3. Uncountably Categorical and $\aleph_0$−stable Theories

In this chapter we will study the results which laid the foundation for stability theory, namely Morley's Categoricity Theorem and the Baldwin-Lachlan Theorem. Some of the concepts arising in their proofs will be redeveloped later for stable theories. We feel, however, that these proofs present an excellent introduction to the key concepts encountered later, and are historically important enough to warrant individual treatment.

In Section 1 a proof of Morley's Categoricity Theorem is given. In the third section of the chapter totally transcendental theories, which arose in Morley's original proof of Morley's Categoricity Theorem, will be studied more deeply. Again, ideas will be introduced which are seen throughout stability theory. In the fourth section these new concepts are applied to prove the Baldwin-Lachlan Theorem. Groups definable in totally transcendental theories are studied in the fifth section.

3.1 Morley's Categoricity Theorem

Throughout this section an arbitrary theory is assumed to be countable and have infinite models. For emphasis this assumption may be repeated within the statements of theorems. Recall that a theory T is said to be *categorical in* κ (or *κ−categorical*), where κ is an infinite cardinal, if T has a unique model of cardinality κ, up to isomorphism. A theory is called *uncountably categorical* if it is categorical in every uncountable cardinality. In the previous chapter theories were exhibited which are:

- categorical in every infinite cardinal;
- categorical in $\aleph_0$ but not in any uncountable cardinal;
- categorical in every uncountable cardinal, but not in $\aleph_0$;
- not categorical in any infinite cardinal.

It was conjectured by Łoś that every countable complete theory satisfies one of these four possibilities. Morley proved this conjecture with

Theorem 3.1.1 (Morley's Categoricity Theorem). *If a countable complete theory T is categorical in some uncountable cardinality then it is categorical in every uncountable cardinality.*

This section is devoted to a proof of this theorem. Some examples of theories categorical in some uncountable cardinality are:

1. the theory in the empty language with only infinite models;
2. infinite Abelian groups in which all elements have order p, for p some prime;
3. divisible torsion-free Abelian groups;
4. algebraically closed fields of a fixed characteristic;
5. the theory of a model (A, σ), where A is an infinite set and σ is a permutation of A with no finite cycles;
6. the theory of the model (ω, S), where S is the successor function.

The uncountable categoricity of the theories in 2, 3 and 4 above follow from well-known classical results. For instance, it is known that any divisible torsion-free Abelian group is a direct sum of copies of $(\mathbb{Q}, +)$. Thus, the isomorphism type of a divisible torsion-free Abelian group G is determined by the number of copies of the rationals used in such a decomposition. If G has cardinality $\kappa > \aleph_0$, then κ copies of the rationals must appear in a decomposition. Hence, any two uncountable divisible torsion-free Abelian groups are isomorphic. Steinitz's Theorem says that the isomorphism type of an algebraically closed field is determined by its characteristic and transcendence degree. For uncountable algebraically closed fields the transcendence degree is the same as the cardinality, hence the theory in 4 is uncountably categorical. The uncountable categoricity of the theories in 1, 5 and 6 follow quickly from quantifier-elimination. Close examination shows that in each of the examples above the isomorphism type of a model is determined by some cardinal invariant. Furthermore, this invariant is the dimension of some subset of the model with respect to a dependence relation. We will see, in fact, that whenever a theory is categorical in an uncountable cardinal the models are determined by the dimension on a definable subset of the model with respect to a particular dependence relation.

Remark 3.1.1. The assumption that T is complete in Morley's Categoricity Theorem was only made to avoid distracting the reader from the main issues. A classical result known as the Łoś-Vaught Test implies that a first-order theory categorical in some $\kappa \geq |T|$ is complete. See Remark 2.3.1.

Definition 3.1.1. *Let λ be an infinite cardinal and T a complete theory (of any cardinality) with an infinite model. T is said to be λ−stable if for all $\mathcal{M} \models T$ and $A \subset M$ of cardinality $\leq \lambda$, $|S_1(A)| \leq \lambda$. The term ω−stable may be used in place of $\aleph_0$−stable. A model $\mathcal{M}$ is called λ−stable if $Th(\mathcal{M})$ is λ−stable.*

A straight-forward induction on n shows that if T is λ−stable, A is a subset of a model of T and $|A| \leq \lambda$, then $|S_n(A)| \leq \lambda$. Observe that an $\aleph_0$−stable theory must be countable and small. While the definition of a

$\lambda-$stable theory could easily be rewritten for possibly incomplete theories, the benefits of the added generality are negligible.

Lemma 3.1.1. *If the countable theory T is categorical in some uncountable cardinal, then T is $\aleph_0-$stable.*

Proof. Here is where Skolem functions come into play. Let T be categorical in $\lambda \geq \aleph_1$. By Lemma 2.5.2, T has a model $\mathcal{M}$ of cardinality λ such that for any countable $A \subset M$, $\mathcal{M}$ realizes only countably many complete types over A. Assuming that T is not $\aleph_0-$stable there is a model $\mathcal{N}$ of T containing a countable set B such that $|S(B)| > \aleph_0$. Without loss of generality, $\mathcal{N}$ is countable. By a simple compactness argument $\mathcal{N}$ has an elementary extension $\mathcal{N}'$ of cardinality λ realizing uncountably many complete types over B. Since T is $\lambda-$categorical $\mathcal{N}'$ must be isomorphic to $\mathcal{M}$. This contradiction proves the lemma.

As stated above, on a model of an uncountably categorical theory there is a dependence relation and a corresponding notion of dimension, which gives rise to an isomorphism invariant. It is the $\aleph_0-$stability of the uncountably categorical theory which gives rise to this dependence relation. These dependence relations are developed in the next few pages.

Definition 3.1.2. *Let $\mathcal{M}$ be a model and φ a nonalgebraic formula (in n variables) over M. We call φ strongly minimal if for every $\mathcal{N} \succ \mathcal{M}$ and every formula ψ (in n variables) over N, $\varphi(\mathcal{N}) \cap \psi(\mathcal{N})$ or $\varphi(\mathcal{N}) \cap \neg\psi(\mathcal{N})$ is finite. We call a complete theory T strongly minimal if the formula $x = x$ is strongly minimal.*

Slightly rewording the definition in terms of definable sets, φ is strongly minimal if for all $\mathcal{N} \succ \mathcal{M}$ every subset of $\varphi(\mathcal{N})$ definable in $\mathcal{N}$ (over N) is finite or cofinite.

Remark 3.1.2. Let $\bar{a}$ be a sequence from a model $\mathcal{M}$, $\varphi(v, \bar{a})$ a formula over $\bar{a}$, and let $\bar{b}$ be a sequence from a model $\mathcal{N}$ such that $tp_{\mathcal{M}}(\bar{a}) = tp_{\mathcal{N}}(\bar{b})$. Then $\varphi(v, \bar{a})$ is strongly minimal if and only if $\varphi(v, \bar{b})$ is strongly minimal. (The proof of this is left to the reader in Exercise 3.1.12.)

Example 3.1.1. (Strongly minimal theories)

(i) (The theory of infinite sets in the empty language) For L the empty language, the theory in L saying that there are infinitely many elements is quantifier-eliminable. Let $\mathcal{M}$ be an arbitrary model of T. A formula in the single variable v over M is equivalent to a boolean combination of formulas $v = a_i$, for some $a_0, \ldots, a_n \in M$. Thus, any subset of M definable over M is finite or cofinite. That is, T is strongly minimal.

(ii) (The theory of vector spaces over a field F) For a field F the theory T of infinite vector spaces over F is quantifier-eliminable (in the natural language). Let $\mathcal{M}$ be an arbitrary model of T. A subset of M defined by a

linear equation over F with coefficients from M consists of a single element or is all of M. Since any subset of M definable over M is a boolean combination of sets defined by linear equations, the formula $x = x$ is strongly minimal.

(iii) (The theory of algebraically closed fields of a fixed characteristic) In this case the theory T is also quantifier-eliminable. Given a model $\mathcal{M}$ of T, a subset of M definable by some equation over M is either finite or the entire field. As in the previous example, it follows that T is strongly minimal.

The properties of strongly minimal formulas are best described using the algebraic closure relation.

Definition 3.1.3. *Let A be a subset of a model $\mathcal{M}$. A finite tuple $\bar{a}$ from M is said to be* algebraic over A *if $tp_{\mathcal{M}}(\bar{a}/A)$ is algebraic. The* algebraic closure *of A (in $\mathcal{M}$), denoted $acl(A)$, is $\{\, a \in M : a$ is algebraic over $A \,\}$. Sequences $\bar{a}$ and $\bar{b}$ are* interalgebraic over A *if $\bar{a} \in acl(A \cup \{\bar{b}\})$ and $\bar{b} \in acl(A \cup \{\bar{a}\})$.*

Remark 3.1.3. Let A be a subset of a model $\mathcal{M}$.
   (i) Notice that the model plays no active role in the definition of $acl(A)$; if $\mathcal{N} \succ \mathcal{M}$ then $a \in N$ is algebraic over A only when $a \in M$ and $tp_{\mathcal{M}}(a/A)$ is algebraic. See Exercise 3.1.4.
   (ii) In the exercises the reader is asked to verify that $|acl(A)| \leq |A| + |T|$, where T is the theory of $\mathcal{M}$.

Algebraic closure is most naturally studied in the context of closure operators.

Definition 3.1.4. *Let S be some set and $c\ell$ a unary operator on the set of subsets of S.*
   (i) $c\ell$ is a closure operator *if for all $X, Y \subset S$:*

   (a) $X \subset c\ell(X)$,
   (b) $c\ell^2(X) = c\ell(X)$, and
   (c) $X \subset Y \implies c\ell(X) \subset c\ell(Y)$.

A closure operator $c\ell$ is called finitary *(or of finite character) if $c\ell(X) = \bigcup\{\, c\ell(Y) : Y \subset X$ and Y is finite$\}$. (Standard terminology uses "algebraic" where we use "finitary", but we feel this leaves too much room for confusion with other uses of the word algebraic.) A subset X of S is called* closed *if $X = c\ell(X)$.*
   (ii) If $c\ell$ is a finitary closure operator on S, then $\mathbf{S} = (S, c\ell)$ is a pregeometry *if it satisfies the* exchange property: *for all $a, b \in S$ and $A \subset S$, if $a \in c\ell(A \cup \{b\}) \setminus c\ell(A)$ then $b \in c\ell(A \cup \{a\})$. A pregeometry is a* geometry *if $c\ell(\emptyset) = \emptyset$ and for all singletons $a \in S$, $c\ell(\{a\}) = \{a\}$.*
   (iii) Let $\mathbf{S} = (S, c\ell)$ be a pregeometry and $A, B \subset S$. We say that A is $c\ell$–independent over B *if for all $a \in A$, $a \notin c\ell(B \cup (A \setminus \{a\}))$. For $X \subset S$ we call A a* basis *of X over B if A is a maximal subset of $c\ell(X \cup B)$ which is $c\ell$–independent over B. A standard argument using the exchange property*

and the transitivity of closure shows that all bases of X over B have the same cardinality, which is called the dimension of X over B *and denoted* $\dim(X/B)$.

(iv) Let $\mathbf{S} = (S, c\ell)$ *be a pregeometry,* X, Y *and* $Z \subset Y$ *subsets of* S. *We say* X *is* dim $-$independent from Y over Z *or simply* independent from Y over Z *if for all finite* $X_0 \subset X$, $\dim(X_0/Y) = \dim(X_0/Z)$.

Remark 3.1.4. Let S be a pregeometry.

(i) Notice that dimension on S is additive:

$$\text{For all } A, B \subset S, \ \dim(A \cup B) = \dim(A/B) + \dim(B).$$

(See Exercise 3.1.8.)

(ii) When $S \neq \emptyset$, $c\ell$−independent is different from dim −independent. (Given a nonempty $X \subset S$, X is dim −independent from X over X, but X is not $c\ell$−independent from X over X.)

Lemma 3.1.2. *Algebraic closure forms a finitary closure operator on the universe of a model.*

Proof. Left to the reader in Exercise 3.1.7.

Elimination of quantifiers can be used to verify that algebraic closure on an algebraically closed field or a vector space is a pregeometry. Actually, this is typical of sets defined by strongly minimal formulas:

Lemma 3.1.3. *Let* $\mathcal{M}$ *be a model of a theory of cardinality* κ, φ *a strongly minimal formula over* $A \subset M$ *and* $D = \varphi(\mathcal{M})$. *Let* B *be a subset of* M *containing* A *and let* $c\ell$ *be the restriction to* D *of algebraic closure over* B. *(That is, for* $X \subset D$ *and* $a \in D$, $a \in c\ell(X)$ *if* $a \in acl(X \cup B) \cap D$.).*

(i) There is a unique nonalgebraic $p \in S(B)$ *containing* φ.
(ii) $(D, c\ell)$ *is a pregeometry.*
(iii) If $\{a_0, \ldots, a_n\}$, $\{b_0, \ldots, b_n\} \subset D$ *are* $c\ell$−independent over B, *then*

$$tp_{\mathcal{M}}(a_0, \ldots, a_n/B) = tp_{\mathcal{M}}(b_0, \ldots, b_n/B).$$

Thus, any subset of D *which is* $c\ell$−independent over B *is an indiscernible set over* B. *Furthermore, if* $I, J \subset D$ *are infinite and* $c\ell$−independent over B, *then* $D(I) = D(J)$.

Proof. (i) For ψ an arbitrary formula over B only one of $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ is nonalgebraic (by the strong minimality of φ). Thus, φ has a unique nonalgebraic completion over B.

(ii) By Lemma 3.1.2 it only remains to verify that $c\ell$ satisfies the exchange property. (To simplify the notation we assume φ to be a formula in one variable; i.e., $D \subset M$ instead of M^n, for some n. The proof is almost identical in general.) First we prove:

Claim. Suppose that, for $i = 0, 1$, a_i, $b_i \in D$, $a_i \notin acl(B)$ and $b_i \notin acl(B \cup \{a_i\})$. Then $tp_{\mathcal{M}}(a_0 b_0/B) = tp_{\mathcal{M}}(a_1 b_1/B)$.

By (i), $tp_{\mathcal{M}}(a_0/B) = tp_{\mathcal{M}}(a_1/B)$. Since the types of elements do not change when passing between a model and an elementary extension, we are free to replace $\mathcal{M}$ by an elementary extension. By Lemma 2.2.9 there is a model $\mathcal{N} \succ \mathcal{M}$ having an automorphism f such that f is the identity on B and $f(a_0) = a_1$. Then $f(b_0)$ is an element of $\varphi(\mathcal{N})$ which is not in the algebraic closure of $B \cup \{a_1\}$. Again by (i), $tp_{\mathcal{N}}(f(b_0)/B \cup \{a_1\}) = tp_{\mathcal{N}}(b_1/B \cup \{a_1\})$. Since $f(a_0) = a_1$ and automorphisms preserve types, $tp_{\mathcal{N}}(a_0 b_0/B) = tp_{\mathcal{N}}(f(a_0)f(b_0)/B) = tp_{\mathcal{N}}(a_1 f(b_0)/B) = tp_{\mathcal{N}}(a_1 b_1/B)$.

Now suppose the assertion in (ii) to fail; i.e., for some $C \subset D$ and $a, b \in D$, $a \in c\ell(C \cup \{b\}) \setminus c\ell(C)$ and $b \notin c\ell(C \cup \{a\})$. Let $B' = B \cup C$. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ in which $D' = \varphi(\mathcal{N})$ has cardinality $> (|B'| + \kappa)^+ = \lambda$. Choose Y, $D \subset Y \subset D'$ of cardinality λ and let $X = acl(B' \cup Y) \cap D'$, which has cardinality $\lambda + \kappa = \lambda$ (by Exercise 3.1.5). Since $|D'| > \lambda$ there is a $d \in D' \setminus X$. By (i), d and b realize the same complete type over $B' \cup \{a\}$ in $\mathcal{N}$. In fact, by the claim, if c is any element of $X \setminus acl(B')$, $tp_{\mathcal{N}}(dc/B') = tp_{\mathcal{N}}(ba/B')$. Thus, $X \subset acl(B' \cup \{d\})$. This contradicts that $|X| = \lambda > |B'| + \kappa = |B' \cup \{d\}| + \kappa = |acl(B' \cup \{d\})|$, to prove (ii).

(iii) This follows directly from the proof of the above claim and an induction on n.

Remark 3.1.5. Let $\mathcal{M}$ be a model of the theory of algebraically closed fields of a fixed characteristic. A subset of M is $acl-$independent if and only if it is algebraically independent. Other notions, like basis and dimension also agree with their standard algebraic interpretations.

If $\mathcal{M}$ is a model of the theory of vector spaces over a field F, then $I \subset M$ is $acl-$independent if and only if I is linearly independent.

Corollary 3.1.1. *If T is a countable strongly minimal theory, then T is categorical in every uncountable cardinal.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be models of T of cardinality $\kappa > \aleph_0$. Let I and J be bases for the closed sets M and N, respectively. Thus, $M = acl(I)$ and $N = acl(J)$. Since $|acl(I)| \leq |I| + \aleph_0$ and κ is uncountable, $|I|$ and $|J|$ must both be κ. Since I and J are indiscernible sets and $D(I) = D(J)$ (by Lemma 3.1.3(iii)) any bijection f from I onto J is an elementary map. An elementary map between two sets can be extended to an elementary map between their algebraic closures (see Exercise 3.1.10). Thus, f extends to an isomorphism of $\mathcal{M}$ onto $\mathcal{N}$.

Strongly minimal formulas enter the proof of Morley's Categoricity Theorem through

Lemma 3.1.4. *Let T be an $\aleph_0-$stable theory and let $\mathcal{M}$ be a countable saturated model of T. Then there is a strongly minimal formula over $\mathcal{M}$.*

Proof. Since T is $\aleph_0-$stable $|S(A)|$ is countable for any countable subset A of a model of T. Hence, T does have a countable saturated model $\mathcal{M}$ and,

for each n, every element of $S_n(M)$ has Cantor-Bendixson rank. First notice that any isolated $p \in S(M)$ is algebraic. (Let $\varphi(\bar{v}) \in p$ isolate p. There is an $\bar{a}$ from M satisfying φ; i.e., $\{\varphi(\bar{v}), \bar{v} = \bar{a}\}$ is consistent. Since φ isolates p, $\bar{v} = \bar{a}$ must be in p.) Let p be a type in $S_1(M)$ of least Cantor-Bendixson rank among the nonisolated complete $1-$types, and let $\varphi \in p$ isolate p relative to the nonisolated types. Hence,

$$p \text{ is the unique nonalgebraic element of } S_1(M) \text{ containing } \varphi. \qquad (3.1)$$

Claim. φ is strongly minimal.

For ψ any formula over M (in one variable) one of $\varphi \wedge \psi$ or $\varphi \wedge \neg\psi$ is algebraic (by (3.1)). To prove that φ is strongly minimal, however, this condition must be true when ψ is a formula over an arbitrary elementary extension of $\mathcal{M}$. Let $\mathcal{N} \succ \mathcal{M}$ and let ψ be a formula over N in one variable. Let $\bar{a}$ be the parameters in φ and $\psi = \psi(x, \bar{b})$, where $\psi(x, \bar{y})$ is over $\emptyset$. Suppose, towards a contradiction, that $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are both nonalgebraic. Then for each n, $\mathcal{N}$ is a model of the sentences saying that there are $\geq n$ elements satisfying $\varphi \wedge \psi$ and there are $\geq n$ elements satisfying $\varphi \wedge \neg\psi$. Since $\mathcal{M}$ is saturated there is a $\bar{c}$ in M such that $tp_{\mathcal{M}}(\bar{c}/\bar{a}) = tp_{\mathcal{N}}(\bar{b}/\bar{a})$, hence $\mathcal{M} \models \exists^{\geq n}x(\varphi(x, \bar{a}) \wedge \psi(x, \bar{c}))$ and $\mathcal{M} \models \exists^{\geq n}x(\varphi(x, \bar{a}) \wedge \neg\psi(x, \bar{c}))$, for each n. Thus, both $\varphi(x, \bar{a}) \wedge \psi(x, \bar{c})$ and $\varphi(x, \bar{a}) \wedge \neg\psi(x, \bar{c})$ are nonalgebraic. Since $\bar{c}$ is from M this contradicts (3.1) to prove the lemma.

Examining the proof of the lemma yields:

Corollary 3.1.2. *Let T be $\aleph_0-$stable and $\psi(\bar{v})$ any nonalgebraic formula over a countable saturated model $\mathcal{M}$. Then, over M, there is a strongly minimal formula φ which implies ψ.*

Another property of $\aleph_0-$stable theories used critically below is the existence of prime models, even over uncountable sets.

Lemma 3.1.5. *Let T be $\aleph_0-$stable, $\mathcal{M}$ a model of T and $A \subset M$. Then there is a model $\mathcal{N}$ which is prime over A and atomic over A.*

Proof. The bulk of the work is contained in

Claim. For every $B \subset M$ the isolated types are dense in $S(B)$.

(By our conventions for dealing with parameters, "the isolated types are dense in $S(B)$" simply means that the isolated types are dense in $Th(\mathcal{M}_B)$.) The proof of this claim is quite similar to the proof of Lemma 2.1.1, where it was shown that the isolated types are dense in a small countable complete theory. Here, B is potentially uncountable. However, we will show that if the isolated types are not dense in $S(B)$ there is a countable $B' \subset B$ with $S(B')$ uncountable, contradicting the $\aleph_0-$stability of T.

Suppose to the contrary that there is a formula φ over B not contained in an isolated element of $S(B)$. Let X be the set of finite sequences of 0's and

1's and Y the set of sequences of length ω from $\{0, 1\}$. Define by recursion a family of formulas φ_s, for $s \in X$, with the properties: (a) $\varphi_\emptyset = \varphi$, (b) φ_s is not contained in an isolated complete type over B, (c) if t is an initial segment of s then $\mathcal{M} \models \forall \bar{v}(\varphi_s \longrightarrow \varphi_t)$ and (d) if t is not an initial segment of s and s is not an initial segment of t, then $\varphi_s \wedge \varphi_t$ is inconsistent with $Th(\mathcal{M}_B)$. (This is possible since given φ_s there is a ψ such that $\varphi_s \wedge \psi$ and $\varphi_s \wedge \neg \psi$ are consistent, neither of which can be contained in an isolated complete type over B.) For $f \in Y$ let $p_f = \{ \varphi_s : s$ is an initial segment of $f \}$. Then each p_f is consistent and for distinct f and g in Y, $p_f \cup p_g$ is inconsistent. Since there are countably many φ_s's there is a countable set $B' \subset B$ such that each p_f is over B'. Consistent completions of the p_f's form $2^{\aleph_0}$ many elements of $S(B')$, contradicting the $\aleph_0$−stability of the theory to prove the claim.

A prime model over A is constructed as follows. Let $\lambda = |A| + \aleph_0$. Define by recursion a set $\{ c_\alpha : \alpha < \lambda \} \subset M$ such that, letting $C_\alpha = \{ c_\beta : \beta < \alpha \}$,

- for $\alpha < \lambda$, $tp_{\mathcal{M}}(c_\alpha / A \cup C_\alpha)$ is isolated;
- if $\varphi(x)$ is a consistent formula over $A \cup C_\alpha$, then $\varphi \in tp_{\mathcal{M}}(c_\beta / A \cup C_\beta)$ for some $\beta \geq \alpha$.

In detail, the construction proceeds as follows. Without loss of generality, $\mathcal{M}$ has cardinality λ. Let ψ_α, $\alpha < \lambda$, be a list of all consistent formulas in one variable over M. Assume c_β to be defined for each $\beta < \alpha$, and let $\Gamma = \{ \gamma < \lambda : \psi_\gamma$ is over $A \cup C_\alpha$ and is not satisfied by some element of $C_\alpha \}$. If $\Gamma \neq \emptyset$ let φ be ψ_γ for γ the least element of Γ; if $\Gamma = \emptyset$, let φ be $v = v$. Since the isolated types are dense in $S_1(A \cup C_\alpha)$ there is some element d in M such that $\mathcal{M} \models \varphi(d)$ and $tp_{\mathcal{M}}(d / A \cup C_\alpha)$ is isolated. Let $c_\alpha = d$. This completes the definition of the set C_λ. It follows quickly from the Tarski-Vaught Test that C_λ is the universe of an elementary submodel $\mathcal{N}$ of $\mathcal{M}$.

It is easy to see that $\mathcal{N}$ is an atomic model over A. Let $\mathcal{M}'$ be an arbitrary model containing A such that $\mathcal{M}'_A \equiv \mathcal{M}_A$. To show that $\mathcal{N}$ is a prime model over A we need to prove

Claim. There are elements d_α in M', for $\alpha < \lambda$, such that the map f which is the identity on A and takes c_α to d_α is an elementary.

Assume that d_β has been defined, for $\beta < \alpha$, so that the mapping g which fixes A pointwise and takes c_β to d_β, for $\beta < \alpha$, is elementary. Let $\varphi(x, \bar{c})$ be a formula which isolates $tp_{\mathcal{M}}(c_\alpha / A \cup C_\alpha)$. Let $\bar{d} = g(\bar{c})$ and choose d_α to be any element of M' which satisfies $\varphi(x, \bar{d})$. Since $\varphi(x, \bar{c})$ isolates a complete type over $A \cup C_\alpha$, $\varphi(x, \bar{d})$ isolates a complete type over $A \cup D_\alpha$, and the extension of g which takes c_α to d_α is elementary. This proves the claim.

Clearly, $N' = \{ d_\alpha : \alpha < \lambda \}$ is the universe of a model $\mathcal{N}' \prec \mathcal{M}'$ and f is an isomorphism of $\mathcal{N}$ onto $\mathcal{N}'$ fixing A. This proves that $\mathcal{N}$ is a prime model over A.

Remark 3.1.6. As with countable theories, it follows quickly from the previous lemma that *any* prime model over a set in an $\aleph_0$−stable theory is atomic.

It is also the case that there is a *unique* (up to isomorphism) prime model over a set in an $\aleph_0$−stable theory. Unlike the corresponding result for countable theories, however, this is rather difficult to prove (see Corollary 5.5.1).

Most readers of this book will have heard the term "prime field". The notion of a prime element of a class of structures is found throughout mathematics, especially algebra. Likewise in model theory we will occasionally speak of the prime model relative to a nonelementary class of models. Next we develop the notion of a prime model over a set, relative to the $\aleph_0$−saturated models.

Notation.   If p is a type over a set A and $B \subset A$ we let $p \upharpoonright B$ denote $\{\varphi \in p : \varphi$ is over $B\}$, called the *restriction of p to B*.

Definition 3.1.5. *Let $\mathcal{M}$ be a model and $A \subset M$.*
   (i) We say that $p \in S_n(A)$ is $\aleph_0$−isolated over B if $B \subset A$ is finite and p is the only extension of $p \upharpoonright B$ in $S_n(A)$, in which case we say that $p \upharpoonright B$ $\aleph_0$−isolates p. p is $\aleph_0$−isolated if it is $\aleph_0$−isolated over some finite subset of A.
   (ii) A set $B \subset M$ is said to be $\aleph_0$−atomic over A if for each finite sequence $\bar{a}$ from B, $tp_{\mathcal{M}}(\bar{a}/A)$ is $\aleph_0$−isolated.
   (iii) $\mathcal{M}$ is $\aleph_0$−prime over A if

− *$\mathcal{M}$ is $\aleph_0$−saturated, and*
− *for any $\aleph_0$−saturated model $\mathcal{N}$ containing A with $\mathcal{N}_A \equiv \mathcal{M}_A$, there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$ which is the identity on A.*

In a sense, the definition of an $\aleph_0$−prime model over a set can be obtained from the definition of a prime model over a set by uniformly replacing formulas by complete types over finite sets. Continuing the parallel development:

Lemma 3.1.6. *Let T be $\aleph_0$−stable, $\mathcal{M}$ a model of T and $A \subset M$. Then there is a model $\mathcal{N}$ which is $\aleph_0$−prime over A and $\aleph_0$−atomic over A.*

Proof. Only an outline of the proof is given, leaving the details (which are very similar to those found in Lemma 3.1.5) to the reader. The first claim in that previous proof is replaced by the following.

Claim. Let B be a subset of M and q a complete 1−type over a finite subset of B. Then there is $p \in S_1(B)$ which extends q and is $\aleph_0$−isolated.

In the verification that the constructed model $\mathcal{N}$ is $\aleph_0$−prime we use the fact that if $\mathcal{M}'$ is $\aleph_0$−saturated, $B \subset M'$ and $p \in S(B)$ is $\aleph_0$−isolated, then p is realized in $\mathcal{N}$. (p is $\aleph_0$−isolated by a complete type q over a finite set, which is realized by some $a \in N$. Then $tp_{\mathcal{N}}(a/B) = p$.)

We now turn towards properties more specific to theories categorical in some uncountable cardinal.

Definition 3.1.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be models and T a countable complete theory.*

(i) We say that $(\mathcal{M}, \mathcal{N}, \varphi)$ is a Vaughtian triple *if $\mathcal{N}$ is a proper elementary submodel of $\mathcal{M}$, φ is a nonalgebraic formula in one free variable over N and $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$.*

(ii) The pair $(\mathcal{M}, \mathcal{N})$ is called a Vaughtian pair *if for some φ, $(\mathcal{M}, \mathcal{N}, \varphi)$ is a Vaughtian triple.*

(iii) T is said to admit *a Vaughtian pair (or triple) if there is a Vaughtian triple $(\mathcal{M}, \mathcal{N}, \varphi)$, where $\mathcal{M}$ and $\mathcal{N}$ are models of T, in which case we call $(\mathcal{M}, \mathcal{N}, \varphi)$ a Vaughtian triple for T.*

Proposition 3.1.1. *Let T be categorical in some uncountable power. Then there is no Vaughtian pair consisting of models of T.*

Remark 3.1.7. This proposition, which is proved in the next few lemmas, comprises a major portion of the work in the proof of Morley's Categoricity Theorem. Let κ be an uncountable cardinal in which T is categorical. To prove the proposition we assume, to the contrary, that T has a Vaughtian pair. We then prove (making heavy use of the $\aleph_0$−stability of T) that there is a Vaughtian pair $(\mathcal{M}, \mathcal{N})$ with $|M| = \kappa$ and $|N| = \aleph_0$. Thus some N−definable subset of M is countable. However a relatively straightforward elementary chain argument shows that T (in fact any countable theory) has a model $\mathcal{M}'$ of cardinality κ such that every M'−definable relation in $\mathcal{M}'$ is uncountable. This contradiction proves the proposition.

Lemma 3.1.7. *If T is a small theory and admits a Vaughtian pair, then T admits a Vaughtian pair $(\mathcal{M}, \mathcal{N})$ where $\mathcal{N}$ and $\mathcal{M}$ are countable saturated models.*

Proof. Let $(\mathcal{M}_0, \mathcal{N}_0, \varphi)$ be a Vaughtian triple for T. Let L be the language of T, let $\bar{a}$ be the parameters in φ and $L(\bar{a})$ the expansion of L obtained by adding constants for the elements of $\bar{a}$. The proof centers on finding a theory (in a larger language) expressing that a triple is a Vaughtian triple for T. Specifically, we show

Claim. There is a theory $T' \supset Th(\mathcal{N}_0, \bar{a})$ in a language containing $L(\bar{a})$ and a new unary relation P such that whenever $\mathcal{M}' \models T'$,

> (a) $P(\mathcal{M}')$ is the universe of an elementary submodel $\mathcal{N}$ of $\mathcal{M}' \upharpoonright L = \mathcal{M}$,
> (b) $\bar{a}$ is from N, and
> (c) $(\mathcal{M}, \mathcal{N}, \varphi)$ is a Vaughtian triple.

Let P be a new unary relation symbol and $L' = L(\bar{a}) \cup \{P\}$. Let T' be the set of sentences in L' expressing the following:

− $T' \supset T$;

− for $\psi(v_0, \ldots, v_n)$ a formula of L,

$$\forall v_0 \ldots v_{n-1}(\bigwedge_{i<n} P(v_i) \wedge \exists v\psi(v_0, \ldots, v_{n-1}, v) \longrightarrow$$

$$\exists v(\psi(v_0, \ldots, v_{n-1}, v) \wedge P(v)));$$

− every element of $\bar{a}$ satisfies P;
− $\forall v(\varphi(v) \longrightarrow P(v))$ and
− there is an element not satisfying P.

Interpreting P by N_0 gives an expansion of $(\mathcal{M}_0, \bar{a})$ which is a model of T', so the theory is consistent. To verify that T' satisfies the requirements of the claim let $\mathcal{M}'$ be any model of T'. The second item in the definition implies that $N = P(\mathcal{M}')$ is the universe of a model $\mathcal{N}$ (in L) which is an elementary submodel of $\mathcal{M} = \mathcal{M}' \upharpoonright L$. The last three items verify (c) of the claim.

Claim. There is a countable model $\mathcal{M}'$ of T' in which $\mathcal{M} = \mathcal{M}' \upharpoonright L$ and $P(\mathcal{M}')$ are each a countable saturated model of T.

The targeted model $\mathcal{M}'$ is constructed using a standard elementary chain argument. Let $\mathcal{M}'_0$ be any countable model of T. Assuming $\mathcal{M}'_n$ to be defined, let $\mathcal{M}'_{n+1}$ be a countable elementary extension of $\mathcal{M}'_n$ with the property:

> If p is a complete type in L over a finite subset A of M'_n then p is realized in $\mathcal{M}'_{n+1}$. Furthermore, if $A \subset P(\mathcal{M}'_n)$, then there is a realization of p in $P(\mathcal{M}'_{n+1})$.

Then $\mathcal{M}' = \bigcup_{i<\omega} \mathcal{M}'_i$ satisfies the requirements of the claim.

Letting $\mathcal{M}$ be the restriction of $\mathcal{M}'$ to L and $\mathcal{N}$ the elementary submodel of $\mathcal{M}$ with universe $P(\mathcal{M}')$ gives the required Vaughtian pair.

The next step in the proof of Proposition 3.1.1 is to show that we can "stretch" the larger model in a Vaughtian pair of $\aleph_0$−saturated models while fixing the smaller one. Continuing this through κ steps results in a Vaughtian pair which contradicts the κ−categoricity of T, as described immediately after the statement of the proposition. This stretching of a Vaughtian pair is accomplished using the nonsplitting relation on types. The importance of the nonsplitting relation to the study of arbitrary $\aleph_0$−stable theories justifies this rather lengthy diversion.

Definition 3.1.7. *Let $\mathcal{M}$ be a model, $A \subset B \subset M$, and $p \in S(B)$.*
(i) We say that p does not split over A if for all tuples $\bar{a}$, $\bar{b}$ from B and formulas $\varphi(\bar{x}, \bar{v})$ over $\emptyset$,

$$\text{if } tp_{\mathcal{M}}(\bar{a}/A) = tp_{\mathcal{M}}(\bar{b}/A), \text{ then } (\varphi(\bar{x}, \bar{a}) \in p \Longleftrightarrow \varphi(\bar{x}, \bar{b}) \in p).$$

The negation of "p does not split over A" is p splits over A.
(ii) Suppose that p does not split over A, $B \subset C \subset M$, and $q \in S(C)$. Then q is called a strong heir of p if $q \supset p$ and q does not split over A.

Remark 3.1.8. The reader should supply proofs for the following elementary facts about the splitting relation. Let $\mathcal{M}$ be a model, $A \subset B \subset M$, and $p \in S(B)$.

(i) If p does not split over A and $A \subset A' \subset B$, then p does not split over A'.

(ii) If p does not split over A and $\varphi(\bar{x}, \bar{v})$ is a formula over A, then for all tuples $\bar{a}, \bar{b}$ from B realizing the same complete type over A,

$$\varphi(\bar{x}, \bar{a}) \in p \Longleftrightarrow \varphi(\bar{x}, \bar{b}) \in p.$$

(In other words, the clause defining the nonsplitting relation holds for formulas over A as well as formulas over $\emptyset$.)

(iii) p does not split over B.

Example 3.1.2. (i) Let T be the theory of a single equivalence relation E having infinitely many infinite classes and no finite classes.

Claim. Let $\mathcal{M}$ be a model of T. Each complete 1—type over $\mathcal{M}$ does not spilt over some element of M. Furthermore, each element of $S_1(M)$ has a strong heir in $S_1(A)$ for any $A \supset M$.

The theory T is complete and has elimination of quantifiers. If $p \in S_1(M)$ is algebraic there is a $b \in M$ such that $x = b \in p$. In this case, p does not split over b, and for any $A \supset M$, a subset of an elementary extension of $\mathcal{M}$, the unique extension of p in $S_1(A)$ is a strong heir of p. Now let $\mathcal{N}$ be a proper elementary extension of $\mathcal{M}$, $a \in N \setminus M$ and $p = tp_{\mathcal{N}}(a/M)$. Suppose that $E(a, b)$ holds for some b in M. We claim that p does not split over b. Every formula is equivalent in T to a boolean combination of instances of E and equality. Since $a \notin M$, $x \neq c \in p$, for all $c \in M$. Let $c, d \in M$ realize the same complete type over b. Then, $E(x, c) \in p \Longleftrightarrow \mathcal{M} \models E(c, b) \Longleftrightarrow \mathcal{M} \models E(d, b) \Longleftrightarrow E(x, d) \in p$. Using the elimination of quantifiers we conclude that p does not split over b. (Notice that p does split over $\emptyset$: for c an element not E—equivalent to b, $tp_{\mathcal{N}}(c) = tp_{\mathcal{N}}(b)$ while $E(x, b) \in p$ and $\neg E(x, c) \in p$.) Now let $A \supset M$ be a subset of the model $\mathcal{M}'$. There is a $q \in S_1(A)$ containing p such that for all $c \in A$, $x \neq c \in q$, and $E(x, c) \in q \Longleftrightarrow \mathcal{M}' \models E(c, b)$. The type q is a strong heir of p.

Supposing that $a \in N \setminus M$ is not E—equivalent to any element of M, a simple argument shows that $p = tp_{\mathcal{N}}(a/M)$ does not split over $\emptyset$. Let $\mathcal{M}' \prec \mathcal{M}$, $\mathcal{M}' \supset A \supset M$ and $q \in S_1(A)$ a type such that for all $c \in A$, $x \neq c \in q$ and $\neg E(x, c) \in q$. Then, q is a strong heir of p, proving the claim.

(ii) Let $\mathcal{M}$ be a dense linear order without endpoints; i.e., a model of $Th(\mathbb{Q}, <)$. A *cut* in $\mathcal{M}$ is a subset J of M such that, whenever $a \in J$ and $b < a$, $b \in J$. For $B \cup \{a\} \subset M$, $\sup B = a$ if every $b \in J$ is $\leq a$ and, for any $a' \in M$ such that $b \in J \implies b \leq a'$, $a \leq a'$. For subsets B and C of M we say that $\sup(B) = \sup(C)$ if for all $b \in B$ there is a $c \in C$ such that $b > c$ and for all $c \in C$ there is a $b \in B$ greater than c. The relation $\inf(B) = \inf(C)$ is defined similarly.

Let A be a cut in $\mathcal{M}$ such that $\sup(A)$ does not exist in M and $M \setminus A$ is nonempty. Let p be the unique element of $S_1(M)$ containing $\{\, x > a : a \in A \,\} \cup \{\, x < b : b \in M \setminus A \,\}$. We show that there are very strict limitations on the subsets of M over which p does not split in

Claim. Given $B \subset M$, p does not split over B if and only if $\sup(B \cap A) = \sup(A)$ or $\inf(B \setminus A) = \inf(M \setminus A)$.

First suppose that $\sup(B \cap A) = \sup(A)$ and c, $d \in M$ have the same type over $B \cap A$. If $b \in B \cap A$ and $c < b$, then $d < b$, so $x > c \in p$ and $x > d \in p$. If $c > b$ for all $b \in B \cap A$, then $x < c$ and $x < d$ are both in p. It follows quickly from elimination of quantifiers that p does not split over $B \cap A$. Similarly, p does not split over B if $\inf(B \setminus A) = \inf(M \setminus A)$. Suppose, on the other hand, that there are $a \in A$ greater than every element of $B \cap A$ and $c \in M \setminus A$ less than each element of $B \setminus A$. Then, $tp_{\mathcal{M}}(b/B) = tp_{\mathcal{M}}(c/B)$, $x > b \in p$ and $x < c \in p$, proving that p splits over B.

Among other things, we conclude that p splits over any finite subset of M.

These definitions reflect the following view of types. Let $\mathcal{M}$ be a model, $B \subset M$, $p \in S_1(B)$ and a a realization of p in $\mathcal{M}$. The formulas in p define the relations holding on $(a, \bar{b})$ for sequences $\bar{b}$ from B. If p does not split over $A \subset B$, a definable relation holding on a and sequences from B is determined by A in the following sense. For any formula $\varphi(x, \bar{y})$ there is $P_\varphi \subset S(A)$ such that $\varphi(x, \bar{b})$ belongs to p if and only if $tp_{\mathcal{M}}(\bar{b}/A) \in P_\varphi$. We think of the family $\mathcal{P} = \{\, P_\varphi : \varphi \text{ a formula} \,\}$ as being a kind of oracle which tells us which formulas go into p. If $q \in S(C)$ is a strong heir of p no essentially new relations are being defined using the elements of C; the family $\mathcal{P}$ still determines which formulas enter the type. A strong heir is a "freest" possible extension in that only the traits which are inherited from p are found in q. (We use the term strong heir as "heir" has been reserved for a different but closely related concept defined in [LP79] (see Definition 5.1.13). Such notions of "free" extensions of types are the foundation of stability theory.)

Definition 3.1.8. *Let $\mathcal{M}$ be a model, $A \subset M$.*

(i) Let p be a type over A and f an elementary map whose domain contains A. Then $f(p)$ denotes $\{\, \varphi(\bar{v}, f(\bar{a})) : \varphi(\bar{v}, \bar{a}) \in p \,\}$, a set of formulas over $f(A)$.

(ii) If $\bar{b}$ and $\bar{c}$ are sequences and there is an elementary map f such that f is the identity on A and $f(\bar{b}) = \bar{c}$, then we say $\bar{b}$ is conjugate to $\bar{c}$ over A. This terminology is applied to infinite sequences $\bar{b}$ and $\bar{c}$ as well as finite sequences. Occasionally we will say, e.g., "B and C are conjugate over A", leaving the relevant ordering of B and C to be understood.

(iii) When p and q are types over a model $\mathcal{M}$ and there is an elementary map f whose domain contains A such that $f(p) = q$, we say that p and q are conjugate over A.

Let p, A, and f be as in (i) of the definition. Because f is elementary, $f(p)$ is itself a type (i.e., it is consistent) and $f(p)$ is complete whenever p is complete. It is easily verified that if, e.g., f is an automorphism of a model $\mathcal{M}$ containing $B \cup \{\bar{a}\}$ and $p = tp(\bar{a}/B)$, then $f(p) = tp(f(\bar{a})/f(B))$.

If $\mathcal{M}$ is a model and $p \in S_1(M)$, the automorphisms of $\mathcal{M}$ act on p to produce other elements of $S_1(M)$. This action is effected by nonsplitting in the following way.

Lemma 3.1.8. *Let $B \subset M$, where $\mathcal{M}$ is a model, p an element of $S(B)$ which does not split over $A \subset B$, $A \subset B_0 \subset B$, and f an elementary map fixing A pointwise and taking B_0, to $B_1 \subset B$. Then, $f(p \upharpoonright B_0) = p \upharpoonright B_1$. In particular, if f maps B onto B, then $f(p) = p$.*

Proof. Let $\bar{b}$ be a sequence from B_0 and $\bar{b}' = f(\bar{b})$. Since f is the identity on A, $tp(\bar{b}/A) = tp(\bar{b}'/A)$. Since p does not split over A, $\varphi(\bar{x}, \bar{b}) \in p \iff \varphi(\bar{x}, \bar{b}') \in p$ for any formula $\varphi(\bar{x}, \bar{y})$. However,

$$\varphi(\bar{x}, \bar{b}) \in p \upharpoonright B_0 \iff \varphi(\bar{x}, \bar{b}') \in f(p \upharpoonright B_0),$$

from which we conclude that $p \upharpoonright B_1 = f(p \upharpoonright B_0)$.

In other words, for $\mathcal{M}$ a model and $p \in S_1(M)$ which does not split over $A \subset M$, if $q \in S_1(M)$ is conjugate to p over A, then $q = p$.

To set the stage for the next lemma, let $\mathcal{M}$ be a countable saturated model of an $\aleph_0$–stable theory and $p \in S_1(M)$. There are continuum many automorphisms of $\mathcal{M}$, each generating a conjugate of p in $S_1(M)$. Assuming that p splits over every finite subset of M we prove (in the next lemma) that continuum many of these conjugates of p are distinct (a contradiction).

Lemma 3.1.9. *Let T be $\aleph_0$–stable, A a subset of a model of T and $p \in S(A)$. Then there is a finite set $B \subset A$ such that p does not split over B.*

Proof. Assume, to the contrary, that there is no finite $B \subset A$ over which p does not split. The $\aleph_0$–stability of T will be contradicted by constructing continuum many types over some countable set. Let $\mathcal{M}$ be an $\aleph_0$–saturated model of T containing A, and let X be the set of all finite sequences of 0's and 1's.

Claim. For s, $t \in X$ there are: $A_s \subset A$ finite, $B_s \subset M$, $q_s \in S(B_s)$, and elementary maps f_s from A_s onto B_s such that

    (a) $f_s(p \upharpoonright A_s) = q_s$,
    (b) if t is an initial segment of s then $f_t \subset f_s$,
    (c) if t is not an initial segment of s and s is not an initial segment of t, then $q_s \cup q_t$ is inconsistent.

To begin let $A_\emptyset = B_\emptyset = f_\emptyset = \emptyset$. Assume that A_t, B_t and f_t have been defined for all $t \in X$ of length k. For an arbitrary $s \in X$ of length k we show how to define A_r, B_r and f_r for $r = s^\frown\langle i\rangle = si$, $i = 0, 1$. By assumption, p splits over the finite set A_s; i.e., there are $\bar{a}$ and $\bar{b}$ from A such that $tp(\bar{a}/A_s) = tp(\bar{b}/A_s)$ and a formula φ, such that $\varphi(\bar{x}, \bar{a}) \in p$ and $\neg\varphi(\bar{x}, \bar{b}) \in p$. Let $A_{s0} = A_s \cup \bar{a}$ and $A_{s1} = A_s \cup \bar{b}$. Since $\mathcal{M}$ is $\aleph_0$−saturated there is a $\bar{c}$ in M and elementary maps f_{s0}, f_{s1} extending f_s such that $f_{s0}(\bar{a}) = \bar{c}$ and $f_{s1}(\bar{b}) = \bar{c}$. Let $B_{si} = B_s \cup \bar{c}$ and $q_{si} = f_{si}(p \upharpoonright A_{si})$, for $i = 0, 1$. Since $\varphi(\bar{x}, \bar{c}) \in q_{s0}$ and $\neg\varphi(\bar{x}, \bar{c}) \in q_{s1}$ all of the required conditions are satisfied, completing the proof of the claim.

Let $B = \bigcup_s B_s$, Y the set of all sequences of 0's and 1's of length ω, and for $s \in Y$ let $q_s = \bigcup\{q_t : t = s \upharpoonright k$, for some $k\}$. By conditions (a) and (b) in the claim, when t is an initial segment of $r \in X$, $q_t = f_t(p \upharpoonright A_t) = f_r(p \upharpoonright A_t) \subset f_r(p \upharpoonright A_r) \subset q_r$, hence q_s is consistent for each $s \in Y$. For $s \in Y$ let q'_s be a completion of q_s in $S(B)$. There are continuum many such q'_s's by (c). Since B is countable this contradicts the $\aleph_0$−stability of T to prove the lemma.

Corollary 3.1.3. *Let T be an $\aleph_0$−stable theory.*
(i) T is κ−stable for all $\kappa \geq \aleph_0$.
(ii) For every regular cardinal κ, T has a saturated model of cardinality κ.

Proof. (i) Let A_0 be a subset of a model with $|A_0| = \kappa$, and let $\mathcal{M}$ be an $\aleph_0$−saturated model of cardinality κ which contains A_0. Since distinct elements of $S(A_0)$ extend to distinct elements of $S(M)$, it suffices to show that $|S(M)| = \kappa$. Every element of $S(M)$ does not split over some finite subset of M. Since there are κ many finite subsets of M it suffices to prove

Claim. For all finite $A \subset M$ there are only countably many elements of $S(M)$ which do not split over A.

Suppose to the contrary that there are distinct $p_i \in S_n(M)$, for $i < \omega_1$, such that each p_i does not split over A. Let $\mathcal{N}$ be a countable saturated elementary submodel of $\mathcal{M}$ containing A and let $q_i = p_i \upharpoonright N$, for $i < \omega_1$. We claim that $q_i \neq q_j$, for $i \neq j < \omega_1$. Let i, j be distinct ordinals $< \omega_1$, and $\varphi(\bar{x}, \bar{a})$ a formula such that $\varphi(\bar{x}, \bar{a}) \in p_i$ and $\neg\varphi(\bar{x}, \bar{a}) \in p_j$. Let $q = tp(\bar{a}/A)$ and $\bar{c}$ a realization of q in $\mathcal{N}$. Since p_i and p_j both do not split over A, $\varphi(\bar{x}, \bar{c}) \in p_i$ and $\neg\varphi(\bar{x}, \bar{c}) \in p_j$. Thus, the q_i's form an uncountable set of complete types over the countable set N, contradicting the $\aleph_0$−stability of the theory. This completes the proof of the claim and this part of the lemma.
(ii) This follows from (i) and Lemma 2.2.6.

Lemma 3.1.10. *Let T be an $\aleph_0$−stable theory and $\mathcal{M} \prec \mathcal{N}$ models of T with $\mathcal{M}$ $\aleph_0$−saturated. Then every element of $S(M)$ has a strong heir in $S(N)$.*

Proof. Let $p \in S(M)$ and let A be a finite subset of M over which p does not split. Define a set of formulas Γ over N by the scheme: Given a formula $\varphi(x, \bar{v})$ over $\emptyset$ and tuple $\bar{a}$ from N, $\varphi(x, \bar{a})$ is in Γ if there is a sequence $\bar{b}$ from M such that $tp_N(\bar{a}/A) = tp_N(\bar{b}/A)$ and $\varphi(x, \bar{b}) \in p$. This set Γ is well-defined since p does not spilt over A. (Given $\varphi(x, \bar{a})$, if there is a sequence $\bar{b}$ from M such that $q = tp_N(\bar{a}/A) = tp_N(\bar{b}/A)$ and $\varphi(x, \bar{b}) \in p$, then for any sequence $\bar{c}$ from M realizing q, $\varphi(x, \bar{c}) \in p$.) Also, $\Gamma \supset p$. To see that Γ is consistent, let $\{\varphi_0(x, \bar{a}_0), \ldots, \varphi_n(x, \bar{a}_n)\}$ be a finite subset of Γ. Since $\mathcal{M}$ is $\aleph_0$-saturated there are $\bar{b}_i \in M$ with $tp_N(\bar{a}_0 \ldots \bar{a}_n/A) = tp_N(\bar{b}_0 \ldots \bar{b}_n/A)$. The definition of Γ implies that $\{\varphi_0(x, \bar{b}_0), \ldots, \varphi_n(x, \bar{b}_n)\} \subset p$, hence Γ is consistent. It also follows quickly from its definition that Γ is a complete type which does not split over A.

Lemma 3.1.11 (Stretching a Vaughtian pair). *Let T be an $\aleph_0$-stable theory with a Vaughtian triple $(\mathcal{M}, \mathcal{N}, \varphi)$, where $\mathcal{M}$ and $\mathcal{N}$ are $\aleph_0$-saturated models of T and $\mathcal{N}$ is countable. Then there is a proper elementary extension $\mathcal{M}'$ of $\mathcal{M}$ which is $\aleph_0$-saturated such that $(\mathcal{M}', \mathcal{N}, \varphi)$ is a Vaughtian triple.*

Proof. Let a be any element of $M \setminus N$ and $p = tp_\mathcal{M}(a/N)$. There is a finite $A \subset N$ over which p does not split (by Lemma 3.1.9) and there is a $q \in S(M)$ which is a strong heir of p (by Lemma 3.1.10). Without loss of generality, A contains the parameters in φ. Let b be a realization of q in some elementary extension of $\mathcal{M}$ and let $\mathcal{M}'$ be an $\aleph_0$-prime model over $M \cup \{b\}$ (which exists by Lemma 3.1.6). It remains to verify that $(\mathcal{M}', \mathcal{N}, \varphi)$ is a Vaughtian triple. Assume to the contrary that there is a $c \in M' \setminus N$ satisfying φ. Since $(\mathcal{M}, \mathcal{N}, \varphi)$ is a Vaughtian triple, $c \in M' \setminus M$, hence $r = tp_{\mathcal{M}'}(c/M \cup \{b\})$ is $\aleph_0$-isolated. We will contradict that $(\mathcal{M}, \mathcal{N}, \varphi)$ is a Vaughtian triple by finding a $c_0 \in M \setminus N$ satisfying φ.

Let B, $M \supset B \supset A$ be a finite set such that r is $\aleph_0$-isolated over $B \cup \{b\}$. Since B is finite there is a countable saturated model $\mathcal{N}_0 \prec \mathcal{M}$ containing B. Since $tp_{\mathcal{M}'}(c/N_0 \cup \{b\})$ is also $\aleph_0$-isolated (over $B \cup \{b\}$) there is an $\mathcal{M}'_0 \prec \mathcal{M}$ which is $\aleph_0$-prime over $N_0 \cup \{b\}$ and contains c. Since $\mathcal{N}$ and $\mathcal{N}_0$ are both countable saturated models there is an isomorphism f_0 of $\mathcal{N}_0$ onto $\mathcal{N}$ which is the identity on A. Since $q = tp(b/M)$ does not split over A, $f_0(q \upharpoonright N_0) = q \upharpoonright N = p = tp(a/N)$ (by Lemma 3.1.8). Thus, f_0 extends to an elementary map f_1 of $N_0 \cup \{b\}$ onto $N \cup \{a\}$. Since $\mathcal{M}'_0$ is $\aleph_0$-prime over $N_0 \cup \{b\}$ and $\mathcal{M}$ is an $\aleph_0$-saturated model containing $N \cup \{a\}$ there is an elementary embedding f of $\mathcal{M}'_0$ into $\mathcal{M}$ which extends f_1. Then, $f(c)$ is an element of $M \setminus N$ which satisfies φ. This contradicts that $(\mathcal{M}, \mathcal{N}, \varphi)$ is a Vaughtian triple, completing the proof of the lemma.

Proof of Proposition 3.1.1. Let κ be an uncountable cardinal in which T is categorical and assume the proposition to fail. By Lemma 3.1.7 there is a Vaughtian triple $(\mathcal{M}, \mathcal{N}, \varphi)$ with $\mathcal{M}$ and $\mathcal{N}$ countable saturated models.

Claim. There is an elementary chain $\mathcal{M}_\alpha$, $\alpha < \kappa$, such that

(1) $\mathcal{M}_\alpha$ is an $\aleph_0-$saturated model of cardinality $< \kappa$,
(2) $(\mathcal{M}_\alpha, \mathcal{N}, \varphi)$ is a Vaughtian triple,
(3) if $\beta < \alpha$, $\mathcal{M}_\beta \neq \mathcal{M}_\alpha$.

This chain is defined by recursion using the previous lemma. Let $\mathcal{M}_0 = \mathcal{M}$. Suppose that $\mathcal{M}_\beta$ has been defined for all $\beta < \alpha$ and (1)-(3) hold relative to these models. If α is a limit ordinal let $\mathcal{M}_\alpha = \bigcup_{\beta < \alpha} \mathcal{M}_\beta$. If $\alpha = \beta + 1$ apply Lemma 3.1.11 to the Vaughtian triple $(\mathcal{M}_\beta, \mathcal{N}, \varphi)$ to obtain an $\aleph_0-$saturated proper elementary extension $\mathcal{M}_\alpha$ such that $(\mathcal{M}_\alpha, \mathcal{N}, \varphi)$ is a Vaughtian triple. Of course, we can assume that $|M_\alpha| = |M_\beta|$. Notice that (1)-(3) hold for all $\alpha < \kappa$, proving the claim.

Let $\mathcal{M}' = \bigcup_{\alpha < \kappa} \mathcal{M}_\alpha$. Then $\mathcal{M}'$ is a model of cardinality κ (by (1) and (3)) and $(\mathcal{M}', \mathcal{N}, \varphi)$ is a Vaughtian triple. Since $\mathcal{N}$ is countable, $\varphi(\mathcal{M}')$ is countable. However, as noted in Remark 3.1.7 the $\kappa-$categoricity of T implies that every relation definable over M' is uncountable. This contradiction proves the proposition.

The proof of Morley's Categoricity Theorem is completed by showing

Theorem 3.1.2. *Let T be a countable complete $\aleph_0-$stable theory with no Vaughtian pair. Then, T is categorical in every uncountable cardinal.*

The proof of the theorem is separated into the following three steps.

(a) First it is shown that there is a strongly minimal formula φ (in one variable) over a prime model $\mathcal{M}_0$ of T.
(b) Next, given a model $\mathcal{M} \succ \mathcal{M}_0$ of T we prove that $\mathcal{M}$ is prime over $\varphi(\mathcal{M}) \cup$ (the parameters in φ).
(c) Finally, we show that the theorem follows from (b).

Presently we only know that there is a strongly minimal formula over a countable saturated model (by Lemma 3.1.4). This is improved to obtain (a) in:

Proposition 3.1.2. *Let T be a countable complete $\aleph_0-$stable theory with no Vaughtian pair. Then there is a strongly minimal formula over a prime model.*

The proposition will be proved using

Lemma 3.1.12. *Suppose that T is $\aleph_0-$stable, has no Vaughtian pair and $\varphi(\bar{x}, \bar{v})$ is a formula. Then there is a natural number n such that for all models $\mathcal{M}$ of T and all $\bar{a}$ from M, $\varphi(\mathcal{M}, \bar{a})$ is infinite or of cardinality at most n.*

Proof. For notational simplicity we assume $\varphi(\bar{x}, \bar{v}) = \varphi(x, \bar{v})$; i.e., $\bar{x}$ has length 1. Suppose, to the contrary, that there is no such n for $\varphi(x, \bar{v})$. If ψ is an algebraic formula over a model $\mathcal{N}$ and $\mathcal{N}' \succ \mathcal{N}$, then $\psi(\mathcal{N}') \subset N$. Thus,

($*$) For all $n < \omega$ there is a model $\mathcal{N}$ and a sequence $\bar{a}$ from N such that
$$|\varphi(\mathcal{N}, \bar{a})| \geq n \text{ and for all } \mathcal{N}' \succ \mathcal{N}, \ \varphi(\mathcal{N}', \bar{a}) \subset N.$$

We will define a theory T' such that a model of T' represents a pair of models which is a Vaughtian pair for T. The consistency of T' will follow from ($*$). This contradiction to Proposition 3.1.1 will prove the lemma.

Let L be the language of T and $L' = L \cup \{P, c_1, \ldots, c_k\}$, where P is a unary predicate symbol and $c_1, \ldots, c_k$ are new constant symbols ($k = $ the length of $\bar{v}$). Let $T' \supset T$ be a theory in L' such that for any $\mathcal{M}' \models T'$,

- $P(\mathcal{M}')$ is the universe of a proper elementary submodel (with respect to L) of $\mathcal{M}' \restriction L$, and $\bar{c} \subset P(\mathcal{M}')$,
- for each n, $|\varphi(\mathcal{M}', \bar{c})| \geq n$ and
- $\varphi(\mathcal{M}', \bar{c}) \subset P(\mathcal{M}')$.

(The formulation of the actual sentences in L' comprising T' is left to the reader.) By ($*$) and compactness, T' is consistent. Let $\mathcal{M}'$ be a model of T', $\mathcal{M} = \mathcal{M}' \restriction L$, $\mathcal{N} = $ the proper elementary submodel of $\mathcal{M}$ with universe $P(\mathcal{M}')$ and $\bar{b}$ the interpretation of $\bar{c}$. The second and third items say that $\varphi(\mathcal{M}, \bar{b})$ is infinite and contained in $N \neq M$. This contradicts that T does not have a Vaughtian pair, completing the proof of the lemma.

Proof of Proposition 3.1.2. The argument is like the proof of Lemma 3.1.4 but we work over a prime model instead of an $\aleph_0$–saturated model. Let $\mathcal{M}$ be a prime model of T. Let $\varphi(v)$ be a formula which has a unique non-algebraic extension in $S(M)$ and let $A \subset M$ be the set of parameters in φ. The strong minimality of φ is proved as follows. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ and $\psi(v, \bar{b})$ a formula over N, where $\psi(v, \bar{x})$ is over $\emptyset$. Assume, towards a contradiction, that both $\sigma(v, \bar{b}) = \varphi(v) \wedge \psi(v, \bar{b})$ and $\tau(v, \bar{b}) = \varphi(v) \wedge \neg\psi(v, \bar{b})$ are nonalgebraic. By Lemma 3.1.12 there is an integer n such that for all elementary extensions $\mathcal{M}'$ of $\mathcal{M}$ and $\bar{c}$ from M', if $\sigma(\mathcal{M}', \bar{c})$ and $\tau(\mathcal{M}', \bar{c})$ both have cardinality $\geq n$, then $\sigma(x, \bar{c})$ and $\tau(x, \bar{c})$ are nonalgebraic. Since $\mathcal{N} \models \exists^{\geq n} v \sigma(v, \bar{b}) \wedge \exists^{\geq n} v \tau(v, \bar{b})$, there is a $\bar{d}$ from M such that $\mathcal{M} \models \exists^{\geq n} v \sigma(v, \bar{d}) \wedge \exists^{\geq n} v \tau(v, \bar{d})$. By the choice of n, both $\sigma(v, \bar{d}) = \varphi(v) \wedge \psi(v, \bar{d})$ and $\tau(v, \bar{d}) = \varphi(v) \wedge \neg\psi(v, \bar{d})$ are nonalgebraic. Since φ has a unique nonalgebraic completion over M we have obtained a contradiction which proves the proposition.

The following example shows that in Proposition 3.1.2 we cannot eliminate the hypothesis that the theory does not have a Vaughtian pair.

Example 3.1.3. Let E be a binary relation and T the theory expressing that (a) E is an equivalence relation and (b) for all $n < \omega$, there is an E–class containing exactly n elements. T is quantifier-eliminable, complete, and $\aleph_0$–stable. By the Omitting Types Theorem there is a model $\mathcal{M}$ of T (namely the prime model) such that each E–class in $\mathcal{M}$ is finite. By compactness there is an $\mathcal{N} \succ \mathcal{M}$ containing infinitely many infinite classes. It is easy

to verify from the elimination of quantifiers that when the class of $a \in N$ is infinite, $E(x, a)$ is a strongly minimal formula. Of course, when $a \in M$, $E(x, a)$ is not strongly minimal, and neither is $\neg E(x, a)$ (since $E(x, b)$, for $b \in N \setminus M$, defines an infinite and coinfinite subset of $\neg E(\mathcal{N}, a)$). There are other possibilities for strongly minimal formulas over $\mathcal{M}$, however they all basically reduce to $E(x, a)$ or $\neg E(x, a)$ by the elimination of quantifiers. That is, there is no strongly minimal formula over the prime model of T.

Note: There are models $\mathcal{N}$ and $\mathcal{N}'$, with $\mathcal{N}'$ a proper elementary extension of $\mathcal{N}$, such that for some $a \in N$, $E(\mathcal{N}', a)$ is an infinite subset of N. Thus, T has a Vaughtian pair.

Item (b) in the outline of the proof of Theorem 3.1.2 is handled in

Corollary 3.1.4. *Let T be a countable complete $\aleph_0$−stable theory with no Vaughtian pair, $\mathcal{M}$ a model of T and φ a strongly minimal formula over some finite set $A \subset M$. Then,*
(i) $\mathcal{M}$ is prime over $\varphi(\mathcal{M}) \cup A$ and minimal over $\varphi(\mathcal{M}) \cup A$, and
(ii) if $\mathcal{M}$ is uncountable, $\dim(\varphi(\mathcal{M})/A) = |M|$.

Proof. (i) Simply by the $\aleph_0$−stability of the theory there is $\mathcal{N} \prec \mathcal{M}$ which is a prime model over $\varphi(\mathcal{M}) \cup A$ (see Lemma 3.1.5). Since T does not have a Vaughtian pair and $\varphi(\mathcal{N}) = \varphi(\mathcal{M})$, $\mathcal{M}$ cannot be a proper elementary extension of $\mathcal{N}$. Thus $\mathcal{M}$ is a prime model over $\varphi(\mathcal{M}) \cup A$, which is also minimal over this set by the same reasoning.

(ii) Let $\kappa = |M|$ be uncountable and I a basis for $\varphi(\mathcal{M})$ over A. Then $|\varphi(\mathcal{M})| \leq |acl(I \cup A)| \leq |I| + \aleph_0$ (since $\varphi(\mathcal{M}) \subset acl(I \cup A)$ and A is finite). Suppose, towards a contradiction, that $|I| < \kappa$. Then $|\varphi(\mathcal{M})| < \kappa$, in which case there is a proper elementary submodel $\mathcal{N}$ of $\mathcal{M}$ containing $\varphi(\mathcal{M}) \cup A$. This is impossible because T does not have a Vaughtian pair, proving the corollary.

Proof of Theorem 3.1.2. Let $\mathcal{M}$ and $\mathcal{N}$ be two models of T of the same uncountable cardinality κ. By Lemma 3.1.2 there is a formula $\varphi(v, \bar{x})$ over $\emptyset$ and an $\bar{a}$ from M such that $tp_{\mathcal{M}}(\bar{a})$ is isolated and $\varphi(v, \bar{a})$ is strongly minimal. Since $tp_{\mathcal{M}}(\bar{a})$ is isolated there is a sequence $\bar{b}$ from N such that $tp_{\mathcal{N}}(\bar{b}) = tp_{\mathcal{M}}(\bar{a})$. Then, $\varphi(v, \bar{b})$ is also strongly minimal (by Remark 3.1.2). Let I be a basis for $\varphi(\mathcal{M}, \bar{a})$ over $\bar{a}$ and J a basis for $\varphi(\mathcal{N}, \bar{b})$ over $\bar{b}$. By (ii) of the previous corollary, I and J both have cardinality κ. Let f be a mapping which takes $\bar{a}$ to $\bar{b}$ and is a bijection from I onto J. If we could assume that both $\bar{a}$ and $\bar{b}$ are empty, so that I and J are from the same strongly minimal set, then Lemma 3.1.3 would imply that f is an elementary map (see the proof of Corollary 3.1.1). In general, a slight adaptation of Lemma 3.1.3, whose proof is left to the reader in Exercise 3.1.13, shows that f is elementary. Since $\varphi(\mathcal{M}, \bar{a})$ and $\varphi(\mathcal{N}, \bar{b})$ are contained in the algebraic closures of $I \cup \bar{a}$ and $J \cup \bar{b}$, respectively, f extends to an elementary map from $\varphi(\mathcal{M}, \bar{a})$ onto $\varphi(\mathcal{N}, \bar{b})$ (see Exercise 3.1.10). By Corollary 3.1.4(i), $\mathcal{M}$ is

prime over $\varphi(\mathcal{M}, \bar{a}) \cup \bar{a}$, so f extends to an elementary embedding g of $\mathcal{M}$ into $\mathcal{N}$ which maps $\bar{a}$ to $\bar{b}$ and $\varphi(\mathcal{M}, \bar{a})$ onto $\varphi(\mathcal{N}, \bar{b})$. Since $\mathcal{N}$ is a minimal model over $\varphi(\mathcal{N}, \bar{b}) \cup \bar{b}$, g maps $\mathcal{M}$ onto $\mathcal{N}$. Thus, g is an isomorphism from $\mathcal{M}$ onto $\mathcal{N}$ proving the theorem.

Proof of Theorem 3.1.1(**Morley's Categoricity Theorem**). Combining Theorems 3.1.1 and 3.1.2 shows that a countable theory T categorical in some uncountable power is categorical in every uncountable cardinality.

Corollary 3.1.5. *For T a countable complete theory the following are equivalent.*

(i) T is uncountably categorical.

(ii) T is $\aleph_0$—stable and does not have a Vaughtian pair.

(iii) For every regular $\kappa > \aleph_0$, every model of T of cardinality κ is saturated.

(The restriction to regular κ in (iii) will be removed in Lemma 3.4.10.)

The proof of Morley's Categoricity Theorem yields the following information about an uncountably categorical theory T. Over any model $\mathcal{M}$ of T there is a strongly minimal formula φ. Remember that $\varphi(\mathcal{M})$ is a pregeometry under algebraic closure. Letting A be the set of parameters in φ, $\mathcal{M}$ is prime over $\varphi(\mathcal{M}) \cup A$. In this way $\mathcal{M}$ is represented by a pregeometry. When $\mathcal{M}$ is uncountable assign to $\mathcal{M}$ a cardinal number $\mathcal{I}(\mathcal{M})$ = the dimension of $\varphi(\mathcal{M})$, for φ any strongly minimal formula over $\mathcal{M}$. Given any two uncountable models $\mathcal{M}$ and $\mathcal{N}$ of T, $\mathcal{M} \cong \mathcal{N}$ if and only if $\mathcal{I}(\mathcal{M}) = \mathcal{I}(\mathcal{N})$. The number $\mathcal{I}(\mathcal{M})$ is called a *cardinal isomorphism invariant*, or simply a *cardinal invariant*, for $\mathcal{M}$. (When T is the theory of infinite vector spaces over a field F the dimension of $\mathcal{M} \models T$ is a cardinal invariant for $\mathcal{M}$. When T is the theory of algebraically closed fields of a fixed characteristic and $\mathcal{M} \models T$ the transcendence degree of $\mathcal{M}$ is a cardinal invariant of $\mathcal{M}$.) Such an assignment of cardinal invariants is known as a *structure theorem*. (See page 326 for further discussion.) An important feature of the proof of Theorem 3.1.2 is that this cardinal invariant is independent of the choice of strongly minimal formula: if $\varphi(v)$ and $\varphi'(v)$ are strongly minimal formulas (over the finite sets $A \subset M$ and $A' \subset M$, respectively), then $\dim(\varphi(\mathcal{M})/A) = |M| = \dim(\varphi'(\mathcal{M})/A')$. What we have not yet proved is that there are also cardinal invariants for the countable models of T. This is basically the Baldwin-Lachlan Theorem which is proved in Section 3.4 using the more powerful machinery developed in Section 3.3.

Historical Notes. Morley's Categoricity Theorem was proved by Morley in [Mor65]. His proof involved Morley rank and other tools developed in Section 3.3. The term "λ—stable" was introduced by Rowbottom in [Row64]. Strongly minimal formulas were defined by Marsh [Mar66], and developed further by Baldwin and Lachlan [BL71]. It was in this later paper where

Theorem 3.1.2 and its key component, Proposition 3.1.2, were proved. The concept of splitting was developed in the early papers of Shelah (see, e.g., [She69]).

Exercise 3.1.1. Let T be a theory which is categorical in some $\kappa \geq |T|$. Prove that T is complete.

Exercise 3.1.2. Show that if T is λ-stable and $|A| \leq \lambda$, then $|S_n(A)| \leq \lambda$.

Exercise 3.1.3. Let T be a countable uncountably categorical theory and M an uncountable model of T. Show that M is $\aleph_0$-saturated.

Exercise 3.1.4. Prove: If $\mathcal{M}$ is a model, $\mathcal{N} \prec \mathcal{M}$ and $a \in N$ is algebraic over $A \subset M$, then $a \in M$ and $tp_\mathcal{M}(a/A)$ is algebraic.

Exercise 3.1.5. Let A be a subset of a model $\mathcal{M}$ in a theory of cardinality κ. Prove that $|acl(A)| \leq |A| + \kappa$.

Exercise 3.1.6. Prove: If the type of $\bar{a} = (a_1, \ldots, a_n)$ over A (in some model) is algebraic, then $a_1, \ldots, a_n \in acl(A)$.

Exercise 3.1.7. Prove Lemma 3.1.2.

Exercise 3.1.8. Let S be a pregeometry. Show that for all A, $B \subset S$,

$$\dim(A \cup B) = \dim(A/B) + \dim(B).$$

Exercise 3.1.9. Let T be an $\aleph_0$-stable theory in the language L and let T_0 be the restriction of T to a sublanguage of L. Show that T_0 is $\aleph_0$-stable.

Exercise 3.1.10. Let $\mathcal{M}$ and $\mathcal{N}$ be models, $A \subset M$, $B \subset N$ and f an elementary mapping from A onto B. Prove that f can be extended to an elementary mapping from $acl(A)$ onto $acl(B)$.

Exercise 3.1.11. Find an example of a model $\mathcal{M}$ and formula $\varphi(x)$ such that every subset of $\varphi(\mathcal{M})$ definable over M is finite or cofinite, but φ is not strongly minimal. (Thus, in the definition of strongly minimal we cannot avoid checking elementary extensions of $\mathcal{M}$. HINT: There is an example in this section.)

Exercise 3.1.12. Prove Remark 3.1.2.

Exercise 3.1.13. Let $\mathcal{M}$ be a model, $\bar{a}$ and $\bar{b}$ sequences from M realizing the same complete type and $\varphi(x, \bar{a})$ strongly minimal. By Remark 3.1.2, $\varphi(x, \bar{b})$ is also strongly minimal. Let I and J be bases for $\varphi(\mathcal{M}, \bar{a})$ over $\bar{a}$ and $\varphi(\mathcal{M}, \bar{b})$ over $\bar{b}$, respectively. Show that if $|I| = |J|$ and f is an elementary mapping from $\bar{a}$ onto $\bar{b}$ then f can be extended to an elementary mapping from $I \cup \bar{a}$ onto $J \cup \bar{b}$.

Exercise 3.1.14. Suppose that T is a countable theory which is categorical in $\aleph_1$, but not categorical in $\aleph_0$. Prove that the prime model of T is minimal.

Exercise 3.1.15. Let $\mathcal{M}$ be a model, $A \subset M$, $\varphi(x)$ a strongly minimal formula over A and $p \in S(M)$ the unique nonalgebraic extension of φ. Prove that p does not split over A.

Exercise 3.1.16. Suppose that $\mathcal{M}$ is a model, $\mathcal{N} \succ \mathcal{M}$, $a \in N \backslash M$ and A is a subset of M over which $tp_{\mathcal{N}}(a/M)$ does not split. Show that for all sequences $\bar{b}$, $\bar{c}$ from M, $tp_{\mathcal{N}}(\bar{b}/A) = tp_{\mathcal{N}}(\bar{c}/A) \implies tp_{\mathcal{N}}(\bar{b}/A \cup \{a\}) = tp_{\mathcal{N}}(\bar{c}/A \cup \{a\})$.

Exercise 3.1.17. Let T be $\aleph_0$–stable, $\mathcal{M}$ a countable saturated model of T and let $CB(-)$ denote Cantor-Bendixson rank as computed in $S_1(M)$. Prove that for all sequences $\bar{a}$, $\bar{b}$ from M with $tp(\bar{a}) = tp(\bar{b})$ and formulas $\varphi(x,\bar{y})$, $CB(\varphi(x,\bar{a})) = CB(\varphi(x,\bar{b}))$.

Exercise 3.1.18. Give a detailed proof of Lemma 3.1.3(iii).

Exercise 3.1.19. Give a detailed proof of Lemma 3.1.6.

Exercise 3.1.20. Let T be an uncountably categorical theory and $\mathcal{M}$ a countable model of T. Show that $Th(\mathcal{M}_M)$, the theory of $\mathcal{M}$ with constants added for the elements of $\mathcal{M}$, is also uncountably categorical.

Exercise 3.1.21. Let T be any complete countable theory and κ an uncountable cardinal. Prove that there is a model $\mathcal{M}$ of T of cardinality κ such that for any formula φ over $\mathcal{M}$, $\varphi(\mathcal{M})$ is finite or of cardinality κ.

3.2 A Universal Domain

Before proceeding with our study of totally transcendental theories we introduce some conventions to simplify the notation when dealing with the models of a fixed complete theory.

In Section 2.2 we introduced saturated models and proved some of their basic properties. One of the more useful properties of a saturated model $\mathcal{M}$ is its universality; i.e., if $\mathcal{N}$ is a model elementarily equivalent to $\mathcal{M}$ and $|N| \leq |M| = \kappa$ then $\mathcal{N}$ is isomorphic to an elementary submodel of $\mathcal{M}$. Thus, any property invariant under isomorphism and possessed by a model of the theory of cardinality $\leq \kappa$ holds in some elementary submodel of $\mathcal{M}$.

Suppose that a theory T has saturated models of arbitrarily large cardinality. Many theorems in model theory assert that a property holds for all models of a theory. If we fix a saturated model $\mathcal{M} \models T$ of cardinality κ and prove the theorem relative to the elementary submodels of $\mathcal{M}$, we seem to have limited ourselves to the models of cardinality $\leq \kappa$. However, with few exceptions it is possible to give a proof which does not depend on a particular κ, in the following sense. Let τ denote a theorem concerning the models

of T and for $\mathcal{M}$ a model of T, let $\tau \restriction \mathcal{M}$ denote the relativization of τ to the elementary submodels of $\mathcal{M}$. For $\mathcal{M}'$ another saturated model of T, the proof of $\tau \restriction \mathcal{M}'$ can be obtained from a proof of $\tau \restriction \mathcal{M}$ simply by replacing a reference to $\mathcal{M}$ to $\mathcal{M}'$. Thus, to prove τ (which is equivalent to "$\tau \restriction \mathcal{M}'$ holds for all saturated models of T") it suffices to prove $\tau \restriction \mathcal{M}$. For these reasons we adopt the following conventions without changing the validity of any theorems.

We assume that any theory under discussion has saturated models of arbitrarily large cardinality. (As noted in Section 2.1 this will be true assuming that there are arbitrarily large strongly inaccessible cardinals. The reader who is uncomfortable with such an assumption can reword any of the subsequent proofs to see that they do not depend on any additional set-theoretic assumptions.)

Definition 3.2.1. *Given a complete theory T we let $\mathfrak{C}$ denote a saturated model of T of arbitrarily large cardinality. $\mathfrak{C}$ is called the* universal domain of T *or simply the* universe of T. *(Such a model is called the "monster model" of T in some sources.)*

From hereon, we will say "$\mathcal{M}$ is a model of T" only when $\mathcal{M}$ is an elementary submodel of T and $|M| < |\mathfrak{C}|$. (The restriction on the cardinality of $\mathcal{M}$ is discussed below.) Narrowing our attention to elementary submodels of $\mathfrak{C}$ eliminates the need to specify an ambient model whenever we speak of the type of an element. If a is an element (of $\mathfrak{C}$) and $\mathcal{M}$ is a model (which is, by fiat, an elementary submodel of $\mathfrak{C}$) containing a, then $\mathcal{M} \models \varphi(a)$ if and only if $\mathfrak{C} \models \varphi(a)$. Thus, we may as well drop the reference to the model altogether and write $\models \varphi(a)$ instead of $\mathfrak{C} \models \varphi(a)$. Also, simply being told that the set M is the universe of a model completely determines the model. Since the term "model" is only applied to an elementary submodel of $\mathfrak{C}$, the interpretation on M of the elements of the language is uniquely determined. The full list of our conventions follows.

- $\mathfrak{C}$ denotes a model of the relevant complete theory which is saturated and of arbitrarily large cardinality.
- The term "A is a set" is interpreted as: $A \subset \mathfrak{C}$ and $|A| < |\mathfrak{C}|$. Similarly, "a is an element" means $a \in \mathfrak{C}$.
- By the term "formula" we always mean a formula over $\mathfrak{C}$. Similarly, a "type" is a type over $\mathfrak{C}$.
- For $\bar{a}$ a tuple and $\varphi(\bar{v})$ a formula we write $\models \varphi(\bar{a})$ for $\mathfrak{C} \models \varphi(\bar{a})$, and say "$\bar{a}$ satisfies $\varphi(\bar{v})$". If A is a set $tp(\bar{a}/A)$ denotes $tp_{\mathfrak{C}}(\bar{a}/A)$.
- We say "M is a model" if M is a set (hence of cardinality $< |\mathfrak{C}|$) and M is the universe of an elementary submodel of $\mathfrak{C}$. (Note: for M a model and φ a formula $M \models \varphi(\bar{a})$ if and only if $\bar{a}$ a tuple from M and $\models \varphi(\bar{a})$.) Models are denoted by M, N, M', etc.

The main reason for restricting to sets of cardinality $< |\mathfrak{C}|$ is so that types over sets can be realized. (If $A \subset \mathfrak{C}$, $|A| < |\mathfrak{C}|$ and $p \in S(A)$, then p is realized

in $\mathfrak{C}$, which may not be true if $|A| = |\mathfrak{C}|$.) A closely related benefit if the restriction is:

− Every elementary map extends to an element of $\mathrm{Aut}(\mathfrak{C})$.

(Let f be an elementary map from A onto B. Then, by assumption, A and B are subsets of $\mathfrak{C}$ with $|A| = |B| < |\mathfrak{C}|$. By the homogeneity of $\mathfrak{C}$, f is the restriction to A of an automorphism of $\mathfrak{C}$.)

Definition 3.2.2. *Given a complete theory with universal domain $\mathfrak{C}$, X is a definable set if $X = \varphi(\mathfrak{C})$ for some formula φ.*

Remark 3.2.1. (i) Comparing Definition 3.2.2 with the definition of "definable in $\mathcal{M}$" in Definition 1.1.1 we see that X is a definable set exactly when it is a definable set in $\mathfrak{C}$.

In most sources the term "definable set" means any set of the form $\varphi(M)$, where M is a model and φ is a formula over M. For the subject matter of this book it is most natural to reserve the unqualified term for sets of the form $\varphi(\mathfrak{C})$, and to say explicitly "definable set in M" when working in a particular model M.

(ii) For φ a nonalgebraic formula the saturation of $\mathfrak{C}$ forces $\varphi(\mathfrak{C})$ to have the same cardinality as $\mathfrak{C}$, thus strict adherence to our terminology prohibits $\varphi(\mathfrak{C})$ from being called a set. However, "definable set" is the established term for an object of the form $\varphi(\mathfrak{C})$ and since its usage should cause no confusion we will stick with it.

Notation. Given a universal domain $\mathfrak{C}$, formula φ and $X = \varphi(\mathfrak{C})$, we call X a *strongly minimal set* if φ is strongly minimal.

By convention an element of $S(\mathfrak{C})$ is not called a type since its domain is not a set. However, these "ideal types" or "limit types" do provide us with a convenient way to discuss all possible extensions of a type. For example, a formula $\varphi(x)$ over A is strongly minimal if for all sets $B \supset A$ there is a unique nonalgebraic $p \in S(B)$ which contains φ. A quicker way to say this is: φ is strongly minimal if there is a unique nonalgebraic $p \in S(\mathfrak{C})$ containing φ. Elements of $S(\mathfrak{C})$ will only be used to streamline notation in this manner.

As an exercise to familiarize the reader with this notation we recommend rewriting several definitions, lemmas and proofs from the previous section under the assumption that we are working in a universal domain.

3.3 Totally Transcendental Theories

The totally transcendental theories were defined and used by Morley in his proof of Morley's Categoricity Theorem (Theorem 3.1.1). Instead of total

transcendentality our proof used $\aleph_0$−stability, which we will show is equivalent to total transcendentality for countable theories. Here, totally transcendental theories are investigated without any categoricity or cardinality assumptions and results are proved which go far beyond those obtained in Section 3.1 for $\aleph_0$−stable theories. Besides shedding light on a rich class of theories this section introduces many of the central concepts of stability theory. These results will be applied in the next two sections to prove the Baldwin-Lachlan Theorem for uncountably categorical theories (mentioned at the end of Section 3.1), and to begin the study of t.t. groups (i.e., those totally transcendental theories which are the theory of a group).

Our proof of Morley's Categoricity Theorem depended heavily on the following properties of a strongly minimal set (i.e., a set defined by a strongly minimal formula) in an uncountably categorical theory. Let T be uncountably categorical and let $\varphi(v)$ be a strongly minimal formula which, for simplicity, we assume to be over $\emptyset$. Let M be a model of T and $D = \varphi(M)$.

(1) $c\ell(-) = $ the restriction of algebraic closure to D, defines a pregeometry on D.
(2) Let I be a basis for $\varphi(M)$, $N \models T$ and J a basis of $\varphi(N)$. Then any elementary map from J into I extends to an elementary map of $\varphi(N)$ into $\varphi(M)$.
(3) M is prime over $\varphi(M)$.

In this way the pregeometry on a strongly minimal set represents a model of T. In an arbitrary $\aleph_0$−stable theory T there is a strongly minimal formula φ (over $\mathfrak{C}$), but

− there may not be a strongly minimal formula over the prime model, and
− even when φ is over M property (3) may fail for M.

Thus, we must look beyond strongly minimal formulas to find a dependence relation which influences the whole model. We will find a relation on all subsets of the universe (of an $\aleph_0$−stable theory) which meets the following conditions. The term "free" is used instead of "independent" to avoid confusion with later specific interpretations of the term.

Definition 3.3.1. *A ternary relation $\mathcal{F}$ on subsets of the universe is called a freeness relation if it satisfies the following conditions. ($\mathcal{F}(A, B; C)$ is read A is free from B over C.)*

(1) *(finite character and monotonicity) A is free from B over C if and only if for all finite $A_0 \subset A$ and $B_0 \subset B$, A_0 is free from B_0 over C.*
(2) *For any $\bar{a}$ and B there is a $B_0 \subset B$ of cardinality $\leq |T|$ such that $\bar{a}$ is free from B over B_0.*
(3) *(transitivity of independence) If $A \subset B \subset C$ then $\bar{a}$ is free from C over A if and only if $\bar{a}$ is free from C over B and $\bar{a}$ is free from B over A.*

(4) *(symmetry) If A is free from B over C, then B is free from A over C.*

(5) *If A is free from B over C and $f \in Aut(\mathfrak{C})$ then $f(A)$ is free from $f(B)$ over $f(C)$.*

(6) *There is a cardinal λ such that for $B \supset A$ and $p \in S(A)$, the set of types $\{ tp(\bar{a}/B) : \bar{a}$ is free from B over A and $tp(\bar{a}/B) \supset p \}$ is nonempty and of cardinality $\leq \lambda$.*

(7) *(reflexivity) If $b \notin acl(A)$ then b is not free from $A \cup \{b\}$ over A.*

No formal abstract properties of freeness relations will be proved. The conditions (1)–(7) will only be used as a basic minimal list of properties that any usable notion of freeness must satisfy. As a first example of a freeness relation, let D be a strongly minimal set (in the universe of some theory) and define, for sets A, B, C,

$$\mathcal{F}_0(A, B; C) \iff \text{for all finite } A' \subset A, \dim(A'/B \cup C) = \dim(A'/C).$$

Previously proved properties of strongly minimal sets show that $\mathcal{F}_0$ is a freeness relation. The axioms for an abstract dependence relation given by van der Waerden in [VdW49] include the transitivity of dependence: If a depends on X and each $x \in X$ depends on Y, then a depends on Y. However, van der Waerden's notion is formulated as: "the point a depends on the set $X \ldots$". Virtually any dependence relation applying to all subsets of the universe fails to satisfy transitivity. Indeed transitivity fails for the freeness relation $\mathcal{F}_0$ defined above. (Let a, b, c and d be four independent nonzero elements of a vector space. Let $A = \{a, b\}$, $B = \{b, c\}$ and $C = \{c, d\}$. Then A depends on B over $\emptyset$ and B depends on C over $\emptyset$, but A is independent from C over $\emptyset$.)

Item (5) in the definition says that freeness is determined by types; i.e., if $tp(\bar{a}/B \cup C) = tp(\bar{b}/B \cup C)$, then $\bar{a}$ is free from B over C if and only if $\bar{b}$ is free from B over C. Thus, it makes sense to define a freeness relation as a relation on types, which is done below with Morley rank.

Some of the properties of $\aleph_0$−stable theories proved in Section 3.1 (for example, the existence of prime models) used the fact that for A any countable set every element of $S(A)$ has Cantor-Bendixson rank. Morley rank is basically Cantor-Bendixson rank computed over universe, instead of over a fixed set. Remember that the term "formula" means "formula over $\mathfrak{C}$".

Definition 3.3.2. *Let T be a complete theory. The relation $MR(\varphi) = \alpha$, for φ a formula in n variables and α an ordinal or -1, is defined by the following recursion.*

(1) $MR(\varphi) = -1$ *if φ is inconsistent;*
(2) $MR(\varphi) = \alpha$ *if*

$$\{ p \in S_n(\mathfrak{C}) : \varphi \in p \text{ and } \neg\psi \in p \text{ for all formulas}$$
$$\psi \text{ with } MR(\psi) < \alpha \}$$

is nonempty and finite.

For p any n−type, $MR(p)$ is defined to be

$$\inf\{MR(\varphi) : \varphi \text{ a formula implied by } p\}.$$

(Thus, for $p \in S(\mathfrak{C})$, $MR(p)$ is $\inf\{MR(\varphi) : \varphi \in p\}$.) When $MR(p) = \alpha$ we say that the Morley rank of p is α. If there is no α with $MR(p) = \alpha$ we write $MR(p) = \infty$ and say that the Morley rank of p does not exist.

Extend the scope of $<$ so that $-1 < \alpha < \infty$ for all ordinals α. Then, $MR(p) \geq \alpha$ is a quick way to express that $MR(p) \neq \beta$ for all $\beta < \alpha$. Using these conventions (2) in the definition can be restated as: $MR(\varphi) = \alpha$ if $\{p \in S_n(\mathfrak{C}) : \varphi \in p \text{ and } MR(p) \geq \alpha\}$ is finite and nonempty.

The only reason Morley rank is not simply Cantor-Bendixson rank in $S(\mathfrak{C})$ is because CB-rank is computed for types over a fixed *set A*, which by convention must have cardinality $< |\mathfrak{C}|$. This difference is strictly due to a notational convention. All properties of CB-rank proved in Section 2.2 hold for Morley rank with virtually identical proofs. The statements of the most critical properties will be repeated for ease of reference.

It follows immediately from the definition that for p and q types with $q \models p$, $MR(p) \geq MR(q)$. We leave it as an exercise to the reader to show that conjugate types have the same Morley rank.

Remark 3.3.1. Let $\mathfrak{C}$ be the universal domain of a complete theory. Given a type p and formula φ such that $p \models \varphi$ there is a finite $p_0 \subset p$ such that $\bigwedge p_0 \models \varphi$, hence $MR(\varphi) \geq MR(\bigwedge p_0)$. Thus, if p is closed under finite conjunctions there is a formula $\varphi \in p$ such that $MR(\varphi) = MR(p)$.

The following is an almost literal restatement of Lemma 2.2.3.

Lemma 3.3.1. *Let T be a complete theory, p an $n-$type and α an ordinal.*
   (i) If $p \in S(\mathfrak{C})$ then $MR(p) = 0$ if and only if p is algebraic.
   (ii) $MR(p) = \alpha$ if and only if there is a formula φ implied by p such that $\{q \in S_n(\mathfrak{C}) : \varphi \in q \text{ and } MR(q) = \alpha\}$ is finite and nonempty, and this set is equal to $\{q \in S_n(\mathfrak{C}) : p \subset q \text{ and } MR(q) = \alpha\}$.
   (iii) If $MR(p) = \alpha$ there is a $q \in S_n(\mathfrak{C})$ such that $q \supset p$ and $MR(q) = \alpha$.
   (iv) If $p \in S(\mathfrak{C})$ and $MR(p) = \alpha$ there is a $\varphi \in p$ such that p is the only element of $\{q \in S_n(\mathfrak{C}) : \varphi \in q \text{ and } MR(q) \geq \alpha\}$.
   (v) $MR(p) \geq \alpha$ if and only if, for all $\beta < \alpha$ and all φ implied by p, $\{q \in S_n(\mathfrak{C}) : \varphi \in q \text{ and } MR(q) \geq \beta\}$ is infinite.
   (vi)

$$MR(\varphi) \text{ is the least ordinal } \alpha \text{ such that} \qquad (3.2)$$
$$\{q \in S_n(\mathfrak{C}) : \varphi \in q \text{ and } MR(q) \geq \alpha\} \text{ is finite.}$$

Proof. A complete type over a model is isolated if and only if it is algebraic. Thus, (i) is really a restatement of Lemma 2.2.3(i). A proof of each remaining part can be obtained from the proof of the corresponding part in Lemma 2.2.3 simply by changing the notation.

Definition 3.3.3. *Let p be an n−type in a complete theory and suppose $MR(p) = \alpha < \infty$. By (ii) of the previous lemma $\{\, q \in S_n(\mathfrak{C}) : p \subset q$ and $MR(q) = \alpha\,\}$ is finite. The* Morley degree of p, *denoted* $\deg(p)$, *is defined to be $|\{\, q \in S_n(\mathfrak{C}) : p \subset q$ and $MR(q) = \alpha\,\}|$. If $\deg(p) = 1$ we say that p is* stationary.

When $p = tp(\bar{a}/B)$ we may write $MR(\bar{a}/B)$ for $MR(p)$.

Notation. Given a universal domain $\mathfrak{C}$ and definable set $X = \varphi(\mathfrak{C})$, we may write $MR(X)$ for $MR(\varphi)$ and $\deg(X)$ for $\deg(\varphi)$.

Definition 3.3.4. *A complete theory T is called* totally transcendental (t.t.) *if every $p \in S(\mathfrak{C})$ has Morley rank.*

Notice that a complete theory T is t.t. if and only if for every n and $\bar{v} = (v_1, \ldots, v_n)$, $MR(\bar{v} = \bar{v}) < \infty$. (See Remark 3.3.2.)

Definition 3.3.5. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory. For A, B, C subsets of $\mathfrak{C}$ we say A is* Morley rank independent *from B over C and write $A \underset{C}{\perp} B$ if for all finite tuples $\bar{a}$ from A, $MR(\bar{a}/B \cup C) = MR(\bar{a}/C)$; writing $A \underset{C}{\not\perp} B$ for the negation of Morley rank independence (called* Morley rank dependence*). We write $A \perp B$ and $A \not\perp B$ for $A \underset{\emptyset}{\perp} B$ and $A \underset{\emptyset}{\not\perp} B$, respectively. If p is a complete type over $B \supset C$ and $q = p \restriction C$ we call p a* free extension *of q if $MR(p) = MR(q)$. When p is a free extension of $p \restriction C$ we may say p is* free over C.

Let T be t.t. Most of the properties of a freeness relation are easily verified for Morley rank independence. That the relation has finite character, is transitive, and is preserved under automorphisms is clear from the basic properties of Morley rank. For $p \in S(A)$ there is a finite $B \subset A$ such that p is a free extension of $p \restriction B$. If $p \in S(A)$ and $B \supset A$ then there is at least one and only finitely many $q \in S(B)$ which are free extensions of p. Reflexivity follows from the fact that $MR(\bar{a}/B) = 0$ if and only if $\bar{a} \in acl(B)$. The symmetry property (Definition 3.3.1(4)) however, will take a significant amount of work to prove.

In this section the term "Morley rank independent" (Morley rank dependent) is shortened to "independent" (dependent).

In Section 2.2 we gave examples of theories formulated using equivalence relations such that, for M any model of the theory, every element of $S(M)$ has CB-rank. Letting $M = \mathfrak{C}$ in each example and rewording the justifications shows that each of the theories is t.t.

Here are some basic facts about Morley rank and independence in a t.t. theory.

Lemma 3.3.2. *Let T be totally transcendental.*

(i) For $\bar{a}'$ a subsequence of the sequence $\bar{a}$, $MR(\bar{a}'/B) \leq MR(\bar{a}/B)$.

(ii) For all $\bar{a}$, $\bar{b}$ and A, $\bar{a} \in acl(A \cup \{\bar{b}\}) \implies MR(\bar{a}\bar{b}/A) = MR(\bar{b}/A)$.

(iii) For all $\bar{a}$ and A, $\bar{a} \underset{A}{\downarrow} acl(A)$.

(iv) For all $\bar{a}$ and sets B there is a finite $B_0 \subset B$ such that $\bar{a}$ is independent from B over B_0.

(v) (Pairs Lemma) For all $\bar{a}$, $\bar{b}$ and $B \supset A$,

$$B \underset{A}{\downarrow} \bar{a}\bar{b} \iff B \underset{A \cup \bar{b}}{\downarrow} \bar{a} \text{ and } B \underset{A}{\downarrow} \bar{b}.$$

Proof. Part (i) is left to the exercises, and (iv) follows immediately from the definition of Morley rank independence.

(ii) Without loss of generality, $A = \emptyset$. By (i) it suffices to show that $MR(\bar{a}\bar{b}) \leq MR(\bar{b}) = \beta$. Below, $tp(\bar{b})$ is viewed as a type in the variables $\bar{y}$ and $tp(\bar{a}\bar{b})$ as a type in $\bar{x}\bar{y}$. For $\bar{v}$ a sequence of variables, let $\Psi_{\bar{v}} = \{\psi(\bar{v}) : MR(\psi(\bar{v})) < \beta\}$. Let $\theta(\bar{x}\bar{y})$ be any formula in $tp(\bar{a}\bar{b})$ such that $MR(\exists\bar{x}\theta(\bar{x},\bar{y})) = \beta$ and, for any $\bar{d}$, $\theta(\bar{x},\bar{d})$ is algebraic. Then, $\{q(\bar{x}\bar{y}) \in S(\mathfrak{C}) : \theta(\bar{x}\bar{y}) \in q \text{ and } \neg\psi(\bar{x}\bar{y}) \in q \text{ for all } \psi \in \Psi_{\bar{x}\bar{y}}\} \subset \{q(\bar{x}\bar{y}) \in S(\mathfrak{C}) : \theta(\bar{x}\bar{y}) \in q \text{ and } \neg\psi(\bar{y}) \in q \text{ for all } \psi \in \Psi_{\bar{y}}\} = Q$. Since $MR(\exists\bar{x}\theta(\bar{x},\bar{y})) = \beta$, $R = \{r(\bar{y}) \in S(\mathfrak{C}) : r \text{ is the restriction to } \bar{y} \text{ of some element of } Q\}$ is finite. Since $\theta(\bar{x},\bar{d})$ is algebraic for any $\bar{d}$, $r(\bar{y}) \cup \{\theta(\bar{x}\bar{y})\}$ has finitely many completions in $S(\mathfrak{C})$, for any $r \in R$. Hence, Q is finite, from which we conclude that $MR(\bar{a}\bar{b}) \leq MR(\theta(\bar{x}\bar{y})) \leq \beta$.

(iii) Suppose, to the contrary, that $\varphi(\bar{v},\bar{y})$ is a formula over A and $\varphi(\bar{v},\bar{e}) \in tp(\bar{a}/acl(A))$ has Morley rank $\beta < MR(\bar{a}/A)$. We may assume that $\exists\bar{v}\varphi(\bar{v},\bar{y})$ isolates $tp(\bar{e}/A)$. Let p be a free extension of $tp(\bar{a}/A)$ in $S(\mathfrak{C})$. Since $\exists\bar{v}\varphi(\bar{v},\bar{y}) \in p$ isolates an algebraic type over A, $\varphi(\bar{v},\bar{e}') \in p$ for some $\bar{e}'$. Since $\varphi(\bar{v},\bar{e}')$ is conjugate to $\varphi(\bar{v},\bar{e})$ it also has Morley rank $\beta < MR(\bar{a}/A) = MR(p)$. This contradiction proves (iii).

(v) ($\implies$) Suppose that B is independent from $\bar{a}\bar{b}$ over A, and let $\bar{c} \subset B$. Since $MR(\bar{c}/A \cup \bar{a}\bar{b}) = MR(\bar{c}/A)$, $MR(\bar{c}/A \cup \bar{b}) = MR(\bar{c}/A)$ and $MR(\bar{c}/A \cup \bar{a}\bar{b}) = MR(\bar{c}/A \cup \bar{b})$.

($\impliedby$) This direction follows immediately from the transitivity of independence.

Remark 3.3.2. Let $\mathfrak{C}$ be the universal domain of a complete theory and let $\varphi(x,\bar{y})$ be a formula over $\emptyset$. If $MR(\exists x\varphi(x,\bar{y})) < \infty$ and for each $\bar{a}$, $MR(\varphi(x,\bar{a})) < \infty$, then $MR(\varphi(x,\bar{y})) < \infty$. In fact a bound on $MR(\varphi(x,\bar{y}))$ can be computed in terms of $MR(\exists x\varphi(x,\bar{y}))$ and a bound on $MR(\varphi(x,\bar{a}))$, as $\bar{a}$ ranges over tuples from $\mathfrak{C}$.

This fact is due to Erimbetov [Eri75]. Proofs can also be found in [Lac80] and [She90, V.7.8]. This lemma is occasionally helpful in showing that a particular theory is t.t. To show that a complete theory T is t.t., instead of showing that every $p \in S(\mathfrak{C})$ has Morley rank, it is enough to show that each

$p \in S_1(\mathfrak{C})$ has Morley rank. Equivalently, T is t.t. if and only if $MR(v = v) < \infty$.

Morley rank and Cantor-Bendixson rank are further connected by

Lemma 3.3.3. *Let T be a complete theory, M an $\aleph_0-$saturated model of T and let $CB(-)$ denote Cantor-Bendixson rank computed in $S_n(M)$. Then, for all $n-$types p over M,*
  (i) $MR(p) = CB(p)$;
  (ii) if $MR(p) < \infty$ and p is complete, then p is stationary.
  (iii) Hence, if every element of $S(M)$ has CB-rank, T is totally transcendental.

Proof. (i) It suffices to consider the case when p is a formula φ. That $CB(\varphi) \geq MR(\varphi)$ is a straight-forward exercise left to the reader. To show the reverse inequality we prove (by induction on α) that $CB(\varphi) \geq \alpha \implies MR(\varphi) \geq \alpha$. Assuming that $MR(\varphi) \not\geq \alpha$ yields the inconsistency of $\{\varphi\}\cup\{\neg\psi : MR(\psi) < \alpha\}$. Thus, there are formulas $\psi_1, \ldots, \psi_n$ such that

$$\models \forall\bar{v}(\varphi(\bar{v}) \to \bigvee_{1\leq i\leq n} \psi_i(\bar{v})) \text{ and } MR(\psi_i) < \alpha, \text{ for } i = 1, \ldots, n.$$

By Exercise 3.3.5, $\psi = \bigvee_{1\leq i\leq n} \psi_i$ has Morley rank $< \alpha$. Letting A be a finite set containing the parameters in φ, the $\aleph_0-$saturation of M yields a formula ψ' over M conjugate over A to ψ. By induction, $CB(\psi') = MR(\psi') < \alpha$. Since φ implies ψ', $CB(\varphi) < \alpha$; i.e., $CB(\varphi) \not\geq \alpha$.

  (ii) Since $MR(p) = CB(p) = \alpha < \infty$ there is a formula $\varphi \in p$ such that $\{q \in S(M) : \varphi \in q$ and $MR(q) \geq \alpha\} = \{p\}$ (by Lemma 2.2.3(iv)). Supposing that $\deg(p) > 1$ produces two contradictory formulas ψ_1 and ψ_2 over $\mathfrak{C}$ such that $MR(\varphi \wedge \psi_i) = \alpha$, for $i = 1, 2$. Arguing as in (i) yields contradictory formulas ψ'_1 and ψ'_2 over M such that $MR(\varphi \wedge \psi'_i) = \alpha$, for $i = 1, 2$. This contradicts that p is the unique completion of φ over M having Morley rank α.

  (iii) T is t.t. since, letting $\bar{v} = (v_1, \ldots, v_n)$, $MR(\bar{v} = \bar{v}) = CB(\bar{v} = \bar{v}) = \sup\{CB(p) : p \in S_n(M)\}$, for each $n < \omega$.

  Let φ be a strongly minimal formula (over $\emptyset$) in a t.t. theory and $D = \varphi(\mathfrak{C})$. Remember that D and $c\ell(-)$, the restriction of $acl(-)$ to D, form a pregeometry and the resulting dependence relation defines a freeness relation (called dim $-$independence above). Since $\mathfrak{C}$ is t.t., Morley rank defines another freeness relation on D.

  The next lemma connects dim $-$independence and Morley rank independence in a strongly minimal D. (Note: In the lemma the ambient theory is not required to be totally transcendental.)

Lemma 3.3.4. *Let φ be a strongly minimal formula over A^* (in some complete theory) and $D = \varphi(\mathfrak{C})$. If $\bar{a}$ is a tuple from D and A is any set containing A^*, then $MR(\bar{a}/A) = \dim(\bar{a}/A)$. Thus, for all subsets X, Y and*

$Z \subset Y$ of D, X is Morley rank independent from Y over Z if and only if X is $\dim$ $-$independent from Y over Z.

Furthermore, if $\bar{a} \cap acl(A^*) = \emptyset$, $\bar{a}$ is acl-independent over A if and only if $\bar{a}$ is Morley rank independent from A and A^* and $\bar{a}$ is Morley rank independent over A.

Proof. Without loss of generality, $A^* = \emptyset$. The lemma is proved by induction on $n = \dim(\bar{a}/A)$. Let $p = tp(\bar{a}/A)$. By Lemma 3.3.1(i), $MR(p) = 0$ if and only if $\bar{a} \in acl(A)$, hence the result is true when $n = 0$. Without loss, $\bar{a} = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_k)$ where, $\dim(\{a_1, \ldots, a_n\}/A) = n > 0$. We first show that $MR(p) \geq n$. Let $B = \{b_1^i : i < \omega\} \subset D$ be a set which is acl-independent over A. Since $tp(b_1^i/A) = tp(a_1/A)$, for all i, there is a tuple $\bar{b}^i$ realizing p such that b_1^i is the first entry in $\bar{b}^i$. We can furthermore assume that $\bar{b}^i$ is acl-independent from $A \cup B$ over $A \cup \{b_1^i\}$, hence, $\dim(\bar{b}^i/A \cup B) = \dim(\bar{b}^i/A \cup \{b_1^i\}) = n - 1$ (by additivity, Remark 3.1.4) and $tp(\bar{b}^i/A \cup B) \neq tp(\bar{b}^j/A \cup B)$ when $i \neq j$. By induction $MR(\bar{b}^i/A \cup B) = n - 1$, hence $\{tp(\bar{b}^i/A \cup B) : i < \omega\}$ forms an infinite set of extensions of p, each having Morley rank $n - 1$. Thus $MR(p) \geq n$.

To prove that $MR(p) \leq n$ it suffices to show that p does not have infinitely many contradictory extensions of Morley rank $\geq n$. Assume, to the contrary, that there is a set $B \supset A$ and, for each $i < \omega$, there is a $\bar{b}^i$ realizing p such that for all $i < j < \omega$, $MR(\bar{b}^i/B) \geq n$ and $tp(\bar{b}^i/B) \neq tp(\bar{b}^j/B)$. By the first paragraph, $n \leq \dim(\bar{b}^i/B) \leq \dim(\bar{a}/A) = n$, hence $\dim(\bar{b}^i/B) = n$ (for $i < \omega$). Then, letting $\bar{d}^i = (b_1^i, \ldots, b_n^i)$, the fact that $tp(\bar{b}^i/A \cup \bar{d}^i)$ is algebraic forces $\{b_1^i, \ldots, b_n^i\}$ to be acl-independent over B. Since an acl-independent subset of a strongly minimal set is indiscernible (by Lemma 3.1.3(iii)), $tp(\bar{d}^i/B) = tp(\bar{d}^j/B)$ for all i and j. Hence, for each $i < \omega$ there is an $f_i \in \text{Aut}(\mathfrak{C})$ fixing B pointwise and taking $\bar{d}^i$ to $\bar{d}^0$. Let $\bar{a}' = (a_1, \ldots, a_n)$ and let $\varphi(\bar{x}, \bar{a}')$ be an algebraic formula satisfied by $\bar{a}$. Then, $\models \varphi(\bar{b}^i, \bar{d}^i)$, hence $\models \varphi(f_i(\bar{b}^i), \bar{d}^0)$, for all i. Since $\varphi(\bar{x}, \bar{d}^0)$ is an algebraic formula there are distinct i and j such that $f_i(\bar{b}^i) = f_j(\bar{b}^j)$. This implies (since the f_i's are automorphisms fixing B) that $tp(\bar{b}^i/B) = tp(\bar{b}^j/B)$. This contradiction completes the first part of the proof.

The furthermore clause in the lemma follows from the definitions and the first part of the lemma.

Notation. For A, B, C subsets of a strongly minimal set D we can use A *is independent from B over C* for "A is Morley rank independent from B over C" or "A is $\dim$ $-$independent from B over C". The acl-independence relation on D will be termed "algebraically independent".

In Lemma 2.2.4 the existence of Cantor-Bendixson rank was connected to the number of complete types in the theory. A similar connection exists for Morley rank.

Proposition 3.3.1. *Let T be a complete theory. If T is t.t. then for all sets A, $|S(A)| \leq |A| + |T|$; i.e., T is λ−stable for all infinite cardinals $\lambda \geq |T|$. If T is countable, then T is $\aleph_0$−stable if and only if T is totally transcendental.*

Proof. Let A be an arbitrary subset of $\mathfrak{C}$. For any formula φ over A let $U_\varphi = \{ p \in S(A) : \varphi \in p$ and $MR(p) = MR(\varphi) \}$. Every element of $S(A)$ is in some U_φ and each U_φ is finite, hence $|S(A)|$ is equal to the number of formulas over A, which is $|A| + |T|$.

If T is countable and $\aleph_0$−stable then T has a countable saturated model M and every element of $S(M)$ has CB-rank by Lemma 2.2.4. By Lemma 3.3.3, T is t.t., completing the proof.

Warning: There is an uncountable theory T which is $|T|$−stable but not t.t.

We saw in previous sections the usefulness of indiscernible sequences in the construction of uncountable models with special properties. Indiscernible *sets* also played an important role in our proof of Morley's Categoricity Theorem, e.g., if M is a model of an uncountably categorical theory and φ is a strongly minimal formula (over say $\emptyset$) then a basis for $\varphi(M)$ is an indiscernible set over which the model is prime. Indiscernible sets will also play a vital role in the analysis of Morley rank independence in a t.t. theory. Preliminarily, we show that every infinite indiscernible sequence is actually an indiscernible set. This is an instance of a basic theme in stability theory: the presence of an order gives rise to many types over sets.

Lemma 3.3.5. *Let T be a complete theory which is λ−stable for some $\lambda \geq |T|$, A a set and $(I, <)$ an infinite indiscernible sequence over A. Then*
    (i) I is an indiscernible set over A.
    (ii) For any formula $\varphi(\bar{x}, \bar{y})$ over A there is an $n < \omega$ such that for all $\bar{a}$,

$$|\{ \bar{b} \in I : \models \varphi(\bar{a}, \bar{b})\}| \leq n \ \ or \ \ |\{ \bar{b} \in I : \models \neg\varphi(\bar{a}, \bar{b})\}| \leq n.$$

Proof. (i) Suppose, to the contrary, that there is a formula $\varphi(\bar{v}_1, \ldots, \bar{v}_n)$ over A satisfied by increasing n−tuples from I, but not satisfied by some n−tuple of distinct elements from I. Let P_n denote the group of permutations of $\{1, \ldots, n\}$. Let P_n^+ be the set of elements σ of P_n such that for $\bar{a}_1 < \ldots < \bar{a}_n$ from I, $\models \varphi(\bar{a}_{\sigma 1}, \ldots, \bar{a}_{\sigma n})$. By assumption, $P_n^+ \neq P_n$. Every element of P_n can be written as a product of transpositions of the form $(k, k+1)$ (which denotes the permutation fixing every $i \notin \{k, k+1\}$ and switching k and $k+1$). Thus, there are $\sigma \in P_n^+$, $\tau \in P_n \setminus P_n^+$ and k such that $\tau = (k, k+1) \cdot \sigma$. Letting $\psi(\bar{v}_1, \ldots, \bar{v}_n)$ be the formula $\varphi(\bar{v}_{\sigma 1}, \ldots, \bar{v}_{\sigma n})$ we have

$$\models \psi(\bar{a}_1, \ldots, \bar{a}_n) \ \text{ and } \ \models \neg\psi(\bar{a}_1, \ldots, \bar{a}_{k-1}, \bar{a}_{k+1}, \bar{a}_k, \bar{a}_{k+2}, \ldots, \bar{a}_n),$$
$$\text{for } \bar{a}_1 < \ldots < \bar{a}_n \text{ from } I.$$

Let $(Y, <)$ be a dense linear order without endpoints of cardinality $> \lambda$ containing a dense subset X of cardinality λ. (First let μ be the least cardinal

such that $2^\mu > \lambda$. Let X_0 be the set of sequences of 0's and 1's of length $< \mu$, ordered lexicographically by $<$. Then $(X_0, <)$ is a dense linear order with 2^μ cuts. Let $(Y, <)$ be an order of cardinality $> \lambda$ in which X_0 is dense and let X be any subset of Y of cardinality λ which contains X_0.) By Corollary 2.4.1 we may assume $(Y, <)$ to be an indiscernible sequence of tuples with $D(Y) = D(I)$. Let $y, y' \in Y$ with $y < y'$. Then, there are $x_i \in X$, for $1 \leq i \leq n$ and $i \neq k$, such that

$$x_1 < \ldots < x_{k-1} < y < x_{k+1} < y' < x_{k+2} < \ldots < x_n.$$

Consequently,

$$\models \psi(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n) \text{ and } \models \neg\psi(x_1, \ldots, x_{k-1}, y', x_{k+1}, \ldots, x_n).$$

Thus, distinct elements of Y realize distinct types over X. Since $|X| = \lambda$ and $|Y| > \lambda$ this contradicts the λ-stability of T, proving (i).

Turning to (ii), by (i) we may assume I to be an indiscernible set. Suppose (ii) to fail for $\varphi(\bar{x}, \bar{y})$. Then for all n there are $I_0, I_1 \subset I$, each of cardinality $> n$, such that $\{\varphi(\bar{x}, \bar{b}) : \bar{b} \in I_0\} \cup \{\neg\varphi(\bar{x}, \bar{b}) : \bar{b} \in I_1\}$ is consistent. By compactness and the indiscernibility of I, for I_0, I_1 any two disjoint subsets of I, $\{\varphi(\bar{x}, \bar{b}) : \bar{b} \in I_0\} \cup \{\neg\varphi(\bar{x}, \bar{b}) : \bar{b} \in I_1\}$ is consistent. Again, by Corollary 2.4.1, there is an indiscernible set J with $D(J) = D(I)$ and $|J| = \lambda$. Thus, for any disjoint pair of subsets J_0, J_1 of J, $\{\varphi(\bar{x}, \bar{b}) : \bar{b} \in J_0\} \cup \{\neg\varphi(\bar{x}, \bar{b}) : \bar{b} \in J_1\}$ is consistent. Thus, $|S(J)| = 2^{|J|} = 2^\lambda$. This contradiction of the λ-stability of T, proves the lemma.

Let φ be a strongly minimal formula (over some set A). Assuming that a_β has been defined for $\beta < \alpha$ let a_α be a realization of the unique nonalgebraic extension of φ over $\{a_\beta : \beta < \alpha\} \cup A$. Then, for any α, $\{a_\beta : \beta < \alpha\}$ is an indiscernible set over A. The following concept is a generalization of this construction to an arbitrary stationary type. If I is an ordered set and $C = \{c_i : i \in I\}$ is a set indexed by I, then for $i \in I$, C_i denotes $\{c_j : j < i\}$.

Definition 3.3.6. *Let T be a totally transcendental theory, $p \in S(C)$ a stationary type and $B \supset C$. We call $A = \{\bar{a}_\beta : \beta < \alpha\}$ a Morley sequence over B in p if for all $\beta < \alpha$, $tp(\bar{a}_\beta / B \cup A_\beta)$ is the unique free extension of p in $S(B \cup A_\beta)$.*

As with bases of strongly minimal sets such sequences are indiscernible:

Lemma 3.3.6. *Let T be a totally transcendental theory, $p \in S(C)$ a stationary type, $B \supset C$ and $A = \{\bar{a}_\beta : \beta < \alpha\}$ a Morley sequence over B in p. Then A is an indiscernible set over B.*

Proof. The following argument is a slight generalization of the proof of Lemma 3.1.3(iii). Define an order $<$ on A by: $\bar{a}_\beta < \bar{a}_\gamma$ if $\beta < \gamma$. By

Lemma 3.3.5 it suffices to show that $(A, <)$ is an indiscernible sequence. To accomplish this we prove by induction on k,

$$\text{for all } \bar{a}_{\beta_1} < \ldots < \bar{a}_{\beta_k} \text{ and } \bar{a}_{\gamma_1} < \ldots < \bar{a}_{\gamma_k} \text{ from } A,$$
$$tp(\bar{a}_{\beta_1} \ldots \bar{a}_{\beta_k}/B) = tp(\bar{a}_{\gamma_1} \ldots \bar{a}_{\gamma_k}/B).$$

Assuming this is true for $k < n$ let $\bar{a}_{\beta_1} < \ldots < \bar{a}_{\beta_n}$ and $\bar{a}_{\gamma_1} < \ldots < \bar{a}_{\gamma_n}$ be from A. By induction there is an automorphism f of $\mathfrak{C}$ fixing B pointwise and mapping $\bar{a}_{\beta_i}$ to $\bar{a}_{\gamma_i}$, for $1 \leq i < n$. Since f fixes B pointwise and preserves Morley rank, $MR(f(\bar{a}_{\beta_n})/B \cup \{\bar{a}_{\gamma_1}, \ldots, \bar{a}_{\gamma_n}\}) = MR(p)$ and $f(\bar{a}_{\beta_n})$ realizes p, hence $f(\bar{a}_{\beta_n})$ and $\bar{a}_{\gamma_n}$ both realize the unique free extension of p over $B \cup \{\bar{a}_{\gamma_1}, \ldots, \bar{a}_{\gamma_{n-1}}\}$. We conclude that $(\bar{a}_{\beta_1}, \ldots, \bar{a}_{\beta_n})$ and $(\bar{a}_{\gamma_1}, \ldots, \bar{a}_{\gamma_n})$ have the same type over B, proving the lemma.

As with independent subsets of a strongly minimal set, if I is a Morley sequence over $B \supset A$ in the stationary $p \in S(A)$, then (by Lemma 3.3.5) for all $\bar{a} \in I$, $\bar{a}$ is independent from $B \cup (I \setminus \{\bar{a}\})$ over A.

Example 3.3.1. (Two examples of Morley sequences.) Let E be a binary relation and T the theory saying that E is an equivalence relation with infinitely many infinite classes and no finite classes. Let $M \models T$. The unique $p \in S_1(\emptyset)$ is stationary. To obtain a Morley sequence in p simply take any $I \subset M$ such that $a \neq b \in I \implies \models \neg E(a, b)$. Now let a be any element of M and $q \in S(a)$ the unique nonalgebraic completion of $E(x, a)$ (which also happens to be stationary). Then, any $J \subset E(M, a) \setminus \{a\}$ is a Morley sequence in q.

With these tools in hand we can complete the proof that Morley rank independence satisfies all of the properties of a freeness relation:

Proposition 3.3.2 (Symmetry Lemma). *For all sets A, B and C,*

$$A \underset{C}{\downarrow} B \implies B \underset{C}{\downarrow} A.$$

Proof. Assuming the symmetry property to fail, the finite character of dependence yields a set C and (finite sequences) $\bar{a}$ and $\bar{b}$ such that $\bar{b}$ depends on $\bar{a}$ over C and $\bar{a}$ is independent from $\bar{b}$ over C; i.e.,

$$MR(\bar{b}/C \cup \bar{a}) < MR(\bar{b}/C) = \alpha \text{ and } MR(\bar{a}/C \cup \bar{b}) = MR(\bar{a}/C) = \beta. \quad (3.3)$$

Claim. There are $\bar{a}, \bar{b}$ and C satisfying (3.3) with C the universe of an $\aleph_0$−saturated model M.

Let $M \supset C$ be an $\aleph_0$−saturated model. Let $\bar{b}'$ be a realization of $tp(\bar{b}/C)$ which is independent from M over C, and choose $\bar{a}'$ such that $tp(\bar{b}'\bar{a}'/C) = tp(\bar{b}\bar{a}/C)$ and $\bar{a}'$ is independent from $M \cup \bar{b}'$ over $C \cup \bar{b}'$. Then, $MR(\bar{b}'/M) = MR(\bar{b}'/C) = \alpha$, $MR(\bar{b}'/M \cup \bar{a}') \leq MR(\bar{b}'/C \cup \bar{a}') < \alpha$ and $MR(\bar{a}'/M \cup \bar{b}') =$

$MR(\bar{a}'/C \cup \bar{b}') = MR(\bar{a}'/C) = \beta$. Replacing $\bar{b}$ by $\bar{b}'$ and $\bar{a}$ by $\bar{a}'$ proves the claim.

By Lemma 3.3.3(ii), both $p = tp(\bar{b}/M)$ and $q = tp(\bar{a}/M)$ are stationary. Let $\varphi(\bar{x}, \bar{y})$ be a formula in $tp(\bar{b}\bar{a}/M)$ such that $MR(\varphi(\bar{x}, \bar{a})) < \alpha = MR(p)$. The formula φ will be used to contradict Lemma 3.3.5(ii). First, let I be an infinite Morley sequence over M in q and let $\bar{b}'$ be a realization of p which is independent from I over M. For any $\bar{a}' \in I$, $\varphi(\bar{x}, \bar{a}')$ has Morley rank $< \alpha$ (since it is conjugate to $\varphi(\bar{x}, \bar{a})$) hence $\models \neg\varphi(\bar{b}', \bar{a}')$. Now let J be an infinite Morley sequence over $M \cup \bar{b}' \cup I$ in q. The unique free extension of q over $M \cup \bar{b}$ (considered as a type in $\bar{y}$) is $tp(\bar{a}/M \cup \bar{b})$, hence contains $\varphi(\bar{b}, \bar{y})$. Since $tp(\bar{b}/M) = tp(\bar{b}'/M)$, $q \cup \{\varphi(\bar{x}, \bar{b}')\}$ also has Morley rank β, hence the unique free extension of q over $M \cup \bar{b}'$ contains $\varphi(\bar{b}', \bar{y})$. Thus, for every $\bar{a}' \in J$, $\models \varphi(\bar{b}', \bar{a}')$. Checking the definitions of I and J, $I \cup J$ is a Morley sequence over M in q, hence an indiscernible set over M. Since $\{\bar{c} \in I \cup J : \models \varphi(\bar{b}', \bar{c})\} = J$ and $\{\bar{c} \in I \cup J : \models \neg\varphi(\bar{b}', \bar{c})\} = I$ are both infinite we have contradicted Lemma 3.3.5(ii), proving the lemma.

This lemma allows us to say, for instance, "A and B are independent over C," instead of A is independent from B over C or B is independent from A over C. A collection of sets (or elements) I is called *independent over A* or *A−independent* if for each $X \in I$, X is independent from $I \setminus X$ over A. Note: a Morley sequence over A is independent over A.

Symmetry is such a basic property of independence that the lemma will usually be applied tacitly.

Corollary 3.3.1. *Let T be t.t., $p \in S(A)$ a stationary type, I a Morley sequence over A in p, and $\bar{b}$ a finite sequence. Then there is a finite $J \subset I$ such that $I \setminus J$ is a Morley sequence over $A \cup J \cup \bar{b}$ in p.*

Proof. Let J be a finite subset of I such that $MR(\bar{b}/A \cup I) = MR(\bar{b}/A \cup J) = \beta$. Let $\bar{a} \in I \setminus J$ and $I' = I \setminus \{\bar{a}\}$. Then $MR(\bar{b}/A \cup I' \cup \bar{a}) = MR(\bar{b}/A \cup I')$ (since $I' \supset J$), so by the Symmetry Lemma, $MR(\bar{a}/A \cup I' \cup \bar{b}) = MR(\bar{a}/A \cup I') = MR(\bar{a}/A)$. Thus, $I \setminus J$ is a Morley sequence over $A \cup J \cup \bar{b}$ in p.

In Section 3.1 the relation "$p = tp(\bar{a}/B)$ does not split over $A \subset B$" was promoted as a way of expressing that $\bar{a}$ is (at least intuitively) free from B over A. We now introduce a strengthening of the nonsplitting relation called definability. The relation "$p \in S(B)$ does not split over $A \subset B$" is equivalent to: for all formulas $\varphi(\bar{x}, \bar{y})$ there is a set $P_\varphi \subset S(A)$ such that for any $\bar{a}$ from B, $\varphi(\bar{x}, \bar{a}) \in p$ if and only if $tp(\bar{a}/A) \in P_\varphi$. Then p will be definable over A if p does not split over A *and* for each φ, P_φ is a basic open set in the appropriate Stone Space. Explicitly,

Definition 3.3.7. *Let p be a complete type over B and A a set. We say that p is* definable over A *if for every formula $\varphi(\bar{x}, \bar{y})$ over $\emptyset$ there is a formula $\psi(\bar{y})$ over A such that for all sequences $\bar{a}$ from B, $\varphi(\bar{x}, \bar{a}) \in p \iff \models \psi(\bar{a})$.*

For p and φ as above, let $p \upharpoonright \varphi$ be the type $\{ \varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{b}) \in p \} \cup \{ \neg\varphi(\bar{x}, \bar{b}) : \neg\varphi(\bar{x}, \bar{b}) \in p \}$. When ψ has the property that for all sequences $\bar{a}$ from B, $\varphi(\bar{x}, \bar{a}) \in p \iff \models \psi(\bar{a})$, we say that ψ defines $p \upharpoonright \varphi$.

If $p \in S(B)$ is definable there is a function d such that for each formula $\varphi(\bar{x}, \bar{y})$ over $\emptyset$, $d\varphi$ is a formula $\psi(\bar{y})$ which defines $p \upharpoonright \varphi$. Both d and the collection of formulas $\{ d\varphi : \varphi$ a formula over $\emptyset \}$ are called a defining scheme *for p.*

Clearly, if p is definable over A then p does not split over A.

If $p \in S(\mathfrak{C})$ and d is a defining scheme for p consisting of formulas over A, then d is also a defining scheme for any complete $q \subset p$. For this reason, many of the results stated below for elements of $S(\mathfrak{C})$ can actually be applied to any complete type.

Definition 3.3.8. *For A a set and φ a formula, φ is* almost over A *if the set $\{ f(\varphi) : f \in \mathrm{Aut}(\mathfrak{C})$ and f fixes A pointwise $\}$ contains finitely many formulas, up to equivalence; i.e.,*

$$\{ f(\varphi(\mathfrak{C})) : f \in \mathrm{Aut}(\mathfrak{C}) \text{ and } f \text{ fixes } A \text{ pointwise } \}$$

contains finitely many sets.

Much of the remainder of the section is devoted to the proof of the following theorem which ties together freeness, definability and nonsplitting.

Theorem 3.3.1. *Let T be a t.t. theory, $p \in S(\mathfrak{C})$ and A a set.*
(i) The following are equivalent.

  (1) *p is a free extension of $p \upharpoonright A$ and $p \upharpoonright A$ is stationary.*
  (2) *p is definable over A.*
  (3) *p does not split over A.*

(ii) The following are also equivalent.

  (1) *p is a free extension of $p \upharpoonright A$.*
  (2) *There is a defining scheme for p consisting of formulas which are almost over A.*
  (3) *p is definable over any model containing A.*

This theorem is actually a compilation of many lemmas and propositions, which we have collected as a focal point for the remainder of the section. One part of the theorem, namely (3) $\implies$ (2) of (i), will not be proved until Section 4.1.1. With the additional tools developed in Section 4.1 the proof of this implication is easier than it would be if we forced it into this section. It is stated as part of a theorem in this section to present a coherent picture of the relationship between freeness and definability. The most difficult part of the theorem is (1) $\implies$ (2) of (i), which is handled in

Lemma 3.3.7 (Definability Lemma). *Suppose that T is t.t., A is a set, $A \subset B \subset \mathfrak{C}$, $p \in S(B)$ is a free extension of $p \restriction A$ and $p \restriction A$ is stationary. Then, p is definable over A.*

Part (i) of the following elementary result is used in the proof of the lemma.

Lemma 3.3.8. *(i) Let φ be a formula and A a set such that for all automorphisms f of $\mathfrak{C}$ fixing A pointwise, φ is equivalent to $f(\varphi)$. Then φ is equivalent to a formula over A.*

(ii) Let $\varphi(\bar{x}, \bar{y})$ be a formula over A. If $\varphi(\bar{x}, \bar{c})$ is almost over A there is a formula E over A which defines an equivalence relation with finitely many classes and satisfies: for all $\bar{d}$ realizing $tp(\bar{c}/A)$, $\models E(\bar{d}, \bar{d}') \iff \varphi(\mathfrak{C}, \bar{d}) = \varphi(\mathfrak{C}, \bar{d}')$.

Proof. (i) Let $p \in S(A)$ be the type over A of some tuple $\bar{a}$ satisfying φ. If $\bar{b}$ also realizes p there is an $f \in \mathrm{Aut}(\mathfrak{C})$ fixing A and taking $\bar{a}$ to $\bar{b}$. Since φ is equivalent to $f(\varphi)$, $\bar{b}$ also satisfies φ. Thus, by compactness there is a formula $\psi \in p$ such that $\models \forall \bar{x}(\psi \to \varphi)$. A further compactness argument, left to the reader, shows that there is a formula ψ' over A equivalent to φ.

(ii) Let $E_0(\bar{y}, \bar{y}')$ be the equivalence relation defined by:

$$E_0(\bar{y}, \bar{y}') \iff \forall \bar{x}(\varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{y}')).$$

Since $\varphi(\bar{x}, \bar{y})$ is over A, so is E_0. Let $p = tp(\bar{c}/A)$. Since $\varphi(\bar{x}, \bar{c})$ is almost over A there are finitely many, say k, E_0-classes containing a realization of p. By compactness there is a formula $\psi \in p$ such that

$$\models \exists \bar{y}_1 \cdots \bar{y}_k \forall \bar{y}[\psi(\bar{y}) \longrightarrow \bigvee_{1 \leq i \leq k} E_0(\bar{y}, \bar{y}_i)].$$

The equivalence relation $E(\bar{y}, \bar{y}') = (E_0(\bar{y}, \bar{y}') \wedge \psi(\bar{y}) \wedge \psi(\bar{y}')) \vee (\neg \psi(\bar{y}) \wedge \neg \psi(\bar{y}'))$ has the desired properties.

Proof of the Definability Lemma. First suppose that B is a set; i.e., $|B| < |\mathfrak{C}|$. We may assume that $B = M$ is a $\kappa-$saturated model, where $\kappa > |A| + |T|$. (Let $M \supset B$ be such a $\kappa-$saturated model and p' the unique free extension of p over M. If p' is definable over A then so is p.) Let $q = p \restriction A$. Since M is $\kappa-$saturated and q is stationary M contains an infinite Morley sequence I in q. Let $\bar{a}$ be a realization of p and note that $I \cup \{\bar{a}\}$ is also a Morley sequence in q (since $p \restriction (I \cup A)$ is the unique free extension of q over $I \cup A$). Let $\varphi(\bar{x}, \bar{y})$ be a formula over $\emptyset$.

Claim. For any sequence $\bar{b}$ from M of length $\ell = lh(\bar{y})$, $\varphi(\bar{x}, \bar{b}) \in p$ if and only if $\{\bar{c} \in I : \models \varphi(\bar{c}, \bar{b})$ is cofinite in $I\}$.

By Corollary 3.3.1 there is a cofinite set $I' \subset I$ which is a Morley sequence over $A \cup \bar{b}$ in q, in fact, $I' \cup \{\bar{a}\}$ is a Morley sequence over $A \cup \bar{b}$ in q. Thus, for $\bar{b}$ a sequence from M of length ℓ,

$$\varphi(\bar{x}, \bar{b}) \in p \iff \models \varphi(\bar{a}, \bar{b}) \iff (\models \varphi(\bar{a}', \bar{b}) \text{ for every } \bar{a}' \in I').$$

This proves the claim.

Bringing Lemma 3.3.5(ii) into play, there is an n such that for all $\bar{b} \in M^\ell$, $|\{\bar{a}' \in I : \models \varphi(\bar{a}', \bar{b})\}| < n$ or $|\{\bar{a}' \in I : \models \neg\varphi(\bar{a}', \bar{b})\}| < n$. Thus,

$$
\begin{aligned}
\varphi(\bar{x}, \bar{b}) \in p \iff & \ (\text{ there is } J \subset I \text{ of cardinality } \geq n \text{ such that} \\
& \quad \bar{a}' \in J \implies \models \varphi(\bar{a}', \bar{b})) \qquad\qquad (3.4) \\
\iff & \ (\text{ there is no } J \subset I \text{ of cardinality } \geq n \text{ such that} \\
& \quad \bar{a}' \in J \implies \models \neg\varphi(\bar{a}', \bar{b})). \qquad\quad (3.5)
\end{aligned}
$$

A formula defining p relative to $\varphi(\bar{x}, \bar{y})$ is found through this equivalence. Let $J = \{\bar{a}_0, \ldots, \bar{a}_{2n}\}$ be a subset of I of cardinality $2n+1$ and $\bar{c}$ an enumeration of J. Let

$$\psi(\bar{y}, \bar{c}) = \bigvee \left\{ \bigwedge_{i \in w} \varphi(\bar{a}_i, \bar{y}) : w \subset 2n+1, |w| = n \right\}.$$

If $\bar{b}$ is any sequence from M which satisfies $\psi(\bar{y}, \bar{c})$, there are certainly n elements of I satisfying $\varphi(\bar{x}, \bar{b})$, hence $\varphi(\bar{x}, \bar{b}) \in p$ (by (3.4)). Conversely, if $\models \neg\psi(\bar{b}, \bar{c})$ there must be n elements of I satisfying $\neg\varphi(\bar{x}, \bar{b})$, hence $\neg\varphi(\bar{x}, \bar{b}) \in p$ (by (3.5)).

To prove the lemma we must obtain a defining formula for $p \upharpoonright \varphi$ which is over A, while the defining formula $\psi(\bar{y}, \bar{c})$ is over $A \cup I$.

Claim. For any $\bar{d}$ realizing $tp(\bar{c}/A)$ in M, $\psi(\bar{y}, \bar{d})$ is equivalent to $\psi(\bar{y}, \bar{c})$.

Let $\bar{d}$ in M realize $tp(\bar{c}/A)$. Then $\bar{d}$ is also an enumeration of a Morley sequence $\{\bar{d}_0, \ldots, \bar{d}_{2n}\}$ in q, and there is an infinite Morley sequence $J \subset M$ in q containing $\{\bar{d}_0, \ldots, \bar{d}_{2n}\}$. In obtaining $\psi(\bar{y}, \bar{c})$ above we can take I to be any infinite Morley sequence in q which is contained in M and $\bar{c}$ any subset of I of cardinality $2n+1$. Thus, for all $\bar{b}$ from M, $\varphi(\bar{x}, \bar{b}) \in p \iff \models \psi(\bar{b}, \bar{d})$. Hence, $\models \forall \bar{y}(\psi(\bar{y}, \bar{c}) \leftrightarrow \psi(\bar{y}, \bar{d}))$, proving the claim.

Thus, all conjugates over A of $\psi(\bar{y}, \bar{c})$ which are over M are equivalent. Since M is κ−saturated, this implies that all conjugates over A of $\psi(\bar{y}, \bar{c})$ are equivalent. By Lemma 3.3.8, $\psi(\bar{y}, \bar{c})$ is equivalent to a formula $\psi'(\bar{y})$ over A, proving the lemma in the case when B is a set.

Now let $B \subset \mathfrak{C}$ be arbitrary and suppose, towards a contradiction, that there is some formula $\varphi(\bar{x}, \bar{y})$ such that $p \upharpoonright \varphi$ is not defined by a formula over A. Then there is a $p' \subset p$ whose domain is a set $B' \supset A$ such that $p' \upharpoonright \varphi$ is not defined by a formula over A, in contradiction to the first part of the proof. This completes the proof of the Definability Lemma.

Theorem 3.3.1(i), (2)$\Longleftrightarrow$(3) is proved in

Lemma 3.3.9. *Let T be t.t. and $p \in S(\mathfrak{C})$. Then p does not split over A if and only if p is definable over A.*

Proof. If p is definable over A, then clearly p does not split over A. Now suppose that p does not split over A. By the Definability Lemma, p is definable over some set. Let $\varphi(\bar{x}, \bar{y})$ be an arbitrary formula and let $\psi(\bar{y})$ be a defining formula of $p \upharpoonright \varphi$. If $f \in \text{Aut}(\mathfrak{C})$ fixes A pointwise, then $f(p) = p$ since p does not split over A. Also,

$$\begin{aligned} \models \psi(\bar{a}) \quad &\text{if and only if} \quad \varphi(\bar{x}, \bar{a}) \in p \\ &\text{if and only if} \quad \varphi(\bar{x}, f(\bar{a})) \in f(p) = p \\ &\text{if and only if} \quad \models \psi(f(\bar{a})) \end{aligned}$$

That is , ψ is preserved by the automorphisms of $\mathfrak{C}$ which fix A. Thus ψ is equivalent to a formula over A, (by Lemma 3.3.8(i)), as required.

The remainder of the proof of part (i) of the main theorem is delayed until Lemma 4.1.4.

Proof of Theorem 3.3.1(ii). (1) $\implies$ (2). First notice that if ψ and ψ' are formulas which define $p \upharpoonright \varphi$ then ψ and ψ' are equivalent.

Since p is a free extension of $q = p \upharpoonright A$,

$$\{ p' \in S(\mathfrak{C}) : p' \text{ is conjugate to } p \text{ over } A \}$$

is a finite set of types, which we enumerate as $\{p_0, \ldots, p_k\}$. (Every conjugate of p is an extension of q in $S(\mathfrak{C})$ with the same Morley rank as q.) Suppose that ψ defines $p \upharpoonright \varphi$ (where φ is some formula over $\emptyset$).

Claim. If $f, g \in \text{Aut}(\mathfrak{C})$ fix A pointwise and $f(p) = g(p)$, then $f(\psi)$ is equivalent to $g(\psi)$.

For f and g as hypothesized and $r = f(p) = g(p)$, both $f(\psi)$ and $g(\psi)$ define $r \upharpoonright \varphi$. Hence, $f(\psi)$ is equivalent to $g(\psi)$.

Since there are finitely many elements of $S(\mathfrak{C})$ conjugate over A to p, ψ has finitely many conjugates over A. This proves (1) $\implies$ (2).

(2) $\implies$ (3). It suffices to show that a formula $\psi(\bar{y}, \bar{c})$ which is almost over A is equivalent to a formula over any model $M \supset A$. By Lemma 3.3.8(ii) there is an $A-$definable equivalence relation E with finitely many classes such that, for all $\bar{c}'$, $\models E(\bar{c}, \bar{c}')$ if and only if $\psi(\mathfrak{C}, \bar{c}') = \psi(\mathfrak{C}, \bar{c})$. Since E has only finitely many classes every class has a representative in M, in particular, there is a sequence $\bar{c}'$ from M such that $\models E(\bar{c}, \bar{c}')$. Then $\psi(\bar{y}, \bar{c}')$ is the desired formula over M equivalent to $\psi(\bar{y}, \bar{c})$.

(3) $\implies$ (1). Suppose p is definable over any model containing A and, to the contrary, there is a formula $\varphi(\bar{x}, \bar{a}) \in p$ (where $\varphi = \varphi(\bar{x}, \bar{y})$ is over $\emptyset$) such that

$$MR(\varphi(\bar{x}, \bar{a})) < MR(p \upharpoonright A) = \alpha.$$

Let $M \supset A$ be an $\aleph_0$−saturated model independent from $\bar{a}$ over A. By hypothesis, there is a formula ψ over M defining $p \restriction \varphi$. Since $r = tp(\bar{a}/M)$ is stationary there is an infinite I which is a Morley sequence in r over M. Let $\bar{b}$ realize $p \restriction M \cup I$ and let $I' \subset I$ be a finite set such that $\bar{b}$ is independent from $M \cup I$ over $M \cup I'$. Let $\bar{a}' \in I \setminus I'$.

Claim. $\bar{a}'$ and $\bar{b}$ are independent over A.

Simply because $\bar{a}' \in I$, $\bar{a}'$ is independent from $\bar{b}$ over $M \cup I'$. Since I is a Morley sequence over M and $\bar{a}' \in I \setminus I'$, $\bar{a}'$ is independent from I' over M. Furthermore, since $tp(\bar{a}'/M) = tp(\bar{a}/M)$ and $\bar{a}$ is independent from M over A, $\bar{a}'$ is also independent from M over A. Applying the transitivity of independence we conclude that $\bar{a}'$ is independent from $M \cup I' \cup \bar{b}$ over A, hence $\bar{a}'$ and $\bar{b}$ are independent over A. This proves the claim.

Since $\models \psi(\bar{a}')$ and ψ defines $p \restriction \varphi$, $\varphi(\bar{x}, \bar{a}') \in p$. Hence, $\models \varphi(\bar{b}, \bar{a}')$. Thus, $MR(\bar{b}/\bar{a}') \leq MR(\varphi(\bar{x}, \bar{a}')) = MR(\varphi(\bar{x}, \bar{a})) < \alpha = MR(\bar{b}/A)$, contradicting the claim and completing the proof of Theorem 3.3.1(ii).

Most properties of Morley rank independence in a t.t. theory can be derived quickly from Theorem 3.3.1. We include a few corollaries for ease of reference.

The following corollary is left as an exercise to the reader.

Corollary 3.3.2. *Let T be t.t. and $p \in S(\mathfrak{C})$. If p is a free extension of $q = p \restriction A$, then for all models $M \supset A$ there is a finite $B \subset M$ such that p is definable over B and $p \restriction B$ is stationary.*

The following observation lets us pick the parameters in the formulas of a defining scheme to have a very particular form.

Corollary 3.3.3. *Let T be t.t. and $p \in S(\mathfrak{C})$. Let $q \in S(A)$ be any complete stationary type such that p is the unique free extension of q and let I be an infinite Morley sequence in q over A. Then, p is definable over I.*

Proof. This follows immediately from the proof of the Definability Lemma.

The following consequence of definability will play a minor role in the proof of the Baldwin-Lachlan Theorem. This is occasionally called "The Open Mapping Theorem", although it is actually a corollary of that result proved for stable theories in Lemma 5.1.11.

Lemma 3.3.10. *If T is t.t., $tp(\bar{a}/A \cup \{\bar{b}\})$ is isolated and $tp(\bar{a}/A)$ is non-isolated, then $\bar{a} \mathop{\smile\hskip-0.8em\diagup}\limits_{A} \bar{b}$.*

Proof. Suppose, to the contrary, that $\bar{a}$ is independent from $\bar{b}$ over A. Let $\varphi(\bar{x}, \bar{b})$ isolate $tp(\bar{a}/A \cup \bar{b})$, where $\varphi(\bar{x}, \bar{y})$ is a formula over A.

Claim. There is a formula $\psi(\bar{x})$ over A such that whenever $\models \psi(\bar{c})$, $q = tp(\bar{b}/A)$ has a free extension over $\bar{c}$ containing $\varphi(\bar{c}, \bar{y})$.

To see this, let $p(\bar{y}) \in S(\mathfrak{C})$ be a free extension of q and $\psi_0(\bar{x})$ a formula almost over A defining $p \upharpoonright \varphi$. Let $\psi_0, \ldots, \psi_k$ be a list of the conjugates of ψ_0 over A (up to equivalence), and let $\psi = \psi_0 \vee \ldots \vee \psi_k$. Since ψ is invariant under any automorphism which fixes A pointwise, ψ is equivalent to a formula over A (by Lemma 3.3.8). Suppose $\models \psi(\bar{c})$. Then $\models \psi_i(\bar{c})$, for some i, and for p' a conjugate over A to p such that ψ_i defines $p' \upharpoonright \varphi$, $\varphi(\bar{c}, \bar{y}) \in p'$. This proves the claim.

A contradiction will be reached by showing that ψ isolates $tp(\bar{a}/A)$. Suppose that $\bar{c}$ and $\bar{c}'$ satisfy ψ. By the claim there are $\bar{d}$ and $\bar{d}'$ realizing q such that $\models \varphi(\bar{c}, \bar{d})$ and $\models \varphi(\bar{c}', \bar{d}')$. Since $tp(\bar{c}/\bar{d})$ and $tp(\bar{c}'/\bar{d}')$ are isolated by $\varphi(\bar{c}, \bar{d})$ and $\varphi(\bar{c}', \bar{d}')$, respectively, there is an automorphism which is the identity on A and takes $\bar{c}\bar{d}$ to $\bar{c}'\bar{d}'$. Hence $tp(\bar{c}/A) = tp(\bar{c}'/A)$. This contradiction proves the lemma.

The proof of the following is left to the exercises.

Corollary 3.3.4. *Suppose that T is t.t., M is a model and N is a prime model over $M \cup A$. Then $b \in N \setminus M \implies b \mathop{\smile\hspace{-0.9em}/}\limits_{M} A$.*

The final topic to be covered in this section could be called "relativization". Suppose that T is t.t., $\varphi(x)$ is a formula over A and $D = \varphi(\mathfrak{C})$. In some studies it is natural to "restrict the universe to D", defined formally as follows.

Definition 3.3.9. *Let $\mathfrak{C}$ be the universal domain of a complete theory and $D = \varphi(\mathfrak{C})$ an $A-$definable subset of $\mathfrak{C}^k$ for some k. The relativization of $\mathfrak{C}$ to D is the model N (in a language L^*) defined as follows.*

> (1) *The universe of N is D.*
> (2) *For each $A-$definable relation $X \subset D^n$ (for some n) there is a relation symbol R in L^* whose interpretation on N is X.*

We may alternatively call N the relativization of $Th(\mathfrak{C})$ to φ or the restriction of $\mathfrak{C}$ to D.

When N is a relativization of $\mathfrak{C}$ to some definable D it is natural to ask: What is the difference between N and the structure on D induced by all definable relations in $\mathfrak{C}$? The next proposition says that there is no difference when T is t.t.

Proposition 3.3.3. *Let T be t.t. and D a subset of $\mathfrak{C}^n$ (for some n) which is definable over A. Then, for any k and definable $H \subset D^k$ there is a $B \subset D$ such that H is definable over $A \cup B$.*

Preliminarily, we prove

Lemma 3.3.11. *(i) If T is complete and there is a defining scheme for $p \in S(A)$ consisting of formulas almost over A, then there is a defining scheme for p consisting of formulas over A.*
(ii) If T is t.t., then every $p \in S(A)$ is definable over A.

Proof. (i) Let $\psi(\bar{y}, \bar{a})$ be a formula almost over A which defines $p \upharpoonright \varphi$ (where $\psi(\bar{y}, \bar{z})$ is over A). Since $\psi(\bar{y}, \bar{a})$ is almost over A there is a formula $\theta(\bar{z})$ over A (by compactness) such that $\models \theta(\bar{b})$ implies $\psi(\bar{y}, \bar{b})$ is equivalent to a conjugate over A of $\psi(\bar{y}, \bar{a})$. If $\psi(\bar{y}, \bar{a}')$ is conjugate over A to $\psi(\bar{y}, \bar{a})$, then for $\bar{c}$ any tuple from A, $\models \psi(\bar{c}, \bar{a}) \iff \models \psi(\bar{c}, \bar{a}')$. Thus, $\psi'(\bar{y}) = \exists \bar{z}(\theta(\bar{z}) \wedge \psi(\bar{y}, \bar{z}))$ defines $p \upharpoonright \varphi$.

(ii) Let $q \in S(\mathfrak{C})$ be a free extension of p. By Theorem 3.3.1(ii) there is a defining scheme d for q consisting of formulas almost over A. Since d is also a defining scheme for p, (i) implies that p is definable over A.

Remark 3.3.3. A was not assumed to be a set in this last lemma; when $|A| = |\mathfrak{C}|$ the same proof works.

Proof of Proposition 3.3.3. Let $H \subset D^k$ be $\psi(\mathfrak{C}, \bar{a})$. By the previous lemma, there is a defining scheme for $tp(\bar{a}/D \cup A)$ consisting of formulas over $D \cup A$. Thus, there is a formula $\theta(\bar{x})$ over $D \cup A$ such that $\models \psi(\bar{b}, \bar{a})$ if and only if $\models \theta(\bar{b})$, for all $\bar{b}$ from $D \cup A$; i.e., θ defines H. This proves the proposition.

The proof of the following is left to the reader.

Corollary 3.3.5. *Suppose that T is t.t., D is a definable set in $\mathfrak{C}$ and T_D is the theory of the relativization of $\mathfrak{C}$ to D. Then, the Morley rank of D in T is the same as the Morley rank of the universe in T_D.*

Corollary 3.3.6. *Suppose that T is t.t., D is a definable set in $\mathfrak{C}$ and T_D is the theory of the relativization of $\mathfrak{C}$ to D. If T is uncountably categorical then so is T_D.*

Proof. See Exercise 3.3.22.

Corollary 3.3.7. *Let M be a model of a t.t. theory and $p \in S_n(M)$. Let φ be a formula over M such that any realization of p is a tuple from $\varphi(\mathfrak{C})$. Then, $p \upharpoonright \varphi(M)$ implies p.*

Proof. Let A be a finite set such that φ is over A and let $\psi(\bar{v}, \bar{b}) \in p$ be such that $\psi(\mathfrak{C}, \bar{b})$ is a set of tuples from $D = \varphi(\mathfrak{C})$. By Proposition 3.3.3 there is a formula $\theta(\bar{v}, \bar{a})$ equivalent to $\psi(\bar{v}, \bar{b})$, where $\bar{a} \subset D$ and $\theta(\bar{v}, \bar{w})$ is over A. Since M is a model there is such an $\bar{a}$ from $\varphi(M)$. In other words there is a formula in $p \upharpoonright \varphi(M)$ (namely $\theta(\bar{v}, \bar{a})$) which implies $\psi(\bar{v}, \bar{b})$. This proves the corollary.

Much more can be said about the relationships between definability, independence and stationarity (see Section 5.1) but this basic foundation is sufficient to prove the results in the next chapter and the remainder of this one.

Historical Notes. Morley rank and the concept of a totally transcendental theory were developed by Morley in [Mor65], where we also find the notion of a Morley sequence. Definability of types was introduced by Shelah [She71],

although not in the setting of a t.t. theory. Overall, the treatment of t.t. theories given here was motivated by properties of the forking dependence relation on stable theories as proved by Shelah; [She90] is the definitive source.

Exercise 3.3.1. Let T be the theory in the language with two binary relations E_1, E_2 which says that the E_i's define equivalence relations with infinitely many infinite classes and no finite classes, E_2 refines E_1 and each E_1-class contains infinitely many E_2-classes. Compute $MR(x = x)$.

Exercise 3.3.2. Let T be the theory in the first exercise, M a model of T and b an element such that $E_1(b, a)$ for some $a \in M$, but b is not E_2-equivalent to any element of M. Show that there is a defining scheme d over a for $p = tp(b/M)$. What are dE_1 and dE_2?

Exercise 3.3.3. Prove (i) of Lemma 3.3.2.

Exercise 3.3.4. Write out the proof that Morley rank independence satisfies properties (1)-(3) and (5)-(7) in the definition of a freeness relation.

Exercise 3.3.5. Let $\psi_0(x)$ and $\psi_1(x)$ be formulas in some complete theory. Show that $MR(\psi_0 \vee \psi_1) = \max\{MR(\psi_0), MR(\psi_1)\}$.

Exercise 3.3.6. Let X be a definable set of Morley rank $\alpha < \infty$ in the universal domain of some theory. Show that $\deg(X) = 1$ if and only if for all definable $Y \subset X$, $MR(Y) < \alpha$ or $MR(X \setminus Y) < \alpha$.

Exercise 3.3.7. Show that for T a countable complete theory, T is $\aleph_0-$stable if and only if $MR(x = x) < \infty$. (See Remark 3.3.2.)

Exercise 3.3.8. Let $\mathfrak{C}$ be the universal domain of a t.t. theory and let A be a subset of $\mathfrak{C}$. Let $T = Th(\mathfrak{C}_A)$, the theory of $\mathfrak{C}$ in the language with a constant symbol for each element of A. Show that T is also t.t.

Exercise 3.3.9. Prove: If T is t.t., then for all sets A and B there is $C \subset B$ such that $A \underset{C}{\downarrow} B$ and $|C| \leq |A|$.

Exercise 3.3.10. Show that if T is t.t., I is independent and $I \downarrow A$, then I is $A-$independent.

Exercise 3.3.11. Let T be a complete theory, M an $\aleph_0-$saturated model of T and let $CB(-)$ denote Cantor-Bendixson rank computed in $S_n(M)$. Prove that for all $n-$types p over M, $MR(p) \geq CB(p)$.

Exercise 3.3.12. Let I be an infinite indiscernible set in a t.t. theory. Show that there is a type $p \in S(I)$ such that $\bar{b}$ realizes p if and only if $I \cup \{\bar{b}\}$ is indiscernible. Conclude that in a $\kappa-$saturated model M any infinite set of indiscernibles is contained in an indiscernible set $J \subset M$ of cardinality κ.

Exercise 3.3.13. Suppose that T is a t.t. theory, $tp(a/A)$ and $tp(b/A \cup \{a\})$ are both stationary. Then $tp(ba/A)$ is stationary.

Exercise 3.3.14. Give an example of a t.t. theory and a $p \in S_1(\mathfrak{C})$ which is a free extension of $p \upharpoonright \emptyset$ but not definable over $\emptyset$.

Exercise 3.3.15. Let T be a t.t. theory, $\varphi(u, v)$ a formula and $A = \{a_i : i < \omega\}$ a set such that $\models \varphi(a_i, a_j)$ if and only if $i \leq j$. Then A is finite. (See Lemma 5.1.6 for a proof of this property in stable theories.)

Exercise 3.3.16. Suppose that T is t.t., $p \in S(A)$ is a stationary type and I and J are Morley sequences over A in p with $|I| = |J|$. Then there is an automorphism of $\mathfrak{C}$ fixing A and mapping I onto J. Also, if f is an elementary map from A onto A' and I' is a Morley sequence in $p' = f(p)$ with $|I'| = |I|$, then f can be extended to an elementary map g taking I onto I'.

Exercise 3.3.17. Let T be the theory of infinite vector spaces over a field (formulated in the usual language so that T is strongly minimal). Show that for all tuples $\bar{a}$ and sets A, $tp(\bar{a}/A)$ is stationary.

Exercise 3.3.18. Let T be an uncountably categorical theory, M a model of T, p, $q \in S(M)$ strongly minimal types and a a realization of p. Then there is a realization b of q which is interalgebraic with a over M.

Exercise 3.3.19. Complete the proof of Lemma 3.3.8(i).

Exercise 3.3.20. Prove Corollary 3.3.2.

Exercise 3.3.21. Prove Corollary 3.3.4.

Exercise 3.3.22. Prove Corollary 3.3.6.

3.4 The Baldwin-Lachlan Theorem

In this section we prove that an uncountably categorical theory has 1 or $\aleph_0$ many countable models. This is the Baldwin-Lachlan proof of a conjecture due to Vaught. We will actually prove the following stronger result.

Theorem 3.4.1 (Baldwin-Lachlan). *Let T be a countable theory which is uncountably categorical but not $\aleph_0$-categorical. Then, T has $\aleph_0$ many countable models. Moreover,*

(i) If M is a countable model and $\varphi(v, \bar{a})$ is a strongly minimal formula with $tp(\bar{a})$ isolated, and $\bar{b}$ is a sequence from M with $tp(\bar{b}) = tp(\bar{a})$, then

$$\dim(\varphi(M, \bar{a})/\bar{a}) = \dim(\varphi(M, \bar{b})/\bar{b}).$$

(ii) Every model of T is homogeneous.

Throughout this section

- T denotes a countable uncountably categorical theory;
- $\varphi(v, \bar{x})$ is a formula and $\bar{a}^*$ is a tuple with $\varphi(v, \bar{a}^*)$ strongly minimal and
- $q = tp(\bar{a}^*)$ isolated;
- when $N \models T$ and $\bar{b} \subset N$ realizes q, $\mathrm{Dim}_{\bar{b}}(N)$ denotes $\dim(\varphi(N, \bar{b})/\bar{b})$.

(We do not assume here that T is not $\aleph_0$−categorical.) By Corollary 3.1.4(i), any model M is prime over $\varphi(M, \bar{a}) \cup \bar{a}$, where $\bar{a}$ is any realization of q in M. In fact M is prime over $I \cup \bar{a}$, for I any basis for $\varphi(M, \bar{a})$ over $\bar{a}$.

The results in the proof of Morley's Categoricity Theorem are enough to prove the requisite upper bound:

Lemma 3.4.1. *T has countably many countable models.*

Proof. Let M and N be countable models. By Corollary 3.1.4(i), M and N are isomorphic if there are $\bar{a} \subset M$ and $\bar{b} \subset N$ realizing q such that $\mathrm{Dim}_{\bar{a}}(M) = \mathrm{Dim}_{\bar{b}}(N)$. Since $\{\, \mathrm{Dim}_{\bar{c}}(M') \; : \; M' \text{ is a countable model and } \bar{c}$ realizes q in $M' \,\}$ is countable, T has countably many countable models.

For convenience all parts of Theorem 3.4.1 are stated for theories which are not categorical in $\aleph_0$, however both (i) and (ii) are true for *all* uncountably categorical theories. (Theorem 3.4.1(i) is true for theories which are also $\aleph_0$−categorical since $\mathrm{Dim}_{\bar{a}}(M) = |M|$ for any model M and $\bar{a} \in q(M)$. Part (ii) is proved for *all* uncountably categorical theories in Lemma 3.4.10 below.)

The next lemma ties categoricity in $\aleph_0$ to properties of strongly minimal sets in uncountably categorical theories.

Lemma 3.4.2. *The following are equivalent:*

(1) *T is $\aleph_0$−categorical.*
(2) *For $\bar{a}$ a realization of q, $acl(A) \cap \varphi(\mathfrak{C}, \bar{a})$ is finite for all finite sets $A \supset \bar{a}$.*
(3) *For all $M \models T$ and all $\bar{a}$ realizing q in M, $\mathrm{Dim}_{\bar{a}}(M)$ is infinite.*

Proof. Assume (1), let $\bar{a}$ be a realization of q and $A \supset \bar{a}$ a finite set. Since there are only finitely many formulas over A in one free variable, $acl(A) \cap \varphi(\mathfrak{C}, \bar{a})$ must be finite, proving (2). Assume (2), let $M \models T$, $\bar{a} \subset M$ a realization of q and I be a basis for $\varphi(M, \bar{a})$ over $\bar{a}$. Since $\varphi(M, \bar{a})$ is infinite, (2) forces I to be infinite. Thus (3) holds.

The proof that (3) $\Longrightarrow$ (1) is virtually identical to the proof of Theorem 3.1.2. For M and N countable models and $\bar{a}$, $\bar{b}$ realizations of q in M, N, respectively, let I be a basis for $\varphi(M, \bar{a})$ and J a basis for $\varphi(N, \bar{b})$. By assumption, $|I| = |M| = |N| = |J|$, so there is an elementary map f taking $I \cup \bar{a}$ onto $J \cup \bar{b}$. Since M is prime over $I \cup \bar{a}$ and N is prime over $J \cup \bar{b}$, f extends to an isomorphism of M onto N. This proves the $\aleph_0$−categoricity of T.

- **From hereon we assume that T is not $\aleph_0$−categorical.**

By the previous lemma, the prime model M of T contains a realization $\bar{a}$ of q in M for which $\mathrm{Dim}_{\bar{a}}(M)$ is finite. (In fact, since M is homogeneous, $\mathrm{Dim}_{\bar{b}}(M)$ is finite for all $\bar{b}$ from M.) Let I be a (finite) basis for $\varphi(M, \bar{a})$ over $\bar{a}$, and $p \in S(I \cup \bar{a})$ the unique nonalgebraic completion of $\varphi(v, \bar{a})$ over this set. Since p is not realized in M it must be nonisolated. No previously set conditions are invalidated by incorporating I into $\bar{a}$, hence we can assume that the unique nonalgebraic completion of $\varphi(v, \bar{a})$ in $S(\bar{a})$ is nonisolated. Notice that the corresponding behavior carries over to other realizations of q, so we can require

- for any $\bar{b}$ realizing q the nonalgebraic completion of $\varphi(v, \bar{b})$ in $S(\bar{b})$ is non-isolated.

The assumption that T is not $\aleph_0$−categorical is first used to prove

Lemma 3.4.3. *If M is a countable model and there is an $\bar{a}$ from M such that $\mathrm{Dim}_{\bar{a}}(M)$ is infinite, then M is saturated and not prime over a finite set.*

Proof. Suppose that M and $\bar{a}$ are as hypothesized and let N be a countable saturated model. Repeating he proof of $(3) \Longrightarrow (1)$ in the last lemma produces an isomorphism from M onto N. Furthermore, since T is not $\aleph_0$−categorical, the countable saturated model cannot be prime over a finite set.

Remark 3.4.1. If M is a countable model and there is some $\bar{a} \subset M$ realizing q such that $\mathrm{Dim}_{\bar{a}}(M)$ is finite, then $\mathrm{Dim}_{\bar{b}}(M)$ is finite for all $\bar{b}$ realizing q in M.

It will be important in this section to understand the behavior of non-isolated complete strongly minimal types. The next lemma sheds much light on the situation. Notice that the theory is not required to be uncountably categorical here.

Lemma 3.4.4. *Let T_0 be a complete theory, B a set and θ a strongly minimal formula over B. Let $p \in S(B)$ be the unique nonalgebraic type in $S(B)$ containing θ.*
   (i) Then, p is isolated if and only if $\mathrm{acl}(B) \cap \theta(\mathfrak{C})$ is finite.
   (ii) If p is nonisolated and $A \supset B$, then the unique nonalgebraic extension of p in $S(A)$ is nonisolated.

Proof. Since (ii) follows immediately from (i), we only need to prove the first part. Assuming that $\mathrm{acl}(B) \cap \theta(\mathfrak{C}) = X$ is finite, there is a formula ψ over B such that X is $\psi(\mathfrak{C})$. Since there is a unique nonalgebraic element of $S(B)$ containing θ, p is isolated by $\theta \wedge \neg\psi$.
   Conversely, suppose $\mathrm{acl}(B) \cap \theta(\mathfrak{C}) = X$ is infinite. To prove that p is nonisolated it suffices to show that any $\psi \in p$ is satisfied by an element of X. Let

$$\Sigma = \{\, \sigma : \sigma \text{ is an algebraic formula over } B \,\}.$$

Since there is a unique nonalgebraic element of $S(B)$ containing θ the set of formulas $\{\theta \wedge \neg\psi\} \cup \{\neg\sigma : \sigma \in \Sigma\}$ is inconsistent. By compactness there are $\sigma_0, \ldots, \sigma_n \in \Sigma$ such that $\models \forall v(\theta(v) \wedge \bigwedge_{i \leq n} \neg\sigma_i(v) \longrightarrow \psi(v))$. Since X is infinite it contains an element satisfying $\bigwedge_{i \leq n} \neg\sigma_i$, hence ψ. This proves the lemma.

We proved in Lemma 2.3.1 that a theory with finitely many but more than one countable model has a countable model which is prime over a finite set, not saturated, and realizes every complete $n-$type over $\emptyset$. In the next lemma we show that such a model also exists under the (apparently) weaker assumption that (i) of the theorem fails.

Lemma 3.4.5. *Suppose that M is a countable model of T containing realizations $\bar{a}$ and $\bar{b}$ of q such that $\mathrm{Dim}_{\bar{a}}(M) \neq \mathrm{Dim}_{\bar{b}}(M)$. Then M realizes every complete $n-$type over $\emptyset$, for all n, and M is prime over some finite set.*

Proof. For some $\bar{a}$ from M realizing q, $\mathrm{Dim}_{\bar{a}}(M)$ is finite. Furthermore, $\mathrm{Dim}_{\bar{c}}(M)$ must be finite for any realization $\bar{c}$ of q in M by Remark 3.4.1. Fix $\bar{a}$ realizing q in M such that $i = \mathrm{Dim}_{\bar{a}}(M)$ is minimal and let I be a basis for $\varphi(M, \bar{a})$. By Corollary 3.1.4(i) M is prime over $I \cup \bar{a}$. The main step in the proof is
    Claim. For every n there is a $\bar{c}$ with $\mathrm{Dim}_{\bar{c}}(M) \geq n$.
    Assume, to the contrary, that there is a bound on these dimensions. Let $\bar{b}$ be a realization of q in M such that $j = \mathrm{Dim}_{\bar{b}}(M)$ is maximal and let J be a basis for $\varphi(M, \bar{b})$. By hypothesis, $i < j$. Let J' be a subset of J of cardinality i and $N \subset M$ a prime model over $J' \cup \bar{b}$. Since there is an elementary map from $I \cup \bar{a}$ onto $J' \cup \bar{b}$, M and N are isomorphic. Thus, there is a sequence $\bar{d}$ from N with $\mathrm{Dim}_{\bar{d}}(N) = j$. Since T does not have any Vaughtian pairs $\varphi(M, \bar{d}) \not\subseteq N$, in fact, $\mathrm{Dim}_{\bar{d}}(M) > \mathrm{Dim}_{\bar{d}}(N)$. This contradicts the maximality assumption on j to prove the claim.
    Now let p be an arbitrary element of $S(\emptyset)$, $\bar{c}$ a realization of p and N a prime model over $\bar{c}$. Let $\bar{d}$ be a realization of q in N, in which case N is prime over $\varphi(N, \bar{d}) \cup \bar{d}$. There is a finite $\bar{d}-$independent set $J \subset \varphi(N, \bar{d})$ such that $r = tp(\bar{c}/J \cup \bar{d})$ is isolated. By the claim there is a $\bar{b}$ in M realizing q and a $\bar{b}-$independent set $K \subset \varphi(M, \bar{b})$ of cardinality $|J|$. Hence there is an elementary map f taking $\bar{d}$ to $\bar{b}$ and J onto K. Since $f(r)$ is isolated it is realized by some $\bar{c}'$ in M. This sequence $\bar{c}'$ is a realization of p in M.

The combination of the previous lemma and the next proposition will lead quickly to a contradiction.

Proposition 3.4.1. *Let T' be any uncountably categorical theory and $\bar{a}$ any tuple. Then there is a k such that whenever $\{\bar{b}_0, \ldots, \bar{b}_k\}$ is independent, there is some $\bar{b}_i$ independent from $\bar{a}$.*

The bound k obtained in the proposition is given the formal name "pre-weight":

Definition 3.4.1. *Let T' be t.t., A a set and $p \in S(A)$. The pre-weight of p, denoted $pwt(p)$, is*

$$\sup\{\,\kappa: \text{ there is an } \bar{a} \text{ realizing } p \text{ and } A-\text{independent set}$$
$$I \text{ of cardinality } \kappa \text{ such that } \bar{b} \in I \implies \bar{a} \underset{A}{\not\downarrow} \bar{b}\,\}.$$

We may write $pwt(\bar{a}/A)$ for $pwt(tp(\bar{a}/A))$.

(When the supremum of a class X of cardinals does not exist we write $\sup X = \infty$ and extend the order on cardinals so that $\kappa < \infty$ for all cardinals κ.)

We will prove

Proposition 3.4.2. *If T' is uncountably categorical then every complete type has finite pre-weight.*

Remark 3.4.2. Let T' be t.t. and $p = tp(\bar{a}/A)$. If I is an A-independent set such that each $\bar{b} \in I$ depends on $\bar{a}$ over A, then I is finite. (Assuming there is an infinite such I let $J \subset I$ be a finite set such that $\bar{a}$ is independent from $I \cup A$ over $J \cup A$. Then any $\bar{b} \in I \setminus J$ is independent from $\bar{a}$ over A; contradiction.) That there is a finite bound to $|I|$, as I ranges over all such sets, is not immediately clear.

Proposition 3.4.1 follows immediately from Proposition 3.4.2 by letting $k = pwt(tp(\bar{a}))$. Pre-weight (and weight) will be studied extensively in stable theories (after replacing Morley rank independence by forking independence) in Sections 5.6 and 6.3. Proposition 3.4.2 is implicit in Theorem 5.6.1.

Lemma 3.4.6. *Let T be a t.t. theory and p, q complete types with q a free extension of p. Then $pwt(p) \leq pwt(q)$.*

Proof. Suppose that $p \in S(A)$, $q \in S(B)$, $\bar{a}$ realizes p and I is an A-independent set such that each $\bar{b} \in I$ depends on $\bar{a}$ over A. Let $\bar{a}'$ realize q. Let J be a set such that $\{\bar{a}\} \cup I$ is conjugate to $\{\bar{a}'\} \cup J$ over A and $\{\bar{a}'\} \cup J$ is independent from B over A. Using standard properties of independence the reader can show that J is B-independent and each $\bar{b} \in J$ depends on $\bar{a}'$ over B. Thus, $pwt(q) \geq |J| = |I|$. This inequality is true for any A-independent set I such that each element of I depends on $\bar{a}$ over A, hence $pwt(q) \geq pwt(p)$, to complete the proof.

The proof of Proposition 3.4.2 requires some preliminary lemmas involving the following notion.

Definition 3.4.2. *For A, B and C sets in a t.t. theory we say that A is dominated by B over C if for all sets D,*

$$D \underset{C}{\downarrow} B \implies D \underset{C}{\downarrow} A.$$

Notice that for all sets A and B, $acl(A)$ is dominated by A over B.

Lemma 3.4.7. *Let T' be a t.t. theory, M an $\aleph_0-$saturated model of T' and A a set which is atomic over $M \cup B$. Then $A \cup B$ is dominated by B over M.*

Proof. Without loss of generality, $A = \bar{a}$ and $B = \bar{b}$ are finite. Suppose, towards a contradiction, that there is a $\bar{c}$ such that

$$\bar{c} \underset{M}{\,\downarrow\,} \bar{b} \text{ and } \bar{c} \underset{M}{\,\not\downarrow\,} \bar{a} \cup \bar{b}.$$

Then (by symmetry and the transitivity of independence) $\bar{a}$ depends on $\bar{c}$ over $M \cup \bar{b}$. Let $A' \subset M$ be a finite set such that $tp(\bar{a}/M \cup \bar{b})$ is isolated over $A' \cup \bar{b}$, and $\bar{a}$ depends on $\bar{c}$ over $A' \cup \bar{b}$. We can require, furthermore, that $\bar{b}$ is independent from M over A' and $tp(\bar{b}/A')$ is stationary. Let $\varphi(\bar{x})$ be a formula over $A' \cup \bar{b}$ which isolates $tp(\bar{a}/M \cup \bar{b})$ and observe that $MR(\varphi) = MR(\bar{a}/M \cup \bar{b}) = MR(\bar{a}/A' \cup \bar{b}) = \alpha$. Let $\psi(\bar{x}, \bar{c}, \bar{b})$ be a formula in $tp(\bar{a}/A' \cup \bar{b} \cup \bar{c})$ which implies φ and has Morley rank $< \alpha$.

Since $\bar{b}$ is independent from $\bar{c}$ over M and $tp(\bar{b}/A')$ is stationary, $tp(\bar{b}/M \cup \bar{c})$ does not split over A' (by Theorem 3.3.1(i)). Since M is $\aleph_0-$saturated it contains a realization $\bar{c}'$ of $tp(\bar{c}/A')$, in fact, $tp(\bar{c}'/A' \cup \bar{b}) = tp(\bar{c}/A' \cup \bar{b})$. By the conjugacy over A of $\psi(\bar{x}, \bar{c}, \bar{b})$ and $\psi(\bar{x}, \bar{c}', \bar{b})$, $\psi(\bar{x}, \bar{c}', \bar{b})$ is consistent, implies φ and has Morley rank $< \alpha$. Since $\bar{c}'$ is from M this contradicts that φ isolates a complete type over $M \cup \bar{b}$ of Morley rank α.

Corollary 3.4.1. *Let M be a countable $\aleph_0-$saturated model of an uncountably categorical theory, φ a strongly minimal formula over M and $\bar{a}$ a tuple of elements. Then there is a sequence $\bar{c} \subset \varphi(\mathfrak{C})$ such that $\bar{a} \cup \bar{c}$ is dominated by $\bar{c}$ over M and $\bar{a} \cup \bar{c}$ is dominated by $\bar{a}$ over M.*

Proof. By Exercise 3.1.20, $T' = Th(M_M)$, the theory of M in the language with constant symbols for the elements of M, is also uncountably categorical. By Corollary 3.1.4(i) there is a tuple $\bar{c}$ from $\varphi(\mathfrak{C})$ such that $tp(\bar{a}/\bar{c})$ is isolated in T'. In fact, $\bar{c}$ can be chosen in the prime model over $\bar{a}$ (in T'), hence $tp(\bar{c}/\bar{a})$ in T' is also isolated. Returning to the original theory $T = Th(M)$, $tp(\bar{a}/M \cup \bar{c})$ and $tp(\bar{c}/M \cup \bar{a})$ are isolated. By Lemma 3.4.7, $\bar{a} \cup \bar{c}$ is dominated by $\bar{c}$ over M and $\bar{a} \cup \bar{c}$ is dominated by $\bar{a}$ over M, proving the corollary.

Two more facts about pre-weight will be enough to prove the finiteness of pre-weight in uncountably categorical theories.

Lemma 3.4.8. *Let T be a t.t. theory.*
   (i) Let D be a strongly minimal set over a set A and $\bar{c}$ a sequence from D. Then $pwt(\bar{c}/A) = \dim(\bar{c}/A)$.
   (ii) If $\bar{a}$ is dominated by $\bar{b}$ over A, then $pwt(\bar{a}/A) \leq pwt(\bar{b}/A)$.

Proof. (i) It is easy to see that $pwt(\bar{c}/A) \geq \dim(\bar{c}/A)$. That $pwt(\bar{c}/A) \leq \dim(\bar{c}/A)$ is proved by induction on $\dim(\bar{c}/A)$ (uniformly for all A). Let I be

a nonempty A–independent set such that any $\bar{b} \in I$ is A–dependent on $\bar{c}$. Let $\bar{b}_0 \in I$ and $I' = I \setminus \{\bar{b}_0\}$. By Lemma 3.3.4, $\dim(\bar{c}/A \cup \bar{b}_0) < \dim(\bar{c}/A)$. Since I' is independent over $A \cup \bar{b}_0$ and any $\bar{b} \in I'$ depends on $\bar{c}$ over $A \cup \bar{b}_0$, the inductive hypothesis implies $pwt(\bar{c}/A \cup \bar{b}_0) \leq \dim(\bar{c}/A \cup \bar{b}_0)$. Thus, $|I| = |I'| + 1 \leq pwt(\bar{c}/A \cup \bar{b}_0) + 1 \leq \dim(\bar{c}/A \cup \bar{b}_0) + 1 \leq \dim(\bar{c}/A)$. This proves (i).

(ii) Let I be an A–independent set such that each $\bar{d} \in I$ depends on $\bar{a}$ over A. Since $\bar{a}$ is dominated by $\bar{b}$ over A, each $\bar{d} \in I$ must depend on $\bar{b}$ over A. Thus, $|I| \leq pwt(\bar{b}/A)$.

Proof of Proposition 3.4.2. Let A be a set and $p \in S(A)$ a complete type. Let M be an $\aleph_0$–saturated model containing A and $q \in S(M)$ a free extension of p. By Lemma 3.4.6 $pwt(p) \leq pwt(q)$, hence it suffices to show that $pwt(q)$ is finite. Let $\bar{a}$ be a realization of q, φ a strongly minimal formula over M and $\bar{c}$ a (finite) tuple from $\varphi(\mathfrak{C})$ such that $\bar{a} \cup \bar{c}$ is dominated by $\bar{c}$ over M (see Lemma 3.4.6). By Lemma 3.4.8, $pwt(q) = pwt(\bar{a}/M) \leq pwt(\bar{c}/M) = \dim(\bar{c}/M)$, proving the proposition.

As noted above, this also proves Proposition 3.4.1.

The following lemma draws together several results of the section to prove Theorem 3.4.1(i).

Lemma 3.4.9. *Let M be a countable model of T. Then for all sequences $\bar{a}$, $\bar{b}$ from M realizing q,*
$$\mathrm{Dim}_{\bar{a}}(M) = \mathrm{Dim}_{\bar{b}}(M).$$

Proof. Suppose the lemma fails for M. By Lemma 3.4.5 M is prime over a finite sequence $\bar{c}$ and realizes every element of $S(\emptyset)$. Let k $(= pwt(\bar{c}/\emptyset))$ be as guaranteed by Proposition 3.4.1 for $\bar{c}$. Since T is not $\aleph_0$–categorical there is a nonisolated $p \in S(\emptyset)$. Let $\{\bar{b}_0, \ldots, \bar{b}_k\}$ be an independent set of realizations of p and $r = tp(\bar{b}_0, \ldots, \bar{d}_k)$. Since r is realized in M we may as well assume that $\{\bar{b}_0, \ldots, \bar{b}_k\} \subset M$. By our choice of k there is some i such that $\bar{c} \perp \bar{b}_i$. However, since M is prime over $\bar{c}$, $tp(\bar{b}_i/\bar{c})$ is isolated, contradicting Lemma 3.3.10 (since p is nonisolated). This proves the lemma.

Remark 3.4.3. This lemma is also true for an uncountably categorical theory T' which is $\aleph_0$–categorical. (If M is a countable model of T' and $\varphi'(x, \bar{a})$ is a strongly minimal formula over M then for all $\bar{b}$ from M realizing $tp(\bar{a})$, $\dim(\varphi'(M, \bar{b})/\bar{b}) = \aleph_0$.)

From Lemma 3.4.9 we quickly obtain the main assertion of the Baldwin-Lachlan Theorem:

Corollary 3.4.2. *T has $\aleph_0$ many countable models.*

Proof. Let $\bar{a}$ be a realization of q and for each $k < \omega$ let I_k be a subset of $\varphi(\mathfrak{C}, \bar{a})$ of dimension k. We have assumed that the unique nonalgebraic completion of $\varphi(\bar{x}, \bar{a})$ in $S(\bar{a})$ is nonisolated. For each $k < \omega$ let M_k be a prime model over $I_k \cup \bar{a}$.

Claim. For $l \neq k < \omega$, $M_l \not\cong M_k$.

By Lemma 3.4.4(ii), the unique nonalgebraic extension of $\varphi(\bar{x}, \bar{a})$ in $S(I_j \cup \bar{a})$ is nonisolated, hence not realized in M_j. That is, $Dim_{\bar{a}}(M_j) = j$ for each $j < \omega$. Suppose, towards a contradiction, that for some $k \neq l$ there is an isomorphism f from M_l onto M_k. Then, for $\bar{b} = f(\bar{a})$, $Dim_{\bar{b}}(M_k) = l$, contradicting Lemma 3.4.9 to prove the claim and the corollary.

It remains only to prove (ii) of the Baldwin-Lachlan Theorem. We showed in Corollary 3.1.5 that for T' an uncountably categorical countable theory and $M \models T'$, if $|M|$ is uncountable and regular then M is saturated. Included in the next lemma is the removal of the restriction to regular cardinals in that earlier corollary.

Lemma 3.4.10. *If T' is an uncountably categorical countable theory then every model of T' is homogeneous and every uncountable model of T' is saturated.*

Proof. The reader should first verify:

(∗) Let $M \supset N$ be models, φ' a strongly minimal formula defined over $A \subset N$, I a basis for $\varphi'(N)$ over A and J a basis for $\varphi'(M)$ over N. Then $I \cup J$ is a basis for $\varphi'(M)$ over A, and $\dim(\varphi'(M)/A) = \dim(\varphi'(N)/A) + \dim(\varphi'(M)/N)$.

Let M be a model of T'. Certain parts of the proof are handled differently depending on whether M is countable or uncountable, so we split the argument into two (very similar) cases.

First suppose M to be uncountable. Let $A \subset M$, with $|A| < |M| = \kappa$, and let f be an elementary map with $f(A) = B \subset M$. For $N_0 \subset M$ a prime model over A we can assume f to be an isomorphism from N_0 onto a model $N_1 \subset M$ which is prime over B. There is an isolated type $q' \in S(\emptyset)$ and a $\varphi'(\bar{x}, \bar{y})$ such that for any realization $\bar{c}$ of q', $\varphi'(\bar{x}, \bar{c})$ is strongly minimal. Let $\bar{a}'$ be a realization of q' in N_0 and $\bar{b}' = f(\bar{a}')$. Since f is an isomorphism $\dim(\varphi'(N_0, \bar{a}')) = \dim(\varphi'(N_1, \bar{b}')) = \lambda$. Also $\dim(\varphi'(M, \bar{a}')/\bar{a}') = \dim(\varphi'(M, \bar{b}')/\bar{b}') = \kappa$ by Corollary 3.1.4. By (∗), if I_0 is a basis for $\varphi'(M, \bar{a}')$ over N_0 and I_1 is a basis for $\varphi'(M, \bar{b}')$ over N_1, then $\kappa = \lambda + |I_0| = \lambda + |I_1|$. Since $\lambda < \kappa$, $|I_0| = |I_1|$. Thus, by extending f we can assume f to be an elementary map from $N_0 \cup I_0$ onto $N_1 \cup I_1$. There is a submodel M' of M which is prime over $N_0 \cup I_0$. Since I_0 is a basis for $\varphi'(M, \bar{a}')$ over N_0, $\varphi'(M', \bar{a}') = \varphi'(M, \bar{a}')$. Since T' has no Vaughtian pairs, $M' = M$; i.e., M is prime over $N_0 \cup I_0$. Hence f can be extended to an elementary embedding of M into itself so that $f(M) \supset \varphi'(M, \bar{b}')$. Again, since T' has no Vaughtian pairs, this embedding must be onto M, proving that M is homogeneous.

Now suppose that M is countable. The proof is virtually identical to the uncountable case, but individual steps may be justified differently. Certainly

M is homogeneous if T' is $\aleph_0$−categorical, so we can assume that T' is not $\aleph_0$−categorical. Let A be a finite subset of M and f an elementary map from A onto $B \subset M$. Choose N_0, N_1, φ', $\bar{a}'$ and $\bar{b}'$ exactly as above. Since N_0 and N_1 are isomorphic $\dim(\varphi'(N_0, \bar{a}')) = \dim(\varphi'(N_1, \bar{b}')) = k$, and $k < \aleph_0$ by Lemma 3.4.3. By Lemma 3.4.9 $\dim(\varphi'(M, \bar{a}')/\bar{a}') = \dim(\varphi'(M, \bar{b}')/\bar{b}') = \mu$. Again by $(*)$, if I_0 is a basis for $\varphi'(M, \bar{a}')$ over N_0 and I_1 is a basis for $\varphi'(M, \bar{b}')$ over N_1, then $\mu = k + |I_0| = k + |I_1|$. Since k is finite $|I_0| = |I_1|$. From this point on the proof is just like the uncountable case. This proves that every model of T' is homogeneous.

For M an uncountable model of T', M is $\aleph_0$−saturated (see Exercise 3.1.3) and homogeneous, hence M is saturated (by Corollary 2.2.6).

Some of the results in this section will be generalized later to arbitrary $\aleph_0$−stable (in fact, the more general superstable) theories. For example, a countable superstable theory which is not $\aleph_0$−categorical must have infinitely many countable models. This will be proved by establishing Proposition 3.4.1 for superstable theories. We will also show (in Section 5.4) that an $\aleph_0$−stable theory has a saturated model in every cardinality, not just the regular ones.

Historical Notes. All of the main theorems here are due to Baldwin and Lachlan [BL71], although our exposition owes a lot to Lascar [Las86].

Exercise 3.4.1. Suppose that T is an uncountably categorical theory, $N \subset M$ are models of T, and φ, ψ are strongly minimal formulas over N. Show that $\dim(\varphi(M)/N) = \dim(\psi(M)/N)$.

Exercise 3.4.2. Give a quick proof of the Baldwin-Lachlan Theorem assuming that there is a strongly minimal formula over $\emptyset$.

3.5 Introduction to ω−stable Groups

In this section we study ω−stable theories in which the universe is a group under some definable operation. The goal is to elucidate the degree to which the group-theoretic and stability-theoretic properties of these structures influence each other. The importance of this study lies both in the breadth of the class of groups with an ω−stable theory and the manner in which groups arise in a "geometrical" analysis of ω−stable theories. Results pertaining to this second point will be discussed in Sections 4.4 and 4.5.

Definition 3.5.1. *Let T be an ω−stable theory with universal domain $\mathfrak{C}$ such that $(\mathfrak{C}, \cdot)$ is a group for some definable binary operation $\cdot$. Then $(\mathfrak{C}, \cdot)$ is called an ω−stable group. Adopting more standard notation, ω−stable groups will usually be represented by G, H, G', etc.*

Remark 3.5.1. Let T be an ω−stable theory with universal domain $\mathfrak{C}$.

(i) For $(\mathfrak{C}, \cdot)$ to be an ω−stable group it is not necessary for $\cdot$ to be in the language, being a definable operation is enough. Also, there may be definable relations on $\mathfrak{C}$ which are not definable in the group language; i.e., the restriction of $\mathfrak{C}$ to a language containing only a function symbol for $\cdot$.

(ii) Let X be a definable subset of $\mathfrak{C}$ and $\cdot$ a definable operation on X such that $(X, \cdot)$ is a group. The restriction of $\mathfrak{C}$ to X contains no new definable relations, hence $(X, \cdot)$ is an ω−stable group in this restricted universe. Since the definable relations on X in $\mathfrak{C}$ are the same as the definable relations on X in the restriction there is no loss in calling X an ω−stable group without first restricting to X. From hereon, when referring to an ω−stable group G we always leave open the possibility that G is a definable subset of some larger theory.

(iii) The restriction to countable theories (and the consistent use of ω−stable over $\aleph_0$−stable) is purely a convention adopted by the authors in the area. Virtually anything proved here is true in an uncountable t.t. theory with the same justification.

(iv) Here, the term "ω−stable group" only applies to the universal domain of the relevant theory. This is nonstandard. Most authors call the model G an ω−stable group if there is a definable group operation on G and $Th(G)$ is ω−stable. However, we have found our terminology to be more appropriate for the presentation of the material in this book.

Here are some basic examples.

Example 3.5.1. (i) The universal domain of the theory of vector spaces over a fixed field is an ω−stable group.

(ii) The universal domain K of the theory of algebraically closed fields of a fixed characteristic is an ω−stable group under $+$, and $K \setminus \{0\}$ is an ω−stable group under $\cdot$.

(iii) Let M be the direct sum of $\aleph_0$ many copies of the group $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$, $T = Th(M)$ and G the universal domain of T. The reader can show that T is quantifier-eliminable, from which we conclude:

− $2G$ is a strongly minimal set. In fact, $2G$ is a vector space over the field with two elements and there are no definable relations on $2G$ except those defined in the vector space language.
− $MR(G) = 2$.
− T is totally categorical.

(iv) Consider the special case of an abelian group G in the language containing only the group operation $+$ and 0. Macintyre proved (in [Mac71a])

Theorem. *$Th(G)$ is ω−stable if and only if G is of the form $D \oplus H$, where D is divisible and H is of bounded order.*

Given an arbitrary ω−stable abelian group G let G_0 be the restriction of G to a language containing only the group operation and the identity. By

Exercise 3.1.9 G_0 is also an ω−stable group. Thus, the theorem tells us the underlying group structure of G_0 *and* G. Namely, G is a direct sum of a divisible group and a group of bounded order.

Perhaps the most important class of ω−stable groups is the collection of algebraic groups over an algebraically closed field. Here is a summary of the critical concepts.

Let K be an algebraically closed field which we take to be the universal domain of its theory, for simplicity. Let $n < \omega$ and $K[\bar{x}]$ the ring of polynomials in $\bar{x} = (x_1, \ldots, x_n)$ over K. A set $V \subset K^n$ is called an *affine algebraic subset of K^n* if for some $p_1(\bar{x}), \ldots, p_k(\bar{x}) \in K[\bar{x}]$,

$$V = \{\, \bar{a} \in K^n : p_1(\bar{a}) = 0 \wedge \ldots \wedge p_k(\bar{a}) = 0 \,\}.$$

The affine algebraic subsets of K^n form the closed sets in a topology on K^n called the *Zariski topology* on K^n. This topology is *Noetherian*, meaning that it has the descending chain condition on closed sets. (If $V_i \subset K^n$, $i < \omega$, are affine algebraic sets then the ideal generated by the polynomials defining the V_i's is generated by finitely many such polynomials since $K[\bar{x}]$ is Noetherian. Thus, $\bigcap_{i<\omega} V_i$ is the intersection of finitely many of these sets.) Certainly, any affine algebraic set is definable. A Zariski closed set V is called *irreducible* if it cannot be written as $V_1 \cup V_2$, where $V_i \subsetneq V$, $i = 1, 2$, are Zariski closed sets. An irreducible Zariski closed subset of K^n is called an *affine algebraic variety* or simply an *affine variety*. A set $X \subset K^n$ is a *constructible set* if it is a boolean combination of affine algebraic subsets of K^n. The elimination of quantifiers for algebraically closed fields can be stated as: a subset of K^n is definable if and only if it is constructible.

Turning to groups, the set $M_n(K)$ of $n \times n$ matrices over K is definable ($M_n(K)$ is a subset of K^{n^2}). The operations of addition and matrix multiplication on $M_n(K)$ and the determinant function (from $M_n(K)$ into K) are definable over K. Thus, the *general linear group* $\mathrm{GL}_n(K) = \{\, a \in M_n(K) : \det(a) \neq 0 \,\}$ of invertible $n \times n$ matrices over K under multiplication is definable. Also definable is $\mathrm{SL}_n(K)$, the set of elements of $\mathrm{GL}_n(K)$ with determinant 1. In fact, $M_n(K)$, $\mathrm{GL}_n(K)$ and $\mathrm{SL}_n(K)$ are all affine algebraic sets. Most commonly encountered subgroups of $\mathrm{GL}_n(K)$, such as the upper triangular matrices over K, are definable. What is not so obvious is the definability of the projective linear group $\mathrm{PGL}_n(K) = \mathrm{GL}_n(K)/Z$, where Z is the center of the general linear group. This will be verified after our discussion of T^{eq} in the next chapter.

A group H is an *affine algebraic group* if it is a subgroup of $\mathrm{GL}_n(K)$ (for some n) which is also an affine algebraic set. We will not take the time to define an "algebraic group over K", stating only that every affine algebraic group is an algebraic group and every algebraic group is definable. In fact, using much deeper model theory it can be shown that every connected (see below) definable group over K is definably isomorphic to an algebraic group (due to Hrushovski [Hru90b] and, in part, to van den Dries).

Note: $MR(\mathrm{GL}_n(K)) = n^2$, so any affine algebraic group has finite Morley rank.

The interest in ω−stable groups began with the study of the natural examples, for example, vector spaces, algebraically closed fields and affine algebraic groups over algebraically closed fields. In fact, much of what we know about ω−stable groups has been obtained by generalizing results about affine algebraic groups. Such a generalization is facilitated by replacing Zariski topology dimension theory by Morley rank independence. One such result, Zil'ber's Indecomposability Theorem, is proved below. In Zil'ber's Ladder Theorems he shows that any sufficiently complicated uncountably categorical theory contains definable groups and these groups have a significant influence on the structure of the definable subsets (see Section 4.4). These results have been generalized (by Hrushovski and others) to the extent that almost any problem in stability theory involves groups on some level. For instance, groups played a critical role in the proof of Vaught's conjecture for superstable theories of finite rank [Bue93].

Remember: When X is a definable set of Morley rank α, $\deg(X) = 1$ if and only if whenever $Y \subset X$ is definable in M and has Morley rank α, $MR(X \setminus Y) < \alpha$ (see Exercise 3.3.6). Also, when T is t.t. and $\bar{a}$ and $\bar{b}$ are interalgebraic over A, $MR(\bar{a}/A) = MR(\bar{a}\bar{b}/A) = MR(\bar{b}/A)$.

Definition 3.5.2. *Let G be a group, X a set and $\star$ a map from $G \times X$ into X. The triple $(G, X, \star)$ is a* group action *if*

(1) *$e \star x = x$, for e the identity of G and all $x \in X$, and*
(2) *for all $g, h \in G$ and $x \in X$, $(gh) \star x = g \star (h \star x)$.*

Often the $\star$ is dropped from the notation, the group action is denoted (G, X) and $g \star x$ is written gx.

Let (G, X) be a group action.

− *(G, X) is* faithful *if, whenever $g \in G \setminus \{1\}$, $gx \neq x$ for some $x \in X$.*
− *Given $0 \neq k < \omega$, (G, X) is k−transitive if for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$ such that $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq k$, there is a $g \in G$ such that $gx_i = y_i$ for $1 \leq i \leq k$. (G, X) is* sharply k−transitive *if the g in the previous sentence is unique. Equivalently, (G, X) is k−transitive and only 1 fixes k elements of X.*
− *The term* transitive *is used in place of 1−transitive and* regular *replaces* sharply 1−transitive.
− *For $x \in X$ let $stab(x)$, called the* stabilizer *of x, denote $\{ g \in G : gx = x \}$.*
− *The* orbit *of $x \in X$ is $Gx = \{ gx : g \in G \}$.*
− *If $Y \subset X$ and $g \in G$, the* translation *of Y by g is $gY = \{ gy : y \in Y \}$.*

Clearly, the stabilizer of a point is a subgroup of G and if $H = stab(x)$ and $g, h \in G$, then $gH = hH$ if and only if $gx = hx$. This defines a one-to-one correspondence between the left cosets of H and the orbit of x. Thus,

letting G/H denote the set of left cosets of H in G, $|G/H| = |Gx|$; i.e., $[G : stab(x)] = |Gx|$.

We will see that an ω-stable group G acts on the types over G and the definability of types yields definable stabilizer subgroups. First, some terminology about ω-stable groups acting on types.

Let G be an ω-stable group and p a 1-type over the set A. Let p^{-1} denote $\{ \varphi(x^{-1}) : \varphi \in p \}$. For a an element let $ap = \{ \varphi(a^{-1} \cdot x) : \varphi(x) \in p \}$ and $pa = \{ \varphi(x \cdot a^{-1}) : \varphi(x) \in p \}$, both types over $A \cup \{a\}$. Given $a \in A$, ap is also a type over A and, if $p = tp(b/A)$, $ap = tp(a \cdot b/A)$. Since any b and $a^{-1} \cdot b$ are interalgebraic over a the formulas $\varphi(x)$ and $\varphi(a^{-1} \cdot x)$ have the same Morley rank. In fact, since multiplication defines a function, $\varphi(x)$ and $\varphi(a^{-1} \cdot x)$ also have the same Morley degree (see the exercises). Thus, the types p and ap have the same Morley rank and degree. The types ap and pa are called the *left* and *right translates of p by a*, respectively. The term *translation* may be used in place of *left translation*.

Given an ω-stable group G, $a \in G$ and $p \in S_1(G)$, $ap \in S_1(G)$. In fact, left (or right) translation defines a group action of G on $S_1(G)$. By the above comments the action preserves Morley rank.

Definition 3.5.3. *Let G be an ω-stable group and $p \in S_1(G)$. The* left *stabilizer of p, denoted $stab(p)$, is the stabilizer of p under left translation; i.e., $stab(p) = \{ a \in G : ap = p \}$. The* right *stabilizer of p is the stabilizer of p under right translation. By default, the term "stabilizer of p" refers to the left stabilizer of p.*

Lemma 3.5.1. *Let G be an ω-stable group.*

(i) Let $P \subset S_1(G)$ be the collection of all types of some fixed Morley rank. The action of G on P is definable in the sense that for all p, $q \in P$, there is a formula ψ over G such that

$$\text{for all } a \in G, \ ap = q \text{ if and only if } \models \psi(a).$$

There is a formula equivalent to ψ over any set A such that p and q are both definable over A.

(ii) Given $p \in S_1(G)$, $stab(p)$ is a definable subgroup of G and

$$MR(stab(p)) \leq MR(p).$$

Moreover, if p is definable over A, $stab(p)$ is definable over A.

(iii) If p, $q \in S_1(G)$ and q is a right translate of p then $stab(p) = stab(q)$.

Proof. (i) First notice that G acts on P since translation preserves Morley rank. Let p be in P and $\varphi \in p$ be a formula of Morley rank $\alpha = MR(p)$ and degree 1. If $a \in G$, then $\varphi(a^{-1} \cdot x)$ is also a formula of Morley rank α and degree $1 = \deg(p)$. Let q be another element of P. Since every element of P has Morley rank α, q is ap if and only if $\varphi(a^{-1} \cdot x) \in q$. Let $\varphi'(x, y) = \varphi(y^{-1} \cdot x)$ and let $\psi(y)$ be the formula defining $q \restriction \varphi'$. Then, for all $a \in G$, $q = ap$ if

and only if $\models \psi(a)$. From an inspection of the proof we can choose ψ to be over any set A such that p and q are both definable over A.

(ii) As a special case of (i) notice that $\{\, a \in G :\ ap = p \,\} = stab(p)$ is definable. Furthermore, by the proof of (i), if p is definable over the finite set $A \subset G$ then $stab(p)$ is definable over A.

(iii) Left to the reader as a short exercise.

Notation. Given an ω−stable group G we may write ab for $a \cdot b$ when there is no danger of confusing ab with the pair of elements (a, b).

It is frequently handy to work with restrictions of elements of $S_1(G)$ when showing that one type is a translate of another. The necessary tool is

Corollary 3.5.1. *Let G be an ω−stable group, $p,\, q \in S_1(G)$ and $a \in G$. Suppose that p and q are definable over A. Then,*

(1) *$q = ap$ if and only if*
(2) *there is a b realizing $p \restriction A$ such that b is independent from a over A, ab realizes $q \restriction A$ and ab is independent from a over A.*

Proof. Interpolating a few more equivalences will make the proof easy.

Claim.   (a) $q = ap$
$\Longleftrightarrow$ (b) for all sets $B \supset A \cup \{a\}$, $q \restriction B = ap \restriction B$;
$\Longleftrightarrow$ (c) there is a set $B \supset A \cup \{a\}$, $q \restriction B = ap \restriction B$.

That (a) implies (b) is clear from the definition of translation and (b) $\Longrightarrow$ (c) holds trivially. Suppose (c) holds and $B \supset A \cup \{a\}$ is such that $q \restriction B = ap \restriction B$. Let $MR(p) = \alpha$ and let $\varphi \in p \restriction A$ be a formula of Morley rank α and degree 1. Since $ap \restriction B = q \restriction B$, $\alpha = MR(p \restriction B) = MR(q \restriction B)$ and $\varphi(a^{-1}x) \in q$. Since q is definable over A, $MR(q)$ must also be α. As in the proof of Lemma 3.5.1(i), ap is the unique element of $S_1(G)$ which contains $\varphi(a^{-1}x)$ and has Morley rank α. Thus $ap = q$, proving the claim.

Turning to the main assertion of the corollary, (1) implies (2) is just a matter of unraveling the notation. Now assume that a and b meet the conditions in (2). Let $B = A \cup \{a\}$. Since p and q are both definable over A and both b and ab are independent from a over A, b realizes $p \restriction B$ and ab realizes $q \restriction B$. Since $a \in B$, $tp(a \cdot b/B) = ap \restriction B$; i.e., $ap \restriction B = q \restriction B$. By the claim $ap = q$, proving the corollary.

The most fundamental property of an ω−stable group is

Proposition 3.5.1. *An ω−stable group G has the descending chain condition on definable subgroups. That is, if $G = H_0 \supset H_1 \supset \ldots \supset H_i \supset \ldots$ is a chain of definable subgroups of G, there is an $n < \omega$ such that $H_m = H_n$ for all $m \geq n$.*

Proof. Let I be the collection of pairs (α, m) where α is an ordinal and $1 \leq m < \omega$. Well-order I lexicographically; i.e., $(\alpha, m) < (\beta, k)$ if $\alpha < \beta$ or $\alpha = \beta$ and $m < k$. Associate with any formula ψ the pair $(MR(\psi), \deg(\psi)) \in I$. Let $H \subset K$ be subgroups of G defined by ψ and φ, respectively. If $H \neq K$ there is an $a \in K$ such that $a\psi$ defines a coset $aH \neq H$. Since ψ and $a\psi$ have the same Morley rank and both imply φ, $(MR(\psi), \deg(\psi)) < (MR(\varphi), \deg(\varphi))$; i.e., $(MR(H), \deg(H)) < (MR(K), \deg(K))$. If $G = H_0 \supset H_1 \supset \ldots \supset H_i \supset \ldots$ is an infinite descending chain of subgroups, where $H_{i+1} \neq H_i$, for all $i < \omega$, then $\{ (MR(H_i), \deg(H_i)) : i < \omega \}$ is an infinite descending chain in $(I, <)$, which is impossible since this is a well-ordering.

Corollary 3.5.2. *An ω-stable group G has a minimal definable subgroup G^o of finite index in G. G^o is a normal subgroup of G and is definable over $\emptyset$.*

Proof. Let $\mathcal{H}$ be the collection of all definable subgroups of G having finite index in G. If $H_1, H_2 \in \mathcal{H}$ then $H_1 \cap H_2 \in \mathcal{H}$. Combining this with the previous proposition shows that $\bigcap \mathcal{H} = G^o$ is the intersection of finitely many elements of $\mathcal{H}$, hence G^o is definable. Given $g \in G$, $g^{-1}G^o g = g^{-1} \bigcap \mathcal{H} g = \bigcap \{ g^{-1}Hg : H \in \mathcal{H} \} = \bigcap \mathcal{H} = G^o$, hence G^o is normal. That G^o is definable without parameters is left to the exercises.

Definition 3.5.4. *Let G be an ω-stable group. The minimal definable subgroup of finite index in G is called the* connected component *of G, and is denoted G^o. We call G* connected *if $G = G^o$.*

Remark 3.5.2. Since G is the union of finitely many cosets of G^o, $MR(G^o) = MR(G)$. Also, G^o is itself connected as an ω-stable group.

Definition 3.5.5. *Let G be an ω-stable group. A 1-type p over G is a* generic type *if $MR(p)$ is maximal; i.e., $MR(p) = MR(G)$.*

Warning: p is not assumed to be a complete type in the definition.

The set P of generic types in $S_1(G)$ is nonempty and finite. Since translation preserves Morley rank, $GP = P$ (under the action of G on $S_1(G)$ by translation).

Lemma 3.5.2. *Let G be ω-stable and P the set of generics in $S_1(G)$. Then the action of G on P is transitive.*

Proof. Let p and q be any two elements of $S_1(G)$ and $\alpha = MR(p) = MR(q) = MR(G)$. The goal is to find a c such that $cp = q$. Let G_0 be a model of $Th(G)$ and note that p and q are definable over G_0 (since they are both free extensions of their restrictions to $\emptyset$). Let a and b be realizations of $p_0 = p \upharpoonright G_0$ and $q_0 = q \upharpoonright G_0$, respectively, which are independent over G_0. Let φ be a formula in q_0 of Morley rank α and degree 1, and let $c = ba^{-1}$. Using that c and b are interalgebraic over $G_0 \cup \{a\}$, $MR(c/G_0 \cup \{a\}) = MR(b/G_0 \cup \{a\}) = MR(q) = \alpha$. Since the Morley rank of $tp(c/G_0)$ cannot be $> \alpha$, c and a are

independent over G_0. Similarly, c is independent from b over G_0. Since $ca = b$ Corollary 3.5.1 applies to show that $q = cp$, as desired.

This leads to the following linking of the group-theoretic and model-theoretic structure of an ω−stable group.

Corollary 3.5.3. *Let G be an ω−stable group. Then*
(i) $[G : G^o] = \deg(G)$.
(ii) $p \in S_1(G)$ *is generic if and only if* $stab(p) = G^o$.
(iii) G *is connected if and only if* $\deg(G) = 1$.

Proof. Let P be the generic types in $S_1(G)$, G_0 any model of $Th(G)$ and P_0 the generic elements of $S_1(G_0)$. Each element of P is the unique free extension of some element of P_0, hence $|P_0| = |P| = \deg(G)$. The reader should verify that when $tp(a/G_0)$ and $tp(b/G_0)$ are in P_0 and $aG^o \neq bG^o$, $tp(a/G_0) \neq tp(b/G_0)$. Thus, $[G : G^o] \leq \deg(G)$.

Now let p be an arbitrary element of P. Since the action of G on P is transitive one of the basic facts about group actions gives the equation:

$$[G : stab(p)] = |Gp| = |P| = \deg(G).$$

Thus, the definable subgroup $stab(p)$ has finite index in G. Since G^o is the minimal such group, $[G : stab(p)] \leq [G : G^o]$. We conclude both that $\deg(G) = [G : G^o]$ (i.e., (i) holds) and $stab(p) = G^o$, for any generic p. On the other hand, if $p \in S_1(G)$ and $stab(p) = G^o$ then $MR(p) \geq MR(G^o) = MR(G)$ (by Lemma 3.5.1(ii)), so p is a generic type, proving (ii).

Part (iii) follows immediately from (i). This proves the corollary.

Remark 3.5.3. An ω−stable group $G = G^o \cup a_1 G^o \cup \ldots \cup a_n G^o$, where $\{1, a_1, \ldots, a_n\}$ form a complete set of representatives of the cosets of G^o in G. Each $a_i G^o$ has degree 1 and the same Morley rank as G.

Remark 3.5.4. The above analysis of left translation proceeds in an identical manner for right translation. In particular, given an ω−stable group G, $p \in S_1(G)$ is generic if and only if the stabilizer of p with respect to right translation is G^o.

When F is a field, $F^* = F \setminus \{0\}$.

Corollary 3.5.4. *If F is an ω−stable field, then $(F, +)$ and $(F^*, \cdot)$ are both connected. Thus, F has degree 1.*

Proof. By Corollary 3.5.3, $(F, +)$ is connected if and only if $(F^*, \cdot)$ is connected if and only if $\deg(F) = 1$. Let H be a subgroup of $(F, +)$ of finite index. Let I be the ideal $\bigcap\{ kH : k \in F^* \}$. By Proposition 3.5.1 there are $k_1, \ldots, k_n \in F^*$ such that $I = k_1 H \cap \ldots \cap k_n H$, hence I is a nonzero ideal. Since F is a field, $I = F$. Thus, $H = F$ and $(H, +)$ is connected. This proves the corollary.

As an application of the previous Corollary 3.5.3 we prove that an ω–stable group has a "large" abelian subgroup.

Proposition 3.5.2. *An ω–stable group G contains an infinite definable abelian subgroup.*

The proof uses the following basic group-theoretic fact.

Lemma 3.5.3. *If all elements of a group G have finite order and all elements of $G \setminus \{1\}$ are conjugate, then $|G| \leq 2$.*

Proof. We may assume $G \neq \{1\}$. Let $g \in G \setminus \{1\}$ be of prime order p. By the conjugacy condition all nonidentity elements of G have order p. First suppose p is odd. Choose $h \in G$ such that $h^{-1}gh = g^{-1}$. Then for all n, $(h^{-1})^n g h^n = g^{(-1)^n}$. When $n = p$ this yields $g = g^{-1}$, a contradiction. Thus $p = 2$. By a standard exercise G is abelian, hence G has 2 elements.

Remember, for g an element of a group G the *centralizer of g* is $C(g) = \{ h : h^{-1}gh = g \}$. Let g^G denote the conjugacy class of g, $\{ h^{-1}gh : h \in G \}$.

Proof of Proposition 3.5.2. Suppose the proposition fails and G is a counterexample of minimal Morley rank α and Morley degree d. This minimality condition implies that every proper definable subgroup of G is finite. In particular G is connected, hence $d = 1$ by Corollary 3.5.3(i).

Let Z be the center of G, a proper definable subgroup. We will contradict the finiteness of Z by applying Lemma 3.5.3 to the group G/Z. For any $g \in G \setminus Z$ $C(g)$ is a proper definable subgroup of G, hence is finite. Since $g \in C(g)$, g must have finite order. There is a natural one-one correspondence between the conjugacy class g^G of g and the set of cosets $G/C(g)$. Since $C(g)$ is finite g^G must have rank α. If $g^G \neq h^G$, $g^G \cap h^G = \emptyset$, so the fact that G has degree 1 implies there is only one conjugacy class among the elements of $G \setminus Z$. By Lemma 3.5.3 G/Z contains at most two elements, contradicting that Z is finite. This proves the proposition.

Corollary 3.5.5. *A strongly minimal group is abelian.*

Definition 3.5.6. *An element a of an ω–stable group G is* generic over A *if $tp(a/A)$ is generic.*

It is frequently more appropriate to work with elements than types, calling for an equivalent definition and some additional results.

Lemma 3.5.4. *Let G be an ω–stable group. Then $a \in G$ is generic over A if and only if*
$$\forall b \in G (\, b \underset{A}{\downarrow} a \Longrightarrow b \underset{A}{\downarrow} b \cdot a).$$

Proof. First suppose a is generic over A and $b \in G$ is independent from a over A. Then $MR(G) = MR(a/A \cup \{b\}) = MR(b \cdot a/A \cup \{b\}) \leq MR(b \cdot a/A) \leq MR(G)$. Thus, $MR(b \cdot a/A \cup \{b\}) = MR(b \cdot a/A)$, as required.

Conversely, suppose ba is independent from b over A whenever b is independent from a over A. Let g be generic over $A \cup \{a\}$. As in the previous paragraph, ga is generic over $A \cup \{a\}$. By assumption, ga is independent from g over A, hence ga is generic over $A \cup \{g\}$. Since a is interalgebraic with ga over $A \cup \{g\}$, a is also generic over $A \cup \{g\}$. This proves the lemma.

Remark 3.5.5. Given an ω–stable group G and generic elements a and b, $aG^o = bG^o$ if and only if

$(*)$ for all sets A, if a and b are generic over A, then $tp(a/A) = tp(b/A)$.

(See Exercise 3.5.3.)

Translation provides information not only about $S_1(G)$, but about the formulas over G. The following corresponds to and slightly strengthens Lemma 3.5.2. A definable subset X of an ω–stable group is *generic* if it has maximal Morley rank; i.e., X is defined by a generic formula.

Lemma 3.5.5. *Let G be an ω–stable group and $X \subset G$ definable. Then, X is generic if and only if there are left translates (or right translates) $X_0, \ldots, X_k$ of X such that $G = \bigcup_{i \leq k} X_i$.*

Proof. The proof is written for left translates; it is the same for right translates. Let $\alpha = MR(G)$. First assume $G = \bigcup_{i \leq k} X_i$, where $X_0, \ldots, X_k$ are left translates of X. Then one of the X_i's has Morley rank α, hence X has Morley rank α.

Conversely, suppose $MR(X) = \alpha$ and X is definable over the finite set A. Notice it suffices to prove the lemma with X replaced by some translate of X. The proof will be easier after performing a few reductions. Since $X = (a_1 G^o \cap X) \cup \ldots \cup (a_k G^o \cap X)$ for some $a_1, \ldots, a_k$, there is some a such that $X \cap aG^o$ has Morley rank α. Since $(a^{-1} X) \cap G^o$ has Morley rank α, we may as well assume that $MR(X \cap G^o) = \alpha$. Since G is the union of finitely many translates of G^o it suffices to show that G^o is the union of finitely many translates of $X \cap G^o$. So, we can take G to be connected. By Corollary 3.5.3, G has a unique generic type in $S_1(G)$. Since $MR(X) = \alpha$ this generic type contains the formula defining X; i.e.,

$$\text{whenever } a \in G \text{ is generic over } A, \ a \in X.$$

Let $\{\, a_i : i < \omega \,\}$ be an independent set of elements generic over A and let b be an arbitrary element of G. There is an $i < \omega$ such that a_i is independent from b over A, hence a_i is generic over $A \cup \{b\}$. Then $a_i \cdot b$ is generic over A, hence $a_i \cdot b \in X$, equivalently $b \in a_i^{-1} X$. Thus, $G = \bigcup_{i < \omega} a_i^{-1} X$. By compactness, G is the union of finitely many translates of X, completing the proof.

Corollary 3.5.6. *Let G be a connected ω–stable group and $X \subset G$ a generic definable set. Then $X \cdot X = G$.*

Proof. Let X be A–definable and b an arbitrary element of G. In the proof of the lemma we found an element a, generic over A, such that $b \in a^{-1}X$. Since $MR(a^{-1}/A)$ is also α, $a^{-1} \in X$ (by the connectedness of G). Thus $b \in X \cdot X$.

As stated above, many of the results about ω–stable groups are obtained by generalizing proofs about algebraic groups. Zil'ber's Indecomposability Theorem is one such result. Chevalley proved the following about an algebraic group G over an algebraically closed field. Let X_i, $i \in I$, be a family of constructible (i.e., definable) subsets of G such that for each $i \in I$, the identity element e is in X_i and the Zariski closure $\bar{X}_i$ of X_i is irreducible. Then the subgroup H of G generated by the X_i's is Zariski closed, connected and $H = X_{i_1}^{\epsilon_1} \cdots X_{i_n}^{\epsilon_n}$ for some $i_1, \ldots, i_n \in I$ and $\epsilon_j = \pm 1$. (When $X \subset G$, $X^{+1} = X$ and $X^{-1} = \{\, x^{-1} : x \in X \,\}$.) In the general ω–stable context there is no topology, hence nothing exactly like Zariski closure or irreducibility. Zil'ber's substitute for irreducibility is the following.

Definition 3.5.7. *Given an ω–stable group G and $X \subset G$ definable, X is indecomposable if for any definable subgroup H of G, either $|X/H| = 1$ or X/H is infinite (where $X/H = \{\, xH : x \in X \,\}$).*

When X is a group it is indecomposable exactly when it is connected.

Theorem 3.5.1 (Zil'ber's Indecomposability Theorem). *Suppose that G is an ω–stable group of finite Morley rank and, for $i \in I$, X_i is an indecomposable definable subset of G containing the identity element e. Let H be the subgroup of G generated by $\bigcup_{i \in I} X_i$. Then H is definable, connected and for some $i_1, \ldots, i_n \in I$, $H = X_{i_1} \cdot \ldots \cdot X_{i_n}$.*

Proof. Let $\chi = \{\, Y : Y = \Pi_{j \in J} X_j, \ J \subset I$ is finite $\}$. Since each element of Y is definable and $MR(G)$ is finite there is an $X = X_{i_1} \cdot \ldots \cdot X_{i_k} \in \chi$ which has maximal Morley rank among the elements of χ. Let $MR(X) = m$. Let φ_i be the formula defining X_i, φ the formula defining X and $p \in S_1(G)$ a type containing φ with Morley rank m. Let H be the connected component of $stab(p)$. By Lemma 3.5.1 and Corollary 3.5.2 H is definable, hence to complete the proof it remains to show that $X_i \subset H$ for all $i \in I$, and $H = X_{i_1} \cdot \ldots \cdot X_{i_n}$, for some $i_1, \ldots, i_n \in I$. The first step is handled in

Claim. For all $i \in I$, $X_i \subset H$.

Suppose $X_i \not\subset H$. Since $e \in X_i \cap H$, $|X_i/H| > 1$, hence the indecomposability of X_i forces X_i/H to be infinite. Since H has finite index in $H^* = stab(p)$, X_i/H^* is also infinite. Let $\{\, a_j : j < \omega \,\} \subset X_i$ be such that $a_j H^* \neq a_l H^*$ for $j \neq l < \omega$. Elements $b, c \in G$ have the same coset with respect to $stab(p)$ if and only if $bp = cp$. Thus, $\{\, a_j p : j < \omega \,\}$ is an infinite collection of types of Morley rank m. However, each of these types

contains the formula defining $X_i \cdot X$, which also has Morley rank m (by the maximality of m). This contradiction proves the claim.

Being a group H therefore contains not only X but the group generated by the X_i's. The Morley rank of H is $\leq m = MR(p)$ by Lemma 3.5.1(ii). Since $X \subset H$, $MR(H)$ must equal m, hence X is a generic subset of H. Since H is connected $X \cdot X = H$ (by Corollary 3.5.6). A fortiori, H is the group generated by the X_i's, proving the theorem.

This theorem yields the definability of some typical subgroups, for example, commutator subgroups. The *commutator* of elements a, b in a group G is the element $[a, b] = a^{-1}b^{-1}ab$. For A and B subsets of G, $[A, B]$ denotes the subgroup of G generated by $\{ [a, b] : a \in A, b \in B \}$. Notice that $G' = [G, G]$ is the minimal normal subgroup H of G such that G/H is abelian. A priori, there is no reason to think that G' is definable, however it follows from the next lemma that G' is definable when G is a connected group of finite Morley rank.

Lemma 3.5.6. *Let G be a group of finite Morley rank, H a connected definable subgroup of G and A any subset of G. Then the group $[A, H]$ is definable and connected. Moreover, there are finitely many elements $a_1, \ldots, a_n \in A$ such that $x \in [A, H]$ if and only if there are $h_1, \ldots, h_n \in H$ such that $x = [a_1, h_1] \cdots [a_n, h_n]$.*

Proof. For $a \in A$ let $X_a = \{ h^{-1}ah : h \in H \}$. Then $[A, H]$ is the group generated by $\bigcup \{ a^{-1}X_a : a \in A \}$. The indecomposability of $a^{-1}X_a$ is needed to satisfy the hypotheses of Zil'ber's Indecomposability Theorem. Since indecomposability is invaraint under translation it suffices to show that each set X_a is indecomposable. The first step towards this end is

Claim. X_a is indecomposable if $|X_a/K|$ is 1 or infinite for any definable subgroup K of G which is normalized by H; i.e, $h^{-1}Kh = K$ for all $h \in H$.

Let K be any definable subgroup of G and suppose $1 < |X_a/K| < \aleph_0$. Let K_0 be the group $\bigcap_{h \in H} h^{-1}Kh$ and notice that K_0 is normalized by H. We proceed to show that $1 < |X_a/K_0| < \aleph_0$. The group K_0 is the intersection of finitely many of the groups $h^{-1}Kh$, $h \in H$ (by Proposition 3.5.1) hence is definable. Given $h \in H$, $h^{-1}X_ah = X_a$, so $X_a/h^{-1}Kh$ is also finite. If K_1 and K_2 are any two subgroups such that X_a/K_i is finite (for $i = 1, 2$) then $X_a/(K_1 \cap K_2)$ is also finite. Thus X_a/K_0 is finite. Since $|X_a/K| \leq |X_a/K_0|$, $1 < |X_a/K_0| < \aleph_0$, proving the claim.

Fix $a \in A$ and let K be any definable subgroup of G which is normalized by H. Let $H_0 = \{ x \in H : (x^{-1}ax)K = aK \}$. Since K is normalized by H

− H_0 is a subgroup of H and
− for x, $y \in H$, $x^{-1}axK = y^{-1}ayK$ if and only if $xH_0 = yH_0$.

Thus, when $X_a/K = [H : H_0]$ is finite, $|X_a/K| = 1$ (by the connectedness of H). This proves that X_a is indecomposable.

By Zil'ber's Indecomposability Theorem $[A, H]$ is definable, connected and has the form specified in the last sentence of the lemma.

Remark 3.5.6. Implicit in the previous lemma is a proof that $[A, H]$ is infinite or $\{1\}$ under the stated hypotheses.

In all of the above results G is the universal domain of its theory. Many of the results yield information about the subgroups of an arbitrary model G_0 of $Th(G)$ with little additional work. In many instances the condition used to define a subgroup H of G, when relativized to G_0, defines $H \cap G_0$. Here are two examples.

Corollary 3.5.7. *Let G be an ω−stable group, G_0 a model of $Th(G)$ and $p \in S_1(G_0)$. Then $H_0 = \{ a \in G_0 : ap = p \}$ is a subgroup definable in G_0. In fact, for p' the unique free extension of p in $S(G)$ and $H = stab(p')$, $H_0 = H \cap G_0$.*

Proof. This is assigned as Exercise 3.5.4.

Corollary 3.5.8. *Let G be an ω−stable group and G_0 a model of $Th(G)$. Then $G^o \cap G_0$ is*

$$\bigcap \{ H \subset G_0 : H \text{ is a subgroup of finite index definable in } G_0 \}.$$

Proof. This is Exercise 3.5.5.

Remark 3.5.7. Let G be an ω−stable group, G_0 a model of $Th(G)$ and H a subgroup of G_0 definable in G_0. Suppose $H = \varphi(G_0)$. Then H is called *connected* if $\varphi(G)$ is connected. By the previous corollary, H is connected if and only if there is no proper subgroup K of H definable in G_0 and having finite index in H.

Corollary 3.5.9. *Let G be a group of finite Morley rank, G_0 a model of $Th(G)$, H a connected subgroup definable in G_0 and A any subset of G_0. Then the group $[A, H]$ is definable in G_0 and connected.*

Proof. Left to the reader in Exercise 3.5.6.

The following relative version of Zil'ber's Indecomposability Theorem is somewhat less elementary. First we need a definition of indecomposability that applies to sets definable in a model.

Definition 3.5.8. *Let G be an ω−stable group, G_0 a model of $Th(G)$ and X a set definable in G_0. Then X is* indecomposable *if for any subgroup H of G, definable in G, either $|X/H| = 1$ or X/H is infinite.*

Corollary 3.5.10 (Zil'ber's Indecomposability Theorem (relative)). *Let G be an ω−stable group of finite Morley rank and G_0 a model of $Th(G)$. For each $i \in I$ suppose X_i is an indecomposable set definable in G_0 which contains the identity e. Let H be the subgroup of G_0 generated by $\bigcup_{i \in I} X_i$. Then H is definable in G_0, connected and for some $i_1, \ldots, i_n \in I$, $H = X_{i_1} \cdot \ldots \cdot X_{i_n}$.*

Proof. For $i \in I$, let ψ_i be the formula defining X_i and $Y_i = \psi_i(G)$.

Claim. For $i \in I$, Y_i is indecomposable.

Suppose to the contrary that $K = \varphi(G, \bar{a})$ is a definable group and $|Y_i/K| = k$, where $1 < k < \omega$. The formula $\varphi(x, \bar{y})$ can be chosen so that, given $\bar{b}$ in G_0 satisfying $\exists x \varphi(x, \bar{y})$, $K_0 = \varphi(G_0, \bar{b})$ is a subgroup of G_0 and $|X_i/K_0| = k$. This contradicts the indecomposability of X_i to prove the claim.

By Zil'ber's Indecomposability Theorem the group H_0 generated by $\bigcup_{i \in I} Y_i$ is definable, connected and for some $i_1, \ldots, i_n \in I$, $H_0 = Y_{i_1} \cdot \ldots \cdot Y_{i_n}$. Then (using that H_0 is G_0-definable) $H_0 \cap G_0 = X_{i_1} \cdot \ldots \cdot X_{i_n} = H$ is a connected subgroup definable in G_0 and containing X_i for all $i \in I$. This proves the corollary.

By Lemma 3.5.6, whenever G is a connected group of finite Morley rank each element of the series of derived groups $G = G^{(0)} \supset G' \supset G'' \supset \ldots \supset G^{(n)} \supset \ldots$ is definable and connected (where $G^{(n+1)} = [G^{(n)}, G^{(n)}] = (G^{(n)})'$). Similarly, the series $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots$, where $G_{n+1} = [G, G_n]$, consists of connected definable groups. By Proposition 3.5.1 each of these descending chains of groups terminates.

For an arbitrary group G the statement "G is simple" involves quantification over subsets of G. While it is not possible to give a first-order set of axioms for the class of simple groups, when G is an $\omega-$stable group of finite Morley rank being simple is an elementary property in the following sense.

Proposition 3.5.3. *Let G be a group of finite Morley rank.*
   (i) If a model G_0 of $Th(G)$ is nonabelian and not simple then G_0 has a definable nontrivial normal subgroup.
   (ii) If H is a model of $Th(G)$, then H is simple if and only if G is simple.

Proof. (i) Assume to the contrary that G_0 is nonabelian, not simple and has no definable nontrivial normal subgroup. Since $(G_0)^o$ is a definable normal subgroup G_0 must be connected. The center of G_0, call it Z, is a normal subgroup of G_0, definable in G_0. Since G_0 is nonabelian Z is a proper subgroup of G_0, hence $Z = \{1\}$. Let $x \neq 1$ be an element of a nontrivial normal subgroup N of G_0. Since $x \notin Z$, $H = [x, G_0]$ is not $\{1\}$, while it is definable in G_0 and connected by Lemma 3.5.9. Since H is contained in any normal subgroup containing x, $H \subset N$. The minimal normal subgroup H_0 containing H is the group generated by $\bigcup \{ g^{-1}Hg : g \in G \}$. By the relative version of Zil'ber's Indecomposability Theorem (Corollary 3.5.10) H_0 is definable. Since $\{e\} \neq H_0 \subset N$, H_0 is a definable nontrivial normal subgroup of G, a contradiction which proves (i).

   (ii) By (i) it suffices to show that when G contains a definable nontrivial normal subgroup, so does H. This is left as an exercise to the reader.

3.5.1 A Group Acting on a Strongly Minimal Set

This section is a study of an $\omega-$stable group action (G, X) in which X is strongly minimal. This study will yield a result to be used later (Theorem 3.5.2) and show the strength of the hypotheses on (G, X). Many of the arguments are very group-theoretic and specific to this problem. However, the theorem is important enough to warrant a reasonably complete proof.

Notation. When G and H are groups and there is a an embedding $\theta : H \to$ $\text{Aut}(G)$, $G \rtimes H$ is the corresponding semidirect product of G and H. (The particular automorphism θ is generally understood from context.)

When K is a field $K^\times$ denotes the (multiplicative) group of units of K and K^+ denotes the additive group of K.

For G a group we write $G \rhd N$ when N is a normal subgroup of G.

Remark 3.5.8. In this section, when (G, X) is an $\omega-$stable group action it is understood that multiplication on G and the action of G on X are $\emptyset-$definable operations.

Here are three examples of an $\omega-$stable group acting on a strongly minimal set of increasing complexity (at least of increasing rank).

Example 3.5.2. Let G be a strongly minimal group. By Corollary 3.5.5 G is abelian, so additive notation is used for G. The most natural action on the strongly minimal set G is translation by G itself: $x \mapsto x + a$, $a \in G$. This is a regular action.

Now suppose G has an element of order > 2. Here is another action on G. The map $x \mapsto -x$ produces an embedding of $\mathbb{Z}_2$ into $\text{Aut}(G)$. Furthermore $(a, 0)$ and $(a, 1)$, as elements of $G \rtimes \mathbb{Z}_2$, define bijections on G, $x \mapsto x + a$ and $x \mapsto -x + a$, respectively. This gives a faithful transitive group action of $G \rtimes \mathbb{Z}_2$ on G. Note that $G \rtimes \mathbb{Z}_2$ is nonabelian, has Morley rank 1 and the action is not regular.

Example 3.5.3. Let K be an algebraically closed field. Let G_{aff} be the group of affine transformations on K, $x \mapsto a + bx$, where $a \in K$ and $b \in K^\times$. Then G_{aff} acts sharply 2$-$transitively on K and $G \cong K \rtimes K^\times$. When K is the universal domain of its theory (G_{aff}, K) comprises a group of Morley rank 2 acting on a strongly minimal set.

Example 3.5.4. Let K be an algebraically closed field and $\mathbb{P}^1$ the projective line over K, thinking of $\mathbb{P}^1$ as the set of 2×1 column vectors over K, factored by the equivalence relation: $x \sim y \iff K^\times x = K^\times y$. The group $\text{GL}_2(K)$ of invertible 2×2 matrices over K defines an action $*$ on $\mathbb{P}^1$. Note: for any $g, h \in \text{GL}_2(K)$, $g * x = h * x$ for all $x \in \mathbb{P}^1$ if and only if $g = \lambda h$ for some $\lambda \in K^\times$. Let $\text{PGL}_2(K)$ be the quotient of $\text{GL}_2(K)$ by $\{ \lambda \cdot 1 : \lambda \in K^\times \}$, where 1 is the identity of $\text{GL}_2(K)$. The action of $\text{PGL}_2(K)$ on $\mathbb{P}^1$ is faithful.

Note that this action is sharply 3−transitive and (when K is the universal domain) $\mathrm{PGL}_2(K)$ is a group of Morley rank 3 acting faithfully on a strongly minimal set.

What is quite surprising is that these are essentially the only examples of an ω−stable group action on a strongly minimal set.

Theorem 3.5.2. *Let (G, X) be an ω−stable transitive faithful group action with X strongly minimal. Then $MR(G) \leq 3$ and*

(1) *If $MR(G) = 1$, G^o acts regularly on X.*
(2) *If $MR(G) = 2$, there is a field K definable on X and the action of G on X is definably isomorphic to the affine action of $K \rtimes K^\times$ on K.*
(3) *If $MR(G) = 3$, there is a field K definable on $X \setminus \{a\}$ (for some $a \in X$) and the action of G on X is isomorphic to the action of $\mathrm{PGL}_2(K)$ on $\mathbb{P}^1$, the projective line over K.*

The bulk of this subsection is devoted to the proof of this theorem. The proof will require three additional propositions which will not be proved here. The first two represent a nontrivial amount of work.

Proposition 3.5.4. *A connected group of Morley rank 2 is solvable.*

Proof. This is Theorem 6 of [Che79].

Definition 3.5.9. *Given a group H acting on a set A, $B \subset A$ is H−invariant if $HB = B$; i.e., for all $h \in H$, $b \in B$, $hb \in B$.*

Proposition 3.5.5. *Let $\mathfrak{C}$ be the universal domain of a theory of finite Morley rank; G and A infinite definable abelian groups. Suppose there is a definable faithful action of G on A which induces an embedding of G into the automorphism group of A. Further suppose that A contains no infinite definable proper G−invariant subgroup. Then there is a definable field F such that the additive group of F is definably isomorphic to A and there is a definable embedding of G into the multiplicative group of F so that the action of G on A corresponds to multiplication in F.*

Remark 3.5.9. Here is a more precise statement of the conclusion of the proposition. The definable action $*$ of G on A is assumed to induce an embedding θ of G into $\mathrm{Aut}(A)$ by: $\theta(g)$ is the automorphism α such that $\alpha(a) = g*a$, for all $g \in G$, $a \in A$. (Here $\mathrm{Aut}(A)$ is simply the automorphism group of A as an abelian group.) Use θ to define $A \rtimes G$. Then the proposition yields a definable field F and a definable embedding from $A \rtimes G$ into $F^+ \rtimes F^\times$ which restricts to an isomorphism of A onto F^+.

Proposition 3.5.6. *Let T be an ω−stable theory, F an infinite definable field of finite Morley rank and S a definable group of field automorphisms of F. Then $S = \{1\}$.*

Finally we also make use of the following fact whose proof is omitted in favor of bigger fish.

Lemma 3.5.7. *Let G be an ω—stable group of Morley rank $k + 1 < \omega$. Suppose H is a definable subgroup with $MR(G/H) = 1$. Then $MR(H) = k$.*

The restriction to transitive actions in the theorem does not eliminate many interesting cases:

Lemma 3.5.8. *Let (G, X) be an ω—stable faithful group action with X strongly minimal and G infinite.*
(i) If Y is a finite orbit, $|Y| \leq \deg(G)$.
(ii) There is an $\emptyset$—definable finite set $Y \subset acl(\emptyset) \cap X$ such that G acts transitively on $X \setminus Y$.

Proof. (i) Given $y \in Y$, $|Y| = [G : stab(y]$. Thus, $stab(y)$ has finite index in G and $|Y| = [G : stab(y] \leq [G : G^o] = \deg(G)$.
(ii) If each orbit is finite, $G^o \subset stab(x)$ for all $x \in X$, contradicting that the action is faithful and G is infinite. Thus, X contains an infinite orbit. Since an orbit is a definable set and X is strongly minimal, X contains only one infinite orbit and this orbit is $X \setminus Y$ for some finite Y. Since Y is the union of all finite orbits, Y is $\emptyset$—definable.

Lemma 3.5.9. *Let (G, X) be a transitive faithful group action with G abelian. Then the action is regular.*
If, in addition, (G, X) is ω—stable, $MR(G) = MR(X)$.

Proof. Let $a \in X$ and suppose $g \in G$ is such that $ga = a$. Let b be any element of X. By the transitivity of the action there is an $h \in G$ such that $ha = b$. Then $gb = gha = hga = ha = b$. Since the action is faithful, $g = 1$, proving that the action is regular.
Suppose (G, X) is ω—stable, $a \in X$ and $g \in G$ is generic over a. By the regularity of the action $g \in dcl(a, ga)$. Thus, $MR(G) = MR(g) = MR(g/a) \leq MR(ga/a) \leq MR(X)$. Now suppose $b, c \in X$ are independent with $MR(b) = MR(c) = MR(X)$. By the transitivity of the action there is a $h \in G$ such that $hb = c$. Then $MR(X) = MR(c/b) \leq MR(h/b) \leq MR(G)$, completing the proof.

Notation. From here until the end of the subsection we assume (G, X) to be an ω—stable transitive faithful group action with X strongly minimal and G infinite, unless stated otherwise. These hypotheses may be repeated in key results to make later reference easier.

Lemma 3.5.10. *If $N \lhd G$ is infinite and definable then N acts faithfully and transitively on X.*

Proof. It is clear that the action of N on X is faithful. If N^o acts transitively on X then certainly N acts transitively on X, so we may as well assume N is connected. Suppose the orbit of x under N is finite. Then $Nx = \{x\}$ by Lemma 3.5.8(i). An arbitrary $y \in X$ is gx for some $g \in G$ (by the transitivity of the action of G) hence, since N is normal, $Ny = Ngx = gNx = \{gx\} = \{y\}$. In other words the elements of N fix every element of X, contradicting that the action is faithful and $N \neq \{1\}$. Thus, every orbit under N is infinite. Since X is strongly minimal and N is definable X can only contain one infinite orbit under N. That is, N acts transitively on X, as required.

Proof of Theorem 3.5.2(i). Suppose $MR(G) = 1$. Then G^o is a strongly minimal group, hence abelian by Corollary 3.5.5. By Lemma 3.5.10 the action of G^o on X is transitive, so the action is regular by Lemma 3.5.9, proving (i) of the theorem.

Proof of Theorem 3.5.2(ii). We assume now that $MR(G) = 2$. By Proposition 3.5.4 G^o is solvable. Since $G^o \lhd G$, G^o acts transitively on G (by Lemma 3.5.10). If G^o were abelian Lemma 3.5.9 would contradict that $MR(G^o) = 2$ and $MR(X) = 1$. Hence G^o is nonabelian. Let $A = (G^o)' = [G^o, G^o]$, a definable connected subgroup of G^o by Lemma 3.5.6. Since G^o is nonabelian $A \neq \{1\}$, hence A is infinite (see Remark 3.5.6). The solvability of G^o forces A to be a proper subgroup, so A must be strongly minimal. By Theorem 3.5.2(i) A acts regularly on X.

Fix $x \in X$ and let $G_x = \{\, g \in G : gx = x \,\}$, the stabilizer of x in G. For $g, h \in G$, $gG_x = hG_x$ if and only if $gx = hx$. Thus, the map $g \mapsto gx$ defines a bijection between G/G_x and X. Since $MR(G) = 2$ the only possibility for $MR(G_x)$ is 1.

Claim. Conjugation defines an embedding θ of G_x into Aut(A). Moreover, the action of G_x^o on A by conjugation is faithful and regular on $A \setminus \{1\}$.

Since A acts regularly on X, for any $g \in G$ there is a unique $a \in A$ such that $gx = ax$, hence $g \in aG_x$. Since A is a normal subgroup of G, conjugation defines a homomorphism θ of G_x into Aut(A). Suppose towards a contradiction that θ is not an embedding; i.e., there is a $g \neq 1$ in G_x which commutes with each element of A. Then given $y \in X$ there is an $a \in A$ with $y = ax$, hence $gy = gax = agx = ax = y$. In other words, g fixes each element of X, contradicting that G acts faithfully on X. Thus θ is an embedding of G_x into Aut(A).

Conjugation also defines a group action of G_x on A. The above argument shows this action to be faithful on $A \setminus \{1\}$. By part (i) of the theorem conjugation defines a faithful regular action of G_x^o on $A \setminus \{1\}$, proving the claim.

The embedding θ in the claim can be used to define $A \rtimes G_x$. There is a map ψ from G into $A \rtimes G_x$ defined by $\psi(g) = (a, h)$ if and only if $g = ah$.

(The first sentence in the proof of the claim shows that $G = A \cdot G_x$.) A routine verification shows ψ to be an isomorphism.

Since A has no infinite definable proper subgroups it is G_x^o—invariant. Since $MR(G_x) = 1$, G_x^o is abelian. Thus, Proposition 3.5.5 yields a definable field F and a definable embedding σ of $A \rtimes G_x^o$ into $F^+ \rtimes F^\times$ which restricts to an isomorphism of A onto F^+. This isomorphism guarantees that F has Morley rank 1. By Corollary 3.5.4, F is strongly minimal, hence the embedding of G_x^o into $F^\times$ is surjective.

The field structure can be transferred onto A as follows. Let $1'$ be any element of $A \setminus \{1\}$. For each $a \neq 1$ in A let a'' be the unique element of G_x^o taking $1'$ to a (which exists by the claim). Define a binary relation $\otimes$ on A by: $1 \otimes a = a \otimes 1 = 1$ for all $a \in A$ and $a \otimes b = (a'' \cdot b'')1'$, for $a, b \in A \setminus \{1\}$. The reader can show that $(A, \cdot, \otimes, 1, 1')$ is a field isomorphic to F (via a definable bijection). Since the action of A on X is regular there are also definable operations on X under which X is a field (definably) isomorphic to F. The action of A on X corresponds to the translations $x \mapsto x + a$, $a \in A$, while the action of G_x^o on X corresponds to the dilations $x \mapsto bx$, $b \in G_x^o$. (See Example 3.5.3.) Thus $A \rtimes G_x^o$ acts on X like the affine group of the field F. To finish the Morley rank 2 case we need only show

Claim. $G_x = G_x^o$.

Since G_x^o acts faithfully on A it acts regularly by Lemma 3.5.9. Fix $a \in A$. For any $g \in G_x$ there is an $h \in G_x^o$ such that $gag^{-1} = hah^{-1}$. Since $h^{-1}g \in S = \{ f \in G_x : faf^{-1} = a \}$ the claim will be proved once we show $S = \{1\}$.

Define $\oplus$ on G_x^o by

$$g_0 \oplus g_1 = g_2 \text{ if and only if } (g_0 a g_0^{-1})(g_1 a g_1^{-1}) = g_2 a g_2^{-1}.$$

Just as $(A, \cdot, \otimes)$ was a field isomorphic to F, $(G_x^o, \oplus, \cdot)$ is a field isomorphic to F.

The group S acts on G_x^o by conjugation. We claim this action defines an embedding of S into the group Γ of field automorphisms of $(G_x^o, \oplus, \cdot)$. Conjugation by $s \in S$ is clearly a bijection σ of G_x^o which is an automorphism of $\cdot$. Also, σ is an automorphism of $\oplus$ since

$$\begin{aligned}
(sg_0 s^{-1} a s g_0^{-1} s^{-1})(sg_1 s^{-1} a s g_1^{-1} s^{-1}) &= (sg_0 a g_0^{-1} s^{-1})(sg_1 a g_1^{-1} s^{-1}) \\
&= sg_0 a g_0^{-1} g_1 a g_1^{-1} s^{-1}.
\end{aligned}$$

Thus S is a definable group of automorphisms of the field $(G_x^o, \oplus, \cdot)$. By Proposition 3.5.6 $S = \{1\}$. This completes the proof of the claim and Theorem 3.5.2(ii).

For the cases when $MR(G) > 2$ we need the following two lemmas.

Lemma 3.5.11. *Suppose $MR(G) \geq 2$ and $x \in X$. Then G_x acts faithfully and transitively on a cofinite subset of X. Moreover G_x^o acts transitively on a cofinite subset of X.*

Proof. This follows immediately from Lemma 3.5.8 once we show that G_x is infinite. Suppose to the contrary that G_x is finite. Then for any $y \in X$ there are finitely many $g \in G$ such that $gx = y$. Thus, picking g to be a generic of G independent from x and $y = gx$, $MR(g) \le MR(y/x) \le 1$. This contradicts that $MR(G) \ge 2$ to prove the lemma.

Lemma 3.5.12. *Let $MR(G) = k + 1 \ge 3$. Suppose that any ω–stable transitive faithful action (H, Z) with Z strongly minimal and $MR(H) = k$ acts sharply k–transitively. Given $x \in X$ let Y be the unique infinite orbit under G_x^o. Then $X \setminus Y = \{x\}$ and G acts sharply $k + 1$–transitively on X.*

Proof. First observe

Claim. $MR(G_x) = k$.

Let $X \setminus Y = Z$ and $n = |Z|$. Define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $G_a^o = G_b^o$. Any finite orbit of G_x^o contains a single element (by Lemma 3.5.8) so $G_z \supset G_x^o$ for any $z \in Z$. By the transitivity of the action of G, $MR(G_y) = MR(G_{y'})$ for all $y, y' \in X$. Thus, $G_z^o = G_x^o$ for all $z \in Z$; i.e., the $\sim$–class of x is Z. For any $g \in G$ and $a \in X$, $G_{ga} = gG_ag^{-1}$, so $G_{ga}^o = gG_a^og^{-1}$. Thus, $\sim$ is preserved by the action of G; i.e., $a \sim b$ if and only if $ga \sim gb$. Since G acts transitively on X every $\sim$ class contains n elements.

Claim. If $a \ne b \in Y$ then $a \nsim b$.

Let H denote G_x^o and fix $a \ne b \in Y$. Since there is a definable one-to-one correspondence between X and the cosets of G_x in G, $MR(G/G_x) = 1$. By Lemma 3.5.7, $MR(G_x) = k$, hence H acts sharply k–transitively on Y. Thus, H_a acts sharply $(k - 1)$–transitively on $Y \setminus \{a\}$. Let $\bar{c}$ be a sequence of $k - 1$ distinct elements of $Y \setminus \{a\}$. By the sharpness of the action there is a $\bar{c}$–definable bijection between H_a and $(Y \setminus \{a\})^{k-1}$. Since $\deg(Y \setminus \{a\})^{k-1} = 1$ (by Exercise 3.5.10) H_a also has degree 1. Hence H_a is connected. Since $H_a \subset G_a$ and H_a is connected Exercise 3.5.7 forces H_a to be a subgroup of G_a^o. If $a \sim b$, $H_a \subset G_b$, hence any element of H_a fixes b. The action of H_a on $Y \setminus \{a\}$ is $(k - 1)$–transitive, so this is impossible. This proves the claim.

Thus, each $\sim$ –class contains a single element and $X \setminus Y = \{x\}$. In particular G_x acts transitively and faithfully on $X \setminus \{x\}$. Since $MR(G_x) = k$, G_x acts sharply k–transitively on $X \setminus \{x\}$. Since this is true for each $x \in X$, G acts sharply $(k + 1)$–transitively on X. This proves the lemma.

Corollary 3.5.11. *If G has finite Morley rank $k \ge 2$ then the action of G on X is sharply k–transitive.*

Proof. It follows from Theorem 3.5.2(ii) that a group of Morley rank 2 acting faithfully and transitively on a strongly minimal set acts sharply 2–transitively. From here the corollary follows by induction on k.

Proof of Theorem 3.5.2(iii). In this case $MR(G) = 3$. By the previous corollary G acts sharply 3−transitively on X. Fix a point in X and call it ∞. Then G_∞ acts sharply 2−transitively on $Y = X \setminus \{\infty\}$. By (ii) of the theorem there is a field K defined on Y. Moreover, if $0 \in Y$ is the zero of K, $G_{\infty,0}$ (= the set of elements of G fixing both ∞ and 0) is isomorphic (via a definable map) to the multiplicative group of K. Among other things, $G_{\infty,0}$ is strongly minimal.

Since the action of G on X is sharply 3−transitive there is a unique $\alpha \in G$ such that α maps the triple $(0, 1, \infty)$ to $(\infty, 1, 0)$. Since α^2 fixes $\{0, 1, \infty\}$, α^2 must be 1.

Claim. Conjugation by α defines an automorphism σ of $G_{\infty,0}$, $\sigma \neq 1$, $\sigma^2 = 1$ and $\sigma g = g^{-1}$ for all $g \in G_{\infty,0}$.

Given $g \in G_{\infty,0}$, $\alpha g \alpha^{-1}$ is also in $G_{\infty,0}$, hence conjugation by α defines an automorphism σ of $G_{\infty,0}$. Let $a \in X \setminus \{0, 1, \infty\}$ and $g \in G_{\infty,0}$ such that $g1 = a$. If $\sigma = 1$, then $\alpha g \alpha^{-1} = g$, hence $\alpha a = \alpha g1 = g1 = a$, a contradiction since G acts faithfully and $\alpha \neq 1$. Since $\alpha^2 = 1$, $\sigma^2 = 1$. It remains to show that σ is inversion. Let $B = \{\, a \in G_{\infty,0} : \sigma a = a \,\}$ and $C = \{\, a \in G_{\infty,0} : \sigma a = a^{-1} \,\}$. Since $G_{\infty,0}$ is strongly minimal and B is a proper definable subgroup, B is finite. Consider the map $\tau : G_{\infty,0} \longrightarrow G_{\infty,0}$ defined by $\tau(x) = \sigma(x)x^{-1}$. Then, for any $x \in G_{\infty,0}$,

$$\sigma(\tau(x)) = \sigma^2(x)\sigma(x^{-1}) = x\sigma(x)^{-1} = \tau(x)^{-1},$$

so τ maps $G_{\infty,0}$ to C. If $\tau(x) = \tau(y)$, then $\sigma(xy^{-1}) = xy^{-1}$ and $x \in yB$. Since B is finite, x is algebraic over y. This shows that the kernel of τ is finite, hence C contains a generic. Thus, C is all of $G_{\infty,0}$, completing the proof of the claim.

In other words, σ is inversion on $G_{\infty,0}$. Given $a \in K^\times$, let $h \in G_{\infty,0}$ be such that $h1 = a$. Then $\alpha a = \alpha h1 = h^{-1}\alpha 1 = h^{-1}1 = a^{-1}$. Thus, α acts like inversion on $K^\times$. It follows that G contains the group of automorphisms of $\mathbb{P}^1$ generated by all affine maps $x \mapsto cx+d$ and $x \mapsto x^{-1}$. Thus $\mathrm{PGL}_2(K)$ embeds into G. Since $\mathrm{PGL}_2(K)$ itself acts 3−transitively on X and the action of G on X is sharply 3−transitive, this embedding of $\mathrm{PGL}_2(K)$ into G is surjective. That is, the action of G on X is isomorphic to the action of $\mathrm{PGL}_2(K)$ on $\mathbb{P}^1$.

This proves Theorem 3.5.2(iii).

To complete the proof of Theorem 3.5.2 it remains to show that $MR(G) \leq 3$.

Claim. $MR(G) \neq 4$.

Suppose to the contrary that $MR(G) = 4$. By Lemma 3.5.12 G acts sharply 4−transitively on X. Fix two points ∞_1 and ∞_2 in X. Then G_{∞_1,∞_2} acts sharply 2−transitively on $X \setminus \{\infty_1, \infty_2\}$ so there is a field structure K

definable on $X \setminus \{\infty_1, \infty_2\}$. Moreover, the action of the multiplicative group of K (on itself) is isomorphic to the action of $G_{\infty_1, \infty_2, 0} = H$.

There are σ_1, $\sigma_2 \in G$ such that σ_1 maps $(0, 1, \infty_1, \infty_2)$ to $(\infty_1, 1, 0, \infty_2)$ and σ_2 maps $(0, 1, \infty_1, \infty_2)$ to $(\infty_2, 1, \infty_1, 0)$. As in the proof of Theorem 3.5.2(iii), $\sigma_i H \sigma_i^{-1} = H$ and $\sigma_i \notin C(H)$ (= the centralizer of H) for $i = 1, 2$. Repeating the proof of the previous claim shows that $\sigma_i h \sigma_i^{-1} = h^{-1}$, for all $h \in H$. Let $\omega = \sigma_1 \sigma_2$. For any $h \in H$, $\omega h = \sigma_1 h^{-1} \sigma_2 = h \sigma_1 \sigma_2 = h\omega$. If $x \in K^\times \setminus \{1\}$ and $h \in H$ is such that $h1 = x$ then

$$\omega x = \omega h 1 = h \omega 1 = h 1 = x.$$

Since $\omega 1 = 1$, ω is the identity on $K^\times$. It follows that $\omega = 1$, contradicting the fact that $\omega 0 = \infty_2$. This proves the claim.

Let k be the minimal natural number > 3 such that there is an ω–stable group G of Morley rank k and a faithful, transitive, ω–stable action of G on X. Given $x \in X$, G_x acts faithfully and transitively on a strongly minimal subset of X and $MR(G_x) = k - 1$. Thus, k must be 4, contradicting the claim. This proves that $MR(G) \leq 3$ when $MR(G)$ is finite.

To finish the proof we must show that any ω–stable group G acting faithfully and transitively on a strongly minimal set must have finite Morley rank. We know

$$\{1\} = \{ g \in G : \forall x \in X, \ gx = x \} = \bigcap_{x \in X} \{ g \in G : \ gx = x \}.$$

By the descending chain condition on subgroups there are $x_1, \ldots, x_n \in X$ such that $\{1\} = \{ g \in G : \ gx_1 = x_1, \ldots, gx_n = x_n \}$. It follows that the map $g \mapsto (gx_1, \ldots, gx_x)$ is a one-to-one map of G into X^n. Thus $MR(G) \leq MR(X^n) = n$.

This completes the proof of Theorem 3.5.2.

3.5.2 $\bigwedge$ –definable Groups and Actions

Occasionally (most notably in Section 4.5) a theorem giving the existence of a group will not immediately yield a definable group, but a group on the set of realizations of a type in the universe. We show in Theorem 3.5.3 that any such group in a t.t. theory is actually definable. The relevant definitions are as follows.

Definition 3.5.10. *Let $\mathfrak{C}$ be the universal domain of a theory. A subset X of $\mathfrak{C}$ is called* infinity-definable over A *(abbreviated $\bigwedge$ –definable over A) if for some type p over A, $X = p(\mathfrak{C})$. X is $\bigwedge$ –definable if it is $\bigwedge$ –definable over some set A.*

Every definable subset of $\mathfrak{C}$ is $\bigwedge$ –definable.

Given $\mathfrak{C}$ the universe of a t.t. theory and D an $\bigwedge$ –definable set, specifically, $D = p(\mathfrak{C})$;

− $MR(D)$ and $\deg(D)$, the Morley rank and degree of D, are defined to be $MR(p)$ and $\deg(p)$, respectively.

Definition 3.5.11. *Let T be a complete theory.*

(i) We call $(G, \cdot)$ an $\bigwedge$−definable group over A in $\mathfrak{C}$ if

− $(G, \cdot)$ *is a group,*
− G *is a subset of $\mathfrak{C}$, $\bigwedge$−definable over A and*
− *there is a function f, definable over A in $\mathfrak{C}$, such that $f \restriction G \times G$ defines the binary operation $\cdot$ on G under which G is a group.*

(ii) Similarly, a group action $(G, \cdot, X, \star)$ is an $\bigwedge$−definable group action over A in $\mathfrak{C}$ if $(G, \cdot)$ is an $\bigwedge$−definable group over A in $\mathfrak{C}$, X is a subset of $\mathfrak{C}$, $\bigwedge$−definable over A, and $\star$ is the restriction to $G \times X$ of an A−definable function.

(iii) An $\bigwedge$−definable ω−stable group (action) is an $\bigwedge$−definable group (action) in an ω−stable theory.

Theorem 3.5.3. *An $\bigwedge$−definable ω−stable group is definable.*

Remark 3.5.10. Let G be an ω−stable $\bigwedge$−definable group over $\emptyset$. Since G may not be definable we cannot relativize the universe to G and retain a full description of the definable relations on G. That is, Proposition 3.3.3 may fail when relativizing to a set which isn't definable. So, we have to keep sight of the ambient theory.

We say $X \subset G^n$ is a *locally definable relation on G* if $X = Y \cap G^n$ for Y some definable n−ary relation.

Remark 3.5.11. The reader is asked to verify the following basic facts. Let G be an $\bigwedge$−definable ω−stable group over A. An element $a \in G$ is *generic over $B \supset A$* if $MR(a/B) = MR(G)$. A type $p \in S(\mathfrak{C})$ is *generic* if p extends the type defining G and $MR(p) = MR(G)$.

(i) If $a, b \in G$ and a is generic over $B \cup \{b\}$, then $a \cdot b$ and $b \cdot a$ are generic over $B \cup \{b\}$.

(ii) For any formula $\psi(x, y)$ there is a formula $\theta(x$ such that for all a, $\models \theta(a)$ if and only if $\psi(a, y)$ is in every generic type.

The theorem will follow quickly from previously proved results and

Lemma 3.5.13. *Let G be an $\bigwedge$−definable ω−stable group over A. Then there is a definable group H such that G is a subgroup of H.*

Proof. Let $\cdot$ be a definable function whose restriction to $G \times G$ is the group operation on G. Let Φ be a set of definable sets such that $G = \bigcap \Phi$. Pick (by compactness) $X_0 \in \Phi$ such that for all $x, y, z \in X_0$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ and $x \cdot 1 = x$. For any $Y \in \Phi$ let

$$X_Y = \{\, x \in X_0 : \text{for all } y \in G \text{ generic over } x, \ x \cdot y \in Y \,\}.$$

Then
$$G = \bigcap \{ \, X_Y : Y \in \Phi \, \}.$$

(Simply because G is a group, $G \subset X_Y$, for all $Y \in \Phi$. If $x \in X_Y$ for each $Y \in \Phi$, then for any $y \in G$ generic over x, $x \cdot y \in G$ (because $G = \bigcap \Phi$). Since $x, y \in X_0$, $x = x \cdot 1 = x \cdot (y \cdot y^{-1}) = (x \cdot y) \cdot y^{-1} \in G$.) We can furthermore assume that for any finite $\Phi_0 \subset \Phi$ there is a $Y \in \Phi$ such that $Y \subset \bigcap \Phi_0$.

Claim. There is a $Z \in \Phi$ such that $X_0 \supset Z$ and for all $x, y \in X_Z$, $x \cdot y \in X_0$.

If $x \in X_{X_0}$ and $y \in G$, then $x \cdot y \in X_{X_0}$. (Let $z \in G$ be generic over $\{x, y\}$. Then $y \cdot z$ is generic over $\{x, y\}$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z) \in X_0$.) Let $Z_1 \in \Phi$ be such that for all $x \in X_{X_0}$ and $y \in Z_1$, $x \cdot y \in X_0$. A set $Z \in \Phi$ such that $X_Z \subset X_{X_0} \cap Z_1$ satisfies the requirements of the claim.

Let $X_1 = X_Z$.

Claim. If $x \in X_1$ and $y \in G$ then $x \cdot y \in X_1$.

Simply because $x, y \in X_Z$, $x \cdot y \in X_0$. Choose a $z \in G$ generic over $\{x, y\}$. Then $y \cdot z \in G$ is generic over x. Since $x \in X_Y$, $x \cdot (y \cdot z) \in Z$. Thus, $(x \cdot y) \cdot z \in Z$, proving that $x \cdot y \in X_Z$ as required.

Let $X_2 = \{ y \in X_1 : \forall x (x \in X_1 \Rightarrow x \cdot y \in X_1) \}$. Then X_2 is a definable set closed under $\cdot$, and $\cdot$ is associative on X_2. We proved in the claim that $G \subset X_2$. Thus, the invertible elements of X_2 form a definable group containing G.

Proof of Theorem 3.5.3. Let G be an ω−stable $\bigwedge$−definable group over A. Let Φ be a collection of definable sets such that $G = \bigcap \Phi$. By Lemma 3.5.13 there is a definable group $H \supset G$. In fact, from the proof of the lemma we see that for any $X \in \Phi$ there is a definable group H_X, $X \supset H_X \supset G$. Hence, $G = \bigcap_{X \in \Phi} H_X$. The descending chain condition on definable groups in an ω−stable theory (Proposition 3.5.1) yields a finite $\Psi \subset \Phi$ such that $G = \bigcap_{X \in \Psi} H_X$. This proves the theorem.

Historical Notes. Proposition 3.5.1 is due to Macintyre [Mac71b]. Groups of finite Morley rank were studied by Zil'ber [Zil77b] (translated in [Zil91]) and [Zil77a] and independently by Cherlin [Che79]. The notion of a generic type came out of these papers, [CS80] and Poizat's [Poi83a]. Zil'ber's Indecomposability Theorem is found in [Zil77b]. Theorem 3.5.3 is due to Hrushovski [Hru90b, Theorem 2], although Poizat had earlier proved that an $\bigwedge$−definable group which is contained in a definable group is the intersection of its definable supergroups [Poi81].

Exercise 3.5.1. Let $\varphi(x)$ be a formula in an ω−stable group G and let a be an element. Show: $\varphi(x)$ and $\varphi(a^{-1} \cdot x)$ have the same Morley rank and degree.

Exercise 3.5.2. Verify that the connected component of an ω−stable group G is definable without parameters.

Exercise 3.5.3. Prove Remark 3.5.5.

Exercise 3.5.4. Prove Corollary 3.5.7. HINT: Use Corollary 3.5.1.

Exercise 3.5.5. Prove Corollary 3.5.8.

Exercise 3.5.6. Prove Corollary 3.5.9.

Exercise 3.5.7. Let G be an ω−stable group and H a definable subgroup of G. Assuming that H is connected show that $H \subset G^o$.

Exercise 3.5.8. Let T be an ω−stable theory and G, H infinite groups definable in the universal domain of T. Let $K = G \times H$. Then c is a generic of K if and only if $c = (a, b)$, where $a \in G$ and $b \in H$ are generics and a is independent from b.

Exercise 3.5.9. Give a proof of Proposition 3.5.3(ii).

Exercise 3.5.10. Let T be ω−stable and (H, Y) a definable group action, where Y is infinite. Suppose H acts sharply k−transitively on Y and H is connected. Then $Y, Y^2, \ldots, Y^k$ all have degree 1.