (\widehat{m}_n - m_n)/n \end{pmatrix} \xrightarrow{\mathcal{D}} N_2\Big(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2 A^{-1}(\alpha_0, \gamma)\Big), \tag{1.17}$$

where $A^{-1}(\alpha_0, \gamma)$ denotes the inverse matrix to $A(\alpha_0, \gamma)$. If $\alpha_0 = 1$, then, as $n \to \infty$,

$$\delta_n \sqrt{n}(\widehat{\alpha}_n - 1) \xrightarrow{\mathcal{D}} \min(M_1, 1)$$

and

$$\delta_n \sqrt{n}(\widehat{m}_n - m_n)/n \xrightarrow{\mathcal{D}} M_2,$$

where M_1 has distribution $N(0, \sigma^2 a_{22}^{-1}(1, \gamma))$ and M_2 has distribution $\frac{1}{2}N(0, \sigma^2 a_{22}^{-1}(1, \gamma)) + \frac{1}{2}N(0, \sigma^2 a^{22}(1, \gamma))$.

The proofs are postponed to the next section. Here we discuss various consequences of the stated results.

Note that the rate of consistency of $\widehat{\alpha}_n$ depends neither on α_0 nor m_n while the rate of consistency $\widehat{m}_n$ depends on α_0.

The best rate of consistency of $\widehat{\alpha}_n$ is reached for $\alpha_0 = 0$, and the worst one for $\alpha_0 \in (1/2, 1]$. This is in accordance with the results of Ibragimov and Hasminski (1981) concerning consistency of estimators in regular, almost regular and singular cases.

If α_0 is known then comparing the results of the present paper with Theorems 1.1–1.2 in Hušková (1999) we realize that in case $\alpha_0 \in [0, 1/2]$ the limit behavior of $\widehat{m}_n$ is the same as if α_0 is unknown. In case $\alpha_0 \in (1/2, 1]$ by Theorem 1.3 in Hušková (1999), as $n \to \infty$,

$$\delta_n \sqrt{n}(\widehat{m}_n - m_n)/n \xrightarrow{\mathcal{D}} N(0, \sigma^2 a_{11}^{-1}(\alpha_0, \gamma)),$$

which means that the rate of consistency and the type of limit distribution remains the same, however, the variance is smaller in case of α_0 known.

If the change point m_n is known then it can be proved along the line of the proofs of Theorems 1.1–1.3 that (1.13) remains true for $\alpha_0 \in [0, 1]$ (not only for $\alpha_0 \in [0, 1/2]$ when m_n is unknown).

It can be shown that if α_0 is unknown then the estimator $\widetilde{m}_n(\alpha)$ defined as the maximizer over $k = 1, \ldots, n-1$ of

$$\frac{|\sum_{i=1}^n x_{ik}(\alpha) Y_i|}{\left(\sum_{i=1}^n (x_{ik}(\alpha))^2\right)^{1/2}},$$

with $\alpha \in [0, 1]$ prechosen, does not even need to be consistent. It should be pointed out that if $\alpha_0 \in [0, 1/2]$ the estimator $\widehat{m}_n$ is not generally asymptotically optimal; there exists a Bayesian like estimator that performs better.

These results together with some related problems will be discussed in a separate paper.

The present results can be extended to other types of estimators, e.g. to M- and R-estimators. Other type of gradual changes can also be considered, e.g. $Y_i = \mu + \delta_n g((m_n - k)_+/n; \theta) + e_i, i = 1, ..., n$, where $g(t; \theta)$ is a known nonincreasing smooth function in $t \in [0, 1)$ and a smooth function in θ; and θ is a nuisance parameter. The type of smoothness then determines the limit behavior of the estimators.

Concerning the limit behavior of the estimators $\widehat{\mu}_n$ and $\widehat{\delta}_n$, it can be shown, using standard techniques, that their limit behavior is not affected by the fact that α_0 and m_n are known or unknown.

2 Proofs

We start with some lemmas. The proofs of Theorems 1.1–1.3 are at the end of the section. Some of the lemmas are quite technical and require tedious calculations. We try to shorten them.

Lemma 2.1 *Let assumptions (1.1)–(1.4) be satisfied. Then, for $\alpha \in [0, 1]$, as $n \to \infty$,*

$$\max_{1 \le k < n, \alpha \in [0,1]} \frac{|\sum_{i=1}^n x_{ik}(\alpha) e_i|}{\left(\sum_{i=1}^n x_{ik}^2(\alpha)\right)^{1/2}} = O_P(\sqrt{\log \log n}), \tag{2.1}$$

$$\max_{1\le k<n(1-\epsilon),\alpha\in[0,1]} \frac{|\sum_{i=1}^{n} x_{ik}(\alpha)e_i|}{\left(\sum_{i=1}^{n} x_{ik}^2(\alpha)\right)^{1/2}} = O_P(1) \tag{2.2}$$

for any $\epsilon \in (0,1)$.

Proof This is a slight modification of Lemma 2.5 in Hušková (1999) and therefore the proof is omitted. ■

Lemma 2.2 *Let assumptions (1.1)–(1.3) with* $\alpha \in [0,1]$ *and (1.4) be satisfied. Then, as* $n \to \infty$,

$$\widehat{\alpha}_n \xrightarrow{P} \alpha_0.$$

$$(\widehat{m}_n - m_n)/n \xrightarrow{P} 0.$$

Proof Clearly, for $k = 1, ..., n-1$, and $\alpha \in [0,1]$,

$$\frac{\sum_{i=1}^{n} x_{ik}(\alpha)Y_i}{\left(\sum_{i=1}^{n} x_{ik}^2(\alpha)\right)^{1/2}} = \frac{\sum_{i=1}^{n} x_{ik}(\alpha)e_i}{\left(\sum_{i=1}^{n} x_{ik}^2(\alpha)\right)^{1/2}} + \delta_n \frac{\sum_{i=1}^{n} x_{ik}(\alpha)x_{im_n}(\alpha)}{\left(\sum_{i=1}^{n} x_{ik}^2(\alpha_0)\right)^{1/2}}, \tag{2.3}$$

The first term on the right hand side of 2.3 is stochastic one and by Lemma 2.1 the maximum of its absolute values over k and α is of order $(\log\log n)^{1/2}$. The second term on the right hand side is nonstochastic and elementary calculations give

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \frac{\sum_{i=1}^{n} x_{i,[nt]}(\alpha)x_{i,m}(\alpha_0)}{\left(\sum_{i=1}^{n}(x_{i,[nt]}(\alpha))^2\right)^{1/2}} \\
&\quad = \frac{\int_0^1 (x-\gamma)_+^{\alpha_0}(x-t)_+^{\alpha}dx - \int_0^1 (x-\gamma)_+^{\alpha_0}dx(x-t)_+^{\alpha}dx}{\int_0^1 (x-t)_+^{2\alpha} - (\int_0^1 (x-t)_+^{\alpha}dx)^{1/2}} \\
&\quad = Q(\alpha, \alpha_0, t, \gamma),
\end{aligned}$$

(say), for $t \in [0,1)$, $\gamma \in (0,1)$, and $\alpha, \alpha_0 \in [0,1]$. By the Schwarz inequality, for $t \in (0,1)$, $\gamma \in (0,1)$, $\alpha, \alpha_0 \in [0,1]$,

$$\frac{|Q(\alpha,\alpha_0,t,\gamma)|}{Q^{1/2}(\alpha,\alpha,t,t)} \le Q^{1/2}(\alpha_0,\alpha_0,\gamma,\gamma),$$

where equality holds only with $\alpha = \alpha_0$, $t = \gamma$. The function $Q(.)$ is continuous in all variables. This implies for $\alpha_0, \in [0,1]$, and $\gamma \in (0,1)$,

$$\max_{\substack{t\in(0,\gamma-\epsilon)\cup(\gamma+\epsilon,1)\\ \alpha\in(0,\alpha_0-\epsilon)\cup(\alpha_0+\epsilon,1)\\ \alpha\in[0,1]}} \left\{\frac{|Q(\alpha,\alpha_0,t,\gamma)|}{Q^{1/2}(\alpha,\alpha,t,t)}\right\} < Q^{1/2}(\alpha_0,\alpha_0,\gamma,\gamma),$$

for $\epsilon > 0$ small enough. This in combination with the property of the stochastic term on the right hand side of 2.3 implies the assertion of our lemma. ■

To obtain stronger properties than the above consistency we have to investigate both stochastic and nonstochastic terms in more details.

The estimators $\widehat{m}_n$ and $\widehat{\alpha}_n$ can be equivalently defined as the maximizers over $k \in \{0, 1 \ldots, n-1\}$ and $\alpha \in [0,1]$ of

$$\frac{\left(\sum_{i=1}^n x_{ik}(\alpha) Y_i\right)^2}{\sum_{i=1}^n x_{ik}^2(\alpha)} - \frac{\left(\sum_{i=1}^n x_{im_n}(\alpha_0) Y_i\right)^2}{\sum_{i=1}^n x_{im_n}^2(\alpha_0)}. \tag{2.4}$$

It is useful to decompose the single terms as follows

$$\begin{aligned} &\frac{\left(\sum_{i=1}^n x_{ik}(\alpha) Y_i\right)^2}{\sum_{i=1}^n x_{ik}^2(\alpha)} - \frac{\left(\sum_{i=1}^n x_{im}(\alpha_0) Y_i\right)^2}{\sum_{i=1}^n x_{im}^2(\alpha_0)} \\ &\quad = A_k + 2\delta_n (B_{k1} + B_{k2} + B_{k3} + B_{k4}) - \delta_n^2 (C_{k1} + 2C_{k2} + C_{k3} + C_{k4}), \end{aligned} \tag{2.5}$$

for $k = 1, \ldots, n,\ \alpha \in [0,1]$, where

$$\begin{aligned}
A_k &= \frac{\left(\sum_{i=1}^n x_{ik}(\alpha) e_i\right)^2}{\sum_{i=1}^n x_{ik}^2(\alpha)} - \frac{\left(\sum_{i=1}^n x_{im}(\alpha_0) e_i\right)^2}{\sum_{i=1}^n x_{im_n}^2(\alpha_0)}, \\
B_{k1} &= \sum_{i=1}^n (x_{im_n}(\alpha) - x_{im_n}(\alpha_0)) e_i, \\
B_{k2} &= \sum_{i=1}^n (x_{ik}(\alpha) - x_{im_n}(\alpha)) e_i, \\
B_{k3} &= -\sum_{i=1}^n x_{ik}(\alpha) e_i \frac{\sum_{i=1}^n x_{im_n}(\alpha)(x_{im_n}(\alpha) - x_{im_n}(\alpha_0))}{\sum_{i=1}^n x_{ik}^2(\alpha)}, \\
B_{k4} &= -\sum_{i=1}^n x_{ik}(\alpha) e_i \frac{\sum_{i=1}^n x_{ik}(\alpha)(x_{ik}(\alpha) - x_{im_n}(\alpha))}{\sum_{i=1}^n x_{ik}^2(\alpha)}, \\
C_{k1} &= \sum_{i=1}^n (x_{im_n}(\alpha) - x_{im_n}(\alpha_0))^2 - \frac{\left(\sum_{i=1}^n x_{ik}(\alpha)(x_{im_n}(\alpha) - x_{im}(\alpha_0))\right)^2}{\sum_{i=1}^n x_{ik}^2(\alpha)}, \\
C_{k2} &= \sum_{i=1}^n (x_{ik}(\alpha) - x_{im_n}(\alpha))(x_{im_n}(\alpha) - x_{im_n}(\alpha_0)) \\
&\quad - \left(\sum_{i=1}^n x_{ik}(\alpha)(x_{ik}(\alpha) - x_{im_n}(\alpha))\right) \frac{\left(\sum_{i=1}^n x_{ik}(\alpha)(x_{im_n}(\alpha) - x_{im_n}(\alpha_0))\right)}{\sum_{i=1}^n x_{ik}^2(\alpha)},
\end{aligned}$$

$$C_{k3} = \sum_{i=1}^{n} (x_{ik}(\alpha) - x_{im_n}(\alpha))^2,$$

$$C_{k4} = -\frac{\Big(\sum_{i=1}^n x_{ik}(\alpha)(x_{ik}(\alpha) - x_{im_n}(\alpha))\Big)^2}{\sum_{i=1}^n x_{ik}^2(\alpha)}.$$

Notice that

$$C_{k1} + ... + C_{k4} \geq 0, \quad k = 1, ..., n-1,$$

$$\text{var}\{B_{k1} + ... + B_{k4}\} = \sigma^2 (C_{k1} + ... + C_{k4}).$$

In the next few lemmas we investigate the single terms C_k's and B_k's for α close to α_0 and k close m_n. We start with C_k's.

Lemma 2.3 *Let assumptions (1.1)–(1.4) with $\alpha_0 \in [0,1]$ be satisfied. Then, as $\alpha \to \alpha_0$ and $|m_n - k|/n \to 0$, $n \to \infty$:*

$$(2.6) \quad \frac{1}{n} C_{k1} = (\alpha - \alpha_0)^2 a_{11}(\alpha_0, \gamma)(1 + o(|\alpha - \alpha_0|^\kappa + 1))$$

$$(2.7) \quad \frac{1}{n} C_{k2} = (\alpha - \alpha_0) \frac{m_n - k}{n} a_{12}(\alpha_0, \gamma)(1 + o(|\alpha - \alpha_0|^\kappa + 1)),$$

$$(2.8) \quad \frac{1}{n} C_{k4} = -\Big(\frac{m_n - k}{n}\Big)^2 \alpha_0^2 \frac{\Big(\int_0^1 (x-\gamma)^{2\alpha_0 - 1} dx - \int_0^1 (y-\gamma)^{\alpha_0 - 1} dy \int_0^1 (z-\gamma)^{\alpha_0} dz\Big)^2}{\int_0^1 (x-\gamma)^{2\alpha_0} dx - (\int_0^1 (x-\gamma)^{\alpha_0 - 1} dy)^2} \times \big(1 + o(|\alpha - \alpha_0|^\kappa + 1)\big).$$

uniformly for $|k - m_n| = o(n)$ for some $\kappa > 0$.

Proof We derive the assertion for one term only, since others are treated in the same way. We have

$$\frac{1}{n} \sum_{i=1}^n \Big(\Big(\frac{i - m_n}{n}\Big)_+^\alpha - \Big(\frac{i - m_n}{n}\Big)_+^{\alpha_0}\Big)^2 = \int_{\alpha_0}^\alpha \Big(\frac{1}{n} \sum_{i=1}^n \Big(\frac{i - m_n}{n}\Big)_+^{2\beta} \log^2 \Big(\frac{i - m_n}{n}\Big)_+\Big) d\beta = \int_{\alpha_0}^\alpha \Big(\int_0^1 (x-\gamma)_+^{2\beta} \log(x-\gamma)_+ dx\Big) d\beta$$
$$+ \int_{\alpha_0}^\alpha \Big(\sum_{i=1}^n \int_{(i-1)/n}^{i/n} \Big(\Big(\frac{i - m_n}{n}\Big)_+^{2\beta} \log^2 \Big(\frac{i - m_n}{n}\Big)_+ - (x-\gamma)_+^{2\beta} \log(x-\gamma)_+\Big) dx\Big) d\beta$$
$$= (\alpha - \alpha_0) \int_0^1 (x-\gamma)_+^{2\alpha_0} \log^2 (x-\gamma)_+ dx (1 + o(|\alpha - \alpha)|^\kappa + 1)),$$

where we used elements of classical calculus. ■

Lemma 2.4 *Let assumptions (1.1)–(1.4) be satisfied. Then, as $\alpha \to \alpha_0$ and $n \to \infty$,*
(1) for $\alpha \in (1/2, 1]$:
(2.9)
$$\frac{1}{n}C_{k3} = \Big(\frac{m_n - k}{n}\Big)^2 \alpha_0^2 \Big(\int_0^1 (x-\gamma)^{2\alpha_0 - 2}dx - (\int_0^1 (y-\gamma)^{\alpha_0 - 1}dy)^2\Big) \times (1 + o(|\alpha - \alpha_0|^\kappa + 1)),$$
uniformly for $|k - m_n| = o(n)$ for some $\kappa > 0$.

(2) for $\alpha = 1/2$:
$$\frac{1}{n}C_{k3} = \Big(\frac{k - m_n}{n}\Big)^2 \frac{\log(n - m_n)}{4}(1 + o(|\alpha - \alpha_0|^\kappa + 1)), \tag{2.10}$$
uniformly for $|k - m_n| = o(n)$ for some $\kappa > 0$.

(3) for $\alpha \in [0, 1/2)$:
$$\frac{1}{n}C_{k3} = \Big|\frac{k - m_n}{n}\Big|^{2\alpha_0 + 1} \int_{-\infty}^{\infty} ((x+1)_+^{\alpha_0} - x_+^{\alpha_0})^2 dx (1 + o(|\alpha - \alpha_0|^\kappa + 1)), \tag{2.11}$$
uniformly for $|k - m_n| = o(n)$ for some $\kappa > 0$.

Proof The lemma is a slight generalization of Lemmas 2.2-2.4 in Hušková (1999) and therefore the proof is omitted. ■

Lemma 2.5 *Let assumptions (1.1)–(1.4) be satisfied. Then for $\alpha \in [0, 1]$, as $\alpha \to \alpha_0$ and $n \to \infty$,*
$$B_{k1} + B_{k3} = (\alpha - \alpha_0)Y_{n1}\Big(1 + o_p(|\alpha - \alpha_0|^\kappa + 1)\Big)$$
uniformly for $|k - m_n|/n = o(1)$ for some $\kappa > 0$, where
$$Y_{n1} = \sum_{i=1}^{n} \Big(\frac{i - m_n}{n}\Big)_+^{\alpha_0} \log\Big(\frac{i - m_n}{n}\Big)_+ (e_i - \bar{e}_n) - \sum_{i=1}^{n} x_{im_n}(\alpha_0) e_i \times \tag{2.12}$$
$$\frac{\int_0^1 (x-\gamma)_+^{2\alpha_0} \log(x-\gamma)_+ dx - (\int_0^1 (y-\gamma)_+^{\alpha_0} dy)(\int_0^1 (z-\gamma)_+^{\alpha_0}\log(z-\gamma)_+ dz)}{\int_0^1 (x-\gamma)^{2\alpha_0}dx - (\int_0^1 (y-\gamma)_+^{\alpha_0}dy)^2}.$$

Proof The crucial part of the proof is to show that
$$\sum_{i=1}^{n}\Big(\Big(\frac{i - m_n}{n}\Big)_+^{\alpha} - \Big(\frac{i - m_n}{n}\Big)_+^{\alpha_0}\Big)e_i = (\alpha - \alpha_0)\sum_{i=1}^{n}\Big(\Big(\frac{i - m_n}{n}\Big)_+^{\alpha_0} \times \log\Big(\frac{i - m}{n}\Big)_+\Big)e_i\Big(1 + o_p(|\alpha - \alpha_0|^\kappa + 1)\Big).$$

Clearly,

$$S_n(\alpha) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(\Big(\frac{i-m_n}{n}\Big)_+^{\alpha} \log\Big(\frac{i-m_n}{n}\Big)_+\Big) e_i, \quad \alpha \in [0,1],$$

is continuous in α and we have for all $0 \le \alpha_1, \alpha_2 \le 1$ and n large enough

$$\begin{aligned} &E(S_n(\alpha_2) - S_n(\alpha_1))^2 \\ &\le (\alpha_2-\alpha_1)^2 \frac{1}{n} \sum_{i=1}^{n} \Big(\Big(\frac{i-m_n}{n}\Big)_+^{\alpha_2} - \Big(\frac{i-m_n}{n}\Big)_+^{\alpha_2 1}\Big)^2 \times \log^2\Big(\frac{i-m_n}{n}\Big)_+ \\ &\le (\alpha_2-\alpha_1)^2 C, \end{aligned}$$

for some $C > 0$. Hence by Theorem 12.3 in Billingsley (1968) we have that the sequence $S_n = \{S_n(\alpha), \alpha \in [0,1]\}$ is tight and also

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} \Big(\Big(\frac{i-m_n}{n}\Big)_+^{\alpha} \log\Big(\frac{i-m_n}{n}\Big)_+\Big) e_i = \sum_{i=1}^{n} \Big(\Big(\frac{i-m_n}{n}\Big)_+^{\alpha_0} \log\Big(\frac{i-m_n}{n}\Big)_+\Big) e_i \\ \times (1 + o_p(|\alpha-\alpha_0|^{\kappa} + 1)) \end{aligned}$$

uniformly for $\alpha \to \alpha_0$ and $n \to \infty$. The lemma is proved. ∎

The following lemmas are slight generalizations of Lemma 2.6–2.9 in Hušková (1999); their proofs are omitted.

Lemma 2.6 *Let assumptions (1.1)–(1.4) be satisfied. Then for $\alpha_0 \in [0,1]$, as $\alpha \to \alpha_0$ and $n \to \infty$,*

$$B_{k4} = -\frac{m_n - k}{n} Y_{2n} (1 + o_p(|\alpha-\alpha_0|^{\kappa} + 1))$$

uniformly for $|k - m_n|/n = o(1)$ for some $\kappa > 0$, where

$$Y_{2n} = \alpha_0 \sum_{i=1}^{n} x_{im_n}(\alpha_0) e_i \times \frac{\int_0^1 (x-\gamma)_+^{2\alpha_0-1} dx - \int_0^1 (y-\gamma)_+^{\alpha_0-1} dy \int_0^1 (z-\gamma)_+^{\alpha_0} dz}{\int_0^1 (x-\gamma)_+^{2\alpha_0} dx - (\int_0^1 (x-\gamma)_+^{\alpha_0-1} dy)^2}. \tag{2.13}$$

Lemma 2.7 *($\alpha_0 \in [1/2,1]$) Let assumptions (1.1)–(1.4) be satisfied. Then, as $\alpha \to \alpha_0$ and $n \to \infty$,*
(1) for $\alpha_0 \in (1/2,1]$:

$$B_{k2} = \frac{m_n - k}{n} Y_{3n} (1 + o_p(|\alpha-\alpha_0|^{\kappa} + 1)),$$

uniformly in $|k - m_n|/n = o(1)$ *for some* $\kappa > 0$.

$$Y_{3n} = \alpha_0 \sum_{i=1}^{n} \Big(\frac{i - m_n}{n}\Big)_+^{\alpha_0 - 1} (e_i - \bar{e}_n). \tag{2.14}$$

(2) *for* $\alpha_0 = 1/2$

$$B_{k2} = \frac{m_n - k}{n} \frac{1}{2} \sum_{i=1}^{n} \Big(\frac{i - m_n}{n}\Big)_+^{-1/2} e_i (1 + o_p(|\alpha - \alpha_0|^\kappa + 1)),$$

uniformly for $|k - m_n|/n = o(1)$ *for some* $\kappa > 0$.

Next, we introduce the process $V_{n\alpha,T} = \{V_{n,\alpha}(t);\ |t| \le T\}$ with $\alpha \in [0, 1/2)$, T being a positive number and

$$V_{n\alpha}\Big(\frac{k - m_n}{n} (n\delta_n^2)^{1/(2\alpha+1)}\Big) = \delta_n B_{k2}, \quad k = 1, ..., n,$$

and piecewise linear otherwise.

Lemma 2.8 ($\alpha_0 \in [0, 1/2)$) *Let assumptions (1.1)–(1.4) be satisfied. Then for any* $\epsilon > 0$ *and* $\eta > 0$ *there exist* $H_{j\eta} > 0$, $j = 1, 2$, *and* n_η *such that for* $n \ge n_\eta$

$$P\Big(\max_{|\alpha - \alpha_0| \le H_{1\eta}} \ \max_{\epsilon > |k - m_n|/n \ge H_{2\eta}(\delta_n^2 n)^{1/(2\alpha_0+1)}} \frac{|B_{k2}|}{|\delta_n| C_{k3}} \ge \eta\Big) < \eta$$

and, as $\alpha \to \alpha_0$ *and* $n \to \infty$,

$$V_{n\alpha,T} \xrightarrow{D} \sigma W_{\alpha_0,T},$$

where $W_{\alpha_0,T} = \{W_{\alpha_0};\ |t| \le T\}$ *is a Gaussian process with zero mean and the covariance function given by (1.15).*

Proof of Theorem 1.3 The proof is divided into two steps. First we show that, as $n \to \infty$ and $\alpha \to \alpha_0$

$$(\widehat{m}_n - m_m)/n = O_P((\delta_n \sqrt{n})^{-1}), \tag{2.15}$$

$$\widehat{\alpha}_n - \alpha_0 = O_P((\delta_n \sqrt{n})^{-1}), \tag{2.16}$$

and then the limit distribution of $\widehat{m}_n$ and $\widehat{\alpha}$ will be derived.

By Lemma 2.2 it suffices to investigate

$$\max_{|\alpha - \alpha_0| < \epsilon, |k - m_n|/n < \epsilon} \{A_k + 2\delta_n (B_{k1} + ... + B_{k4}) - \delta_n^2 (C_{k1} + C_{k2} + 2C_{k3} + C_{k4})\}$$

for $\epsilon > 0$ small enough. Denoting

$$G_D = \{(\alpha, k); |\alpha - \alpha_0| \le D(|\delta_n|\sqrt{n})^{-1}, |k - m_n|/n \le D(|\delta_n|\sqrt{n})^{-1}\}$$

and

$$H_D = \{(\alpha, k); |\alpha - \alpha_0 < \epsilon, \ |k - m_n|/n < \epsilon\} - G_D$$

we notice that for any $D > 0$ and $\epsilon > 0$ small enough

$$\max_{G_D}\{\delta_n^2(C_{k1} + 2C_{k2} + C_{k3} + C_{k4})\} = 0,$$

and by Lemmas 2.3–2.4

$$\max_{H_D}\{-\delta_n^2(C_{k1} + 2C_{k2} + 2C_{k3} + C_{k4})\} = \max(-D^2(a_{11}(\alpha_0, \gamma) + 2a_{12}(\alpha_0, \gamma) + a_{22}(\alpha_0, \gamma)) - D^2 a_{11}(\alpha_0, \gamma), -D^2 a_{22}(\alpha_0, \gamma)\} \times (1 + o(1))$$

where $a_{ij}(\alpha_0, \gamma)$ are defined by (1.9)–(1.11). Clearly, $a_{11}(\alpha_0, \gamma)+2a_{12}(\alpha_0, \gamma)+a_{22}(\alpha_0, \gamma) > 0$. By assumption (1.4) and by Lemma 2.1 we find that the terms A_k's are negligible and also that for $D > 0$ large enough the non-stochastic terms $\delta_n^2(C_{k1} + C_{k2} + 2C_{k3} + C_{k4})$ dominate the stochastic terms $\delta_n(B_{k1} + B_{k2} + B_{k3} + B_{k4})$ for $(\alpha, k) \in H_D$ with probability close to 1. Since

$$\max_{H_D} |\delta_n||B_{k1} + \ldots + B_{k4}| = O_p(|\delta_n|\sqrt{n})$$

and since D can be chosen arbitrarily large we find that (2.15)–(2.16) hold true.

In order to obtain the limit behavior of our estimators we investigate the maximum of

$$2\delta_n(B_{k1} + \ldots + B_{k4}) - \delta_n^2(C_{k1} + 2C_{k2} + C_{k3} + C_{k4}) \tag{2.17}$$

over the set G_D.

Writing $\alpha = \alpha_0 + t_1(\delta_n\sqrt{n})^{-1}$ and $k = m + t_2 n(\delta_n\sqrt{n})^{-1}$ we get by Lemmas 2.3–2.7 that our problem reduces to investigating the maximum of

$$-(t_1^2 a_{11}(\alpha_0, \gamma) + 2t_1 t_2 a_{12}(\alpha_0, \gamma) + t_2^2 a_{22}(\alpha_0, \gamma)) + 2t_1 \frac{1}{\sqrt{n}} Y_{1n} + 2t_2 \frac{1}{\sqrt{n}}(Y_{2n} + Y_{3n})(1 + o(1)).$$

with respect to t_1 and t_2. Since $A(\alpha_0, \gamma)$, defined in (1.8) is a positive definite matrix and since by the CLT $(Y_{1n}/\sqrt{n}, (Y_{2n} + Y_{3n})/\sqrt{n})$ has asymptotically $N((0,0)^T, \sigma^2 A(\alpha_0, \gamma))$ distribution, we find after some standard steps that the assertion of Theorem 1.1 holds true. ∎

Proof of Theorem 1.2 We proceed similarly as in the proof of Theorem 1.3. Checking the behavior of B_k's and C_k's for $\alpha_0 = 1/2$ (Lemmas 2.3–2.8) we realize that, as $n \to \infty$ and $\alpha \to \alpha_0$,

$$\frac{1}{n} C_{k3} = O\Big(\Big(\frac{|k - m_n|}{n}\Big)^2 \log(n - m_n)\Big), \quad C_{k4} = o(C_{k3}),$$

$$\frac{1}{\sqrt{n}} B_{k2} = O_p\Big(\frac{|k - m_n|}{n}(\log(n - m_n))^{1/2}\Big), \quad B_{k4} = o_p(B_{k2})$$

uniformly for $k - m_n = o(n)$. The terms C_{k1}, C_{k2} and B_{k1}, B_{k3} are not affected in this way, which leads to the conclusion that the rate of consistency of $\widehat{\alpha}_n$ is the same as in case $\alpha_0 \in (1/2, 1]$ while for $\widehat{m}_n$ we have, as $n \to \infty$ and $\alpha \to \alpha_0$

$$(\widehat{m}_n - m_m)/n = O_P\Big((\delta_n\sqrt{n})^{-1} \log^{-1/2}(n - m_n)\Big).$$

Moreover, it is enough to study the maximum of

$$2\delta_n(B_{k1} + B_{k2} + B_{k3}) - \delta_n^2(C_{k1} + C_{k3})$$

over a properly modified set H_D. The proof is now finished in the same way as of Theorem 1.3. ∎

Proof of Theorem 1.1 This is omitted since it is in principle the same as that of Theorem 1.2. ∎

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