

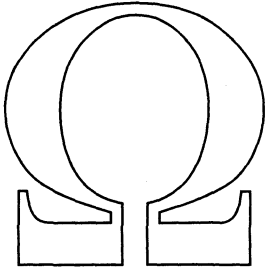
Perspectives in Mathematical Logic

Petr Hájek
Pavel Pudlák

**Metamathematics
of First-Order
Arithmetic**



Springer



Perspectives
in
Mathematical Logic

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R. O. Gandy, H. Hermes, A. Levy, G. H. Müller,
G. E. Sacks, D. S. Scott

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Metamathematics of First-Order Arithmetic



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Dedicated to our wives, Marie and Věra

Preface to the Series

Perspectives in Mathematical Logic

(Edited by the “ Ω -group for Mathematical Logic” of the Heidelberg Akademie der Wissenschaften)

On Perspectives. *Mathematical logic arose from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematise the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly in the last two decades, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps or guides to this complex terrain. We shall not aim at encyclopaedic coverage: nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought; and that, by developing certain themes, it will be of greater interest than a mere assemblage of results and techniques.*

The books in the series differ in level: some are introductory, some highly specialised. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their books with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of value, the credit will be theirs.

History of the Ω -Group. During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R.O. Gandy, A. Levy, G.H. Müller, G. Sacks, D.S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F.K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans

Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of an infinite discussion we found our name, and that of the series. We established our centre in Heidelberg. We agreed to meet twice a year together with authors, consultants and assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the over all plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors' ideas and our own. Often the battle ended with a detailed plan for a better book which the author was keen to write and which would indeed contribute a perspective.

Oberwolfach, September 1975

Acknowledgements. In starting our enterprise we essentially were relying on the personal confidence and understanding of Professor Martin Barner of the Mathematisches Forschungsinstitut Oberwolfach, Dr. Klaus Peters of Springer-Verlag and Dipl.-Ing. Penschuck of the Stiftung Volkswagenwerk. Through the Stiftung Volkswagenwerk we received a generous grant (1970–1973) as an initial help which made our existence as a working group possible.

Since 1974 the Heidelberger Akademie der Wissenschaften (Mathematisch-Naturwissenschaftliche Klasse) has incorporated our enterprise into its general scientific program. The initiative for this step was taken by the late Professor F.K. Schmidt, and the former President of the Academy, Professor W. Doerr.

Through all the years, the Academy has supported our research project, especially our meetings and the continuous work on the Logic Bibliography, in an outstandingly generous way. We could always rely on their readiness to provide help wherever it was needed.

Assistance in many various respects was provided by Drs. U. Felgner and K. Gloede (till 1975) and Drs. D. Schmidt and H. Zeitler (till 1979). Last but not least, our indefatigable secretary Elfriede Ihrig was and is essential in running our enterprise.

We thank all those concerned.

Heidelberg, September 1982

R.O. Gandy

H. Hermes

A. Levy

G.H. Müller

G. Sacks

D.S. Scott

Authors' Preface

After having finished this book on the metamathematics of first order arithmetic, we consider the following aspects of it important: first, we pay much attention to subsystems (fragments) of the usual axiomatic system of first order arithmetic (called Peano arithmetic), including weak subsystems, i.e. so-called bounded arithmetic and related theories. Second, before discussing proper metamathematical questions (such as incompleteness) we pay considerable attention to positive results, i.e. we try to develop naturally important parts of mathematics (notably, some parts of set theory, logic and combinatorics) in suitable fragments. Third, we investigate two notions of relative strength of theories: interpretability and partial conservativity. Fourth, we offer a systematic presentation of relations of bounded arithmetic to problems of computational complexity.

The need for a monograph on metamathematics of first order arithmetic has been felt for a long time; at present, besides our book, at least two books on this topic are to be published, one written by R. Kaye and one written by C. Smoryński. We have been in contacts with both authors and are happy that the overlaps are reasonably small so that the books will complement each other.

This book consists of a section of preliminaries and of three parts: A – Positive results on fragments, B – Incompleteness, C – Bounded arithmetic. Preliminaries and parts A, B were written by P. H., part C by P. P. We have tried to keep all parts completely compatible.

The reader is assumed to be familiar with fundamentals of mathematical logic, including the completeness theorem and Herbrand's theorem; we survey the things assumed to be known in the Preliminaries, in order to fix notation and terminology.

Acknowledgements. Our first thanks go to the members of the Ω -group for the possibility of publishing the book in the series Perspectives in mathematical logic and especially to Professor Gert H. Müller, who invited P. H. to write a monograph with the present title, agreed with his wish to write the book jointly with P. P. and continuously offered every possible help. We

are happy to recognize that we have been deeply influenced by Professor Jeff Paris. Soon after the famous independence results of Paris, Kirby and Harrington, Jeff Paris repeatedly visited Prague and gave talks about the research of his Manchester group. Since then, he has come to Prague many times and we always learn much from him. On various occasions we met other mathematicians working in this field (Adamowicz, Buss, Clote, Dimitracopoulos, Feferman, Kaye, Kossak, Kotlarski, Lindström, Montagna, Ressayre, Simpson, Smoryński, Solovay, Takeuti, Wilkie, Woods and others) and many of them visited Czechoslovakia. Discussions with them and preprints of their papers have been an invaluable source of information for us. We have profited extremely much from our colleagues J. Krajíček and V. Švejdar and other members of our Prague seminar. The Mathematical Institute of the Czechoslovak Academy of Sciences has been a good working place. Several people have read parts of the manuscript and suggested important improvements. Our thanks especially to Peter Clote, William Eldridge, Richard Kaye, Juraj Hromkovič and Jiří Sgall for their help. Mrs. K. Trojanová and Mrs. D. Berková helped us considerably with typing; and D. Harmanec provided valuable technical help with the preparation of the bibliography on a computer. Last but not least, our families have got used to sacrifice for our scientific work. They deserve our most cordial thanks.

November 1990

Petr Hájek
Pavel Pudlák

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