7. Selected Topics

As the title suggests this chapter is a collection of various more advanced topics. The first section, on bounded and unbounded theories, both contains useful facts about a natural class of theories and illustrates how the regular type machinery can be used to classify the models of a theory with relatively simple invariants. The second section delves more deeply into the properties of our notions of rank in some very special theories such as the uncountably categorical ones.

7.1 Bounded and Unbounded Theories

We work in a stable theory throughout the section.

Definition 7.1.1. *(i) The theory T is called* bounded *if there are $< |\mathfrak{C}|$ domination equivalence classes of nonalgebraic stationary types; T is* unbounded *if it is not bounded.*

(ii) The theory T is unidimensional *if any two nonalgebraic types are nonorthogonal.*

Shelah (and many others) call unbounded theories *multidimensional* and bounded theories *nonmultidimensional* or *nmd*, for short. There are many examples of such theories:

Lemma 7.1.1. *The theory of any infinite module is bounded.*

Proof. Let $\mathfrak{C}$ be the universal domain of the relevant theory. By Proposition 5.3.2 and Lemma 5.3.9 an element p of $S_1(\mathfrak{C})$ is the translate of the generic type in *stab*(p), a group $\bigwedge-$definable over $\emptyset$. Certainly p is domination equivalent to this generic type. Since there are $\leq 2^{|T|}$ many such groups this is a bound on the number of domination equivalence classes. The same argument establishes this bound for types in other sorts (i.e., $n-$types in the $1-$sorted theory of the module), proving the lemma.

We will say little here about bounded theories which are properly stable. The superstable ones become easier to handle using

Lemma 7.1.2. *The following are equivalent for a superstable theory T:*

(1) *Every nonalgebraic type is nonorthogonal to* $\emptyset$.
(2) *Every regular type is nonorthogonal to* $\emptyset$.
(3) *For every stationary type* p *and* $f \in \mathrm{Aut}(\mathfrak{C})$ *fixing* $acl(\emptyset)$ *pointwise,* $p \sqsupseteq\!\sqsubseteq f(p)$.
(4) T *is bounded.*

Proof. (1) $\Longrightarrow$ (2) holds trivially.

(2) $\Longrightarrow$ (3) Suppose (2) holds. Let $p \in S(A)$ be stationary and $f \in \mathrm{Aut}(\mathfrak{C})$ which fixes $acl(\emptyset)$ pointwise.

Claim. To prove (3) it suffices to consider the case when A is independent from $f(A)$.

Suppose the property in the claim to be true. For the given stationary $p \in S(A)$ and f there is a $g \in \mathrm{Aut}(\mathfrak{C})$ fixing $acl(\emptyset)$ such that $g(A) \perp A$ and $g(A) \perp f(A)$, hence p, $g(p)$ and $f(p)$ are all domination equivalent, proving the claim.

Assuming now that $A \perp f(A)$ let $q_0 \otimes \ldots \otimes q_n$ be a product of regular types domination equivalent to p. We can take A large enough so that $q_i \in S(A)$, for $i \leq n$, while still assuming that $A \perp f(A)$. Since each q_i is nonorthogonal to $\emptyset$, Proposition 5.6.2 says that q_i is nonorthogonal to $f(q_i)$, hence $q_i \sqsupseteq\!\sqsubseteq f(q_i)$ (since they have weight 1). Domination equivalence is preserved under $\otimes$ so p, $q_0 \otimes \ldots \otimes q_n$, $f(q_0) \otimes \ldots \otimes f(q_n)$ and $f(p)$ are all domination equivalent, as desired.

(3) $\Longrightarrow$ (4) This is clear since there are $\leq 2^{|T|}$ many nonalgebraic stationary types up parallelism and conjugacy over $acl(\emptyset)$.

(4) $\Longrightarrow$ (1) This is left as an exercise.

Definition 7.1.2. *For* T *a bounded superstable theory let* $DIM(T)$, *called the* set of dimensions of T *denote the set of equivalence classes of regular types with respect to nonorthogonality. The cardinality of* $DIM(T)$ *is called the* width of T *(or the* number of dimensions of T*) and denoted* $ND(T)$.

Corollary 7.1.1. *Given a bounded superstable theory* T, *the width of* T *is* $\leq 2^{|T|}$ *and* $\leq |T|$ *when* T *is totally transcendental.*

Proof. This follows immediately from Lemma 7.1.2.

These definitions explain the term unidimensional theory; it is a bounded theory with 1 dimension (when the theory is superstable). As the following shows we have already encountered numerous examples of unidimensional theories (for countable theories the right hand side is simply the definition of "uncountably categorical").

Proposition 7.1.1. *A t.t. theory* T *is* $\lambda-$*categorical for all* $\lambda > |T|$ *if and only if* T *is unidimensional.*

Proof. ($\Longrightarrow$) The categoricity assumption and Proposition 6.4.4(iii) implies that all SR types are nonorthogonal, hence all nonalgebraic stationary types are nonorthogonal.

($\Longleftarrow$) Let $N \supsetneq M$ be models of T and φ is a nonalgebraic formula over M. Corollary 6.4.2 and the unidimensionality of T directly yield an $a \in N \setminus M$ satisfying φ (with $tp(a/M)$ strongly regular). Thus, T does not have a Vaughtian pair. For countable theories Theorem 3.1.2 immediately implies the uncountable categoricity of T. For arbitrary t.t. theories we simply repeat the proof of that earlier theorem using deeper results about t.t. theories (such as the existence of prime models over sets) when necessary.

There are, however, rather simple countable unidimensional theories which are not uncountably categorical:

Example 7.1.1. (A weakly minimal unidimensional theory) We begin by letting G be the direct product of $\aleph_0$ many copies of the group $\mathbb{Z}_2$. Let H_i, for $1 \leq i < \omega$, be the subgroup consisting of the elements whose first i coordinates are 0. Let $M = (G, +, 0, H_i)_{i<\omega}$ in the language consisting of $+$, 0 and predicate symbols P_i interpreted by the H_i's, and let $T = Th(M)$ (as a 1$-$sorted theory). It is easy to show that this theory is quantifier eliminable. Thus, for $\mathfrak{C}$ the universe, $\bigcap_{i<\omega} P_i(\mathfrak{C}) = \mathfrak{C}^o$ is a vector space over $\mathbb{Z}_2$ and there is no other structure on this group induced by the formulas (every vector space automorphism of $\mathfrak{C}^o$ extends to an automorphism of $\mathfrak{C}$). Thus the type $\bigcap_i P_i(x)$ is minimal. For an arbitrary element a, the set of realizations of $stp(a)$ is simply $a + \mathfrak{C}^o$. It follows immediately that $\mathfrak{C}$ is a weakly minimal set and any nonalgebraic element of $S_1(\mathfrak{C})$ is a translate of the generic in $\mathfrak{C}^o$. Thus, all nonalgebraic stationary types are nonorthogonal to the generic type of $\mathfrak{C}^o$. Since the generic of $\mathfrak{C}^o$ is minimal, all nonalgebraic stationary types are nonorthogonal.

Example 7.1.2. The theory of the group of integers, $(\mathbb{Z}, +)$ is also weakly minimal, unidimensional and not t.t. (See the analysis in [BBGK73].)

Remark 7.1.1. Hrushovski showed in [Hru90b] that every stable unidimensional theory is superstable.

For bounded theories the Decomposition Theorem (Theorem 6.3.6) can be strengthened by limiting the collection of needed regular types.

Lemma 7.1.3. *If T is a bounded superstable theory and M is any a$-$model, then for all stationary types p there are regular types $q_0, \ldots, q_n \in S(M)$ such that $p \sqsubseteq q_0 \otimes \ldots \otimes q_n$.*

Proof. In a bounded superstable theory every regular type is nonorthogonal to $\emptyset$. By Lemma 5.6.5 every regular type is nonorthogonal to one in $S(M)$, from which the lemma follows.

This leads us to what can be viewed as a basis for an arbitrary $a-$model (or model in the t.t. case) with respect to a fixed set of regular types.

Proposition 7.1.2. *(i) Let M be an $a-$model of a bounded superstable theory T and $N \subset M$ an $a-$prime model over $\emptyset$. If $C \subset M$ is a maximal $N-$independent set of realizations of regular types over N then M is $a-$prime over $N \cup C$ (in fact, there is no $a-$model M', $N \cup C \subset M' \subsetneq M$).*

(ii) Let M be a model of a bounded t.t. theory and $N \subset M$ a prime model over $\emptyset$. If $C \subset M$ is a maximal $N-$independent set of realizations of strongly regular types over N then M is prime over $N \cup C$ and minimal over $N \cup C$.

Proof. (i) Suppose, to the contrary, that $M' \subsetneq M$ is an $a-$model containing $N \cup C$. There is an $a \in M \setminus M'$ with $p = tp(a/M')$ regular. Since T is bounded we can take p to be based on N, hence a is independent from $M' \supset C$ over N. This contradicts the maximality of C to prove that $M' = M$. Since M contains a model $a-$prime over $N \cup C$ we also conclude that M is $a-$prime over $N \cup C$.

(ii) This is proved exactly like (i), using Proposition 6.4.1 and Corollary 6.4.2 when necessary.

Part (ii) of the proposition gives a decomposition theorem for bounded t.t. theories which generalizes Theorem 6.3.6 more directly than Proposition 6.4.2.

Corollary 7.1.2. *In a bounded t.t. theory a complete nonalgebraic stationary type over a model is RK-equivalent to a finite $\otimes-$product of SR types.*

A *representation theorem* for the models in a class $\mathcal{K}$ is a result yielding a function $\mathcal{I}(-)$ from $\mathcal{K}$ into a collection of sets such that for all $M, N \in \mathcal{K}$, $M \cong N \iff \mathcal{I}(M) = \mathcal{I}(N)$. The set $\mathcal{I}(M)$ is called an *isomorphism invariant of M in $\mathcal{K}$*. A representation theorem is a *structure theorem* when the isomorphism invariant is set-theoretically "simple". The reader is referred to [She85] for a discussion of the complicated matter of making the term "simple" more precise. Here we accept as simple invariants: cardinal numbers, sequences of cardinals of length $\leq 2^{|T|}$ and quotients of such objects by equivalence relations. Following are some examples of structure theorems.

1. A vector space is determined up to isomorphism by its dimension.
2. A divisible abelian group can be written as a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ (for various primes p). The isomorphism type of the group is determined by the number of copies of each of these groups in any such decomposition.
3. Fixing a strongly minimal formula φ over the prime model the isomorphism type of a model M of an uncountably categorical theory T is determined by the dimension in M of a conjugate of φ. (This is Morley's Categoricity Theorem when M is uncountable, and the Baldwin-Lachlan Theorem in general.)

Proposition 7.1.2(ii) is a good first approximation of a structure theorem for the models of a bounded t.t. theory T. Let $M \models T$ and fix a copy of the prime model M_0 in M. For $i \in DIM(T)$ let $d_i(M)$ be $\dim(p, M)$, where p is any SR type in $S(M_0)$ whose nonorthogonality class is i (the choice of p does not effect this dimension). Let $\mathcal{I}_{M_0}(M)$ be $\langle d_i(M) : i \in DIM(T) \rangle$. Now let N, N' be two models containing M_0 such that $\mathcal{I}_{M_0}(N) = \mathcal{I}_{M_0}(N')$. Then there are M_0−independent sets $C \subset N$ and $C' \subset N'$ such that $C = \bigcup_{i \in DIM(T)} C_i$, where C_i is a basis for a regular type over M_0 whose nonorthogonality class is i, and similarly for C'. For each class $i \in DIM(T)$ we can choose C_i and C_i' to be sets of realizations of the same type over M_0. Thus, there is an elementary map f fixing M_0 and taking C onto C'. By Proposition 7.1.2, f extends to an isomorphism of N onto N'. Summarizing these statements, when N, N' are models containing M_0 with $\mathcal{I}_{M_0}(N) = \mathcal{I}_{M_0}(N')$, N is isomorphic to N' over M_0. However, these invariants only characterize the models up to isomorphism over M_0 rather than over $\emptyset$. We might tag a model M with the set of all $\mathcal{I}_{M_0}(M)$, as M_0 ranges over all copies of the prime model in M, however there may be $|M|^{|T|}$ many such models leading to a set-theoretically complicated (hence not very useful) "invariant". The situation is analogous to our picture of the models of an uncountably categorical theory prior to the Baldwin-Lachlan Theorem. For M a countable model of such a theory there is a realizing an isolated type and strongly minimal formula φ over a such that $\dim(\varphi(M)) = \dim(\varphi(N)) \implies M \cong N$ for any N containing a. It is conceivable, though, that there is a countable model N containing a with $\dim(\varphi(M)) \neq \dim(\varphi(N))$ which is still isomorphic to M (by a map not fixing a). This will happen exactly when there is an a' in M realizing $tp(a)$ such that the dimension in M of the conjugate of φ over a' is different from the dimension of φ in M. This is shown to be impossible in the proof of the Baldwin-Lachlan Theorem. Proposition 7.1.2(ii) generalizes Morley's Categoricity Theorem, while the following refined study of dimensions generalizes the Baldwin-Lachlan Theorem (which actually follows from the next lemma).

7.1.1 Bounded ω−stable Theories

In this subsection we restrict our attention to the models of a bounded t.t. theory. Similar results can be proved for the class of a−models in a superstable theory (left to the reader).

Lemma 7.1.4. *Let T be a t.t. theory and $p \in S(a)$ an SR type nonorthogonal to $\emptyset$ with $tp(a)$ isolated. Then for any model M of T containing a and conjugate $p' \in S(a')$ of p over M which is nonorthogonal to p,* $\dim(p, M) = \dim(p', M)$. *In particular, when $stp(a) = stp(a')$, p and p' have the same dimension in M.*

Proof. We adopt the notation p_b for the conjugate of p over b when b realizes $tp(a)$. Notice that every conjugate of p is nonorthogonal to $\emptyset$. By Lemma 7.1.2, all conjugates of p over $acl(\emptyset)$ are nonorthogonal, hence the last sentence of the lemma follows from the first part.

Let M be a model containing a and choose a' realizing $tp(a)$.

Claim. If a' realizes $stp(a)$, $\dim(p, M) = \dim(p_{a'}, M)$.

First consider the case when $a \downarrow a'$. Let $N \subset M$ be a prime model over $\{a, a'\}$. Since a and a' are independent realizations of the same strong type the pairs aa' and $a'a$ have the same type over $\emptyset$. Any elementary map on $\{a, a'\}$ extends to an automorphism of N. Thus, $\dim(p, N) = \dim(p_{a'}, N)$. Since p and $p_{a'}$ are nonorthogonal SR types, $\dim(p|N, M) = dim(p_{a'}|N, M)$. By the additivity of dimension for SR types (Lemma 6.4.3), $\dim(p, M) = \dim(p_{a'}, M)$, as required.

Turn to the general case of an arbitrary realization a' of $stp(a)$. Let b be a realization of $stp(a)$ which is independent from M and N a prime model over $M \cup \{b\}$. By the previous paragraph, $\dim(p, N) = \dim(p_b, N) = \dim(p_{a'}, N)$. Since any SR type in $S(M)$ has finite dimension in N (bounded by $wt(b)$) we derive the equality of $\dim(p, M)$ and $\dim(p_{a'}, M)$ from the equations $\dim(p|M, N) = \dim(p_{a'}|M, N)$ and $\dim(p|M, N) + \dim(p, M) = \dim(p_{a'}|M, N) + \dim(p_{a'}, M)$.

Now take a' to be an arbitrary realization of $tp(a)$ in M such that $p \not\perp p_{a'}$. Since a and a' realize an isolated type over $acl(\emptyset)$, as well as over $\emptyset$, there is a prime model $N \subset M$ containing elements b and b' realizing $stp(a)$ and $stp(a')$, respectively. Since there is an automorphism of N taking b to b', $\dim(p_b, N) = \dim(p_{b'}, N)$. Since p and $p_{a'}$ are nonorthogonal and both are nonorthogonal to $\emptyset$, the conjugates over b and b' are nonorthogonal. Thus, the additivity of dimension again gives the equality of $\dim(p_b, M)$ and $\dim(p_{b'}, M)$. By the first part of the proof, $\dim(p, M) = \dim(p_{a'}, M)$.

For the remainder of the subsection, T is a bounded ω−stable theory.

For $i \in DIM(T)$ we want to give meaning to the term "the dimension of i in a model M". To do this we make a selection of representative types q_i satisfying:

1. q_i is SR of nonorthogonality class i and $dom(q_i)$ realizes an isolated type;
2. if i, $j \in DIM(T)$ and there is a conjugate of q_i nonorthogonal to q_j, then q_i is conjugate to q_j.

Also fix $\{q_i : i \in DIM(T)\}$ for the remainder of the subsection. For $i \in DIM(T)$ and M a model we let $\dim(i, M)$ denote $\dim(r_i, M)$ where r_i is a conjugate over $acl(\emptyset)$ of q_i which is over M. By Lemma 7.1.4 this dimension is independent of the choice of r_i (any r, r' conjugate over $acl(\emptyset)$ to q_i have the same dimension in any model containing $dom(r) \cup dom(r')$). If N is a submodel of M we define $\dim(i|N, M)$ to be $\dim(r_i|N, M)$, where r_i is a conjugate over $acl(\emptyset)$ to q_i which is over N. As a preliminary step to finding invariants for models with respect to isomorphism (over $\emptyset$) we define $\mathcal{I}_1(M)$, called the *pre-invariant of* M, to be the function f from $DIM(T)$ into the class of cardinals such that $f(i) = dim(i, M)$, for $i \in DIM(T)$. It is immediate from the definition that $\mathcal{I}_1(M) = \mathcal{I}_1(N)$ when M and N are models isomorphic over $acl(\emptyset)$, but not necessarily when M and N are isomorphic over $\emptyset$. We will deal with this deficiency later, first taking care of

Lemma 7.1.5. *Given models M, N of T, if $\mathcal{I}_1(M) = \mathcal{I}_1(N)$, then $M \cong N$, in fact, M and N are isomorphic over $acl(\emptyset)$.*

Proof. The basic approach is fairly clear. We choose prime models $M_0 \subset M$ and $N_0 \subset N$ and bases J_i, K_i for the distinguished representative of $i \in DIM(T)$ over these prime models in M, N. Since the dimensions of the regular types match up in M and N we should be able to lift an isomorphism between M_0 and N_0 to one taking J_i onto K_i. However, it's possible for $\dim(i, M) = dim(i, N)$ to be $\aleph_0$, while $\dim(i|M_0, M) = \aleph_0$ and $\dim(i|N_0, N)$ is finite. By choosing the prime models carefully (using the following claim) we will eliminate this irregularity.

Claim. For M_1 a prime model there is a prime model $M_1' \supset M_1$ such that for all $i \in DIM(T)$ with $\dim(i, M_1)$ infinite, $\dim(i|M_1, M_1')$ is also infinite.

Let $i \in DIM(T)$ be such that $\dim(r_i, M_1)$ is infinite, where r_i is conjugate to q_i over $acl(\emptyset)$. Let a realize $r_i|M_1$, N a prime model over $M_1 \cup \{a\}$ and I a basis for r_i in M_1. For any $b \in M_1$ there is an $a' \in I$ realizing $tp(a/b)$ (since I is infinite) hence $tp(ab)$ is isolated. Thus, N is atomic (hence prime) over $M_1 \cup \{a\}$. Iterating this process infinitely many times for each element of $DIM(T)$ results in M_1'.

Given the original choice of the prime model $M_0 \subset M$ let $M_0' \supset M_0$ be a prime model as in the claim. Let f be an isomorphism of M_0' onto M_0 and $M_0'' = f(M_0)$. Thus, replacing M_0 by M_0'' if necessary we can assume that whenever $\dim(i, M_0)$ is infinite, $\dim(i|M_0, M)$ is infinite. Choose a prime model $N_0 \subset N$ with the same property. Now let $i \in DIM(T)$. Since $\mathcal{I}_1(M) = \mathcal{I}_1(N)$ the additivity of dimensions for SR types implies that $\dim(i, M_0) + \dim(i|M_0, M) = \dim(i, N_0) + \dim(i|N_0, N)$. If $\dim(i, M_0) = \dim(i, N_0)$ is finite we conclude automatically that $\dim(i|M_0, M) = \dim(i|N_0, N)$. If the dimension of i is infinite in a prime model, then the choice of M_0 and N_0 forces $\dim(i|M_0, M) = \dim(i|N_0, N)$.

The models M_0 and N_0 are isomorphic over $acl(\emptyset)$ via some map f. For $i \in DIM(T)$, let J_i be a basis in M for $r_i|M_0$, where r_i is as usual. Let K_i be a basis in N for $f(r_i)|N_0$. Since $r_i \perp r_j$ when $i \neq j \in DIM(T)$, $J = \bigcup_{i \in DIM(T)} J_i$ is M_0−independent. Similarly, $K = \bigcup_i K_i$ is M_0−independent. Thus, f extends to an elementary map g which takes J onto K. Because M is prime over $M_0 \cup J$, and N is prime over $N_0 \cup K$, g extends to an isomorphism of M onto N.

Thus, the pre-invariants characterize the models up to isomorphism over $acl(\emptyset)$. However, the pre-invariants may not be be preserved under arbitrary isomorphisms. For example, given $M \supset dom(q_i)$, $f : M \cong N$ may map q_i to an SR type orthogonal to q_i. This behavior is found in the following class of examples.

Example 7.1.3. Fix $n < \omega$. Let T_0 be the theory of an equivalence relation E with exactly n classes, each infinite, and let X be the set of classes of E. We define T by adding structure to X. Let G be an arbitrary group of permutations of X and add relations on X so that in the resulting universe $\mathfrak{C}$, $\{ f \restriction X : f \in \mathrm{Aut}(\mathfrak{C}) \} = G$. For $x \in X$ the formula $v/E = x$ defines a strongly minimal set and distinct elements of X give rise to orthogonal sets. Thus, there is a one-to-one correspondence between $DIM(T)$ and X. Let G act on $DIM(T)$ through this correspondence. Pre-invariants correspond to isomorphic models if and only if they are conjugate with respect to G.

Motivated by this example we define an action of $\mathrm{Aut}(\mathfrak{C})$ on the pre-invariants so that models are isomorphic when they have conjugate pre-invariants. We will then take as the invariant of M the conjugacy class of a pre-invariant of M. The details follow.

Any conjugate of one of the q_i's is SR, hence lies in one of the nonorthogonality classes that make up $DIM(T)$. Define an action of $\mathrm{Aut}(\mathfrak{C})$ on $DIM(T)$ by: for $f \in \mathrm{Aut}(\mathfrak{C})$ and $i \in DIM(T)$, $f(i)$ is the unique j such that $f(q_i) \not\perp q_j$. Set-theoretically a pre-invariant is a function from $DIM(T)$ into the class of cardinals. The action of $\mathrm{Aut}(\mathfrak{C})$ on $DIM(T)$ can be extended to the class of pre-invariants by:

Given pre-invariants ϕ, ϕ' and $f \in \mathrm{Aut}(\mathfrak{C})$, $f(\phi) = \phi'$ if $\phi = \phi' \circ f$.

Lemma 7.1.6. *If f is an isomorphism from the model M onto the model N, then for $i \in DIM(T)$, $\mathcal{I}_1(M)(i) = \mathcal{I}_1(N)(f(i))$; i.e., $f(\mathcal{I}_1(M)) = \mathcal{I}_1(N)$.*

Proof. By Lemma 7.1.4, when r_i is over M and conjugate to q_i over $acl(\emptyset)$, $\dim(f(r_i), N) = \dim(f(i), N)$. Thus, $\dim(i, M) = \dim(f(i), N)$, proving that $f(\mathcal{I}_1(M)) = \mathcal{I}_1(N)$.

Define an equivalence relation $\sim$ by: given pre-invariants ϕ, ϕ', $\phi \sim \phi'$ if there is an $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(\phi) = \phi'$ that is, the $\sim$ –classes are the orbits under the action of $\mathrm{Aut}(\mathfrak{C})$ on the pre-invariants. Finally,

Definition 7.1.3. *Define the* invariant of M *to be* $\mathcal{I}(M) = \mathcal{I}_1(M)/\sim$.

Remember that to qualify as a structure theorem the assigned invariants must be set-theoretically simple in some intuitive way. Let G be the group of permutations of $DIM(T)$ (and the class of pre-invariants) induced by $\mathrm{Aut}(\mathfrak{C})$ as above. Since any element of $\mathrm{Aut}(\mathfrak{C})$ which is the identity on $acl(\emptyset)$ is the identity on $DIM(T)$, G can be identified with a quotient group of the automorphism group of $acl(\emptyset)$. Thus, $|G| \leq 2^{\aleph_0}$ and for any pre-invariant ϕ, the corresponding invariant is $G\phi$. We accept this as sufficient evidence of the set-theoretic simplicity of the invariants for the models of T.

Theorem 7.1.1 (Structure Theorem). *Models M and N are isomorphic if and only if $\mathcal{I}(M) = \mathcal{I}(N)$.*

Proof. See Lemma 7.1.6 for a proof that isomorphic models have conjugate pre-invariants; i.e., the same invariant. On the other hand, if $\mathcal{I}(M) = \mathcal{I}(N)$, then there is a model N' isomorphic to N such that $\mathcal{I}_1(N') = \mathcal{I}_1(M)$. By Lemma 7.1.5, N' and M are isomorphic.

Since invariants are set-theoretically simple we can call the result a "Structure Theorem".

This completes the assignment of invariants to a bounded t.t. theory. Recall that the spectrum function of T is the function $I(-, T)$ assigning to an infinite cardinal λ the number of models of T of cardinality λ. The assignment of invariants lets us compute exactly the spectrum function for T. This is an example of how the Structure Theorem leads to additional information about the models of the theory. The number $I(\lambda, T)$ may depend on the group G (defined above) and properties of $i \in DIM(T)$ such as "there is a model M and $r \in i$ such that $\dim(r, M)$ is finite". Note: we have not yet discussed the possible values of the pre-invariant functions.

Definition 7.1.4. *Let p be a stationary type over a finite set A in a stable theory. Then p is called* eventually nonisolated *(e.n.i.) if there is a finite $B \supset A$ such that $p|B$ is nonisolated. Otherwise, p is n.e.n.i.*

If T is a bounded $\omega-$stable theory as above we call $i \in DIM(T)$ eventually nonisolated *(e.n.i.) if there is an $r \in i$ which is e.n.i.*

Observe that being e.n.i. is preserved under conjugacy of types.

Lemma 7.1.7. *Let T be a bounded $\omega-$stable theory.*

(i) $i \in DIM(T)$ is e.n.i. if and only if for any $r \in i$ and model $M \supset$ dom(r), $\dim(r, M)$ *is finite.*

(ii) If i is e.n.i. and M_0 is a prime model then, for some $k < \omega$, $\dim(i, M_0)$ $=$ *k and for any cardinal $\kappa \geq k$ there is a model M with* $\dim(i, M) = \kappa$.

Proof. Both (i) and (ii) follow easily from two claims which are at the heart of the connection between dimension and being e.n.i.

Claim. Let r be an SR type over a finite set A, $M \supset A$ a model and q an SR type over a finite set $A' \subset M$ which is nonorthogonal to r. Then $\dim(r, M)$ is infinite if and only if $\dim(q, M)$ is infinite.

Without loss of generality, $A' = A$ and M is the prime model over A. There are a, b realizing $r|M$, $q|M$, respectively, such that $tp(a/M \cup \{b\})$ and

$tp(b/M \cup \{a\})$ are both isolated. Let $B \subset M$, $A \subset B$, be a finite set on which $tp(ab/M)$ is based (in which case $tp(b/B \cup \{a\})$ is isolated). Let I be a basis for r in M, J a basis for q in M and suppose I is infinite. For any finite $C \subset M$, $C \supset B$, there is an $a_0 \in I$ realizing $tp(a/C)$. For any such a_0 there is a $b_0 \in M$ such that $tp(a_0b_0/C) = tp(ab/C)$, in particular, b_0 realizes $q|C$. It follows that J is infinite, proving the claim.

Claim. Let r be an SR type over a finite set A. Then r is e.n.i. if and only if $\dim(r, M)$ is infinite in any model $M \supset A$.

First suppose that r is e.n.i. and $B \supset A$ is a finite set such that $r|B$ is nonisolated. Let M be a prime model over B and I a basis for r in M. If I is infinite there is an $a \in I$ realizing $r|B$. This contradicts that $r|B$ is nonisolated and M is prime over B, so I is finite.

Now suppose that r is n.e.n.i. and $M \supset A$ is a model. If $J \subset M$ is a finite Morley sequence in r over A, then $r|(I \cup A)$ is isolated, hence realized in M. Thus, $\dim(r, M)$ is infinite, proving the claim.

(i) follows immediately from the two claims.

The proof of (ii) is left as Exercise 7.1.4.

Let Δ_1 be the e.n.i. dimensions in $DIM(T)$, Δ_2 the n.e.n.i. dimensions and $\delta_i = |\Delta_i|$, for $i = 1, 2$. For simplicity we will compute the function $I^*(-, T)$, where $I^*(\lambda, T)$ is the number of models of cardinality $\leq \lambda$. The "subtraction" needed to compute $I(-, T)$ is left to the reader.

Lemma 7.1.8. *Let Φ_α be the set of pre-invariants of models of T of cardinality $\leq \aleph_\alpha$ and Φ_α/G the orbits of pre-invariants under G; i.e., the invariants of models of T. Then, $I^*(\aleph_\alpha, T) = |\Phi_\alpha/G|$ and $|\Phi_\alpha| = |\alpha + \omega|^{\delta_1} \times |\alpha + 1|^{\delta_2}$. When $|\alpha|$ is regular and uncountable, $I^*(\aleph_\alpha, T) = |\alpha|$.*

(The proof is left to the reader.) The actual value of $|\Phi_\alpha/G|$ depends on detailed information about G — there is no uniform formula for computing this cardinal in terms of $|G|$ and $|\Phi_\alpha|$ — however, for any particular theory it is easy to determine the value. We can, though, give some rough limits for some $\aleph_\alpha$. The group G is infinite only if $DIM(T)$ is infinite, in which case there are infinitely many conjugacy classes of dimensions. Thus, $I^*(\aleph_\alpha, T) \geq |\alpha + 1|^{\aleph_0}$, in fact $I^*(\aleph_\alpha, T) \geq |\alpha + \omega|^{\aleph_0}$ if δ_1 is infinite. We already know that $I^*(\aleph_\alpha, T) \leq |\alpha + \omega|^{\delta_1} \times |\alpha + 1|^{\delta_2}$, so if δ_1 is infinite, $I^*(\aleph_\alpha, T) = |\alpha + \omega|^{\aleph_0}$. If, say, δ_1 is finite, but nonzero, and δ_2 is infinite, then $I^*(\aleph_\alpha, T) = |\alpha + 1|^{\aleph_0} + \aleph_0$. It is easy to construct examples of theories satisfying each of these conditions. When $DIM(T)$ is finite the group G comes into play. It is natural to ask when T can have finitely many models in some uncountable cardinal. Of course, this is possible when T is uncountably categorical, but the above argument says it's also true exactly when $\delta_1 = 0$ and δ_2 is finite. For example, take the theory of an infinite and coinfinite predicate symbol P. This theory is ω−stable, bounded, has no e.n.i. dimensions, two n.e.n.i. dimensions and the group G consists of the identity. By the above formula, $I^*(\aleph_\alpha, T) = |\alpha + 1|^2$, which is finite exactly when α is finite. The reader who constructs other examples will probably conjecture the following result.

Lemma 7.1.9 (Lachlan). *Let T be a bounded $\omega-$stable theory such that in some uncountable cardinal κ, T has more than one but finitely many models of cardinality κ. Then T is $\omega-$categorical.*

Proof. It suffices to assume that T is not $\omega-$categorical and prove it has an e.n.i. dimension. Let $\{ q_i : i \in DIM(T) \}$ be a family of SR types satisfying the conditions on page 328. Remember, $i \in DIM(T)$ is e.n.i. if and only if q_i is e.n.i. Since T is not $\omega-$categorical there is a nonisolated complete type p_0 over $\emptyset$. Let M be a prime model and a a realization of p with $a \downarrow M$. By Proposition 7.1.2(ii) there is a finite $M-$independent set C of realizations of SR types over M such that $tp(C/M \cup \{a\})$ and $tp(a/M \cup C)$ are isolated. Without loss of generality, for each $c \in C$ there is an $i \in DIM(T)$ such that c realizes $q_i|M$. Let $A \subset M$ be finite such that $tp(aC/M)$ is based on A and $tp(a/M \cup C)$ is isolated over $A \cup C$. By the usual corollary to the Open Mapping Theorem $tp(a/A)$ is nonisolated. Thus, $tp(C/A)$ is nonisolated. Let $C' \subset C$ and $c \in C$ be such that $tp(C'/A)$ is isolated and $p = tp(c/A \cup C')$ is nonisolated. Then p is parallel to some q_i, which is hence e.n.i. This proves the lemma.

This lemma about countable theories with finitely many but more than one model in some uncountable cardinality can be improved significantly. Shelah showed in [She90, VIII,1.7] that any countable theory which is not $\omega-$stable has $\geq \min\{2^{2^{\aleph_0}}, 2^{\lambda}\}$ models in each uncountable cardinality λ. Moreover, Lachlan proved in [Lac75] that if T is an $\omega-$stable theory with finitely many models in some uncountable cardinality, then T is $\omega-$categorical and bounded (see Corollary 7.1.3). The best possible result is

> A countable theory T has finitely many but more than one model in some uncountable power if and only if T is $\omega-$stable, $\omega-$categorical and bounded.

The left-to-right direction uses results by Shelah (to reduce our attention to $\omega-$stable theories) and a theorem by Lachlan. (Later we will complete Lachlan's contribution by reproducing the proof that such theories are bounded.) The right-to-left direction reduces to showing that a bounded $\omega-$stable, $\omega-$categorical theory has finitely many dimensions. This requires the deep "geometrical" results found in [CHL85].

We leave it to the reader to investigate other properties of the spectrum functions of bounded $\omega-$stable theories on his one.

Besides calculating the possible spectrum functions this analysis of SR types in a bounded $\omega-$stable theory leads to Proposition 7.1.3. Some background is needed to understand this result.

As mentioned earlier this study of bounded t.t. theories can be viewed as a generalization of the Baldwin-Lachlan Theorem. One part of their analysis was to prove the homogeneity of all countable models of an uncountably categorical theory. The following trivial example shows that not every bounded ω–stable theory has this property.

Example 7.1.4. (A bounded ω–stable theory with a nonhomogeneous countable model) Let $L = \{ P_i : i < \omega \} \cup \{ E \}$, where E is a binary relation symbol and P_i is unary. Let M be a structure for L in which $\{ P_i(M) : i < \omega \}$ form a pairwise disjoint family of infinite sets, and E defines an equivalence relation on M with two classes, each class containing infinitely many elements of $P_i(M)$ for all $i < \omega$. Then $T = Th(M)$ is a bounded ω–stable theory. A countable model N in which one E–class is contained in $\bigcup_{i<\omega} P_i(N)$ and the other contains an element not in any $P_i(N)$, is not homogeneous. (For a fixed $i \in \omega$, let a, $b \in P_i(N)$ such that $\models \neg E(a, b)$. Then $tp(a) = tp(b)$ but there is no automorphism of N mapping a to b.)

Of course, if we added constants to the language for the E–classes in this example every model would be $\aleph_0$–homogeneous, in other words, every model of the theory is $\aleph_0$–homogeneous over $acl(\emptyset)$. We'll see shortly that this is always true in a bounded ω–stable theory.

Definition 7.1.5. *A model M is* almost κ–homogeneous *(where $\kappa \geq |T|$) if M is κ–homogeneous over $acl(\emptyset)$; i.e., M is κ–homogeneous in the language with constants for $acl(\emptyset)$.*

The usual conventions for κ–homogeneous models are adopted for almost κ–homogeneous models, for example, M is almost homogeneous if it is almost $|M|$–homogeneous.

Proposition 7.1.3. *If T is a bounded t.t. theory, then every model of T is almost $\aleph_0$–homogeneous.*

Proof. Given a, b and c in M with $stp(a) = stp(b)$ we must find a $d \in M$ such that $stp(bd) = stp(ac)$. We can enlarge a and b by adjoining elements realizing isolated types over $\{a\} \cup acl(\emptyset)$ and $\{b\} \cup acl(\emptyset)$ to require that M contains $e = \{e_0, \ldots, e_n\}$ such that e is a–independent, $q_i = tp(e_i/a)$ is SR, both $tp(c/ae)$ and $tp(e/ac)$ are isolated, and $tp(c/a) \models stp(c/a)$. Let f be an automorphism fixing $acl(\emptyset)$ and mapping a to b. Then, q_i is conjugate to $r_i = f(q_i)$ over $acl(\emptyset)$ so they have the same dimension in M (by Lemma 7.1.4). Thus, there is $e' = \{e'_0, \ldots, e'_n\}$ such that $stp(ae) = stp(be')$. Since $tp(c/ae)$ is isolated there is a $d \in M$ with $tp(aec) = tp(be'd)$. Since $tp(c/a) \models stp(c/a)$, $tp(d/b) \models stp(d/b)$. Thus, $stp(bd) = stp(ac)$, as required.

Almost $\aleph_0$–homogeneous models do have some of the same relative uniqueness and universality conditions as $\aleph_0$–homogeneous models:

Lemma 7.1.10. *Let T be an $\omega-$stable theory.*

(i) If M is an almost $\aleph_0-$homogeneous model and N is a countable model such that every type over $\emptyset$ realized in N is realized in M, then N can be elementarily embedded into M.

(ii) If M and N are countable almost homogeneous models realizing the same types over $\emptyset$, then $M \cong N$.

Proof. Both parts of the lemma follow quickly from

Claim. Let N be a countable model and M a model such that every element of $S(\emptyset)$ realized in N is realized in M. Then there is a model $N' \cong N$ such that every element of $S(acl(\emptyset))$ realized in N' is realized in M.

Let Q be the set of elements of $S(acl(\emptyset))$ realized in M. Let $N = \{ a_i : i < \omega \}$, $b_i = (a_0, \ldots, a_i)$ and $p_i = tp(b_i)$, a type in a sequence v_i of $i+1$ variables. Since T is $\omega-$stable each p_i has finitely many extensions over $acl(\emptyset)$. Since each p_i is realized in M there is $q_i \in Q$, an extension of p_i, such that for infinitely many (hence all) $j \geq i$ there is an element of Q extending p_j whose restriction to v_i is q_i. In fact, (by König's Lemma) we can choose the q_i's so that q_i is the restriction to v_i of q_{i+1}. Thus, there is a set $\{ c_i : i < \omega \} \subset M$ such that $(c_0, \ldots, c_i)$ realizes q_i. The model $N' = \{ c_i : i < \omega \}$ is the desired isomorphic copy of N, proving the claim.

Since an almost $\aleph_0-$homogeneous model is $\aleph_0-$homogeneous over $acl(\emptyset)$ both (i) and (ii) follow from Corollaries 2.2.2 and 2.2.4.

7.1.2 Unbounded Theories

This subsection is a continuation of our study of how the isomorphism type of a model M of an $\omega-$theory is tied to the dimensions of the regular types over M. For $p \in S(A)$ and B a set conjugate to A over $\emptyset$, p_B denotes a type over B conjugate to A. In Lemma 7.1.8 we calculated the spectrum function for a bounded $\omega-$stable theory. We showed, for example, that when $|\alpha|$ is regular and uncountable, $I^*(\aleph_\alpha, T) = |\alpha|$, a number significantly smaller than the maximum possible value, $2^{\aleph_\alpha}$ (in general). In other words, having a bounded number of SR types, up to nonorthogonality, leads to relatively few models. In the next proposition we give a comparatively large lower bound to the spectrum function of an *un*bounded $\omega-$stable theory.

Proposition 7.1.4. *If T is an unbounded $\omega-$stable theory, then for all $\alpha >$ 0, $I(\aleph_\alpha, T) \geq |\alpha + \omega|^{|\alpha+1|}$.*

Proof. Let $\kappa_\alpha = |\alpha + \omega|^{|\alpha+1|}$. By Lemma 7.1.2 and Proposition 6.4.1 there is an SR type $p \in S(a)$ which is orthogonal to $\emptyset$. Let $q = stp(a)$. Let Λ be the collection of all functions f from $\{ \beta : \beta \leq \alpha \}$ into $\{ \lambda : \lambda$ is a cardinal $\leq \aleph_\alpha \}$ such that $f(\alpha) = \aleph_\alpha$. Note, $|\Lambda| = \kappa_\alpha$. For each $f \in \Lambda$ we construct as follows a model M_f of cardinality $\aleph_\alpha$ so that for $f \neq g \in \Lambda$, $M_f \not\cong M_g$.

Fix $f \in \Lambda$. For $\beta \leq \alpha$ let J_β be a Morley sequence in q of cardinality $f(\beta)$, chosen so that $J = \bigcup_{\beta \leq \alpha} J_\beta$ is also independent. First let M_0 be a prime model over J (which has cardinality $\aleph_\alpha$). Given $b \in J$ and I a basis for p_b in M_0, I is independent from J over b since p_b is orthogonal to q. Thus, I is indiscernible over J, which is hence countable by Theorem 5.5.1. Thus, $\dim(p_b, M_0)$ is countable for all $b \in J$. By iterated use of Proposition 6.4.4(iii) there is a model $M_1 \supset M_0$ of cardinality $\aleph_\alpha$ such that for each $\beta < \alpha$ and $b \in J_\beta$, $\dim(p_b, M_1) = \aleph_\beta$. Let $X = \{p_b : b \in J\}$. Finally, by (iv) of Proposition 6.4.4 there is a model $M_f \supset M_1$ of cardinality $\aleph_\alpha$ such that $\dim(p'|M_1, M_f) = 0$ for all $p' \in X$ and $\dim(r, M_f) = \aleph_\alpha$, for any SR type r over a finite subset of M_f which is orthogonal to each element of X.

We now verify that distinct elements of Λ give rise to nonisomorphic models. Suppose that $F : M_g \cong M_f$, where $f, g \in \Lambda$, and $J \subset M_f$ is a Morley sequence in q as in the above construction. It suffices to show that $g(\beta) \leq f(\beta)$ for all $\beta \leq \alpha$. Simply by the definition of Λ, $f(\alpha) = g(\alpha)$. Let $\beta < \alpha$ and c a realization of q in M_g with $\dim(p_c, M_g) = \aleph_\beta$. By construction, any conjugate of p over M_f which is orthogonal to each type in $\{p_b : b \in J\}$ has dimension $\aleph_\alpha$. Thus, by Proposition 6.4.4(ii), $F(p_c)$ is nonorthogonal to one of these p_b's, in fact, $F(p_c) \not\perp p_b$, for some $b \in J_\beta$. Since p is orthogonal to $\emptyset$, when $d, d' \in q(\mathfrak{C})$ are independent, $p_d \perp p_{d'}$. Thus, if J' is a Morley sequence in q such that $\dim(p_c, M_g) = \aleph_\beta$ for all $c \in J'$, F induces an injection $\phi : J' \longrightarrow J_\beta$ defined by: for $c \in J'$, $\phi(c)$ is the element b of J_β such that $F(c) \not\downarrow b$. We conclude that $g(\beta) \leq f(\beta)$, as required to prove the proposition.

Corollary 7.1.3. *If T is an ω-stable theory with finitely many models in some uncountable cardinal, then T is ω-categorical and bounded.*

(See Lemma 7.1.9 and subsequent remarks concerning this corollary.)

Historical Notes. Unbounded theories are defined as multi-dimensional theories in [She90, V.5.2], although unidimensional theories are defined earlier in [She90, V.2.2]. Lemma 7.1.2 is stated explicitly as [Las86, 9.7]. Proposition 7.1.1 is [She90, IX.1.8]. The main idea in Proposition 7.1.2 is found in Section 4 of [BL83] (Lemma 4.5, in particular) and is stated more explicitly as [Las86, 9.13]. Most of the results in the subsection on bounded ω-stable theories are found in [She90, IX,2.3], [BL83] and [Las86]. Proposition 7.1.3 is Corollary 5.3 of [BL83]. Lemma 7.1.10(ii) is due to Pillay [Pil82]. Lachlan's Lemma 7.1.9 is found in [Lac75]. Proposition 7.1.4 is implicit in Section 5 of [She90, V].

Exercise 7.1.1. Prove (4) $\Longrightarrow$ (1) in Lemma 7.1.2.

Exercise 7.1.2. Prove the following fact using the same ideas used to prove Proposition 7.1.2.

If T is a bounded t.t. theory, M is an a−model and C is an M−independent set of realizations of SR types over M, then a prime model over $M \cup C$ is an a−model.

(This is due to Pillay. HINT: Take a prime model N over $M \cup C$, an a−prime model N' over $M \cup C$ containing N, and show that N' must equal N.)

Exercise 7.1.3. Following the methods used to analyze bounded ω−stable theories develop a theory of invariants for the class of a−models in a bounded countable superstable theory and prove the resulting structure theorem.

Exercise 7.1.4. Prove (ii) in Lemma 7.1.7.

Exercise 7.1.5. State and prove an analogue of Proposition 7.1.4 which holds relative to the class of a−models in a bounded countable superstable theory.

7.2 More on Ranks

This section is devoted to refining our knowledge of Morley rank, ∞−rank and U−rank in some special superstable theories. In particular, we will prove the "definability of Morley rank" and the equivalence of Morley rank and U−rank in uncountably categorical theories. We will also prove corresponding results about ∞−rank in unidimensional superstable theories.

See Definition 6.1.3 for the definition of a "notion of rank". Given a complete theory T, a map R which takes a formula of T to an ordinal is a *notion of rank on formulas* if the map R' which takes a complete type p to $\inf\{ R(\varphi) : \varphi \in p \}$ is a notion of rank.

Definition 7.2.1. *Given a complete theory T, a notion of rank R on formulas in is said to be* definable *if for all formulas $\varphi(x, a)$,*

$$\text{there is a } \theta \in tp(a) \text{ such that } \models \theta(b) \implies R(\varphi(x,a)) = R(\varphi(x,b)). \quad (7.1)$$

When θ satisfies (7.1) and $R(\varphi(x,a)) = \alpha$ we say θ proves that $R(\varphi(x,a)) = \alpha$.

One instance of the definability of Morley rank in uncountably categorical theories played an important role in our proofs of Morley's Categoricity Theorem and the Baldwin-Lachlan Theorem. Specifically, for T such a theory and $\varphi(x,y)$ a formula there is a formula $\theta(y)$ such that $MR(\varphi(x,a)) = 0 \iff \models \theta(a)$ (see Lemma 3.1.12). This fact was of critical importance in showing that there is a strongly minimal formula over the prime model. The definability of Morley rank has numerous applications in the study of uncountably categorical theories. Of equal importance is the equality of Morley rank and U−rank and the fact that these ranks are always finite (proven below). This implies that Morley rank has the same additivity properties as U−rank (Corollary 6.1.1); i.e., $MR(ab) = MR(a/b) + MR(b)$, for all a and b.

Uncountable categoricity is an important hypothesis in obtaining the definability of Morley rank. Example 3.1.3 produces a simple ω−stable theory in which this definability fails. (There is an a such that $E(x,a)$ is nonalgebraic, but for any $\theta \in tp(a)$ there is a b satisfying θ such that $E(x,b)$ is algebraic.)

Our first major goal is

Theorem 7.2.1. *If T is a unidimensional theory, then*
(i) ∞−rank is definable in T and
(ii) for all complete types p, $R^\infty(p) = U(p) < \omega$.

Part (i) is certainly the hardest. This will follow from the slightly more general

Proposition 7.2.1. *Suppose that T is a superstable theory in which every nonalgebraic type of finite ∞−rank is nonorthogonal to $\emptyset$. Then, for all formulas $\varphi(x,a)$,*

(∗) *if $R^\infty(\varphi(x,a)) = n < \omega$, there is a formula $\theta \in tp(a)$ such that* $\models \theta(b) \implies R^\infty(\varphi(x,b)) = n$.

Fix T satisfying the hypotheses of the proposition until the completion of the proof. In the proof we will use the following which was assigned in Section 6.1 as Exercise 6.1.4 (and proved in [She90, V,7.12(5)]).

Lemma 7.2.1. *If $R^\infty(\varphi(x,a)) \geq n$ (where $n < \omega$), then there is a $p \in S(a)$ containing $\varphi(x,a)$ such that $U(p) \geq n$.*

(The proof is a relatively easy induction on $R^\infty(\varphi(x,a))$.)

The proposition is proved by induction on rank. Assume that (∗) holds for all formulas of ∞−rank $< n$ and $n = R^\infty(\varphi(x,a))$. The proof of (∗) for $\varphi(x,a)$ is divided into two similar but distinct parts. In the proofs we will use the fact that if $a' \supset a$ and there is a formula $\theta' \in tp(a')$ such that $\models \theta'(b) \implies R^\infty(\varphi(x,b)) = R^\infty(\varphi(x,a))$, then there is such a formula over a. (Simply quantify existentially over the variables satisfied by the elements of $a' \setminus a$.) This permits us to expand a to a set having additional properties.

Lemma 7.2.2. *There is a formula $\theta \in tp(a)$ such that*

$$\models \theta(b) \implies R^\infty(\varphi(x,b)) \geq n.$$

Proof. Assume the lemma fails. Then, for each $\theta \in tp(a)$ there is an a' satisfying θ such that $R^\infty(\varphi(x,a')) \leq n-1$. By Lemma 7.2.1 there is a type q over $acl(a)$ containing φ and having U−rank $\geq n$. Since U−rank is $\leq \infty$−rank for any complete type, $U(q) = n$. Since q is nonorthogonal to $\emptyset$, we can pick a sufficiently large so that if c realizes q there is a pair $c'a'$ realizing $stp(ca)$ such that

$$a' \downarrow ca,\ c'a' \downarrow a \text{ and } c' \not\downarrow_{aa'} c$$

(see Propositions 5.6.1 and 5.6.2). Let $k = R^\infty(c/aa'c')$ and $k' = R^\infty(c'/aa'c)$, both of which are $< n$. Without loss of generality, $k \leq k'$. Let $\psi(x, y, x', y') \in tp(cac'a')$ be such that $\models \psi(d, b, d', b')$ implies

(2) $\models \varphi(d, b) \wedge \varphi(d', b')$,
(3) $R^\infty(\psi(x, b, d', b')) \leq k$, and
(4) $R^\infty(\psi(d, b, x', b')) = k'$.

(We can require (3) and (4) by the inductive hypothesis. First pick a formula $\psi_0(x, y, x', y')$ such that $\exists x\psi_0(x, y, x', y')$ proves that $k = R^\infty(\psi_0(x, ac'a'))$. Choose $\psi(x, y, x', y')$ to be a formula implying ψ_0 such that $\exists x'\psi(x, y, x', y')$ proves that $k' = R^\infty(\psi(ac, x', a'))$. While $\exists x\psi(x, y, x', y')$ may not prove that $k = R^\infty(\psi(x, ac'a'))$, the inequality in (3) does hold.) We will obtain a contradiction by showing that $k' + n \leq k + (n - 1)$.

Since a' is independent from ca we can apply the Open Mapping Theorem to find a $\sigma \in tp(a')$ such that every element of $S(\emptyset)$ containing σ has a nonforking extension containing $\exists x'\psi(c, a, x', y)$. Since we assumed that the lemma fails there is a b, $b \downarrow ca$ such that $\exists x'\psi(c, a, x', b)$ and $R^\infty(\varphi(x, b)) \leq n - 1$. From (4) we know that $R^\infty(\psi(c, a, x', b)) = k'$, so by Lemma 7.2.1 there is a d satisfying $\psi(c, a, x', b)$ with $U(d/cab) = k'$. Now compute $U(dc/ab)$ using the U−rank identities of Corollary 6.1.1: $U(dc/ab) = U(d/cab)+U(c/ab) = k'+n$. We also know that $U(c/dab) \leq k$ by (3), so $U(dc/ab) = U(c/dab)+U(d/ab) \leq k+(n-1)$. This is the contradiction which proves the lemma.

To complete the proof of Proposition 7.2.1 we prove

Lemma 7.2.3. *There is a formula $\theta \in tp(a)$ such that*

$$\models \theta(b) \implies R^\infty(\varphi(x, b)) \leq n.$$

Proof. The bulk of the proof is contained in

Claim. For any $p = tp(ca)$, where $\models \varphi(c, a)$, there is an $\hat{a} \supset a$, $c \downarrow_a \hat{a}$, and a $\psi(x, \hat{y}) \in tp(c\hat{a})$ such that for all b, $R^\infty(\psi(x, b)) \leq R^\infty(c/a)$.

If $R^\infty(c/a) < n$ the desired formula is obtained by induction, so we can assume that $R^\infty(c/a) = n$. Assume, to the contrary, that there are no such $\hat{a}$ and ψ. Since $stp(c/a)$ is nonorthogonal to $\emptyset$, there are:

– $\hat{a} \supset a$, $c \downarrow_a \hat{a}$, and
– $c'\hat{a}'$ realizing $stp(c\hat{a})$

such that

$$c\hat{a} \downarrow \hat{a}',\ \hat{a} \downarrow c'\hat{a}' \text{ and } c \not\downarrow_{\hat{a}\hat{a}'} c'.$$

Let $R^\infty(c/c'\hat{a}\hat{a}') = k$ and $R^\infty(c'/c\hat{a}\hat{a}') = k'$, where, without loss of generality, $k' \leq k$. By induction, there is a formula $\psi(x, \hat{y}, x', \hat{y}') \in tp(c\hat{a}c'\hat{a}')$ such that $\models \psi(d, \hat{b}, d', \hat{b}')$ implies

(5) $\models \varphi(d, \hat{b}) \wedge \varphi(d', \hat{b}')$.
(6) $R^\infty(\psi(x, \hat{b}, d', \hat{b}')) = k$, and
(7) $R^\infty(\psi(d, \hat{b}, x', \hat{b}')) \leq k'$.

(More properly, in (5) we mean that $\varphi(d, b)$ holds for an appropriate subset b of $\hat{b}$.) Since $c'\hat{a}' \downarrow \hat{a}$ the open mapping theorem yields a formula $\theta(x', \hat{y}') \in tp(c'\hat{a}') = tp(c\hat{a})$ such that $r \in S(\emptyset)$ has a nonforking extension containing $\exists x \psi(x, \hat{a}, x', \hat{y}')$ if and only if $\theta(x', \hat{y}') \in r$. Since we have assumed the claim to fail for p there are $\hat{b}$ and d such that $\models \exists x \psi(x, \hat{a}, d, \hat{b})$, $\hat{a} \downarrow d\hat{b}$ and $R^\infty(d/\hat{b})$ is $\geq n+1$. Furthermore, by Lemma 7.2.1, we can assume that $U(d/\hat{b}) \geq n+1$. By (6) there is a c_0 satisfying $\psi(x, \hat{a}, d, \hat{b})$ such that $R^\infty(c_0/d\hat{a}\hat{b}) = U(c_0/d\hat{a}\hat{b}) = k$. Now compute $U(c_0 d/\hat{a}\hat{b})$ in two ways. If $U(d/\hat{b})$ is infinite, then $U(c_0 d/\hat{b})$ is infinite, hence $\geq k + (n+1)$. If $U(d/\hat{b}) < \omega$, $U(c_0 d/\hat{a}\hat{b}) = U(c_0/d\hat{a}\hat{b}) + U(d/\hat{a}\hat{b}) \geq k + (n+1)$ (by Corollary 6.1.1). On the other hand, $U(c_0 d/\hat{a}\hat{b}) = U(d/c_0\hat{a}\hat{b}) + U(c_0/\hat{a}\hat{b}) \leq k' + n$. This contradicts that $k' \leq k$, to prove the claim.

We now continue the proof to find the desired formula in $tp(a)$. Let M be a saturated model containing a. Let $r \in S(M)$ be an arbitrary type containing $\varphi(x, a)$. By the claim and the saturatedness of M there is an $\hat{a} \in M$, $\hat{a} \supset a$, and a formula $\psi(x, \hat{a}) \in r$ such that

(8) $R^\infty(\psi(x, \hat{b})) \leq R^\infty(r) \leq n$, for all $\hat{b}$.

By compactness, there is a $k < \omega$ and formulas $\psi_i(x, \hat{a}_i)$, $i \leq k$, each satisfying (8), such that any $r \in S(M)$ containing $\varphi(x, a)$ also contains one of these $\psi_i(x, \hat{a}_i)$'s. Let $\delta(x, \hat{c}a) = \bigvee_{i \leq k} \psi_i(x, \hat{a}_i)$, where $\hat{c} = \cup \hat{a}_i$, and observe that for any $\hat{d}b$, $R^\infty(\delta(x, \hat{d}b)) \leq n$. Let $\sigma(z)$ be the formula

$$\exists \hat{y}(\forall x(\varphi(x, z) \rightarrow \delta(x, \hat{y}, z))).$$

Then $\sigma \in tp(a)$ and for all b, $\models \sigma(b) \implies R^\infty(\varphi(x, b)) \leq n$. This proves the lemma.

This completes the proof of Proposition 7.2.1.

Recall that the major weakness of U−rank is that it is not continuous. In the class of theories presently under consideration this limitation is removed in (i) of

Lemma 7.2.4. *Let T be a superstable theory in which each type of finite U–rank is nonorthogonal to $\emptyset$. Further suppose that each type of U–rank 1 is nonorthogonal to a formula of ∞–rank 1. Then for all complete types p with $U(p) = n < \omega$*

(i) there is a $\varphi \in p$ such that for any complete type q, with $\varphi \in q$, $U(q) \leq n$, and

(ii) $U(p) = R^{\infty}(p)$.

Proof. We prove (i) and (ii) simultaneously by induction on rank. Both (i) and (ii) are clear when $U(p) = 0$. To make the induction work we need to handle the rank 1 case separately.

Claim. If p is a complete type of U–rank 1, then $R^{\infty}(p) = 1$.

Without loss of generality $dom(p) = M$ is an a–model and there are a realizing p and b such that $R^{\infty}(b/M) = 1$ and a depends on b over M, hence $a \in acl(M \cup \{b\})$. Thus, $R^{\infty}(a/M) \leq R^{\infty}(ab/M) = R^{\infty}(b/M) = 1$, implying that $R^{\infty}(a/M) = 1$ (since p is nonalgebraic) to prove the claim.

Turning to (i), if p is a complete type of U–rank 1 and $\varphi \in p$ has ∞–rank 1 then each complete type containing φ has U–rank ≤ 1, hence U–rank is continuous on the types of U–rank 1.

Now suppose that $U(p) = n+1$ and both (i) and (ii) hold for complete types of U–rank $\leq n$. We first prove (i) for p. By the Open Mapping Theorem it suffices to find an appropriate formula in some nonforking extension of p (see Exercise 7.2.3), thus we can assume $dom(p) = M$ to be an a–model. By Proposition 6.3.4 (and our freedom to choose M sufficiently large) there are a realizing p and b dependent on a over M such that $U(b/M) = 1$. By the U–rank identity (Corollary 6.1.1), $U(a/M \cup \{b\}) = n$. By induction, $R^{\infty}(a/M \cup \{b\})$ is also n. Furthermore, (i) holds for $tp(b/M)$. Combining these facts with Proposition 7.2.1 produces a formula $\psi(x,y) \in tp(ab/M)$ such that

$$\text{whenever} \quad \models \exists x \psi(x,b'),\ R^{\infty}(\psi(x,b')) = n \text{ and } \ U(b'/M) \leq 1.$$

Suppose that $\models \exists y \psi(a',y)$. To complete the proof of (i) it suffices to show that $U(a'/M) \leq n+1$. Let b' satisfy $\psi(a',y)$. Then $R^{\infty}(a'/M \cup \{b'\}) \leq n$, so $U(a'/M \cup \{b'\}) \leq n$ by Lemma 6.1.2(ii). Since $U(b'/M) \leq 1$, the U–rank identities imply that $U(a'/M) \leq U(a'b'/M) \leq U(a'/M \cup \{b'\}) + U(b'/M) \leq n+1$, as required.

Turning to (ii), assume $U(p) = n+1$ and $R^{\infty}(q) = U(q)$ whenever $U(q) \leq n$. We must now show that $R^{\infty}(p) = n+1$. Let $\varphi \in p$ be such that $U(q) \leq n+1$ for any complete type q containing φ. Let $Q = \{ q \in S(\mathfrak{C}) : \varphi \in q \text{ and } \neg\psi \in q$ for all ψ with ∞–rank $\leq n \}$, which is nonempty since $R^{\infty}(p) \geq n+1$ (U–rank is always $\leq \infty$–rank). Suppose φ is over A. If $q \in Q$, then $U(q) > n$ since (ii) holds for types of U–rank $\leq n$. Furthermore, $U(q \restriction A) \leq n+1$ by the choice of φ, so each element of Q does not fork over A. Thus, $|Q| < |\mathfrak{C}|$, from which we conclude that $R^{\infty}(\varphi) = n+1$ to complete the proof of the lemma.

Corollary 7.2.1. *If T is unidimensional, then $R^\infty(x = x) < \omega$.*

Proof. Let Φ be the set of all formulas of finite ∞−rank (in the same sort as x). By the previous lemma each type of finite U−rank has finite ∞−rank. Thus, assuming $\{\neg\varphi : \varphi \in \Phi\}$ to be consistent results in an element of $S(\mathfrak{C})$ of infinite U−rank, hence a q of U−rank ω. By Lemma 6.1.3 q is orthogonal to any type of finite U−rank. This contradiction to the unidimensionality of the theory proves the corollary.

This completes the proof of Theorem 7.2.1.

When T is a unidimensional t.t. theory we can add Morley rank to the picture. Sticking to the most interesting case we state the relevant result for countable theories.

Proposition 7.2.2. *If T is an uncountably categorical theory, then for all complete types p, $U(p) = R^\infty(p) = MR(p) < \omega$. Furthermore, Morley rank is definable in T.*

Proof. We have already shown that U−rank and ∞−rank are equal and finite in such theories. If we prove that Morley rank is equal to U−rank in T the definability of Morley rank will follow from Theorem 7.2.1. Our proof will closely parallel the proof of Proposition 7.2.1.

We proved above the continuity of U−rank in T. This is strengthened by showing

Claim. For all complete p there is a $\varphi \in p$ such that $\{q \in S(\mathfrak{C}) : \varphi \in q$ and $U(q) \geq U(p)\}$ is finite.

This is proved by induction on $U(p) = n + 1$. (The result is obviously true when p is algebraic.) Again, we can assume that $dom(p) = M$ is a saturated model. There are a realizing p and b dependent on a over M such that $tp(b/M)$ is strongly minimal. By a now standard argument, $U(a/M \cup \{b\}) = R^\infty(a/M \cup \{b\}) = n$. By the definability of ∞−rank and its equality with U−rank there is a formula $\psi(x, y) \in tp(ab/M)$ such that

(9) $\exists x\psi(x, y)$ is strongly minimal,
(10) $\{q \in S(M \cup \{b\}) : \psi(x, b) \in q$ and $U(q) \geq n\}$ is finite, and
(11) for all b' satisfying $\exists x\psi(x, y)$, $R^\infty(\psi(x, b')) \leq n$ and whenever $\models \psi(a', b')$, $b' \in acl(M \cup \{a'\})$.

Suppose that $\models \exists y\psi(a', y)$ and b' satisfies $\psi(a', y)$. By (9) and (11) $U(a'/M \cup \{b'\}) \leq n$ and $U(a'/M) \leq n + 1$. If $U(b'/M) = 0$, then $U(a'/M) \leq n$. Otherwise, $tp(b'/M) = tp(b/M)$ (by (9)) and a' depends on b' over M (by (11)). Thus, $U(a'/M) = n+1$, implying that $U(a'/M \cup \{b'\}) = n$ and $b' \notin M$ (by the usual U−rank computations). Thus, $tp(a'/M \cup \{b'\})$ is conjugate over M to one of the types over $M \cup \{b\}$ defined in (10). We conclude that there are finitely many types in $S(M)$ containing $\theta(x) = \exists y\psi(x, y)$ with U−rank

$n+1$. Since each such type is stationary there are finitely many elements of $S(\mathfrak{C})$ containing θ and having U−rank $n+1$, completing the proof of the claim.

Using this claim we can prove that $U(p) = MR(p)$ for any complete type p by an argument similar to that used to obtain (ii) of Lemma 7.2.4. The details are left to the reader in Exercise 7.2.1.

Contained within the proof of the previous proposition is a proof of

Corollary 7.2.2. *If T is a unidimensional theory containing a formula of Morley rank 1, then T is totally transcendental (hence uncountably categorical).*

Historical Notes. The finiteness and definability of Morley rank in an uncountably categorical theory is due to Baldwin [Bal73] and independently Zil'ber [Zil74]. Shelah extended these results to superstable unidimensional theories in [She90, IX.1.11]. There is a good exposition of these results by Saffe in [Saf84], although the proof of Theorem 4.5 of that paper does not work as written. After much prodding by Ambar Chowdhury I wrote down the proof that appears here. Bradd Hart and, independently, Predrag Tanovic have also written proofs.

Exercise 7.2.1. Finish the proof of Proposition 7.2.2.

Exercise 7.2.2. Prove Corollary 7.2.2.

Exercise 7.2.3. Let T be superstable, $p \in S(A)$, $A' \supset A$ and $p' \in S(A')$ a nonforking extension of p containing a formula φ' such that $\varphi' \in q \implies U(q) \leq U(p')$, for all $q \in S(A')$. Show that there is a $\varphi \in p$ such that $\varphi \in q \implies U(q) \leq U(p)$, for all $q \in S(A)$.

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