2. Constructing Models with Special Properties

Much of the richness of model theory is attributable to the weakness of first-order logic. In the last chapter it was proved that a first-order theory cannot express, for example, that every model has cardinality $\aleph_0$. In fact, by the Löwenheim-Skolem Theorems an elementary class (in a countable language) containing an infinite model contains models in every infinite cardinality. Thus, a basic property expressible in first-order logic and true in an infinite model is true in a great variety of models. Besides the Löwenheim-Skolem Theorems, the Omitting Types Theorem is an example of such a result. Given a nonisolated type p in a countable theory there is a countable model $\mathcal{M}$ realizing p and a countable model $\mathcal{N}$ omitting p. The theme of this chapter is to continue this program of constructing models of a given theory with widely varying properties. In the first two sections we define several kinds of models (of complete theories having infinite models) distinguished by the elementary embeddings they admit and the types they realize or omit.

2.1 Prime and Atomic Models

The first special kind of model to be considered is one which is, intuitively, the "smallest" model of the theory.

Definition 2.1.1. *Given a theory T, we call $\mathcal{M} \models T$ a* prime *model of T if, for any $\mathcal{N} \models T$, $\mathcal{M}$ can be elementarily embedded into $\mathcal{N}$.*

For example, if T is the theory of algebraically closed fields of characteristic 0, the algebraic closure of the rationals, $\tilde{\mathbb{Q}}$, forms a prime model of T. (Since T has quantifier elimination, any model of T contains a copy of $\tilde{\mathbb{Q}}$ as an elementary submodel.) While the definition of a prime model makes sense for any theory only a complete theory can have a prime model (see the exercises). We will see, moreover, that the uniqueness of prime models and a useful condition sufficient for their existence can only be proved for countable complete theories. As a first observation, if $\mathcal{M}$ is a prime model of the countable theory T and $p \in S_n(\emptyset)$ is realized in $\mathcal{M}$ then p must be isolated. (Suppose to the contrary that p is nonisolated and realized in $\mathcal{M}$ by the $n-$tuple $\bar{a}$. By the Omitting Types Theorem, T has a countable model

$\mathcal{N}$ omitting p. Since $\mathcal{M}$ is prime there is an elementary embedding f of $\mathcal{M}$ into $\mathcal{N}$. The type of any tuple $\bar{b}$ in $\mathcal{M}$ is the same as the type of $f(\bar{b})$ in $\mathcal{N}$ (since f is elementary) hence, the type of $f(\bar{a})$ in $\mathcal{N}$ is p, contradicting the fact that $\mathcal{N}$ omits p.) This observation suggests the following definition.

Definition 2.1.2. *Let T be a complete theory. Given a model $\mathcal{M}$ of T, $A \subset M$ is called* atomic *if for each tuple $\bar{a}$ from A, $tp_{\mathcal{M}}(\bar{a})$ is isolated. We call $\mathcal{M}$ an* atomic model *if M is an atomic set.*

It was proved above that a prime model of a countable theory is necessarily atomic. That a countable atomic model is a prime model of its theory and elementarily equivalent countable atomic models are isomorphic are proved in the next proposition. If $\mathcal{M}$ and $\mathcal{N}$ are models and $A \subset M$, a function $f : A \longrightarrow N$ is called an *elementary map* if for all $a_0, \ldots, a_n \in A$, $tp_{\mathcal{M}}(a_0, \ldots, a_n) = tp_{\mathcal{N}}(f(a_0), \ldots, f(a_n))$. (An elementary embedding is simply an elementary map whose domain is a model.)

Proposition 2.1.1. *Let T be a countable complete theory.*

(i) A countable model $\mathcal{M}$ of T is prime if and only if $\mathcal{M}$ is atomic.

(ii) If $\mathcal{M}$ and $\mathcal{N}$ are both countable atomic models of T, then $\mathcal{M} \cong \mathcal{N}$.

Proof. If T has a finite model then all models of T are isomorphic (by Exercise 1.1.11) and every complete type is isolated, making both (i) and (ii) trivially true. Thus, we can assume T to have only infinite models.

(i) It only remains to show that a countable atomic model $\mathcal{M}$ is prime. Let $\mathcal{N}$ be an arbitrary model of T and $\{ a_i : i < \omega \}$ an enumeration of M.

Claim. There is a sequence $\{ b_i : i < \omega \} \subset N$ such that for each $i < \omega$, $tp_{\mathcal{M}}(a_0, \ldots, a_i) = tp_{\mathcal{N}}(b_0, \ldots, b_i)$.

The elements b_i are found by recursion on i. Suppose that $b_0, \ldots, b_{i-1}$ with the desired property have been selected and let $\varphi(v_0, \ldots, v_i)$ be a formula which isolates p = the type of $(a_0, \ldots, a_i)$ in $\mathcal{M}$. When $i = 0$ the completeness of T yields a $b_0 \in N$ such that $\mathcal{N} \models \varphi(b_0)$. In general the existence of a $b_i \in N$ such that $\mathcal{N} \models \varphi(b_0, \ldots, b_i)$ is guaranteed by the inductive hypothesis: $tp_{\mathcal{M}}(a_0, \ldots, a_{i-1}) = tp_{\mathcal{N}}(b_0, \ldots, b_{i-1})$. Being complete, the theory T expresses the fact that φ isolates a complete type; i.e., for any formula $\psi \in p$, $T \models \forall \bar{v}(\varphi(\bar{v}) \longrightarrow \psi(\bar{v}))$. Thus, for all formulas $\psi(v_0, \ldots, v_i)$, $\mathcal{M} \models \psi(a_0, \ldots, a_i) \iff \mathcal{N} \models \psi(b_0, \ldots, b_i)$, proving the claim.

Let f be the map from M into N defined by: $f(a_i) = b_i$, for each $i < \omega$. The reader can verify that f is an elementary map of $\mathcal{M}$ onto $\{ b_i : i < \omega \}$ and $\{ b_i : i < \omega \}$ is the universe of an elementary submodel of $\mathcal{N}$. This proves that $\mathcal{M}$ is a prime model of T.

(ii) The manner in which the isomorphism between $\mathcal{M}$ and $\mathcal{N}$ is constructed is similar to the construction of the embedding in (i). Simply by quoting (i) we know that $\mathcal{M}$ can be elementarily embedded in $\mathcal{N}$, and vice-versa. The construction needs to be altered slightly to obtain an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$ which is surjective. This is our first example of a *back and forth construction* of a map. This frequently used technique is a generalization of the standard uniqueness proof of countable dense linear orderings without endpoints (see the Historical Notes).

Well-order the universes M and N with order type ω. Let a_0 be the first element of M and $\varphi(v_0)$ the formula which isolates the type of a_0 in $\mathcal{M}$. As $\mathcal{N}$ is also a model of T there is a b_0 satisfying $\varphi(x)$ in $\mathcal{N}$. Now, let b_1 be the first element of $N \setminus \{b_0\}$ and find an $a_1 \in M$ such that $tp_{\mathcal{M}}(a_0, a_1) = tp_{\mathcal{N}}(b_0, b_1)$, as in the proof of (i). Let a_2 be the first element of $M \setminus \{a_0, a_1\}$ and find $b_2 \in N$ with $tp_{\mathcal{M}}(a_0, a_1, a_2) = tp_{\mathcal{N}}(b_0, b_1, b_2)$. Going back and forth ω times gives enumerations $\{ a_i : i < \omega \}$ and $\{ b_i : i < \omega \}$ of M and N, respectively, such that for each n, $tp_{\mathcal{M}}(a_0, \ldots, a_n) = tp_{\mathcal{N}}(b_0, \ldots, b_n)$. The map taking a_i to b_i (for $i < \omega$) then defines an isomorphism from $\mathcal{M}$ onto $\mathcal{N}$.

One question we need answered is: Does every complete theory have a prime model, or can we find a meaningful characterization of those which do? The previous proposition reduces the problem of finding a prime model of a countable complete theory to showing that an atomic model exists. The next example shows is not always possible.

Example 2.1.1. (A countable complete theory with no atomic model) Let $L = \{ P_i : i < \omega \}$, where each P_i is a unary relation symbol. Let X be the set of finite sequences of 0's and 1's. Each $s \in X$ is viewed as a function from $\{0, \ldots, m\}$ (for some m) into $\{0, 1\}$, and the *length of* s = $lh(s)$ is defined to be $m + 1$. The theory T will be defined so that for any model $\mathcal{M}$ of T and $s \in X$, the intersection of the family of sets $\{ P_i(\mathcal{M}) : s(i) = 0 \} \cup \{ M \setminus P_i(\mathcal{M}) : s(i) = 1 \}$ is nonempty. Let $P_i^0(v)$ denote the formula $P_i(v)$, and $P_i^1(v)$ the formula $\neg P_i(v)$. For $s \in X$ let $\varphi_s(v)$ be the formula $\bigwedge_{i<lh(s)} P_i^{s(i)}(v)$, $\sigma_s = \exists v \varphi_s(v)$ and $T = \{ \sigma_s : s \in X \}$. The reader can verify that T is a complete quantifier-eliminable theory. Thus, if $\mathcal{M} \models T$ and $a \in M$, the type of a in $\mathcal{M}$ is implied by $\{ P_i^j(v) : \mathcal{M} \models P_i^j(a),\ i < \omega,$ and $j = 0, 1 \}$. We claim that every complete 1–type in T is nonisolated. If, to the contrary, p is an isolated 1–type, then by the characterization of types just mentioned p is isolated by some $\varphi_s \in p$. However, if $j = lh(s)$, both $\exists v(\varphi_s(v) \wedge P_j(v))$ and $\exists v(\varphi_s(v) \wedge \neg P_j(v))$ are in T, proving that φ_s does not isolate a complete type in T. Since T has no isolated 1–types over $\emptyset$, no model of T can be atomic.

A more mathematically common example of a theory without a prime model is the theory of the model $(\mathbb{Z}, +)$, although this property is more difficult to verify for $Th(\mathbb{Z}, +)$ than the theory in the example. A complete description of the countable models of this theory is found in [BBGK73].

For a theory to have an atomic model it must satisfy the next condition, which we will also show is sufficient for countable theories in the subsequent proposition.

Definition 2.1.3. *For T a complete theory we say that the* isolated types of T are dense *if every formula φ in n variables consistent with T is contained in an isolated complete $n-$type over $\emptyset$.*

Such theories are called *atomic* in [CK73]. Our terminology is a literal description of the topological property which holds for such theories: The isolated types of T are dense when every basic open set in the topology on $S_n(\emptyset)$ contains an isolated point, for each n.

Proposition 2.1.2. *Let T be a countable complete theory. Then, T has a countable atomic model if and only if the isolated types of T are dense.*

Proof. For the left-to-right direction let $\mathcal{M} \models T$ be atomic, φ a formula consistent with T and $\bar{a}$ a tuple from M satisfying φ. Then $tp_{\mathcal{M}}(\bar{a})$ is an isolated type extending φ. (Note that this easy direction of the proposition does not require the theory or the atomic model to be countable.)

Now suppose that the isolated types of T are dense. For $n < \omega$ let $\Gamma_n = \{\neg\psi(v_0,\ldots,v_{n-1}) : \psi(v_0,\ldots,v_{n-1})$ isolates a complete $n-$types in $T\}$. If φ is a formula in n variables consistent with T, there is a formula ψ which isolates a complete type in T such that $\varphi \wedge \psi$ is consistent with T, hence Γ_n is nonisolated. By Corollary 1.1.2 (the Extended Omitting Types Theorem) T has a countable model $\mathcal{M}$ omitting each Γ_n. Then each tuple in M satisfies a formula isolating a complete type in T; i.e., $\mathcal{M}$ is atomic.

The next obvious question is: For which theories are the isolated types of T dense? The isolated types are dense for the theory of dense linear orders without endpoints and algebraically closed fields of a fixed characteristic, but not for the theory in Example 2.1.1. As these examples suggest, there is some connection between the density of the isolated types, and the size of $S(\emptyset)$. Specifically, we prove

Lemma 2.1.1. *If T is a complete theory with $|S(\emptyset)| < 2^{\aleph_0}$ then the isolated types of T are dense.*

Proof. This lemma is a special case of a stronger result (Proposition 2.2.6) proved in the next section using Cantor-Bendixson rank. A different proof is included here to improve our picture of atomic models.

Assume that the isolated types of T are not dense. Then there is some n for which $\{\varphi : \varphi$ is a formula in n variables which is consistent with T and not contained in an isolated type$\} = \Phi$ is nonempty. The fact that any formula in n variables which implies an element of Φ is also in Φ is used to construct continuum many complete types with the following recursion. Let X be the set of finite sequences of 0's and 1's and let $\varphi_\emptyset$ be any formula in Φ. Assuming that $s \in X$ and $\varphi_s \in \Phi$ has been defined (for $s \in X$) choose a formula ψ such that $\varphi_s \wedge \psi = \varphi_{s\hat{}0}$ and $\varphi_s \wedge \neg\psi = \varphi_{s\hat{}1}$ are consistent with T. Since $\varphi_{s\hat{}0}$ and $\varphi_{s\hat{}1}$ are both in Φ the recursion can continue. Let Y be the set of sequences of 0's and 1's of length ω, and for $f \in Y$ let $p_f = \{\varphi_s : s$ is an initial segment of f }. By construction, each p_f is consistent and $f \neq g \in Y \implies p_f \cup p_g$ is inconsistent. Extending each p_f to a complete type shows that $|S(\emptyset)| = 2^{\aleph_0}$, completing the proof of the lemma.

Thus, for a countable complete theory, having fewer than continuum many complete types is sufficient to guarantee the existence of a prime model. To see that this condition is not necessary consider, for example, the theory T of the model $(\mathbb{Z}, +, 1)$, where we have added to the cyclic group mentioned earlier a constant symbol for the generator. The reader will show in the exercises that $|S_1(\emptyset)| = 2^{\aleph_0}$. However, since every element of the model $(\mathbb{Z}, +, 1)$ interprets a term of the language, it is an elementary submodel of any model of T; i.e., $(\mathbb{Z}, +, 1)$ is the prime model. This example exhibits a cheap way to find theories with prime models: Given an arbitrary model $\mathcal{M}$ let $\mathcal{M}' = \mathcal{M}_M$. Then $T = Th(\mathcal{M}')$ has $\mathcal{M}'$ as its prime model. A slightly more general result can be obtained vis-a-vis the following notion.

Definition 2.1.4. *Let T be a theory.*

(i) A formula $\varphi(\bar{v})$ is called algebraic in T *if φ is consistent with T and for all $\mathcal{M} \models T$, $\varphi(\mathcal{M})$ is finite.*

(ii) A type is called algebraic in T *if it implies an algebraic formula.*

Remark 2.1.1. (i) For T a complete theory the algebraicity of φ can be tested with any model since if φ defines a finite set in one model then it defines a finite set in every model.

(ii) An algebraic formula is contained in only finitely many complete types in T, each of which is isolated (and algebraic).

(iii) The reader should be cautioned that a theory T may have a model $\mathcal{M}$ and a complete type p such that $p(\mathcal{M})$ is finite, but p is not algebraic (see the exercises).

(iv) A model realizing only algebraic types is atomic, in fact such a model is prime regardless of the cardinality of the language (see the exercises).

Following our conventions for working with parameters, if $\mathcal{M}$ is a model and $A \subset M$, $\mathcal{M}$ is called a *prime model over A* if $\mathcal{M}_A$ is a prime model of $Th(\mathcal{M}_A)$. Referring strictly to models of T, $\mathcal{M}$ is prime over A if whenever $\mathcal{N} \equiv \mathcal{M}$ and $f : A \longrightarrow N$ is an elementary map, there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$ extending f. If $T = Th(\mathcal{M})$ is countable and has fewer than continuum many complete types and $A \subset M$ is finite then $Th(\mathcal{M}_A)$ also has fewer than continuum many complete types (see Exercise 2.2.3 and Lemma 2.2.4). Thus, a countable theory with fewer than continuum many complete types has a prime model over any finite subset of a model. There are few positive results concerning prime and atomic models which hold on the set of all countable theories. For instance, a countable theory may have a prime model over the empty set but not over some other finite set, and vice-versa (as we saw with $Th(\mathbb{Z}, +)$ which does not have a prime model, but does have a prime model over 1).

For uncountable theories the existence and uniqueness of prime and atomic models are all more complicated. The density of the isolated types does not, in general, guarantee the existence of an atomic model. (Remember that our proof in the countable case used the Omitting Types Theorem.) Harrington showed (unpublished) that a prime model of a theory need not be atomic (see Example 5.5.1), and there may be nonisomorphic prime models of a theory. Many of these issues are discussed in [Kni78]. We will see that within the stable theories sharper results are often possible (see Section 5.5).

A model $\mathcal{M}$ is *minimal* if $\mathcal{N} \prec \mathcal{M} \implies \mathcal{N} = \mathcal{M}$. In many of the examples given above the prime model of a theory is also minimal. For example, $\mathcal{M}$ is minimal and prime if it realizes only algebraic types (Exercise 2.1.8). To find a prime model which is not minimal we need look no further than $(\mathbb{Q}, <)$. It is easy to see that if a complete theory has a minimal model and a prime model then they must be isomorphic, however it is possible for a theory to have a minimal model which is not prime; in fact, $Th(\mathbb{Z}, +)$ has continuum many nonisomorphic minimal models (see [BBGK73]).

Historical Notes. The notions of prime and atomic models and the basic properties proved here are due to Vaught [Vau61]. Cantor proved the uniqueness of dense linear orders without endpoints in [Can95], but back-and-forth arguments were first isolated by Huntington in [Hun04] (as pointed out by Jack Plotkin).

Exercise 2.1.1. Show that a theory with a prime model is complete.

Exercise 2.1.2. Let T be a complete theory and φ a formula in n variables which is contained in only finitely many complete $n-$types of T. Show that every complete $n-$type containing φ is isolated.

Exercise 2.1.3. Suppose that $\bar{a}$ and $\bar{b}$ are sequences from a model $\mathcal{M}$ which have the same complete type in $\mathcal{M}$ and $\varphi(v, \bar{a})$ isolates a complete type over $\bar{a}$ (where $\varphi(v, \bar{x})$ is a formula over $\emptyset$). Show that $\varphi(v, \bar{b})$ isolates a complete type over $\bar{b}$.

Exercise 2.1.4. Let $\bar{a}$ and $\bar{b}$ be finite sequences from the universe of the model $\mathcal{M}$. Prove that $tp_{\mathcal{M}}(\bar{a}\bar{b})$ is isolated if and only if $tp_{\mathcal{M}}(\bar{a}/\bar{b})$ and $tp_{\mathcal{M}}(\bar{b})$ are both isolated. Using this fact show that when $\mathcal{M}$ is an atomic model and $\bar{a}$ is a finite sequence from M, then $\mathcal{M}$ is atomic over $\bar{a}$. Conversely, if $\mathcal{M}$ is atomic over $\bar{a}$ and $tp_{\mathcal{M}}(\bar{a})$ is isolated, then $\mathcal{M}$ is atomic.

Exercise 2.1.5. Show that the complete type realized by 1 in $(\mathbb{Z}, +)$ is nonisolated. (HINT: Use the preceding exercise).

Exercise 2.1.6. Show that $Th(\mathbb{Z}, +)$ has continuum many complete $1-$types over $\emptyset$.

Exercise 2.1.7. The definition of an algebraic type is that it implies an algebraic formula, not that it has finitely many realizations in some model. Give an example of a model $\mathcal{M}$ containing an element a which is the only realization of $tp_{\mathcal{M}}(a)$ in $\mathcal{M}$, although this type is not even isolated. (HINT: There is an example in the language with infinitely many constant symbols.)

Exercise 2.1.8. Let $\mathcal{M}$ be a model such that the type in $\mathcal{M}$ of each tuple from M is algebraic. Prove that $\mathcal{M}$ is a prime and minimal model of its theory.

2.2 Saturated and Homogeneous Models

Prime and atomic models are intuitively "small" models of a theory; they are embedded in every model and realize a restricted set of types. In this section we consider "large" models; i.e., those which realize many types and have many models as elementary submodels. Before discussing these properties in full generality we consider their restrictions to the class of countable models, where the results are easier to understand and prove.

Definition 2.2.1. *Let T be a countable complete theory and $\mathcal{M}$ a countable model.*

(i) $\mathcal{M}$ is saturated *if for all finite $A \subset M$, $\mathcal{M}$ realizes every element of $S_1(A)$.*

(ii) $\mathcal{M}$ is homogeneous *if for all finite $A \subset M$, $a \in M$ and elementary maps $f : A \longrightarrow M$, there is an elementary map $g : A \cup \{a\} \longrightarrow M$ extending f.*

(iii) $\mathcal{M}$ is universal *if every countable model of T can be elementarily embedded into $\mathcal{M}$.*

The reader can verify that, in fact, if the countable model $\mathcal{M}$ is saturated then for all finite $A \subset M$, $\mathcal{M}$ realizes every type in $S(A)$ (see Exercise 2.2.7).

Example 2.2.1. (Theories having a saturated countable model)

(i) Let T be the theory in the empty language expressing that the universe is infinite. Then T has a unique countable model (up to isomorphism) and this model is saturated. Similarly, the theory T' of dense linear orders without endpoints has a unique countable model which is also saturated.

(ii) In the language having constant symbols c_i, $i < \omega$, let $T = \{ c_i \neq c_j : i < j < \omega \}$. A saturated countable model (if one exists) must realize the n-type $p_n = \{ v_i \neq v_j : i < j < n \} \cup \{ v_i \neq c_j : i < n,\ j < \omega \}$, which expresses that $v_0, \ldots, v_{n-1}$ are distinct elements, none of which interprets a constant. Thus, a saturated countable model must contain infinitely many elements which do not interpret a constant. Using the quantifier-eliminability of the theory the reader can see that the countable model which contains infinitely many nonconstants is indeed saturated.

(iii) Let T be the theory in the language with unary relations P_0, $P_1, \ldots$ and constants c_{ij}, $i, j < \omega$, axiomatized by the statements: $\neg\exists x(P_i(x) \wedge P_j(x))$ for $i \neq j$; $P_i(c_{ij})$ for $i, j < \omega$; and $c_{ij} \neq c_{ik}$ for $j < k$. This theory is complete and has elimination of quantifiers. For each i there is an n−type saying that $v_0, \ldots, v_{n-1}$ are distinct elements satisfying P_i and none of the v_j's interpret a constant. There is also an n−type expressing that $v_0, \ldots, v_{n-1}$ are distinct elements not satisfying any of the P_i's. Thus, for $\mathcal{M}$ to be a countable saturated model it must at least have the properties that each $P_i(\mathcal{M})$ has infinitely many nonconstant elements and there are infinitely many elements of M not satisfying any P_i. Using the elimination of quantifiers the reader can verify that, in fact, any countable model satisfying these conditions is saturated.

(iv) Let T be the theory of algebraically closed fields of characteristic 0. For each n there is an n−type expressing that $v_0, \ldots, v_{n-1}$ are algebraically independent transcendental elements. So, for a model to have any hope of being saturated it must have infinite transcendence degree over the rationals. The quantifier-eliminability of the theory implies that every complete type is determined by the equations and inequalities it contains. Using this we see that a countable model with transcendence degree $\aleph_0$ over the rationals is saturated.

As a first result we offer

Lemma 2.2.1. *A saturated countable model $\mathcal{M}$ is homogeneous.*

Proof. Let $\bar{a} = (a_1, \ldots, a_n)$ be a tuple from M, $f : \bar{a} \longrightarrow M$ an elementary map, $(f(a_1), \ldots, f(a_n)) = (b_1, \ldots, b_n) = \bar{b}$ and c an element of M. Enumerate $q = tp_{\mathcal{M}}(c, a_1, \ldots, a_n)$ as $\{\varphi_j(v, v_1, \ldots, v_n) : j < \omega\}$ and let $p = \{\varphi_j(v, b_1, \ldots, b_n) : j < \omega\}$. Since $\bar{a}$ and $\bar{b}$ realize the same complete type over $\emptyset$, p is a complete 1−type over $\bar{b}$. There is a $d \in M$ realizing p since $\mathcal{M}$ is saturated. Then $(d, b_1, \ldots, b_n)$ realizes q, equivalently, the map from $(c, a_1, \ldots, a_n)$ to $(d, b_1, \ldots, b_n)$ which extends f and takes c to d is elementary. This proves that $\mathcal{M}$ is homogeneous.

(Later it is proved in the context of potentially uncountable models that a model is saturated if and only if it is homogeneous and universal.)

We turn now to question: When does a countable complete theory have a saturated countable model?

Definition 2.2.2. *A countable complete theory is called* small *if $S(\emptyset)$ is countable.*

Remark 2.2.1. A countable model can only realize countably many complete types, hence only a small theory can have a countable saturated model.

In fact, we will show that smallness is sufficient for the existence of a saturated countable model. The proof of this fact requires a new concept.

Following, is a natural way to construct the saturated countable algebraically closed field of characteristic 0. Let $\mathcal{M}_0 = \tilde{\mathbb{Q}}$, the algebraic numbers. Let a_0 be a transcendental element and $\mathcal{M}_1$ the algebraic closure of $M_0 \cup \{a_0\}$. Let a_1 be an element transcendental over M_1 and $\mathcal{M}_2$ the algebraic closure of $M_1 \cup \{a_1\}$. Continuing in this manner results in a chain $\mathcal{M}_0 \subset \ldots \subset \mathcal{M}_i \subset \ldots$, for $i < \omega$, of algebraically closed fields whose union has transcendence degree $\aleph_0$ (and hence is a saturated countable model). Our general construction of saturated countable models (and, subsequently, models with other properties) will use a generalization of this notion of the union of a chain of fields.

Definition 2.2.3. *A* chain of models *is an increasing sequence of models*

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_\beta \subset \ldots, \quad \beta < \alpha,$$

whose length is an ordinal α.

The union of the chain *is the model $\mathcal{M} = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ defined as follows:*

- *The universe of $\mathcal{M}$ is $M = \bigcup_{\beta<\alpha} M_\beta$.*
- *If R is a relation or function symbol then the interpretation of R on M is $\bigcup_{\beta<\alpha} R^{\mathcal{M}_\beta}$, and*
- *for c a constant symbol, $c^{\mathcal{M}} = c^{\mathcal{M}_\beta}$ for any (all) $\beta < \alpha$.*

Remark 2.2.2. The union of the chain $\{\mathcal{M}_\beta : \beta < \alpha\}$ is the minimal model $\mathcal{N}$ such that $\mathcal{M}_\beta \subset \mathcal{N}$, for all $\beta < \alpha$.

An important special case is obtained by replacing the submodel relation by the elementary submodel relation.

Definition 2.2.4. *An* elementary chain of models *is an increasing sequence of models*

$$\mathcal{M}_0 \prec \mathcal{M}_1 \prec \ldots \prec \mathcal{M}_\beta \prec \ldots, \quad \beta < \alpha,$$

whose length is an ordinal α.

Lemma 2.2.2. *Let $\mathcal{M}_\beta$, $\beta < \alpha$, be an elementary chain of models and $\mathcal{M} = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$. Then $\mathcal{M}_\beta \prec \mathcal{M}$, for all $\beta < \alpha$, and $\mathcal{M}$ is the minimal such model in the sense that whenever $\mathcal{M}_\beta \prec \mathcal{N}$, for all $\beta < \alpha$, $\mathcal{M} \prec \mathcal{N}$.*

Proof. We prove by induction on formulas:

$$\text{for all } \varphi(v_1, \ldots, v_n), \text{ all } \beta < \alpha \text{ and all } a_1, \ldots, a_n \in M_\beta,$$
$$\mathcal{M}_\beta \models \varphi(a_1, \ldots, a_n) \iff \mathcal{M} \models \varphi(a_1, \ldots, a_n).$$

This is clear when φ is atomic since $\mathcal{M}_\beta \subset \mathcal{M}$. The only case in the induction requiring any work is $\varphi = \exists v \psi(v, v_1, \ldots, v_n)$. Suppose that $a_1, \ldots, a_n \in M_\beta$ and $\mathcal{M}_\beta \models \varphi(a_1, \ldots, a_n)$. Then there is a $b \in M_\beta$ such that $\mathcal{M}_\beta \models \psi(b, a_1, \ldots, a_n)$, hence $\mathcal{M} \models \psi(b, a_1, \ldots, a_n)$ and $\mathcal{M} \models \varphi(a_1, \ldots, a_n)$ (by induction). On the other hand, suppose that $\mathcal{M} \models \varphi(a_1, \ldots, a_n)$. Then, for some $\gamma \geq \beta$ and $b \in M_\gamma$, $\mathcal{M} \models \psi(b, a_1, \ldots, a_n)$. By induction, $\mathcal{M}_\gamma \models \psi(b, a_1, \ldots, a_n)$, hence $\mathcal{M}_\gamma \models \exists v \psi(v, a_1, \ldots, a_n)$. Because $\mathcal{M}_\beta \prec \mathcal{M}_\gamma$, $\mathcal{M}_\beta \models \varphi(a_1, \ldots, a_n)$. Thus, $\mathcal{M}_\beta \prec \mathcal{M}$ for all $\beta < \alpha$.

That $\mathcal{M}$ is the minimal such extension of the $\mathcal{M}_\beta$'s is left as an exercise.

Returning to the study of saturated countable models we prove:

Proposition 2.2.1. *A countable complete theory T has a saturated countable model if and only if it is small.*

Proof. If T has a saturated countable model $\mathcal{M}$ then $|S(\emptyset)| = \aleph_0$, since each complete type over the empty set is realized by one of the countably many finite tuples from M. Now assume T to be small. The converse will be established by constructing a saturated countable model as the union of an elementary chain of length ω defined as follows. Let $\mathcal{M}_0$ be any countable model of T, $i < \omega$, and suppose $\mathcal{M}_i$ has been defined. Since T is small, for any $\mathcal{M} \models T$ and finite $A \subset M$, $|S_1(A)| \leq \aleph_0$ (see Exercise 2.2.3). Thus, there is a countable elementary extension $\mathcal{M}_{i+1}$ of $\mathcal{M}_i$ such that for all finite $A \subset M_i$, every element of $S_1(A)$ is realized in $\mathcal{M}_{i+1}$. The union $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$ is the desired countable saturated model.

The saturated countable model, when it exists, is unique:

Proposition 2.2.2. *Elementarily equivalent saturated countable models are isomorphic.*

Proof. The proof of this result is similar to the uniqueness proof for countable atomic models (Proposition 2.1.1(ii)). In fact, these two propositions are special cases of the same result about homogeneous models (Corollary 2.2.2).

Let $\mathcal{M}$ and $\mathcal{N}$ be saturated countable models of the same complete theory. Well-order the universes M and N with order type ω. Letting a_0 be the first element of M there is (by the saturation of $\mathcal{N}$) a $b_0 \in N$ such that $tp_{\mathcal{M}}(a_0) = tp_{\mathcal{N}}(b_0)$. Now let b_1 be the first element of $N \setminus \{b_0\}$ and find an $a_1 \in M$ such that $tp_{\mathcal{M}}(a_0, a_1) = tp_{\mathcal{N}}(b_0, b_1)$ (see the proof of Lemma 2.2.1 for a similar use of saturation). Let a_2 be the first element of $M \setminus \{a_0, a_1\}$ and find $b_2 \in N$ such that $tp_{\mathcal{M}}(a_0, a_1, a_2) = tp_{\mathcal{N}}(b_0, b_1, b_2)$. Going back and forth ω times we find enumerations $\{ a_i : i < \omega \}$ and $\{ b_i : i < \omega \}$ of M and N, respectively, such that for each n, $tp_{\mathcal{M}}(a_0, \ldots, a_n) = tp_{\mathcal{N}}(b_0, \ldots, b_n)$. The map taking a_i to b_i (for $i < \omega$) then defines an isomorphism from $\mathcal{M}$ onto $\mathcal{N}$.

Homogeneity can be viewed as trading some of the strengths of saturation for less stringent requirements for the existence of models with the property (as the next lemma illustrates).

Proposition 2.2.3. *A countable complete theory has a homogeneous countable model.*

Proof. A homogeneous countable model $\mathcal{M}$ is constructed as the union of an elementary chain of models $\mathcal{M}_i$, $i < \omega$, defined as follows. Let $\mathcal{M}_0$ be any countable model. Assuming $\mathcal{M}_i$ to be defined let $\mathcal{M}_{i+1}$ be a countable elementary extension of $\mathcal{M}_i$ such that

> if $\bar{a}$ and $\bar{b}$ are finite sequences from M_i realizing the same complete type in $\mathcal{M}_i$, then for all $c \in M_i$ there is a $d \in M_{i+1}$ such that $tp_{\mathcal{M}_i}(\bar{a}, c) = tp_{\mathcal{M}_{i+1}}(\bar{b}, d)$.

(A countable such $\mathcal{M}_{i+1}$ exists because there are countably many finite sequences from M_i.) The reader can verify that the model $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$ is a homogeneous countable model of T.

Saturation is only achieved when a countable model realizes all types over finite subsets, however homogeneity can occur when enough types are omitted:

Proposition 2.2.4. *A countable atomic model $\mathcal{M}$ is homogeneous.*

Proof. Suppose that $\bar{a}$ and $\bar{b}$ are finite sequences realizing the same complete types in $\mathcal{M}$, and $c \in M$. Let $\varphi(\bar{y}x) \in tp_{\mathcal{M}}(\bar{a}c) = q$ be a formula isolating q. Since $\bar{b}$ and $\bar{a}$ realize the same complete type in $\mathcal{M}$, there is a $d \in M$ such that $\mathcal{M} \models \varphi(\bar{b}d)$. Since the formula φ isolates a complete type $tp_{\mathcal{M}}(\bar{a}c) = tp_{\mathcal{M}}(\bar{b}d)$, proving the homogeneity of $\mathcal{M}$.

Homogeneous countable models of the same complete theory are not necessarily isomorphic, indeed, many of the above examples have a countable saturated model and a prime model which are not isomorphic. There is, however, a relative uniqueness result:

Proposition 2.2.5. *If $\mathcal{M}$ and $\mathcal{N}$ are countable homogeneous models in the same language which realize the same elements of $S(\emptyset)$, then $\mathcal{M} \cong \mathcal{N}$.*

This proposition is a special case of Corollary 2.2.2, to be proved later, so we omit the proof for the sake of brevity. We recommend, however, that the reader construct an independent proof by adapting the argument used in Proposition 2.2.2.

Let T be a countable complete theory. We proved that T has a countable atomic model when $|S(\emptyset)| < 2^{\aleph_0}$ and T has a countable saturated model when $S(\emptyset)$ is countable. It is natural to ask if there is a countable complete theory with $|S(\emptyset)|$ strictly between $\aleph_0$ and $2^{\aleph_0}$ (in some model of set theory). The Cantor-Bendixson Theorem from point-set topology quickly gives a negative answer: $S_n(\emptyset)$ is countable or has cardinality $2^{\aleph_0}$. We will reproduce the proof of this theorem in the terminology of formulas and types, rather than open sets and points, not just for the sake of completeness but also to introduce Cantor-Bendixson rank which will have further applications. (Recall from Section 1.1 the definition of p implies φ, denoted $p \models \varphi$, where p is a type and φ is a formula.)

Definition 2.2.5. *Let T be a complete theory. The relation $CB(\varphi) = \alpha$, for φ a formula in n variables and α an ordinal or -1 is defined by the following recursion.*

(1) $CB(\varphi) = -1$ *if φ is inconsistent;*
(2) *Let* $\Psi_\alpha = \{\, \psi : CB(\psi) = \beta \text{ for some } \beta < \alpha \,\}$.
$CB(\varphi) = \alpha$ *if* $\{\, p \in S_n(\emptyset) : \varphi \in p \text{ and } \neg\psi \in p \text{ for all } \psi \in \Psi_\alpha \,\}$ *is nonempty and finite.*

For p any $n-$type, $CB(p)$ is defined to be

$$\inf \{CB(\varphi) : \varphi \text{ a formula implied by } p\}.$$

(For complete types $CB(p)$ is then $\inf\{CB(\varphi) : \varphi \in p\}$.*) When $CB(p) = \alpha$ we say that the* Cantor-Bendixson rank of p is α. *If there is no α with $CB(p) = \alpha$ we write $CB(p) = \infty$ and say that the Cantor-Bendixson rank of p does not exist.*

Extend the scope of $<$ so that $-1 < \alpha < \infty$ for all ordinals α. Then, $CB(p) \geq \alpha$ is a quick way to express that $CB(p) \neq \beta$ for all $\beta < \alpha$. Using these conventions (2) in the definition can be restated as: $CB(\varphi) = \alpha$ if $\{\, p \in S_n(\emptyset) : \varphi \in p \text{ and } CB(p) \geq \alpha \,\}$ is finite and nonempty. The term Cantor-Bendixson rank is usually shortened to *CB-rank*. It is clear from the definition that the CB relation defines a function.

The theme in the next basic lemma is the relationship between the CB-ranks of types and the CB-ranks of formulas implied by these types.

Lemma 2.2.3. *Let T be a complete theory, p an $n-$type and α an ordinal.*

(i) If p is complete then $CB(p) = 0$ if and only if p is isolated.

(ii) $CB(p) = \alpha$ if and only if there is a formula φ implied by p such that $\{\, q \in S_n(\emptyset) : \varphi \in q \text{ and } CB(q) = \alpha \,\}$ is finite and nonempty. Moreover, when $CB(p) = \alpha$ we can find a φ implied by φ such that $\{\, q \in S_n(\emptyset) : \varphi \in q \text{ and } CB(q) = \alpha \,\} = \{\, q \in S_n(\emptyset) : p \subset q \text{ and } CB(q) = \alpha \,\}$.

(iii) If $CB(p) = \alpha$ there is a $q \in S_n(\emptyset)$ such that $q \supset p$ and $CB(q) = \alpha$.

(iv) If p is complete and $CB(p) = \alpha$ there is a $\varphi \in p$ such that p is the only element of $\{\, q \in S_n(\emptyset) : \varphi \in q \text{ and } CB(q) \geq \alpha \,\}$.

(v) $CB(p) \geq \alpha$ if and only if, for all $\beta < \alpha$ and all φ implied by p, $\{\, q \in S_n(\emptyset) : \varphi \in q \text{ and } CB(q) \geq \beta \,\}$ is infinite.

(vi)

$$CB(\varphi) \text{ is the least ordinal } \alpha \text{ such that} \quad \{\, p \in S_n(\emptyset) : \varphi \in p \text{ and } CB(p) \geq \alpha \,\} \text{ is finite.} \tag{2.1}$$

Proof. (i) If p is isolated by the formula φ then $CB(\varphi) = 0$, hence $CB(p) = 0$. Assuming, conversely, that p has CB-rank 0, there is a $\varphi \in p$ which is contained in only finitely many complete types, say $q_0, \ldots, q_k$, with $q_0 = p$. Let ψ be a formula in p implying φ and not in any of $q_1, \ldots, q_k$. Then ψ isolates p. (In fact, every completion of φ is isolated.)

(ii) Let Ψ be the set of formulas implied by p which have CB-rank α and $\Theta = \{\neg\theta : \theta$ is a formula in n variables with $CB(\theta) < \alpha\}$. Then $\psi \in \Psi \implies X_\psi = \{q \in S_n(\emptyset) : \psi \in q$ and $q \supset \Theta\}$ is finite and nonempty. Furthermore, if $\psi, \psi' \in \Psi$ and ψ implies ψ', $X_\psi \subset X_{\psi'}$. Thus, there is a $\varphi \in \Psi$ such that $X_\varphi = X_\psi$ for all $\psi \in \Psi$ which imply φ. Since p is implied by Ψ each element of X_φ contains p; i.e., $X_\varphi = \{q \in S_n(\emptyset) : p \subset q$ and $CB(q) = \alpha\}$. Since X_φ is finite and nonempty the proof of this part is complete.

(iii) This is just a repeat of part of (ii) but having an explicit reference is helpful.

(iv) This follows immediately from (ii). The details are left to the reader in Exercise 2.2.1.

(v) ($\Longleftarrow$) First notice that $CB(p) \geq \alpha$ if and only if

$$\text{for all } \varphi \text{ implied by } p,\ \{q \in S_n(\emptyset) : \varphi \in q \text{ and } CB(q) \geq \alpha\} \text{ is nonempty,}$$

which in turn is equivalent to

$$\text{for all } \varphi \text{ implied by } p,\ \{\varphi\} \cup \{\neg\psi : CB(\psi) < \alpha\} \text{ is consistent.}$$

Suppose that φ is implied by p and $\{\varphi\} \cup \{\neg\psi : CB(\psi) < \alpha\}$ is inconsistent. This inconsistency yields formulas $\psi_0, \ldots, \psi_n$ of CB-rank $< \alpha$ such that any complete type over $\emptyset$ containing φ also contains one of the ψ_i's. Since each of the ψ_i's has CB-rank $< \alpha$, $\beta = \max\{CB(\psi_0), \ldots, CB(\psi_n)\}$ is also $< \alpha$. For each $i \leq n$, $X_i = \{q \in S_n(\emptyset) : \psi_i \in q$ and $CB(q) \geq \beta\}$ is finite (simply because $CB(\psi_i) \leq \beta$), hence $X_0 \cup \ldots \cup X_n = \{q \in S_n(\emptyset) : \varphi \in q$ and $CB(q) \geq \beta\}$ is finite. Since $\beta < \alpha$ the right-hand-side of (v) fails, proving this direction.

($\Longrightarrow$) Suppose the right-hand-side of (v) to fail; i.e., there is a $\beta < \alpha$ and a φ implied by p such that $X = \{q \in S_n(\emptyset) : \varphi \in q$ and $CB(q) \geq \beta\}$ is finite. If X is empty then $CB(\varphi) < \beta$ (by (ii)), while if it is nonempty the definition of CB-rank yields $CB(\varphi) = \beta$. We conclude that $CB(\varphi)$, hence $CB(p)$ is not $\geq \alpha$, completing the proof.

(vi) This follows immediately from (v).

Occasionally, (2.1) is used as the definition of Cantor-Bendixson rank. Our definition is handy for proving properties of CB-rank, however (2.1) can be helpful in understanding CB-rank in particular examples (such as the ones given below).

When p is a complete type of CB-rank α and $\varphi \in p$ is such that p is the only completion of φ of CB-rank $\geq \alpha$ (see Lemma 2.2.3(iv)) we say that "φ isolates p relative to the types of CB-rank $\geq \alpha$."

Before applying Cantor-Bendixson rank to the problem of determining the possible cardinalities of $S(\emptyset)$ we give some illustrative examples.

Example 2.2.2. Let $L_1 = \{E\}$, where E is a binary relation. The theory T_1 in L_1 expressing that E is an equivalence relation with two classes, each infinite, is quantifier-eliminable. Let $\mathcal{M}$ be a model of T_1 and $T = Th(\mathcal{M}_M)$.

(So, by our conventions on parameters $S(\emptyset)$ in T is the same as $S(M)$ in T_1.) What does $S_1(\emptyset)$ in T look like? The isolated elements of $S_1(\emptyset)$, hence the elements of CB-rank 0, are exactly those containing $x = a$ for some $a \in M$. Let p be a complete 1−type containing $\{x \neq b : b \in M\}$. In any model of T the only E−classes are the two classes represented in M, so $E(x, a) \in p$ for some $a \in M$. If $b \in M$, then $E(x, b) \in p$ if and only if $\mathcal{M} \models E(a, b)$. By the quantifier-eliminability of T, p is the only element of $S_1(\emptyset)$ containing $\{E(x,a)\} \cup \{x \neq b : b \in M\} = \{E(x,a)\} \cup \{\neg\psi(x) : CB(\psi(x)) = 0\}$, hence $CB(E(x,a)) = CB(p) = 1$ (by (2.1)). Summarizing, $S_1(\emptyset)$ contains infinitely many isolated types and two types of CB-rank 1. (Notice that even though there is not a unique complete type of CB-rank ≥ 1, $CB(x = x) = 1$.)

Example 2.2.3. In the same language L_1 let T_2 be the theory saying that E is an equivalence relation with infinitely many infinite classes and no finite classes. Let $\mathcal{M}$ be a model of T_2, $T = Th(\mathcal{M}_M)$ and notice that T is also quantifier-eliminable. As in the previous example an element of $S_1(\emptyset)$ is isolated if and only if it contains $x = b$ for some $b \in M$. Also, for any $a \in M$, the formula $E(x, a)$ isolates a complete type q_a relative to the nonisolated types, hence $CB(E(x,a)) = CB(q_a) = 1$. Now consider any $p \in S_1(\emptyset)$ containing $\{x \neq b : b \in M\} \cup \{\neg E(x,a) : a \in M\}$, which exists by compactness. For any $\varphi \in p$ there are infinitely many $a \in M$ with $\varphi \in q_a$, so $CB(p) \geq 2$ by Lemma 2.2.3(v). Since p is the only complete 1−type which is nonisolated and not one of the q_a's (by quantifier elimination), $CB(x = x) = CB(p) = 2$ by (1).

Example 2.2.4. Here it is shown that for any ordinal α there is a theory with a type of CB-rank α. The theory is formulated as a chain of refining equivalence relations. Let α be an ordinal, and for $1 \leq \beta \leq \alpha$, let E_β be a binary relation. Let T_1 be the theory saying that each E_β is an equivalence relation with only infinite classes and for $1 \leq \beta < \gamma \leq \alpha$, E_β refines E_γ and each E_γ−class contains infinitely many E_β−classes. These two properties are guaranteed with the axioms:

$$\forall xy(E_\beta(x,y) \to E_\gamma(x,y)) \text{ and, for all } n < \omega,$$
$$\forall x \exists y_0 \cdots y_n \Big(\bigwedge_{i \neq j \leq n} E_\gamma(x, y_i) \wedge \neg E_\beta(y_i, y_j) \Big).$$

For $\mathcal{M} \models T_1$ let $T = Th(\mathcal{M}_M)$. As usual, T has elimination of quantifiers. Let $E_0(x,y)$ denote $x = y$.

Claim. For $\beta \leq \alpha$ and $a \in M$, $CB(E_\beta(x,a)) = \beta$.

This is proved by induction on β, with the case $\beta = 0$ being trivial. Let $\beta > 0$ and fix $\gamma < \beta$. There is an infinite set $B \subset M$ such that $E_\gamma(x,b)$ implies $E_\beta(x,a)$ and $b \neq b' \implies \neg E_\gamma(b,b')$ for all b, $b' \in B$. By induction, $E_\gamma(x,b)$ has CB-rank γ, for each $b \in B$, and extends to a complete 1−type of CB-rank γ by Lemma 2.2.3(iii). Since B is infinite this proves that $\{q \in S_1(\emptyset) :$

$E_\beta(x, a) \in q$ and $CB(q) \geq \gamma\}$ is infinite. By Lemma 2.2.3(v), $CB(E_\beta(x, a)) \geq \beta$. Furthermore, $X = \{q \in S_1(\emptyset) : E_\beta(x, a) \in q \text{ and } CB(q) \geq \beta\} \subset \{q \in S_1(\emptyset) : E_\beta(x, a) \in q \text{ and } \neg E_\gamma(x, b) \in q, \text{ for } \gamma < \beta \text{ and } b \in M\} = Y$, X is nonempty (by Lemma 2.2.3(iii)) and Y contains a unique type by quantifier elimination. We conclude from Lemma 2.2.3(vi) that $CB(E_\beta(x, a)) = \beta$, as claimed.

From the axioms for the theory and the claim we conclude that T contains a complete type of CB-rank β for each $\beta \leq \alpha$.

While we restricted our attention to 1−types in these examples, similar arguments show that every element of $S(\emptyset)$ has CB-rank in each of these examples.

Example 2.2.5. In this final example a theory is specified (also involving equivalence relations) in which no nonalgebraic element of $S_1(\emptyset)$ has CB-rank. For $i < \omega$, let E_i be a binary relation. Let T_1 say that each E_i is an equivalence relation with infinitely many infinite classes and no finite classes and E_{i+1} refines E_i. Furthermore, each E_i−class is partitioned into infinitely many E_{i+1}−classes. Let $\mathcal{M} \models T_1$ and $T = Th(\mathcal{M}_M)$. An easy induction shows that for all nonalgebraic $p \in S_1(\emptyset)$ and ordinals α, $CB(p) \geq \alpha$.

In this section Cantor-Bendixson rank is applied to countable theories, however no assumption of countability is made in the definition or in the above examples.

The connection with the number of types in a countable theory is found in

Lemma 2.2.4. *For T a countable complete theory, the following are equivalent for each n.*

(1) $|S_n(\emptyset)| = \aleph_0$.
(2) $|S_n(\emptyset)| < 2^{\aleph_0}$.
(3) *Every $p \in S_n(\emptyset)$ has CB-rank equal to some $\alpha < \omega_1$.*

Proof. Trivially, (1) implies (2). To prove that (3) implies (1), let $\Phi = \{\varphi : \varphi$ is a formula in n variables having a unique completion p with $CB(p) = CB(\varphi)\}$. For each $\varphi \in \Phi$ let p_φ denote the completion of φ with the same CB-rank. By Lemma 2.2.3(iv) and the assumption that (3) holds, each element of $S_n(\emptyset)$ is p_φ for some $\varphi \in \Phi$. The countability of the theory then forces $S_n(\emptyset)$ to be countable; i.e., (1) holds.

To complete the proof we assume (3) to fail and prove that $|S_n(\emptyset)| = 2^{\aleph_0}$. Let Φ be the set of formulas in n variables which have CB-rank $< \omega_1$, and let $\alpha_{CB} = \sup\{CB(\varphi) : \varphi \in \Phi\}$. Since we have assumed (3) to fail there is a formula not in Φ.

Claim. For any formula φ in n variables not in Φ there is a formula ψ in n variables such that $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are both not in Φ.

Since $CB(\varphi) \geq \omega_1$, for each $\alpha < \omega_1$ there is a formula ψ_α such that $CB(\varphi \wedge \psi_\alpha) \geq \alpha$ and $CB(\varphi \wedge \neg\psi_\alpha) \geq \alpha$ (by Lemma 2.2.3(v)). Since there are countably many formulas there is a single formula ψ which is ψ_α for arbitrarily large $\alpha < \omega_1$. Thus, $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are both not in Φ.

Let X be the set of finite sequences of 0's and 1's.

Claim. There is a family of formulas φ_s, for $s \in X$, with the properties: (a) $\varphi_s \notin \Phi$, (b) if t is an initial segment of s then φ_s implies φ_t and (c) if t is not an initial segment of s and s is not an initial segment of t, then $\varphi_s \wedge \varphi_t$ is inconsistent with T.

The collection of φ_s's is defined by recursion. Given φ_s there is a ψ such that $\varphi_s \wedge \psi$ and $\varphi_s \wedge \neg\psi$ are both not in Φ (by the first claim). Let $\varphi_{s^\frown\langle 0\rangle} = \varphi_s \wedge \psi$ and $\varphi_{s^\frown\langle 1\rangle} = \varphi_s \wedge \neg\psi$. Properties (a), (b) and (c) are easy to verify.

Let Y denote the set of sequences of 0's and 1's of length ω and for $f \in Y$ let $p_f = \{\varphi_s : s \text{ is an initial segment of } f\}$. Each p_f is consistent (by (b)) and for distinct f and g in Y, $p_f \cup p_g$ is inconsistent (by (c)). Consistent completions of the p_f's form $2^{\aleph_0}$ many elements of $S_n(\emptyset)$, completing the proof.

In Lemma 2.1.1 it was proved that a small theory has a countable atomic model. The next proposition generalizes this result to potentially uncountable theories.

Proposition 2.2.6. *Let T be a complete theory in which each $p \in S(\emptyset)$ has CB-rank. Then the isolated types are dense in T.*

Proof. Let $\varphi(\bar{v})$ be a formula consistent with T and $\mathcal{O}_\varphi$ the set of complete types in $\bar{v}$ containing φ. Let $p \in \mathcal{O}_\varphi$ have minimal CB-rank among the elements of $\mathcal{O}_\varphi$ and let $\psi \in p$ be a formula isolating p relative to the types of CB-rank $\geq CB(p) = \alpha$. Without loss of generality, ψ implies φ, equivalently, $\mathcal{O}_\psi \subset \mathcal{O}_\varphi$. The following implications show that ψ isolates p: $q \in \mathcal{O}_\psi \implies q \in \mathcal{O}_\varphi \implies CB(q) \geq \alpha \implies q = p$.

A subset X of a topological space is *perfect* if it is closed and contains no isolated points, while X is *scattered* if every nonempty subset of X contains an isolated point. In topological terms the proof of Proposition 2.2.6 can be extended slightly to show that $S_n(\emptyset)$ is scattered when each complete $n-$type has CB-rank. Thus, by Lemma 2.2.4, a countable complete theory is small if and only if $S_n(\emptyset)$ is scattered, for each $n < \omega$. More generally, the set of complete $n-$types in a countable complete theory is the union of two disjoint sets, one scattered and one which is empty or perfect. The scattered set, which is countable, is the set of complete types having CB-rank, while the set of complete types without CB-rank, if nonempty, is perfect and of cardinality $2^{\aleph_0}$. (The reader is asked to prove this fact in Exercise 2.2.6.)

The remainder of this section is devoted to a complete treatment of saturated, homogeneous and universal models of potentially uncountable cardinalities.

Definition 2.2.6. *Let κ be an infinite cardinal and $\mathcal{M}$ a model.*

*(i) We call $\mathcal{M}$ $\kappa-$*saturated *if for all $A \subset M$ of cardinality $< \kappa$, $\mathcal{M}$ realizes every type in $S_1(A)$.*

*(ii) $\mathcal{M}$ is $\kappa-$*homogeneous *if for all $A \subset M$ with $|A| < \kappa$, $a \in M$ and elementary maps $f : A \longrightarrow M$, there is an elementary map $g : A \cup \{a\} \longrightarrow M$ extending f.*

*(iii) $\mathcal{M}$ is $\kappa-$*universal *if every model $\mathcal{N} \equiv \mathcal{M}$ of cardinality $< \kappa$ can be elementarily embedded into $\mathcal{M}$.*

If $\mathcal{M}$ is $|M|-$saturated, $|M|-$homogeneous or $|M|^+-$universal we call $\mathcal{M}$ saturated, homogeneous *or* universal, *respectively.*

As with countable saturated models, if $\mathcal{M}$ is $\kappa-$saturated then for all $A \subset M$ of cardinality $< \kappa$, $\mathcal{M}$ realizes every type in $S(A)$ (see Exercise 2.2.7).

Example 2.2.6. (Uncountable saturated models)

Let F be a countable field and T the theory of infinite dimensional vector spaces over F (which is complete and quantifier-eliminable). We will show that every uncountable model of T is saturated using the following.

Claim. If $\mathcal{M} \models T$ and $A \subset M$ there is a unique nonalgebraic type in $S_1(A)$.

Let p, $q \in S_1(A)$ be nonalgebraic. Taking an elementary extension of $\mathcal{M}$ if necessary we can assume that p and q are realized by elements a, $b \in M$, respectively. Since a and b are not in the subspace generated by A there is an automorphism of $\mathcal{M}$ fixing A and mapping a to b. Since automorphisms preserve types $p = q$.

Now let $\mathcal{M}$ be a model of T of cardinality $\kappa > \aleph_0$ and let $A \subset M$ have cardinality $< \kappa$. Any algebraic type over A is realized in $\mathcal{M}$, so it suffices to consider the unique nonalgebraic $p \in S_1(A)$. Since $|A| < \kappa$ and the field is countable the subspace generated by A has cardinality $< \kappa$. (The quantifier-eliminability of T implies that $tp(a/A)$ is algebraic if and only if a is in the subspace generated by A.) Thus, there is an $a \in M$ such that $tp(a/A)$ is p. Thus, $\mathcal{M}$ is saturated.

Not every countable complete theory has an $\aleph_0-$saturated model in every infinite power. For example, if the theory has continuum many complete types over $\emptyset$ then every $\aleph_0-$saturated model has cardinality $\geq 2^{\aleph_0}$ (since continuum many tuples from a model are needed to realize all of the types). There are similar (and often more complicated) restrictions on the cardinalities of $\lambda-$saturated models of arbitrary theories when λ is an uncountable cardinal. Consider, for instance, the following theory.

Example 2.2.7. The language L consist of two unary relations, P and Q, and a binary relation R. Let X be a set of cardinality $\aleph_0$, Y the power set of X and $E \subset X \times Y$ the relation which is satisfied by (x, y) exactly when x is an element of y. Let $\mathcal{M}$ be the model in this language with universe $X \cup Y$ where X is the interpretation of P, Y interprets Q and E interprets R. Let $T = Th(\mathcal{M})$. Then, for κ an infinite cardinal, any κ^+−saturated model $\mathcal{N}$ of T must have cardinality $\geq 2^\kappa$. (Let A be a subset of $P(\mathcal{N})$ of cardinality κ. For any $B \subset A$ the set of formulas $p_B = \{ R(b, v) : b \in B \} \cup \{ \neg R(a, v) : a \in A \setminus B \}$ is consistent and realized in $\mathcal{N}$, since $\mathcal{N}$ is κ^+−saturated. The realizations of the types p_B, as B ranges over the subsets of A, form a subset of $Q(\mathcal{N})$ of cardinality 2^κ.) Thus, given λ an infinite cardinal, if a saturated model of cardinality λ exists, then $\lambda \geq \sum_{\mu<\lambda} 2^\mu$.

In conclusion, the existence of a saturated model of cardinality κ may require κ to satisfy a relation of cardinal arithmetic which could fail in some model of set theory.

Recall that a cardinal λ is *regular* if λ is infinite and has cofinality λ. For all infinite cardinals κ, κ^+ is regular.

For an arbitrary theory the most general statement that can be made about the existence of models with some amount of saturation is

Lemma 2.2.5. *Let T be a complete theory and κ an infinite cardinal $\geq |T|$. Then T has a κ^+−saturated model of cardinality 2^κ.*

Proof. The targeted model will be constructed as the union of an elementary chain of models. At successor stages in the recursive definition of the elementary chain we will use

Claim. Let $\mathcal{M}$ be a model of T of cardinality 2^κ. Then there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ of cardinality 2^κ such that for every $A \subset M$ of cardinality κ, $\mathcal{N}$ realizes every element of $S_1(A)$.

First, let's count the number of types involved. If $|A| = \kappa$, then the number of formulas over A is $|A|+|T| = \kappa$, hence $|S_1(A)| \leq 2^\kappa$. The number of subsets of cardinality κ of a set X is $|X|^\kappa$ (or 0), so if $|X| = 2^\kappa$ there are $(2^\kappa)^\kappa = 2^\kappa$ many such subsets. Thus, $P = \bigcup \{ S_1(A) : A \subset M \text{ with } |A| = \kappa \}$ has cardinality 2^κ. Enumerate P as $\{ p_i : i < 2^\kappa \}$ and add to the language new constants c_i, $i < 2^\kappa$. Consider the theory $T' = Th(\mathcal{M}_M) \cup \bigcup \{ p_i(c_i) : i < 2^\kappa \}$. (By $p_i(c_i)$ we mean $\{ \varphi(c_i) : \varphi \in p_i \}$.) Compactness implies the consistency of T', hence it has a model of cardinality 2^κ. The restriction of this model to the original language is the model $\mathcal{N}$ required to prove the claim.

An elementary chain $\mathcal{M}_\beta$ for $\beta < \kappa^+$ is constructed as follows by recursion. Let $\mathcal{M}_0$ be any model of T of cardinality 2^κ. Assuming that $\mathcal{M}_\gamma$ has been defined let $\mathcal{M}_{\gamma+1}$ be an elementary extension of $\mathcal{M}_\gamma$ of cardinality 2^κ such that for every subset $A \subset M_\gamma$ of cardinality κ, $\mathcal{M}_{\gamma+1}$ realizes every element of $S_1(A)$ (the claim guarantees the existence of such a model). If δ is a limit ordinal let $\mathcal{M}_\delta = \bigcup_{\gamma<\delta} \mathcal{M}_\gamma$. Now let $\mathcal{M}$ be the union of the elementary chain, $\mathcal{M}_\beta$, $\beta < \kappa^+$. To verify that $\mathcal{M}$ is κ^+–saturated, let A be a subset of M of cardinality κ and p an element of $S_1(A)$. The elementary chain of $\mathcal{M}_\gamma$'s has length κ^+. Since κ^+ is regular there is a $\beta < \kappa^+$ such that $A \subset M_\beta$, hence p is realized in $\mathcal{M}_{\beta+1}$ (by construction). Since $\mathcal{M}_{\beta+1} \prec \mathcal{M}$, the same element realizes p in $\mathcal{M}$, proving the lemma.

As we will see later, saturated models are more useful than κ–saturated models where κ is less than the cardinality of the model (since they are homogeneous). The last lemma only guarantees the existence of a saturated model when $\kappa^+ = 2^\kappa$, a condition which is independent of the axioms of set theory, for many κ. A natural question is: What properties must a theory T and a cardinal λ possess in order for an adaptation of the above argument to yield a saturated model of T of cardinality λ? The special properties of κ^+ and 2^κ that were used to produce a κ^+–saturated model are: $|S_1(A)| \leq 2^\kappa$ when $|A| = \kappa$, and the cofinality of κ^+ is $> \kappa$. Thus, for the same argument to yield a saturated model of cardinality λ, λ must satisfy the conditions: $|A| < \lambda \implies |S_1(A)| < \lambda$, and the cofinality of λ is λ. (This condition on cofinality is used to guarantee that when $\mathcal{M}_\beta$, $\beta < \lambda$, is an elementary chain (λ a cardinal) and $A \subset \bigcup_{\beta<\lambda} M_\beta$ has cardinality $< \lambda$ there is a single M_β containing A.)

The reasoning in the previous paragraph leads to

Lemma 2.2.6. *Let T be a complete theory and $\kappa \geq |T|$ a regular cardinal such that for all models $\mathcal{M}$ of T and $A \subset M$, $|A| < \kappa \implies |S_1(A)| < \kappa$. Then T has a saturated model of cardinality κ.*

Proof. The proof is a rather straight-forward adaptation of the proof of the previous lemma. Arguing as in the claim a model $\mathcal{M}$ of cardinality $< \kappa$ has an elementary extension $\mathcal{N}$ of cardinality $< \kappa$ which realizes every element of $S_1(M)$. Now define an elementary chain $\mathcal{M}_\beta$, $\beta < \kappa$, such that

- every element of $S_1(M_\beta)$ is realized in $\mathcal{M}_{\beta+1}$, and $|M_{\beta+1}| < \kappa$, for all $\beta < \kappa$, and
- $\mathcal{M}_\delta = \bigcup_{\gamma<\delta} \mathcal{M}_\gamma$, when $\delta < \kappa$ is a limit ordinal.

Let $\mathcal{M}$ be the union of the chain, $\mathcal{M}_\beta$, $\beta < \kappa$ and notice that $|M| = \kappa$. To verify that $\mathcal{M}$ is saturated let A be a subset of M of cardinality $< \kappa$. Since κ is regular there is some $\beta < \kappa$ with $A \subset M_\beta$. Every element of $S_1(A)$ extends to an element of $S_1(M_\beta)$, which is realized in $\mathcal{M}_{\beta+1}$, hence in $\mathcal{M}$. (In the exercises the reader is asked to write out a complete proof of this lemma without referring to Lemma 2.2.5.)

A cardinal κ is called *strongly inaccessible* if κ is regular, uncountable and $2^\lambda < \kappa$ whenever $\lambda < \kappa$. It cannot be proved in ZFC that strongly inaccessible cardinals exist, however if λ is strongly inaccessible the last lemma shows that every theory of cardinality $\leq \lambda$ (with an infinite model) has a saturated model of cardinality λ.

We proved previously that every countable complete theory has a countable homogeneous model. Unfortunately, if we try to construct a homogeneous model of uncountable cardinality, say $\aleph_1$, set-theoretic problems arise similar to those which limit the existence of saturated models. For example, let's try generalizing the elementary chain argument used in the proof of Proposition 2.2.3. For $\mathcal{M}_\alpha$ a model of cardinality $\aleph_1$ and $A \subset M_\alpha$ countable, there are at most $\aleph_1$ many 1−types over A realized in $\mathcal{M}_\alpha$, however there are $2^{\aleph_0}$ many countable subsets of M_α to consider as sets of parameters. Thus, for a model $\mathcal{M}_\alpha$ of cardinality $\aleph_1$ there may not be an elementary extension $\mathcal{M}_{\alpha+1}$ of cardinality $\aleph_1$ (without assuming the continuum hypothesis) such that for all elementary maps $g : A \longrightarrow M_\alpha$, where $A \subset M_\alpha$ is countable, and $a \in M_\alpha$, there is an elementary $f : A \cup \{a\} \longrightarrow M_{\alpha+1}$ extending g.

Definition 2.2.7. *For $\mathcal{M}$ a model, the* type diagram of $\mathcal{M}$, *denoted $D(\mathcal{M})$, is $\{\, p \in S(\emptyset) : p$ is realized in $\mathcal{M}\,\}$.*

Notice that for $\mathcal{M}$ $\aleph_0$−saturated $D(\mathcal{M}) = S(\emptyset)$. The relative uniqueness of homogeneous models, and other results, will follow quickly from the next two lemmas.

Lemma 2.2.7. *Let $\mathcal{M}$ to be a κ−homogeneous model, $\mathcal{N}$ a model elementarily equivalent to $\mathcal{M}$ containing sets $A \subset B$ such that $|A| < |B| \leq \kappa$ and $\{\, tp_{\mathcal{N}}(\bar{b}) : \bar{b}$ is a finite sequence from $B\,\} \subset D(\mathcal{M})$. Then for any elementary map $f : A \longrightarrow M$ there is an elementary map $g : B \longrightarrow M$ which extends f.*

Proof. This is proved by induction on $|B|$. Enumerate $B \backslash A$ as $\{\, b_\alpha : \alpha < \lambda \,\}$, where $\lambda = |B \backslash A|$. The desired extension g of f will be constructed by defining it (recursively) on b_α for successively larger $\alpha < \lambda$. Given $\delta < \lambda$, assume that the elementary map g extending f has been defined on b_α for all $\alpha < \delta$. Because $B_\delta = A \cup \{\, b_\alpha : \alpha \leq \delta \,\}$ has cardinality $< |B|$ there is an elementary map $h : B_\delta \longrightarrow M$ extending f. Since gh^{-1} is an elementary map on $\mathcal{M}$ taking $A \cup \{\, h(b_\alpha) : \alpha < \delta \,\}$ to $A \cup \{\, g(b_\alpha) : \alpha < \delta \,\}$, the κ−homogeneity of $\mathcal{M}$ yields a b such that $gh^{-1} \cup \{(h(b_\delta), b)\}$ is an elementary map. Define $g(b_\delta)$ to be b.

Lemma 2.2.8. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are homogeneous models of the same cardinality, $D(\mathcal{M}) = D(\mathcal{N})$, $A \subset M$ with $|A| < |M|$, and f is an elementary map of A into N. Then, f can be extended to an isomorphism from $\mathcal{M}$ onto $\mathcal{N}$.*

Proof. The isomorphism is constructed with a back and forth argument using the previous lemma. More precisely, a chain of elementary maps, f_α, $\alpha < |M|$, is constructed such that $f_0 = f$, every element of M is in the domain of some f_α, and every element of N is in the range of some f_α. Well-order the sets $M \setminus A$ and $N \setminus f(A)$ and suppose that f_γ has been defined for each $\gamma < \delta$. If δ is a limit ordinal let $f_\delta = \bigcup_{\gamma<\delta} f_\gamma$. Suppose that $\delta = \gamma + 2n$, where γ is a limit ordinal and $n \in \omega$ is > 0. Let a be the least element of

$M \setminus (dom f_{\gamma+2n-1})$ in the well-ordering of $M \setminus A$. By Lemma 2.2.7 there is an elementary map f_δ extending $f_{\gamma+2n-1}$ whose domain contains a. Finally, suppose $\delta = \gamma + 2n + 1$, for γ a limit ordinal and $n \in \omega$, and let b be the least element of $N \setminus (range f_{\gamma+2n})$. By the previous lemma there is an elementary map g extending the inverse of $f_{\gamma+2n}$ whose domain contains b. Let $f_\delta = g^{-1}$. It is clear that $\bigcup_{\alpha<|M|} f_\alpha$ is the desired isomorphism from $\mathcal{M}$ onto $\mathcal{N}$.

Corollary 2.2.1 (Extendibility of elementary maps). *If $\mathcal{M}$ is a homogeneous model, $A \subset M$ with $|A| < |M|$ and f is an elementary map from A into M, then f can be extended to an automorphism of $\mathcal{M}$.*

Corollary 2.2.2 (Relative uniqueness of homogeneous models). *Let $\mathcal{M}$ and $\mathcal{N}$ be homogeneous models of the same cardinality with $D(\mathcal{M}) = D(\mathcal{N})$. Then $\mathcal{M} \cong \mathcal{N}$.*

Since $D(\mathcal{M}) = S(\emptyset)$ for any saturated model $\mathcal{M}$,

Corollary 2.2.3 (Uniqueness of saturated models). *If $\mathcal{M}$ and $\mathcal{N}$ are saturated models of the same cardinality and the same complete theory, then $\mathcal{M} \cong \mathcal{N}$.*

Corollary 2.2.4 (Relative universality of homogeneous models). *Let $\mathcal{M}$ be κ−homogeneous and $\mathcal{N}$ a model of cardinality $\leq \kappa$ such that $D(\mathcal{N}) \subset D(\mathcal{M})$. Then $\mathcal{N}$ is elementarily embeddable into $\mathcal{M}$.*

Corollary 2.2.5 (Saturated=homogeneous+universal). *A model $\mathcal{M}$ which is κ−saturated is κ−homogeneous and κ^+−universal. As a partial converse, if $\mathcal{M}$ is κ−homogeneous and $D(\mathcal{M}) = S(\emptyset)$, then $\mathcal{M}$ is κ−saturated.*

Proof. The κ^+−universality of a κ−saturated model is by the previous corollary while its κ−homogeneity is clear from the definition. Now let $\mathcal{M}$ be κ−homogeneous with $D(\mathcal{M}) = S(\emptyset)$, $A \subset M$ of cardinality $< \kappa$ and $p \in S_1(A)$. By the consistency of p there is a model $\mathcal{N}$ containing A such that $\mathcal{M}_A \equiv \mathcal{N}_A$ and p is realized in $\mathcal{N}$ by some element a. By Lemma 2.2.7 there is an elementary map of $\{a\} \cup A$ into M which is the identity on A. The image of a under this map is the desired realization of p.

Corollary 2.2.6. *A homogeneous model which realizes every element of $S(\emptyset)$ is saturated.*

Example 2.2.8. (A countable universal model which is not saturated) Let T be the theory of the order $(\omega, \leq)$. The reader can show that T is quantifier-eliminable. Every model of T can be obtained in the following manner: Let $\mathcal{M} = (A, \leq)$ be any linear order. Form the model $\mathcal{N}$ of T by adding one copy of the ordering $(\mathbb{Z}, \leq)$ to the end of $(\omega, \leq)$ for each element of A, respecting the order induced by $\mathcal{M}$. More precisely,

(a) $N = \omega \cup (A \times \mathbb{Z})$,

(b) $\leq$ is the standard ordering on ω,
(c) $n \leq (a, m)$ for all $n \in \omega$ and $(a, m) \in (A \times \mathbb{Z})$,
(d) $(a, m) \leq (b, n)$ if and only if $a < b$ or $a = b$ and $m \leq n$, for all (a, m), $(b, n) \in (A \times \mathbb{Z})$.

It is not difficult to see that the model $\mathcal{M}$ obtained by letting $(A, \leq) = (\mathbb{Q}, \leq)$ in this algorithm, is a countable saturated model (hence a universal model). Regardless of the theory, any countable elementary extension of a countable saturated model is also universal. This observation quickly leads to a countable universal model which is not saturated: Let $\mathcal{N}$ be the model obtained by adding one copy of $(\mathbb{Z}, \leq)$ to the end of $\mathcal{M}$. ($\mathcal{N}$ cannot be saturated since it is not homogeneous.)

One of the conditions holding in a κ–saturated model $\mathcal{M}$ of cardinality κ, and perhaps failing in a κ–saturated model of cardinality $> \kappa$ is: any elementary map $f : A \to M$, where $A \subset M$ and $|A| < \kappa$, extends to an automorphism of $\mathcal{M}$. In some uses of saturated models they can be replaced by models which only satisfy this extendibility condition, which is given a name in the following definition.

Definition 2.2.8. *For κ an infinite cardinal, the model $\mathcal{M}$ is called* strongly κ–homogeneous *if for all elementary maps $f : A \to M$, where $A \subset M$ and $|A| < \kappa$, f extends to an automorphism of $\mathcal{M}$.*

Widespread existence of these models is proved through

Lemma 2.2.9. *Let $\mathcal{M}$ be an infinite model, $A \subset M$, and $f : A \to M$ an elementary map. Then there is an $\mathcal{N} \succ \mathcal{M}$ having an automorphism g extending f with $|N| = |M|$.*

Proof. The model $\mathcal{N}$ will be constructed in two stages. First we prove

Claim. Let $\mathcal{N}_0$ be an infinite model, $B \subset N_0$, and $f : B \longrightarrow N_0$ an elementary map. Then there is an $\mathcal{N}_1 \succ \mathcal{N}_0$ with $|N_1| = |N_0|$ and an elementary map $g : N_1 \longrightarrow N_1$ which extends f.

Let $\mathcal{M}_0 = \mathcal{N}_0$. We define, by recursion, an elementary chain of models $\mathcal{M}_i$, $i < \omega$, and elementary maps $g_i : M_i \to M_{i+1}$ such that $f \subset g_0 \subset \ldots \subset g_i \subset \ldots$. To begin, let $\kappa = |M_0|$ and let $\mathcal{M}'_1$ be a κ^+–saturated elementary extension. By Lemma 2.2.7 there is an elementary map $g_0 \supset f$ taking M_0 into M'_1. Let $\mathcal{M}_1$ be an elementary submodel of $\mathcal{M}'_1$ of cardinality κ containing both M_0 and $g_0(M_0)$. In general, let $\mathcal{M}'_{i+1} \succ \mathcal{M}_i$ be a κ^+–saturated model, $g_{i+1} : M_i \to M'_{i+1}$ an elementary map extending g_i, and $\mathcal{M}_{i+1}$ an elementary submodel of $\mathcal{M}'_{i+1}$ containing both M_i and $g_i(M_i)$. It is easily verified that $\mathcal{N}_1 = \bigcup_{i<\omega} \mathcal{M}_i$ and $g = \bigcup g_i$ satisfy the requirements of the claim.

The model $\mathcal{N}$ and automorphism g will be obtained as the limit of ω applications of the claim. Let $\kappa = |M|$, $\mathcal{N}_0 = \mathcal{M}$ and $g_0 = f$. To begin, the claim yields a model $\mathcal{N}_1 \succ \mathcal{N}_0$ of cardinality κ and an elementary map

$g_1 : N_1 \longrightarrow N_1$ extending g_0. Now apply the claim to $\mathcal{N}_1$ and g_1^{-1} to obtain an $\mathcal{N}_2 \succ \mathcal{N}_1$ of cardinality κ with an elementary map $g_2 : N_2 \longrightarrow N_2$ which extends g_1^{-1}. Continuing in this manner results in an elementary chain:

$$\mathcal{N}_0 \prec \mathcal{N}_1 \prec \mathcal{N}_2 \prec \ldots \prec \mathcal{N}_i \prec \ldots$$

and a chain of elementary maps $g_0 \subset g_1 \subset g_2^{-1} \subset g_3 \subset \ldots$ such that if i is odd the domain of g_i is $\mathcal{N}_i$ and if i is even (and > 0) the range of g_i^{-1} is $\mathcal{N}_i$. Then $\mathcal{N} = \bigcup_{i<\omega} \mathcal{N}_i$ is an elementary extension of $\mathcal{M}$ of cardinality κ and $g = \bigcup_{i<\omega} g_{2i+1}$ is an automorphism of $\mathcal{N}$ which extends f.

Proposition 2.2.7. *For κ an infinite cardinal and T a theory of cardinality $\leq \kappa$ there is a strongly κ–homogeneous model of cardinality $\leq 2^\kappa$.*

Proof. Given a model $\mathcal{N}$ of cardinality $\leq 2^\kappa$ let $\Phi = \{ f :$ for some $A \subset N$ of cardinality $\leq \kappa$, $f : A \longrightarrow N$ is elementary $\}$. Generalizing the proof of the previous lemma shows that there is an elementary extension $\mathcal{N}'$ of $\mathcal{N}$ of cardinality $\leq 2^\kappa$ such that every element of Φ extends to an automorphism of $\mathcal{N}'$. Using this fact the reader can construct an elementary chain whose union is the desired model.

Corollary 2.2.7. *Every model $\mathcal{M}$ has a strongly $\aleph_0$–homogeneous elementary extension of the same cardinality.*

Proof. Left to the reader.

Historical Notes. Cantor-Bendixson rank was introduced into model theory by Morley in [Mor65]. The notions of κ–saturated and saturated models go back to the η_α–sets of Hausdorff. (These are basically saturated models of the theory of dense linear orders without endpoints.) Their importance in model theory was not exploited until the late '50's. Universal and homogeneous models were developed by Fraïsé and Jónsson. Most of the results relating universal, saturated and homogeneous models are proved by Morley and Vaught in [MV62]. However, the relative uniqueness of homogeneous models, Corollary 2.2.2, was proved by Keisler and Morley in [KM67].

Exercise 2.2.1. Write out a proof of Lemma 2.2.3(iv).

Exercise 2.2.2. Write out the details in Example 2.2.5.

Exercise 2.2.3. Suppose that T is a complete theory in a countable language L which is not small and $T' \supset T$ is a complete theory in a language $L' \supset L$. Show that T' is also not small. On the other hand, if T is small, $\mathcal{M} \models T$ and $A \subset M$ is finite, then $T' = Th(\mathcal{M}_A)$ is small.

Exercise 2.2.4. Prove that the union of an elementary chain of atomic models is atomic. Use this fact to show that a countable complete theory having a prime model which is not minimal has an uncountable atomic model. (HINT: A prime model which is not minimal is isomorphic to a proper elementary extension of itself.)

Exercise 2.2.5. Show that a countable complete theory which is not small has $2^{\aleph_0}$ many nonisomorphic countable homogeneous models.

Exercise 2.2.6. Let T be a countable complete theory. Prove that $S_n(\emptyset)$ is the union of a scattered set and a set which is perfect or empty.

Exercise 2.2.7. Prove that if $\mathcal{M}$ is κ–saturated and A is a subset of M of cardinality $< \kappa$, then $\mathcal{M}$ realizes every complete n–type over A, for all n. (HINT: Use induction on n.)

Exercise 2.2.8. Let κ be an infinite cardinal, $\mathcal{M}$ a κ–saturated model and $A \subset M$ a set of cardinality $< \kappa$. Show that $\mathcal{M}_A$ is also κ–saturated.

Exercise 2.2.9. Let $\mathcal{M}$ be an $\aleph_0$–saturated model and p a complete 1–type in $Th(\mathcal{M})$ such that $p(\mathcal{M})$ is finite. Prove that p is algebraic.

Exercise 2.2.10. Prove that a countable model $\mathcal{M}$ is saturated if and only if it is universal over every finite subset of M.

Exercise 2.2.11. Prove that the union of an elementary chain of $\aleph_0$–homogeneous models is $\aleph_0$–homogeneous.

Exercise 2.2.12. Show that every reduct of a saturated model to a sublanguage is saturated. (The corresponding result about homogeneous models is false. What do you think goes wrong in the proof?)

Exercise 2.2.13. Prove that if $\mathcal{M}$ is κ–saturated, A is a subset of M of cardinality $< \kappa$ and $p \in S(A)$, then the cardinality of the set of realizations of p in $\mathcal{M}$ is finite or $\geq \kappa$.

Exercise 2.2.14. Write out a complete proof of Lemma 2.2.6 which does not refer to the proof of Lemma 2.2.5.

Exercise 2.2.15. Suppose that κ and λ are infinite cardinals and the cofinality of λ is $\geq \kappa$. Prove that the union of an elementary chain $\mathcal{M}_\alpha$, $\alpha < \lambda$, of κ–saturated models is κ–saturated.

Exercise 2.2.16. Find a union of an elementary chain of $\aleph_1$–saturated models which is not $\aleph_1$–saturated. (HINT: See the last example concerning CB-rank.)

Exercise 2.2.17. Write out the details in the proof of Proposition 2.2.7.

Exercise 2.2.18. Prove Corollary 2.2.7.

Exercise 2.2.19. Prove that if $\mathcal{M}$ is strongly $\aleph_0$−homogeneous and $\bar{a}$ and $\bar{b}$ are sequences from $\mathcal{M}$ realizing the same type, then for all formulas $\varphi(\bar{x}, \bar{y})$, $|\varphi(\mathcal{M}, \bar{a})| = |\varphi(\mathcal{M}, \bar{b})|$.

2.3 Countable Models of Complete Theories

In the preceding section we studied models of complete theories which have special properties with respect to the types they realize and automorphisms they admit. The sharpest existence and uniqueness theorems were obtained for the countable models of complete theories. In this section we apply these results to study the following two problems about countable complete theories:

1. Characterize the countable complete theories which have a unique countable model, up to isomorphism.
2. For T a countable complete theory let $n(T)$ denote the number of countable models of T up to isomorphism. Determine the possible values of $n(T)$, as T ranges over the countable complete theories.

Definition 2.3.1. *For κ an infinite cardinal, a theory T is κ−*categorical *(or* categorical in κ*) if T has a unique model of cardinality κ, up to isomorphism.*

Later extensive attention will be given to countable theories which are categorical in some uncountable cardinality. Problem 1 asks for a characterization of $\aleph_0$−categorical theories. Here are some examples of such theories:

- Let T be the theory in a language with no nonlogical symbols expressing that there are infinitely many elements. The countable models of this theory are sets of cardinality $\aleph_0$, hence are all isomorphic.
- The theory of dense linear orders without endpoints is $\aleph_0$−categorical.
- All vector spaces of dimension $\aleph_0$ over a fixed field F are isomorphic. In the natural language of vector spaces over F there is no set of sentences which express that the dimension is $\aleph_0$ for an arbitrary field. However, assuming that F is a finite field any countable model of the theory T of *infinite* vector spaces over F must have dimension $\aleph_0$. Thus, T is categorical in $\aleph_0$.
- The theory of an equivalence relation with infinitely many infinite classes and no finite classes is $\aleph_0$−categorical.

Remark 2.3.1. Let T be a countable theory with an infinite model which is categorical in some infinite cardinal. Then T is complete. This is a classical result called the Łoś-Vaught Test (for completeness of a theory). (The proof is left to the reader in Exercise 2.3.1.)

The class of countable theories which are categorical in $\aleph_0$ admits the following characterization.

Theorem 2.3.1 (Ryll-Nardzewski, Engeler, Svenonius). *For a countable complete theory T the following are equivalent:*

(1) *T is $\aleph_0$–categorical.*
(2) *For each n, $S_n(\emptyset)$ is finite.*
(3) *For each n, there are finitely many formulas in n free variables, up to equivalence in T.*

Proof. First we establish a couple of preliminary claims about countable complete theories which are of interest in their own right.

Claim. Let T' be a countable complete theory with an infinite model such that every complete n–type in T is isolated. Then there are finitely many complete n–types.

Assuming every complete n–type to be isolated let $\Gamma = \{\neg\varphi : \varphi$ is a formula in n variables which isolates a complete type in $T'\}$. Since Γ is not contained in a complete type it is inconsistent. By compactness there are formulas $\varphi_0, \ldots, \varphi_m$ which isolate distinct complete types in T' such that $T' \models \forall\bar{v}(\varphi_0(\bar{v}) \vee \ldots \vee \varphi_m(\bar{v}))$. Thus, each element p of $S_n(\emptyset)$ contains one of the φ_i's, and for each i only one element of $S_n(\emptyset)$ contains φ_i. We conclude that $|S_n(\emptyset)| = m + 1$, proving the claim.

Claim. Let T' be a countable complete theory such that $S_n(\emptyset)$ is finite. Then there are finitely many formulas in n free variables up to equivalence with respect to T'.

For φ a formula in n variables, $\mathcal{O}_\varphi$ denotes the set of complete n–types in T' which contain φ. There are only finitely many subsets of $S_n(\emptyset)$ which can be $\mathcal{O}_\varphi$ for some formula φ in n variables. Since $\mathcal{O}_\varphi = \mathcal{O}_\psi$ if and only if $T' \models \forall\bar{v}(\varphi \leftrightarrow \psi)$, there are finitely many formulas in n variables in T'.

Turning to the main body of the proof, let T be $\aleph_0$–categorical. A countable theory which is not small has continuum many countable models, hence T is small and has a countable atomic model and a countable saturated model. Thus, the unique countable model $\mathcal{M}$ of T is both atomic and saturated. Since every element of $S_n(\emptyset)$ is realized in $\mathcal{M}$, every complete n–type is isolated. By the first claim, $S_n(\emptyset)$ is finite, proving that (1) $\Longrightarrow$ (2).

That (2) $\Longrightarrow$ (3) is proved in the second claim, so assume (3) to hold. The $\aleph_0$– categoricity of T will follow from the uniqueness of prime models (Proposition 2.1.1) once we show that every model of T is atomic. Let $\mathcal{M}$ be a model of T and $\bar{a}$ a finite tuple from M, and let $\varphi_1, \ldots, \varphi_m$ be a list of the finitely many formulas, up to equivalence in T, which are satisfied by $\bar{a}$ in $\mathcal{M}$. Then $\varphi_1 \wedge \ldots \wedge \varphi_m$ isolates $tp_{\mathcal{M}}(\bar{a})$. Thus, $\mathcal{M}$ is atomic, completing the proof of the theorem.

In the remainder of the section we discuss the progress to date on the second question stated above, namely:

For T a countable complete theory what are the possibilities for $n(T)$, the number of countable models of T up to isomorphism.

Simply for set-theoretic reasons we know that $n(T) \leq 2^{\aleph_0}$. A theory with continuum many complete types has $2^{\aleph_0}$ many countable models up to isomorphism (since every complete type is realized in some countable model). Thus, $n(T) = 2^{\aleph_0}$ is one possibility. (In the previous section we gave an example of a theory which is not small.) We have also seen examples of countable complete theories in which $n(T) = 1$; i.e., $\aleph_0$−categorical theories. We continue by giving examples which delineate less trivial possibilities.

For the remainder of this section an arbitrary theory is assumed to be countable, complete and have infinite models.

Example 2.3.1. (Of a theory with $\aleph_0$ many countable models) Let T be the theory of algebraically closed fields of characteristic 0. By quantifier elimination the isomorphism type of a model is determined by the transcendence degree of the model over the prime field. There are $\aleph_0$ many possibilities for countable transcendence degree.

Example 2.3.2. (Of a simpler theory with $\aleph_0$ many countable models) Consider the theory in a language with constant symbols c_i, $i < \omega$, which says that the constants are distinct. This theory has elimination of quantifiers and is complete. For each $n < \omega$ there is a model of T with exactly n elements which are not the interpretation of some constant. Furthermore, the isomorphism type of any model is determined by the number of such nonconstant elements. Thus, there are $\aleph_0$ many countable models.

Example 2.3.3. (Of a small theory with $2^{\aleph_0}$ many countable models) Example 2.2.1(iii) is an example of such a theory. The fact that the theory has a countable saturated model is equivalent to it being small. For any $X \subset \omega$ there is a countable model $\mathcal{M}_X$ of T such that $P_i(\mathcal{M}_X)$ contains a nonconstant if and only if $i \in X$. Thus, T has continuum many nonisomorphic countable models.

Example 2.3.4. (Of a theory with 3 countable models) This is a classical example due to Ehrenfeucht. Let T be the theory in the language with a binary relation $<$ and constants, c_i, $i < \omega$, saying that $<$ is a dense linear order without endpoints and $c_n < c_{n+1}$, for $n < \omega$. An elimination of quantifiers argument shows that T is complete and has the following 3 countable models. $\mathcal{M}_1$ is the model in which every element is $<$ some c_n. $\mathcal{M}_2$ is the model which contains elements $>$ every c_n and there is a least such. Lastly, $\mathcal{M}_3$ is the model which contains elements greater than every c_n and $\sup_{n<\omega} c_n$ does not exist in the model. Notice that $\mathcal{M}_1$ is the prime model of T and $\mathcal{M}_3$ is the countable saturated model of the theory.

Closer examination of the last example reveals some interesting interplay between saturation, universality and atomicity over a finite set. Since this theory T contains the axioms for dense linear orders without endpoints, if there is an element in a model which is greater than all of the c_i's, then there is a copy of the rationals greater than all of the c_i's in this model. In fact, there is an isomorphic embedding of $\mathcal{M}_3$ into $\mathcal{M}_2$ (viewing $\mathcal{M}_1$ as a submodel of each of these, embed the remaining elements of $\mathcal{M}_3$ into the elements of $M_2 \setminus (M_1 \cup \{\sup_{n<\omega} c_n\})$ in an order-preserving way). Since T has elimination of quantifiers, if $\mathcal{M}$ and $\mathcal{N}$ are models of T and $\mathcal{M} \subset \mathcal{N}$, then $\mathcal{M} \prec \mathcal{N}$. Thus, $\mathcal{M}_3$ is elementarily embedded into $\mathcal{M}_2$. Since $\mathcal{M}_3$ is saturated, we conclude that $\mathcal{M}_2$ is universal. Since $\mathcal{M}_2$ is not saturated, $\mathcal{M}_2$ must fail to be homogeneous (by Corollary 2.2.5) which is verified as follows. Let a be the element of M_2 which is the supremum of the c_i's. All elements of M_2 which are greater than all of the c_i's realize the same complete type in $\mathcal{M}_2$. However, any automorphism of $\mathcal{M}_2$ leaves a fixed.

As a final remark on the properties of the $\mathcal{M}_i$'s notice that $\mathcal{M}_2$ is prime over a. (A model $\mathcal{N}$ is prime over a if whenever $\mathcal{M} \models T$ and f is an elementary map of a into $\mathcal{M}$, f can be extended to an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$. Since $\mathcal{M}_1$ does not realize $tp(a)$ there is no elementary map taking a into $\mathcal{M}_1$. Since $\mathcal{M}_3$ is saturated any elementary $f : \{a\} \longrightarrow \mathcal{M}_3$ extends to an elementary embedding of $\mathcal{M}_2$ into $\mathcal{M}_3$ (by Lemma 2.2.7). Thus, $\mathcal{M}_2$ is prime over a.)

By adding $n-2$ unary relations to the language it is possible to adapt the previous example to obtain a theory having exactly n countable models for each $n \geq 3$ (see the exercises). All such theories have a model behaving much like the model $\mathcal{M}_2$ in the Ehrenfeucht example, with the only difference being that the universality must be weakened.

Lemma 2.3.1. *Let T be a countable complete theory having finitely many but more than one countable model. Then T has a countable model which is prime over some finite set, is not saturated, and realizes every element of $S(\emptyset)$.*

Proof. Let $\mathcal{M}$ be a countable saturated model of T (which exists since the theory is small). Let $p_0, p_1, \ldots$ be an enumeration of $S(\emptyset)$. For $i < \omega$, let $\bar{a}_i$ be a finite sequence from M such that $j \leq i \implies p_j$ is realized by some subsequence of $\bar{a}_i$. Since T is small it has a model $\mathcal{N}_i$ which is prime over $\bar{a}_i$. We claim that $\mathcal{N}_i$ is not saturated, for all $i < \omega$. Assuming to the contrary that $\mathcal{N}_i \cong \mathcal{M}$, $\mathcal{M}$ is prime over some finite sequence $\bar{c}$ in M. Since $(\mathcal{M}, \bar{c})$ is also a saturated model (see Exercise 2.2.8), we conclude that $Th(\mathcal{M}, \bar{c})$ is $\aleph_0$-categorical. However, since T is not $\aleph_0$-categorical, there are infinitely many distinct complete n-types for some n. Distinct complete n-types in T extend to distinct complete n-types in $Th(\mathcal{M}, \bar{c})$, hence there are infinitely many complete n-types in $Th(\mathcal{M}, \bar{c})$, in contradiction to Theorem 2.3.1. Thus, none of the $\mathcal{N}_i$'s are saturated. Since T has finitely many countable models there is an infinite $X \subset \omega$ such that $\mathcal{N}_i \cong \mathcal{N}_j$ for all $i, j \in X$. By the conditions on the $\bar{a}_i$'s, each $\mathcal{N}_j$, for $j \in X$, must realize every complete type of T. Since $\mathcal{N}_j$ is not saturated we have proved the lemma.

The above examples raise the question: Is it possible for a countable complete theory to have exactly two countable models? Vaught, in the seminal paper [Vau61] proved that this is impossible. The bulk of his argument appears in the proof of the preceding lemma.

Proposition 2.3.1. *No complete countable theory has exactly two countable models, up to isomorphism.*

Proof. Suppose T to be a counterexample to the theorem. Since T is not $\aleph_0$−categorical it has a nonisolated complete type p. Since T is small it has a countable saturated model $\mathcal{M}$ and a prime model $\mathcal{N}$, which is not isomorphic to $\mathcal{M}$ (since there is a nonisolated complete type). By the preceding lemma T also has a model $\mathcal{N}'$ which is not saturated but realizes every complete type of T. Since p is realized in $\mathcal{N}'$, $\mathcal{N}'$ cannot be atomic, hence is not isomorphic to $\mathcal{N}$. Thus, the existence of $\mathcal{N}'$ contradicts that T has exactly two countable models, proving the proposition.

Turning to the other end of the spectrum, in each of our examples of a theory having uncountably many countable models we show without appealing to the continuum hypothesis that T has $2^{\aleph_0}$ many countable models. Vaught hypothesized in [Vau61] that this is always the case:

Conjecture 2.3.1 (Vaught's Conjecture). If T is a countable complete theory with fewer than $2^{\aleph_0}$ many countable models, up to isomorphism, then T has countably many countable models.

To date, this conjecture is open. Morley [Mor70] succeeded in showing that $n(T)$ (for T a countable complete theory) is always countable, $\aleph_1$ or $2^{\aleph_0}$. Researchers have approached Vaught's Conjecture by trying to prove it for classes of theories satisfying additional hypotheses. With respect to the stability heirarchy (see later chapters for the relevant definitions), Bouscaren and Lascar proved Vaught's Conjecture for ω−stable theories of finite Morley rank ([Bou83] and [BL83]), with Shelah handling all ω−stable theories in [SHM84]. Several years later a proof was obtained (by Newelski [New90] and Buechler [Bue87]) for properly superstable theories of ∞−rank 1 using significant results from geometrical stability theory. Buechler proved Vaught's conjecture for superstable theories of finite ∞−rank in [Bue93]. An informative discussion of Vaught's conjecture and isomorphism invariants can be found in [Las85].

While calculating $n(T)$ (or the number of models in an uncountable cardinal) is not today a central concern of stability theory, it was through work on these so-called spectrum problems that much of model theory (especially stability theory) was developed. Furthermore, progress on a problem like

Vaught's Conjecture often requires significant new tools which could see use elsewhere.

Historical Notes. Theorem 2.3.1 is due independently to the three researchers mentioned in the statement. See [RN59], [Eng59] and [Sve59]. Lemma 2.3.1 was extracted from Vaught's proof of Proposition 2.3.1 by J. Rosenstein.

Exercise 2.3.1. Prove Remark 2.3.1.

Exercise 2.3.2. Modify Ehrenfeucht's example to find a theory with exactly n countable models for each $n \geq 3$.

Exercise 2.3.3. Suppose that T is not $\aleph_0$−categorical and every countable model of T is homogeneous. Show that T has infinitely many countable models.

Exercise 2.3.4. Suppose that every countable model of T is homogeneous and T has uncountably many countable models. Show that T has $2^{\aleph_0}$ many countable models.

Exercise 2.3.5. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are countable models, $\mathcal{M}$ can be elementarily embedded into $\mathcal{N}$ and $\mathcal{N}$ can be elementarily embedded into $\mathcal{M}$. Does it follow that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic?

2.4 Indiscernible Sequences

In the next section we confront the problem of constructing potentially uncountable models with special properties using Skolem functions. Indiscernibles are introduced here because they play a part in most applications of Skolem functions. However, indiscernibles have applications in the context of stable theories (developed later) which far out distance their uses in conjunction with Skolem functions. In this section we only touch on the most basic properties.

Definition 2.4.1. *Let $\mathcal{M}$ be a model in the language L, and $X \subset M^m$ (for some m) a subset on which there is a linear order $<$. (This order need not be in L.) We call $(X, <)$ an* indiscernible sequence in $\mathcal{M}$ *if for all n and all sequences $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ from X, $tp_{\mathcal{M}}(x_1, \ldots, x_n) = tp_{\mathcal{M}}(y_1, \ldots, y_n)$. X is called an* indiscernible set in $\mathcal{M}$ *if $tp_{\mathcal{M}}(x_1, \ldots, x_n) = tp_{\mathcal{M}}(y_1, \ldots, y_n)$ for any $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in X^n$ such that $x_i \neq x_j$ and $y_i \neq y_j$, for all $1 \leq i < j \leq n$. (In other words, X is an indiscernible set in $\mathcal{M}$ if $(X, <)$ is an indiscernible sequence in $\mathcal{M}$ for any linear ordering $<$ of X.)*

It is important to bear in mind that the ordering of X in the definition may or may not be in the language. When working with specific examples the following lemma (whose proof is assigned in the exercises) is a useful sufficient condition for indiscernibility. Note: If $\mathcal{M}$ is a model, $\bar{x}$ is the $n-$tuple $(x_1, \ldots, x_n)$ from M and f is an elementary map whose domain contains the x_i's, then $f(\bar{x})$ denotes $(f(x_1), \ldots, f(x_n))$.

Lemma 2.4.1. *Let $\mathcal{M}$ be a model and $(X, <)$ a linearly ordered set with $X \subset M^m$ such that for each pair of sequences $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ there is an automorphism f of $\mathcal{M}$ with $f(x_i) = y_i$ for $1 \leq i \leq n$. Then $(X, <)$ is an indiscernible sequence in $\mathcal{M}$.*

Example 2.4.1. Let $\mathcal{M} = (\mathbb{Q}, <)$ be the ordering on the rationals. We claim that $(\mathbb{Q}, <)$ is an indiscernible sequence in $\mathcal{M}$. Let $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ be sequences of rationals. Since the theory of dense linear orders without endpoints has elimination of quantifiers the type of any sequence is determined by the order relations within the sequence, hence $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ have the same type in $\mathcal{M}$.

Example 2.4.2. Let $\mathcal{M}$ be any model of the theory of algebraically closed fields of characteristic 0. Let X be an algebraically independent set in $\mathcal{M}$; i.e., a set of elements in $\mathcal{M}$ such that $x \in X \implies x$ is transcendental over $X \setminus \{x\}$. For $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ two $n-$tuples of distinct elements from X there is an automorphism of $\mathcal{M}$ taking $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_n)$. Thus, by the last lemma, X is an indiscernible set.

Example 2.4.3. Let V be a vector space over a field F and B a linearly independent set in V. Arguing as in the last example shows that B is a set of indiscernibles in V.

In our last example we specify two indiscernible sets A and B in the same model in which the type of an $n-$tuple from A is different from the type of an $n-$tuple from B.

Example 2.4.4. Let T be the theory in a language with a binary relation E saying that E is an equivalence relation with infinitely many infinite classes and no finite classes. Let $\mathcal{M}$ be any model of T. Let A be the set of elements of M comprising a single $E-$class, and let $B = \{b_i : i < \omega\}$ be a set of representatives from different classes; i.e., for $i \neq j$, $\mathcal{M} \models \neg E(b_i, b_j)$. Since T is quantifier eliminable it is easy to verify that both A and B are indiscernible sets.

The last three examples suggest a connection between the indiscernibility of a set and independence with respect to a dependence relation. This is the manner in which indiscernibles usually arise in the context of stable theories. In general, though, the following theorem is required to get an indiscernible sequence.

Theorem 2.4.1. *Let T be a theory with infinite models and $(X, <)$ any linearly ordered set. Then there is a model $\mathcal{M}$ of T with $X \subset M$ such that $(X, <)$ is an indiscernible sequence in $\mathcal{M}$.*

To prove the theorem a result from combinatorics is needed, whose proof can be found elsewhere. (See, for example [Hod93, 11.1.3].) Given a set A let $[A]^n$ denote the set of subsets of A with exactly n elements. Notice that if A is linearly ordered by $<$, there is a one-to-one correspondence between the elements of $[A]^n$ and increasing sequences of n elements from A.

Lemma 2.4.2 (Ramsey's Theorem). *Let I be an infinite set and $n < \omega$. If $\sim$ is an equivalence relation on $[I]^n$ with finitely many classes, then there is an infinite subset $J \subset I$ such that $[J]^n$ is contained in a single $\sim$ −class.*

Proof of Theorem 2.4.1: First expand the language L of T to $L(X)$, where there is a constant symbol for each element of X. Let T' be the result of adding to T all sentences of $L(X)$ of the form $\varphi(x_1, \ldots, x_n) \longleftrightarrow \varphi(y_1, \ldots, y_n)$, where $\varphi(v_1, \ldots, v_n)$ is a formula of L and $x_1 < \ldots < x_n$, $y_1 < \ldots < y_n$ are from X. The restriction of any model of T' to the original language will be the desired model having $(X, <)$ as an indiscernible sequence. The consistency of T' will follow (by compactness) from the consistency of $T \cup \{\varphi(x_1, \ldots, x_n) \longleftrightarrow \varphi(y_1, \ldots, y_n) : \varphi \in \Phi$, and $x_1 < \ldots < x_n$, $y_1 < \ldots < y_n$ from $X_0\}$, where Φ is an arbitrary finite set of formulas of L and X_0 a finite subset of X. We may assume that all formulas in Φ have exactly n free variables. Let $\mathcal{N}$ be any model of T, A an infinite subset of N and $<$ some linear ordering of A. For increasing n−tuples $a_1 < \ldots < a_n$, $b_1 < \ldots < b_n$ from A we define $\{a_1, \ldots, a_n\} \sim \{b_1, \ldots, b_n\}$ to hold if $\mathcal{N} \models (\varphi(a_1, \ldots, a_n) \longleftrightarrow \varphi(b_1, \ldots, b_n))$ for every $\varphi \in \Phi$. Since Φ is a finite set $\sim$ is an equivalence relation on $[A]^n$ with finitely many classes. By Ramsey's Theorem there is an infinite subset A_0 of A such that all increasing n−tuples from A_0 belong to the same $\sim$ −class. Thus, if we interpret the elements of X_0 by an appropriate number of elements of A_0, listed in increasing order, we get a model of $T \cup \{\varphi(x_1, \ldots, x_n) \longleftrightarrow \varphi(y_1, \ldots, y_n) : \varphi \in \Phi$, and $x_1 < \ldots < x_n$, $y_1 < \ldots < y_n$ from $X_0\}$. This proves the consistency of T', hence the theorem.

Notice that for an indiscernible sequence $(X, <)$ in $\mathcal{M}$, there is a unique complete type which is the type in $\mathcal{M}$ of an increasing n−tuple from X. The *type diagram of X*, denoted $D(X)$, is the set $\{p_n : n < \omega$ and p_n is the complete type realized by increasing n−tuples from $X\}$. Making a specific choice for the set A in the proof yields the following. Notice that while the notation in the proof fixed A as a subset of N it works equally well when A is a subset of N^k for some k. Thus, X need not be a subset of M in this corollary.

Corollary 2.4.1. *Let $(X, <)$ be an infinite sequence of indiscernibles in a model $\mathcal{M}$ and $(Y, <)$ any infinite linear ordering. Then there is some model*

$\mathcal{N} \equiv \mathcal{M}$ such that $(Y, <)$ is an indiscernible sequence in $\mathcal{N}$ and $D(Y) = D(X)$.

Historical Notes. Except for Ramsey's theorem (which was proved in [Ram30]) the results in this section are due to Ehrenfeucht and Mostowski [EM56]. The main results in that paper are covered in the next section.

Exercise 2.4.1. Give a proof of Lemma 2.4.1.

Exercise 2.4.2. Let X be an infinite indiscernible set in a model $\mathcal{M}$ and $n < \omega$. Let X' be a collection of $n-$tuples from X which are pairwise disjoint. Show that X' is also a set of indiscernibles.

Exercise 2.4.3. Let κ be an infinite cardinal and $\mathcal{M}$ a $\kappa-$saturated model. Show that for any nonalgebraic formula φ over M there is a countably infinite indiscernible sequence $(X, <)$ contained in $\varphi(\mathcal{M})$. Furthermore, for any such $(X, <)$ there is an indiscernible sequence $(Y, <')$ of size κ such that $Y \supset X$ and $<'$ extends $<$.

Exercise 2.4.4. Give an example of a model $\mathcal{M}$ containing an infinite set X such that any pair of distinct elements from X realize the same complete $2-$type in $\mathcal{M}$, but there are two triples from X with different $3-$types. (HINT: A projective plane)

2.5 Skolem Functions

Skolem functions, used in conjunction with indiscernibles, provide a way to construct uncountable models with various special properties. These theorems differ from those which yield, e.g., a $\kappa-$saturated model (for some κ) in that they may result in uncountable models which omit specified types.

In algebra it is common to speak of the object (e.g., the group or vector space) generated by a subset. Following is our formal definition of the notion of a "submodel generated by a set".

Definition 2.5.1. *Let $\mathcal{M}$ be a model in the language L and $X \subset M$.*

(i) The hull of X, *denoted $H(X)$, is the subset of M obtained by closing $X \cup \{ a \in M : a$ interprets a constant of $L \}$ under $F^{\mathcal{M}}$, for every function F of L.*

(ii) $\mathcal{H}(X)$ is the submodel of $\mathcal{M}$ with universe $H(X)$ (when $H(X) \neq \emptyset$). We also call $\mathcal{H}(X)$ the hull of X.

For any model $\mathcal{M}$ and $X \subset M$, if $X \neq \emptyset$ or the language contains a constant symbol, then $\mathcal{H}(X)$ is the submodel generated by X in the sense that, $X \subset H(X)$, $\mathcal{H}(X) \subset \mathcal{M}$ and $\mathcal{H}(X) \subset \mathcal{N}'$ for every $\mathcal{N}' \subset \mathcal{M}$ containing X.

In general, however, there is no well-defined notion of "elementary submodel generated by a set". Consider, for example, $\mathcal{M} = (\mathbb{Q}, <)$ and X a finite nonempty subset of $\mathbb{Q}$. Then $(X, < \restriction X)$ is the submodel generated by X but it is certainly not an elementary submodel (for one thing, it is finite). In fact, if $\mathcal{N}$ is any elementary submodel of $\mathcal{M}$ containing X there is $\mathcal{N}'$ containing X which is a proper elementary submodel of $\mathcal{N}$. Here, we will show how to expand a theory T to a theory T^* in a larger language so that whenever $\mathcal{M} \models T^*$ and $\mathcal{N} \subset \mathcal{M}$, $\mathcal{N} \prec \mathcal{M}$. The specific goal is

Theorem 2.5.1 (Skolem). *Let T be a theory in a language L. Then there is a theory T^* in a language L^* such that:*

(1) $L \subset L^*$, $T \subset T^*$ *and* $|T^*| = |T|$;
(2) *every model of T can be expanded to a model of T^*, and*
(3) *if $\mathcal{M}^* \models T^*$ and $\mathcal{N}^* \subset \mathcal{M}^*$ then $\mathcal{N}^* \prec \mathcal{M}^*$.*

Proof. The language L^* and theory T^* will be the unions of chains which approximate (3) with increasing precision. The inductive step in the construction of the chain is handled by

Claim. Let T be a theory in a language L. Then there is a theory T' in a language L' such that:

(a) $L \subset L'$, $T \subset T'$ and $|T'| = |T|$;
(b) every model of T can be expanded to a model of T', and
(c) if $\mathcal{M}' \models T'$ and $\mathcal{N}' \subset \mathcal{M}'$ then $\mathcal{N}' \restriction L \prec \mathcal{M}' \restriction L$.

We define $L' \supset L$ to be the language which adds to L a new constant symbol $c_{\exists x\varphi}$ for every sentence $\exists x\varphi$ of L, and a new n−ary function symbol $F_{\exists x\varphi}$ for every formula $\exists x\varphi(x, v_1, \ldots, v_n)$. Let T' be the union of T and the set of all sentences of L' of the form: $\exists x\varphi(x) \longrightarrow \varphi(c_{\exists x\varphi})$ or $\forall v_1 \ldots v_n[\exists x\varphi \longrightarrow \varphi(F_{\exists x\varphi}(v_1, \ldots, v_n), v_1, \ldots, v_n)]$, for all appropriate formulas $\exists x\varphi$ of L. That (a) holds is clear, and it is easy to interpret the new functions and constants on a model of T to ensure (b). The Tarski-Vaught Test guarantees that (c) holds.

Turning to the proof of the theorem define: $L_0 = L$, $T_0 = T$, $L_{n+1} = (L_n)'$, $T_{n+1} = (T_n)'$, $L^* = \bigcup_{n<\omega} L_n$ and $T^* = \bigcup_{n<\omega} T_n$. The verification of (1)-(3) is now easy using the claim and the fact that any formula of L^* is a formula of some L_n.

The functions and constants added to L to obtain L^* are called the *Skolem functions for* L and $\Gamma = T^* \setminus T$ is the set of *Skolem axioms for* L. Notice that Γ depends only on the language; i.e., if T and T_0 are two theories in L, then $T^* = T \cup \Gamma$ and $T_0^* = T_0 \cup \Gamma$ satisfy (1)–(3) of the theorem. A theory T having the property that whenever $\mathcal{M} \models T$ and $\mathcal{N} \subset \mathcal{M}$ then $\mathcal{N} \prec \mathcal{M}$ is said to *have Skolem functions.* This terminology is used even when T is not constructed from some other theory using the theorem.

If $\mathcal{M}$ is a model of a theory having Skolem functions then for any $X \subset M$, $\mathcal{H}(X) \prec \mathcal{M}$, and we call $\mathcal{H}(X)$ the *Skolem hull of* X. As the elements of $H(X)$ interpret terms of the language applied to the elements of X, $X \subset Y \subset M \implies \mathcal{H}(X) \prec \mathcal{H}(Y)$. A model which is the Skolem hull of a sequence of indiscernibles is often called an *Ehrenfeucht-Mostowski model*, after the researchers who developed indiscernibles.

In theories with Skolem functions the Skolem hulls of indiscernible sequences have properties which are rigidly tied to the indiscernibles. This is made explicit in

Lemma 2.5.1. *Let T be a complete theory with Skolem functions. Let $\mathcal{M}$ and $\mathcal{N}$ be models of T and $(X, <)$, $(Y, <)$ infinite indiscernible sequences in $\mathcal{M}$, $\mathcal{N}$, respectively, with $D(X) = D(Y)$.*

(i) $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ have the same type diagrams.

(ii) If h is an order-preserving map of X into Y then h extends uniquely to an elementary embedding h^* of $\mathcal{H}(X)$ into $\mathcal{H}(Y)$. In fact, $h^*(\mathcal{H}(X)) = \mathcal{H}(h(X))$.

(iii) If h is an order-preserving map of X onto itself then h extends uniquely to an automorphism of $\mathcal{H}(X)$.

Proof. Recalling the definition of the hull of X an arbitrary element of $H(X)$ is of the form $t^{\mathcal{M}}(x_1, \ldots, x_n)$ where t is some term of L and $x_1 < \ldots < x_n$ in X. Thus, if $\mathcal{H}(X)$ realizes a 1−type p there is some such $t^{\mathcal{M}}(x_1, \ldots, x_n)$ such that $\mathcal{M} \models \varphi(t(x_1, \ldots, x_n))$, for every $\varphi \in p$. This is to say that $\varphi(t(v_1, \ldots, v_n))$ belongs to the type satisfied by increasing n−tuples from X. Since $D(Y) = D(X)$, it follows that $t^{\mathcal{N}}(y_1, \ldots, y_n)$ realizes p in $\mathcal{H}(Y)$ for any $y_1 < \ldots < y_n$ in Y. The proof for types in n variables is just notationally more complicated, so we have proved (i).

Continuing the notation of the previous paragraph, the extension h^* needed to obtain (ii) is seen to be the map defined by:

$$h^*(t^{\mathcal{M}}(x_1, \ldots, x_n)) = t^{\mathcal{N}}(h(x_1), \ldots, h(x_n)),$$

for $x_1 < \ldots < x_n$ in X and terms t. Part (iii) follows immediately from (ii).

Corollary 2.5.1 (Ehrenfeucht-Mostowski). *Let T be a complete theory with an infinite model and $(X, <)$ some linearly ordered set. Then there is a model $\mathcal{M}$ of T in which $(X, <)$ is an indiscernible sequence and every automorphism of this linear order extends to an automorphism of $\mathcal{M}$.*

Proof. Let T' be an expansion of T to a complete theory with Skolem functions. Let $\mathcal{M}'$ be a model of T' in which $(X, <)$ is an indiscernible sequence. We may choose $\mathcal{M}'$ to be $\mathcal{H}(X)$. By Lemma 2.5.1(iii) every automorphism of $(X, <)$ extends to an automorphism of $\mathcal{M}'$. Such an automorphism is, a fortiori, an automorphism of $\mathcal{M} = \mathcal{M}' \restriction L$, proving the corollary.

Our next application of the lemma illustrates how Skolem functions and indiscernibles can be used to find very special uncountable models.

Theorem 2.5.2. *Let T be a countable complete theory in L having an infinite model. Then T has a countable model $\mathcal{M}$ such that for all cardinals $\kappa \geq \aleph_0$ there is a model $\mathcal{N}$ of cardinality κ with $D(\mathcal{N}) = D(\mathcal{M})$.*

Proof. First, let T' be an expansion of T to a complete theory with Skolem functions. By Theorem 2.4.1 T' has a countable model $\mathcal{M}_0$ containing an infinite indiscernible sequence X. Let $\mathcal{M}^* = \mathcal{H}(X)$ and $\mathcal{M} = \mathcal{M}^* \upharpoonright L$. Given an infinite cardinal κ, let $(Y, <)$ be any linear order of cardinality κ. By Corollary 2.4.1 T' has a model $\mathcal{N}_0$ in which $(Y, <)$ is an indiscernible sequence with $D(X) = D(Y)$. Let $\mathcal{N}^* = \mathcal{H}(Y)$, the hull of Y in $\mathcal{N}_0$, and $\mathcal{N} = \mathcal{N}^* \upharpoonright L$. By Lemma 2.5.1(i) $D(\mathcal{M}^*) = D(\mathcal{N}^*)$. Restricting to the original language, $D(\mathcal{M}) = D(\mathcal{N})$, as desired.

Further results can be obtained by varying the properties of the linear order $(Y, <)$ used in the proof of the theorem. For example, if $\kappa \geq \aleph_0$ there is a dense linear order without endpoints $(Y, <)$ with 2^κ many automorphisms. By (iii) of the lemma each of these extends to an automorphism of the Skolem hull of the indiscernible sequence $(Y, <)$. A result which will see important duty in the next chapter is

Lemma 2.5.2 (Morley). *Let T be a countable complete theory with infinite models. Then for every infinite cardinal κ, T has a model $\mathcal{M}$ of cardinality κ such that for every $A \subset M$, $\mathcal{M}$ realizes at most $|A| + \aleph_0$ many complete types over A.*

Proof. Let T^* be an expansion of T to a theory with Skolem functions, L the language of T and L^* the language of T^*. Let $(X, <)$ be a well-ordering of order type κ, considered as a sequence of indiscernibles in some model of T^*. Let $\mathcal{M}^*$ be the Skolem hull of X and $\mathcal{M} = \mathcal{M}^* \upharpoonright L$. To verify that $\mathcal{M}$ satisfies the requirements, let A be a subset of M. For the purposes of this lemma we may as well require that $A = H(Y)$, for some $Y \subset X$. (For any A there is a Y such that $A \subset H(Y)$ and $|Y| \leq |A| + \aleph_0$.) We call two sequences $x_1 < \ldots < x_n$, $y_1 < \ldots < y_n$ from X *equivalent over* Y if for $1 \leq k \leq n$ and all $z \in Y$, $x_k = z$ if and only if $y_k = z$ and $x_k < z$ if and only if $y_k < z$. (That is, the x_i's and y_i's satisfy the same order relations with the elements of Y.) Because X is an indiscernible sequence, whenever $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ are equivalent over Y they have the same complete type over Y in $\mathcal{M}$. In fact, for any term $t(v_1, \ldots, v_n)$ of $L^*(Y)$ the two elements $t(x_1, \ldots, x_n)$ and $t(y_1, \ldots, y_n)$ realize the same complete type over A. Similarly for n-tuples from M. Thus, to complete the proof it suffices to show that the equivalence relation of being equivalent over Y has at most $|Y| + \aleph_0$ many classes. To see this we define x', for $x \in X \setminus Y$, by

- $x' = \infty$ if there is no $z \in Y$ with $x < z$, and
- $x' =$ the least $z \in Y$ such that $x < z$, if there is such a $z \in Y$.

Then, $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ are equivalent over Y if and only if $x'_1 = y'_1, \ldots, x'_n = y'_n$. Hence there are $\leq |Y| + \aleph_0$ many equivalence classes of $n-$tuples, as required.

Historical Notes. Skolem functions date back to Skolem's 1920 paper [Sko20]. Morley proved Lemma 2.5.2 in [Mor65]. The other results in the section were proved by Ehrenfeucht and Mostowski in [EM56].

Exercise 2.5.1. Let $\mathcal{M}$ and $\mathcal{N}$ be models, X and Y nonempty subsets of M and N, respectively, and f an elementary bijection from X onto Y. Show that f extends to an isomorphism between the submodels generated by X and Y.

Exercise 2.5.2. Let $\mathcal{M} = (\omega, \leq), T = Th(\mathcal{M})$ and T^* the Skolem expansion of T. Show that $\mathcal{M}$ has two expansions to a model of T^* which are not elementarily equivalent. (Thus, T^* is not complete.)

Exercise 2.5.3. Let T be a complete theory with Skolem functions, $(X, <)$ an indiscernible sequence in $\mathcal{H}(X) \models T$. Show that there is an embedding of the automorphism group of $(X, <)$ into the automorphism group of $\mathcal{H}(X)$. Also, if both X and $<$ are definable this embedding is an isomorphism.

Exercise 2.5.4. Elements x and y of a model $\mathcal{M}$ are said to have the *same automorphism type* if there exists an automorphism f of $\mathcal{M}$ such that $f(x) = y$. Show that if T is a countable complete theory with infinite models, then T has a model in each cardinality which has only countably many automorphism types.

Exercise 2.5.5. Suppose that X is an infinite set of indiscernibles, $\mathcal{M}$ is a Skolem hull of X and the language of $\mathcal{M}$ is countable. How many automorphisms are there of $\mathcal{M}$?