Introduction

People have been interested in natural numbers since forever. The ancient mathematicians knew and used the principle of *descente infinie,* which is a form of mathematical induction. The principle is as follows: if you want to show that no number has the property φ, it suffices to show that for each number n having the property φ there is a smaller number $m < n$ having the property φ. (If there were a number having φ we could endlessly find smaller and smaller numbers having φ, which is absurd.) The Greeks used the principle for a proof of incommensurability of segments. The principle was rediscovered in modern times by P. Fermat (1601–1665). The principle of mathematical induction itself (if 0 has the property φ and for each number n having φ also $n+1$ has φ then all numbers have φ) seems to have been first used by B. Pascal (1623–1662) in a proof concerning his triangle. A general formulation appears in a work of J. Bernoulli (1654–1705). (Our source is [Meschkowski 78-81].)

In 1861 Grassman published his *Lehrbuch der Arithmetik;* in our terms, he defines integers as an ordered integrity domain in which each non-empty set of positive elements has a least element. In 1884 Frege's book *Grundlagen der Arithmetik* was published. We can say that Frege's natural numbers are classes; each such class consists of all sets of a certain fixed finite cardinality. (Frege speaks of concepts, not of classes.) The famous Dedekind's work *Was sind und was sollen die Zahlen* appears in 1888. Dedekind's natural numbers are defined as a set N together with an element $1 \in N$ a one-one mapping f of N into itself such that 1 is not in the range of f and N is the smallest set containing 1 and closed under f. Dedekind and Frege agreed that arithmetic is a part of logic, but differed in their opinions on what logic is. They both used the same main device: a one-one mapping and closedness under that mapping.

Dedekind was not interested in finding a formal deductive system for natural numbers; this was the main aim of Peano's investigation of natural numbers (*Arithmetices principia nova methoda exposita,* 1889). Peano's axiom system (taken over from Dedekind, who had it from Grassman) is, in our terminology, second order: it deals with numbers and sets of numbers. Nowadays it is usual to call the first-order axiomatic arithmetic Peano arithmetic; this terminology was probably introduced by Tarski (personal communication by G.H. Müller). Whitehead and Russell published their *Principia mathematica* in 1908; the book also includes a formalization of arithmetic.

Hilbert formulated his programme as follows: *unsere üblichen Methoden der Mathematik samt und sonders als widerspruchsfrei zu erkennen* (to show that our usual methods of mathematics are free from contradictions in their whole). [Hilbert-Bernays 34, Zur Einleitung]. This should have been shown by finitary methods forming a proper part of arithmetic [ibid., p. 42]. Gödel's famous incompleteness results [Gödel 31] showed Hilbert's program in its original formulation to be unrealistic (even if Hilbert denies this in his Einleitung); but it has remained an important source of inspiration for proof theory, see [Kreisel 68]. Related work from the thirties by Tarski (undefinability of truth in arithmetic), Church (undecidability of arithmetic) and Rosser (elimination of the assumption of ω-consistency) is well known. In modern texts these results are proved using the well-known diagonalization (or self-reference) lemma, which is already implicit in Gödel's proof. This lemma first appeared explicitly in [Carnap 34], but, surprisingly, it was neglected by many authors for a long time. Feferman's paper [Feferman 60] is a fundamental paper for modern study of arithmetization of metamathematics. But it is also necessary to mention Volume II of Hilbert-Bernays's monograph [Hilbert-Bernays 39], containing a detailed exposition of arithmetization including the arithmetized completeness theorem. Early results following Feferman's Arithmetization were obtained by Montague, Shepherdson and others. In the sixties, Feferman and Montague worked on a monograph devoted to the arithmetization of metamathematics, but unfortunately the book has never been finished. [Smoryński 81-fifty] is a very readable survey of the development of self-reference.

Non-standard models of arithmetic were first constructed by Skolem [Skolem 34]; in present terms, he used the method of definable ultrapower. In 1952 Ryll-Nardzewski proved that Peano arithmetic PA (first order!) is not finitely axiomatizable. Specker and McDowell showed in 1959 that each (countable) model of PA has an elementary end-extension. Rabin [61] showed that PA is not axiomatizable by any axiom system of bounded quantifier complexity. Further important results were obtained by Friedman, Gaifman and Paris in the early seventies. [Smoryński 82] is a very readable treatise of development of model theory of arithmetic (up to the early eighties).

A result of fundamental importance was obtained by Paris in 1977: he found an arithmetical statement with a clear combinatorial meaning which is true but unprovable in PA; moreover, he was able to show the unprovability by model-theoretical means, without any use of self-reference. His proof used a new method, called the method of indicators, developed by Paris and Kirby. Harrington found an elegant reformulation of Paris's statement; his reformulation is a strengthening of the finite Ramsey's theorem on homogeneous sets

[Paris-Harrington 77]. This was followed by many papers by various authors, among them McAloon, Kotlarski, Murawski.

Later Paris and his students (Kirby, Clote, Kaye, Dimitrocopulos and others) turned to the study of fragments of PA. We shall rely substantially on their work. The first four chapters of the book deal mainly with fragments containing at least induction for Σ_1-formulas. At present let us only say that in such theories we may freely construct recursive functions using primitive recursion. The fifth chapter deals with bounded arithmetic. Parikh seems to have been the first to study bounded arithmetic [Parikh]. He suggested investigating induction for bounded formulas since they are easily decidable (e.g. in linear space). This was developed significantly by Paris, Wilkie and Paris's students. The relation to complexity theory has been known from the beginning of the investigation of bounded arithmetic. Buss's dissertation, which later appeared as a book [Buss 86, Bounded ar.], was a further important impulse. Buss contributed both in finding new connections with complexity theory and in applying proof-theoretical methods. There are various later results; the reader will find such results here.

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The aim of our study of the metamathematics of first-order arithmetic is to give the reader a deeper understanding of the role of the axiom schema of induction and of the phenomenon of incompleteness. In Part A, we develop important parts of mathematics and logic in various fragments of first order arithmetic. The main means are by coding of finite sets, arithmetization of logical syntax and semantics and through a version of König's lemma called the Low basis theorem.

Part B is devoted to incompleteness. Our main question reads: what more can we say about systems of arithmetic than that they are all incomplete? There are at least four directions in which the answer may be looked for:

(1) For each formula φ unprovable and non-refutable in an arithmetic T we may ask, how *conservative* it is over T, i.e. for which formulas ψ the provability of ψ in $(T + \varphi)$ implies the provability of ψ in T.

(2) We may further ask if $(T + \varphi)$ is *interpretable* in T, i.e. whether the notions of T may be redefined in T in such a way that for the new notions all axioms of $(T + \varphi)$ are provable in T.

(3) Given T we may look for *natural* sentences true but unprovable in T (for example, various combinatorial principles).

(4) Moreover, we may investigate *models* of T and look at how they visualize our syntactic notions and features.

Bounded arithmetic is studied in Part C. Various results of Part A are strengthened by showing that constructions done in stronger fragments are possible in some systems of bounded arithmetic and how. For bounded arithmetic we ask, besides questions (1)–(4), also the following:

(5) What is the relationship between provability in fragments and complexity of computation? One of the most important goals (presently inaccessible) is to show independence of some open problems of complexity theory from some fragments.

Details on the structure of the book are apparent from the table of contents.