XV. A More General Iterable Condition Ensuring $\aleph_1$ Is Not Collapsed

§0. Introduction

Chapter XI was restricted to forcing notions not adding reals in a specific way so that under CH, Nm is always permissible. This was used to show that various combinatorical principles of $\aleph_2$ were equiconsistent with the existence of (small) large cardinals. We constructed our models starting from CH without adding reals, so that CH also holds in the final model. But what if we want CH to fail in the final model? Can we phrase a condition preserved by iterations, implying $\aleph_1$ does not collapse and include semiproper forcing and Nm? This, promised in the first version of this book, is carried out here. We start with notions similar to the one in Chapter X, and then move in the direction of semiproperness. Further theorems (which shed light on preservation of not adding reals) will appear elsewhere (see [Sh:311]). The preservation theorems from this chapter are sufficient to prove analogue of some theorems from Chapter XI with the negation of CH. For example adding Cohen reals to the construction of XI 1.4 we can show: If "ZFC+ $\exists$ weakly compact cardinal" is consistent, then so is "ZFC+ $2^{\aleph_0} = \aleph_2$ + for every stationary $S \subseteq S^2_0$ there is a closed copy of ω_1 included in it". Generally the preservation proofs generalize those of Chapter XI, except in the case of "iterating up to a strongly inaccessible and doing one more step (in this case 3.6). We generalize Gitik and Shelah [GiSh:191] which improve the relevant theorem in XI (i.e. [Sh:b, XI]).

Of course we can also add reals to the construction in XI 1.2 and get an extension V^P where $\aleph_1^V = \aleph_1^{V^P}$, $\kappa = \aleph_2^{V^P}$ and the filter generated by the measure on κ in V will include S_0^2, and it is not clear that it will be precipitous, see X §6. As in Chapter XI, we use for proving the preservation, partition theorems and Δ-system theorems on trees: mainly 2.6 and 2.6A, 2.6B, 2.6C. Some of them are from Rubin and Shelah [RuSh:117], see detailed history there, on pages 47, 48.

§1. Preliminaries

The replacement of RCS (revised countable support) by GRCS (defined below) is not essential - it is intended to simplify the preservation theorems (one of the cases in Chapter VI refers to GRCS).

1.1 Conventions. A forcing notion here, P, is a nonempty set (denoted by P too) and two partial orders $\leq_{pr}, \leq$ and a minimal element $\emptyset_P \in P$, $[p \leq_{pr} q \rightarrow p \leq q]$. We call $p \in P$ pure if $\emptyset_P \leq_{pr} p$ and we call q a pure extension of p if $p \leq_{pr} q$. (In Chapter XIV=[Sh:250] this was written $\leq_0$).

We denote forcing notions by P, Q, R. (The forcing relation of course refers to the partial order $\leq$).

1.2 Definition. Let $MAC(P)$ be the set of maximal antichains of the forcing notion P.

1.2A Remark. 1) Note: $|MAC(P)| \leq 2^{|P|}$, P satisfies the $|P|^+$-c.c. and if P satisfies the λ-c.c. then $|MAC(P)| \leq |P|^{<\lambda}$.

2) Note

(*) if Q is a forcing notion, $\lambda = \lambda^{<\lambda} > |Q| + \aleph_0, \Vdash_Q$ "$(\forall \mu < \lambda)\mu^{\aleph_0} < \lambda$" and $Q' = Q * \underset{\sim}{\text{Levy}}(\aleph_1, < \lambda)$ *then* $|MAC(Q')| = |Q'| = \lambda$

1.3 Notation. Car is the class of cardinals.

IRCar is the class of infinite regular cardinals.

RCar = IRCar $\cup \{0\}$.

RUCar is the class of uncountable regular cardinals.

$\mathcal{D}^{\mathrm{cb}}_\lambda$ is the filter of co-bounded subsets of λ.

$\eta^- = \eta \restriction (\ell g(\eta) - 1)$ for $\eta \in V^{<\omega}$ and $\ell g(\eta) > 0$

1.4 Notation. $H(\chi)$ is the family of sets with transitive closure of power $< \chi$; let $<^*_\chi$ be a well ordering of $H(\chi)$.

1.5 Definition. *GRCS* iteration is as defined in Ch X, except that, for each condition all but finitely many of the atomic conditions in it are pure (or as in Chapter XIV §1 for $\kappa = \aleph_1$, $e = 1$).

1.6 Fact. $(*)_1$ if $\bar{Q}$ is a *GRCS* iteration, and for each i $\le^{Q_i}_{pr} = \le^{Q_i}$ *then* $\bar{Q}$ is an *RCS* iteration.

$(*)_2$ if $\bar{Q}$ is an *GRCS* iteration, and for each i the order $\le^{Q_i}_{pr}$ is equality *then* $\bar{Q}$ is essentially a finite support iteration.

$(*)_3$ the distributivity law, etc. (Chapter X 1.5, and §1 generally) holds for GRCS (by Chapter XIV §1).

1.7 Claim. Suppose we want to prove for all generic extensions V^Q of V, that for iteration $\langle P_i, \underaccent{\tilde}{Q}_j : i \le \alpha, j < \alpha\rangle$ as in 3.1 below, for a property φ that:

$(*)$ if Q and each Q_i has the property φ (of course Q in V, $\underaccent{\tilde}{Q}_i$ in $V^{Q * \underaccent{\tilde}{P}_i}$)

then P_α has the property φ (in V^Q).

Then it is enough to prove (*) when (a) and (b) below hold:

(a) for $i < j < \alpha, P_j/P_i$ has the property φ (in $V^{Q * \underaccent{\tilde}{P}_i}$)

(b) $\ell g(\bar{Q})$ is: 2, or ω, or ω_1, or strongly inaccessible $> |P_i|$ for each $i < \ell g(\bar{Q})$.

Remark. You may add:

(c) $(*)$ holds for all $\bar{Q}'$, $\langle P'_i, Q'_j : i \le \alpha', j < \alpha'\rangle$ for which $\alpha' < \alpha$ (not just in V, but in every generic extension of it).

Proof. By induction on α, using 3.7 (later in this chapter) and X §1 (or XIV §1).

§2. Trees of Models and UP

2.1 Definition.

1) A tagged tree is a pair $\langle T, \mathbf{I}\rangle$ such that:

   (a) T is a ω-tree, which here means a nonempty set of finite sequences of ordinals such that if $\eta \in T$ then any initial segment of η belongs to T. T is ordered by initial segments, i.e., $\eta \trianglelefteq \nu$ iff η is an initial segment of ν.

   (b) $\mathbf{I}$ is a partial function from T such that for every $\eta \in T$: if $\mathbf{I}(\eta) = \mathbf{I}_\eta$ is defined then $\mathbf{I}(\eta)$ is an ideal of subsets of some set called the domain of $\mathbf{I}_\eta, \mathrm{Dom}(\mathbf{I}_\eta)$, and

$$\mathrm{Suc}_T(\eta) \stackrel{\mathrm{def}}{=} \{\nu : \nu \text{ is an immediate successor of } \eta \text{ in } T\} \subseteq \mathrm{Dom}(\mathbf{I}_\eta),$$

   and if not said otherwise $\mathrm{Suc}_T(\eta) \notin \mathbf{I}_\eta$. Usually $\mathbf{I}_\eta$ is $\aleph_2$-complete.

   (c) For every $\eta \in T$ we have $\mathrm{Suc}_T(\eta) \neq \emptyset$.

2) We call $(T, \mathbf{I})$ normal if $\eta \in \mathrm{Dom}(\mathbf{I}_\eta) \Rightarrow \mathrm{Dom}(\mathbf{I}_\eta) = \mathrm{Suc}_T(\eta)$.

2.1A Convention. For any tagged tree $(T, \mathbf{I})$ we can define $\mathbf{I}^\dagger$, by:

$$\mathrm{Dom}(\mathbf{I}') = \{\eta : \mathrm{Suc}_T(\eta) \subseteq \mathrm{Dom}(\mathbf{I}_\eta),\ \text{and } \mathrm{Suc}_T(\eta) \notin \mathbf{I}_\eta\} \text{ and}$$

$$\mathbf{I}^\dagger_\eta = \{\{\alpha : \eta\hat{\ }\langle\alpha\rangle \in A\} : A \in \mathbf{I}_\eta\};$$

we sometimes, in an abuse of notation, do not distinguish between $\mathbf{I}$ and $\mathbf{I}^\dagger$ e.g. if $\mathbf{I}^\dagger_\eta$ is constantly I^*, we write I^* instead of $\mathbf{I}$.

2.2 Definition. 1) η will be called a splitting point of $(T, \mathbf{I})$ if $\mathbf{I}_\eta$ is defined and $\mathrm{Suc}_T(\eta) \notin \mathbf{I}_\eta$ (normally this follows but we may forget to decrease the domain of $\mathbf{I}$). Let split$(T, \mathbf{I})$ be the set of splitting points. We will only consider trees where each branch meets split$(T, \mathbf{I})$ infinitely often.

2) For $\eta \in T$, $T^{[\eta]} \stackrel{\text{def}}{=} \{\nu \in T : \ \nu = \eta \text{ or } \nu \triangleleft \eta \text{ or } \eta \triangleleft \nu\}$.

2.3 Definition. We now define orders between tagged trees:

a) $(T_1, \mathbf{I}_1) \leq (T_2, \mathbf{I}_2)$ if $T_2 \subseteq T_1$, and split$(T_2, \mathbf{I}_2) \subseteq$ split$(T_1, \mathbf{I}_1)$, and $\forall \eta \in$ split$(T_2, \mathbf{I}_2)$: $\mathbf{I}_2(\eta) \restriction \mathrm{Suc}_{T_2}(\eta) = \mathbf{I}_1(\eta) \restriction \mathrm{Suc}_{T_2}(\eta)$.

(where $I \restriction A = \{B : B \subseteq A \text{ and } B \in I\}$). (So every splitting point of T_2 is a splitting point of T_1, and $\mathbf{I}_2$ is completely determined by $\mathbf{I}_1$ and split$(T_2, \mathbf{I}_2)$ provided that $\mathbf{I}_2$ is normal.)

b) $(T_1, \mathbf{I}_1) \leq^* (T_2, \mathbf{I}_2)$ *iff* $(T_1, \mathbf{I}_1) \leq (T_2, \mathbf{I}_2)$ and split$(T_2, \mathbf{I}_2) =$ split$(T_1, \mathbf{I}_1) \cap T_2$.

c) $(T_1, \mathbf{I}_1) \leq^{\otimes} (T_2, \mathbf{I}_2)$ if $(T_1, \mathbf{I}_1) \leq^* (T_2, \mathbf{I}_2)$ and $\eta \in T_2 \setminus$ split$(T_1, \mathbf{I}_1) \Rightarrow$ $\mathrm{Suc}_{T_2}(\eta) = \mathrm{Suc}_{T_1}(\eta)$

(d) $(T_1, \mathbf{I}_1) \leq^{\otimes}_{\mu} (T_2, \mathbf{I}_2)$ *if* $(T_1, \mathbf{I}_1) \leq (T_2, \mathbf{I}_2)$ and $\eta \in T_2$ & $|\mathrm{Suc}_{T_1}(\eta)| < \mu \Rightarrow$ $\mathrm{Suc}_{T_2}(\eta) = \mathrm{Suc}_{T_1}(\eta)$ and $\eta \in T_2$ & $|\mathrm{Suc}_{T_1}(\eta)| \geq \mu$ & $\eta \in \mathrm{Sp}(T_1, \mathbf{I}_1) \Rightarrow \eta \in$ $\mathrm{Sp}(T_2, \mathbf{I}_2)$

2.4 Definition. 1) For a set $\mathbb{I}$ of ideals, a tagged tree $(T, \mathbf{I})$ is an $\mathbb{I}$-tree if for every splitting point $\eta \in T$ we have $\mathbf{I}_\eta \in \mathbb{I}$ (up to an isomorphism).

2) For a set $\mathbf{S}$ of regular cardinals, an $\mathbf{S}$-tree T is a tree such that for any point $\eta \in T$ we have: $|\mathrm{Suc}_T(\eta)| \in \mathbf{S}$ or $|\mathrm{Suc}_T(\eta)| = 1$.

3) We omit $\mathbf{I}$ and denote a tagged tree $(T, \mathbf{I})$ by T whenever $\mathbf{I}_\eta = \{A \subseteq \mathrm{Suc}_T(\eta) : |A| < |\mathrm{Suc}_T(\eta)|\}$ and $|\mathrm{Suc}_T(\eta)| \in$ IRCar $\cup \{1\}$ for every $\eta \in T$.

4) For a tree T, $\lim T$ is the set of *branches* of T, i.e. all ω-sequences of ordinals, such that every finite initial segment of them is a member of T: $\lim T = \{s \in {}^{\omega}\mathrm{Ord} : (\forall n)\, s \restriction n \in T\}$.

5) A subset J of a tree T is a *front* if: $\eta \neq \nu \in J$ implies none of them is an initial segment of the other, and every $\eta \in \lim T$ has an initial segment which is a member of J.

6) $(T, \mathbf{I})$ is standard if for every nonsplitting point $\eta \in T$, $|\mathrm{Suc}_T(\eta)| = 1$.

7) $(T, \mathbf{I})$ is full if every $\eta \in T$ is a splitting point.

2.4A Remark. (1) The set lim$T$ is not absolute, i.e., if $V_1 \subseteq V_2$ are two universes of set theory then in general $(\lim T)^{V_1}$ will be a proper subset of $(\lim T)^{V_2}$.

(2) However, the notion of being a front is absolute: if $V_1 \models$ "F is a front in T", then there is a depth function $\rho : T \to \mathrm{Ord}$ satisfying $\eta \triangleleft \nu \,\&\, \forall k \leq \ell g(\eta)[\eta{\restriction}k \notin F] \to \rho(\eta) > \rho(\nu)$. This function will also witness in V_2 that F is a front.

(3) $F \subseteq T$ contains a front iff F meets every branch of T. So if $F \subseteq T$ contains a front of T and $T' \subseteq T$, then $F \cap T'$ contains a front of T'. Also this notion is absolute.

2.4B Notation. In several places in this chapter we will have an occasion to use the following notation: Assume that $(T, \mathbf{I})$ is a tagged tree, and for all $\eta \in T$ there is a family a_η of subsets of $T^{[\eta]}$ such that $\eta \triangleleft \nu \Rightarrow \forall A \in a_\eta \, \exists B \in a_\nu$ $[B \subseteq A]$. Then we can define for all $\alpha \in \mathrm{Ord} \cup \{\infty\}$

$\mathrm{Dp}_\alpha(\eta)$ iff $\forall \beta < \alpha \forall A \in a_\eta \exists \nu \in A \cap \mathrm{split}(T)[\{\rho : \rho \in \mathrm{Suc}_T(\nu) \,\&\, \mathrm{Dp}_\beta(\rho)\} \notin \mathbf{I}_\nu]$.

Then it is easy to see that

$\mathrm{Dp}(\eta) \stackrel{\mathrm{def}}{=} \max\{\alpha \in \mathrm{Ord} \cup \{\infty\} : \mathrm{Dp}_\alpha(\eta)\}$

is well defined, and $\mathrm{Dp}_\alpha(\eta) \Leftrightarrow \mathrm{Dp}(\eta) \geq \alpha$. We call $\mathrm{Dp}(\eta)$ the "depth" of η (with respect to the family $\mathbf{a} = \langle a_\eta : \eta \in T \rangle$ and the tagged tree $(T, \mathbf{I})$). It is easy to check that $\eta \triangleleft \nu \Rightarrow \mathrm{Dp}(\eta) \geq \mathrm{Dp}(\nu)$.

2.5 Definition. 1) An ideal I is λ-complete if any union of less than λ members of I is still a member of I.

2) A tagged tree $(T, \mathbf{I})$ is λ-complete if for each $\eta \in T \cap \mathrm{Dom}(\mathbf{I})$ the ideal $\mathbf{I}_\eta$ is λ-complete.

3) A family $\mathbb{I}$ of ideals is λ-complete if each $I \in \mathbb{I}$ is λ-complete. We will only consider $\aleph_2$-complete families $\mathbb{I}$.

4) A family $\mathbb{I}$ is restriction-closed *if* $I \in \mathbb{I}$, $A \subseteq \mathrm{Dom}(I)$, $A \notin I$ implies $I{\restriction}A = \{B \in I : B \subseteq A\}$ belongs to $\mathbb{I}$.

5) The restriction closure of $\mathbb{I}$, res-cl($\mathbb{I}$) is $\{I{\restriction}A : I \in \mathbb{I}, A \subseteq \mathrm{Dom}(I), A \notin I\}$.

6) I is λ-indecomposable *if* for every $A \subseteq \mathrm{Dom}(I), A \notin I$, and $h : A \to \lambda$ there is $Y \subseteq \lambda, |Y| < \lambda$ such that $h^{-1}(Y) \notin I$. We say $\mathbf{I}$ or $\mathbb{I}$, is λ-indecomposable if each $\mathbf{I}_\eta$ (or $I \in \mathbb{I}$) is λ-indecomposable.

7) I is strongly λ-indecomposable if for $A_i \in I(i < \lambda)$ and $A \subseteq \mathrm{Dom}(I), A \notin I$ we can find $B \subseteq A$ of cardinality $< \lambda$ such that for no $i < \lambda$ does A_i include B.

2.5A Remark. As indicated by the names, if I is strongly λ-indecomposable then I is λ-indecomposable at least when λ is regular. [Why? Given A, h as in 2.5(6), let $A_i = h^{-1}(\{j : j < i\})$; if for some i, $A_i \notin I$ we are done, otherwise by 2.5(7) there is $Y \subseteq A$, $|Y| < \lambda \bigwedge_i Y \nsubseteq A_i$. But as λ is regular $> |Y|$, $i(*) = \sup\{h(x) + 1 : x \in Y\} < \lambda$ hence $Y \subseteq A_{i(*)}$, contradiction.]

2.6 Lemma. Let θ be an uncountable regular cardinal (the main case here is $\theta = \aleph_1$). Let $\mathbb{I}$ be a family of θ^+-complete ideals, $(T_0, \mathbf{I})$ a tagged tree, $A = \{\eta \in T : 0 < |\mathrm{Suc}_{T_0}(\eta)| \leq \theta\}$, $[\eta \in T_0 \setminus A \Rightarrow \mathbf{I}_\eta \in \mathbb{I}\,\&\,\mathrm{Suc}_{T_0}(\eta) \notin \mathbf{I}_\eta]$ and $[\eta \in A \Rightarrow \mathrm{Suc}_{T_0}(\eta) \subseteq \{\eta\hat{}\langle i\rangle : i < \theta\}]$ and $H : T_0 \to \theta$ and $\bar{e} = \langle e_\eta : \eta \in A\rangle$, is such that e_η is a club of θ. *Then* there is a club C of θ such that: for each $\delta \in C$ there is $T_\delta \subseteq T_0$ satisfying:

(a) T_δ a tree.

(b) If $\eta \in T_\delta$, $|\mathrm{Suc}_{T_0}(\eta)| < \theta$, then $\mathrm{Suc}_{T_\delta}(\eta) = \mathrm{Suc}_{T_0}(\eta)$, and if $|\mathrm{Suc}(\eta)| = \theta$, then $\mathrm{Suc}_{T_\delta}(\eta) = \{\eta\hat{}\langle i\rangle : i < \delta\} \cap \mathrm{Suc}_{T_0}(\eta)$ and $\delta \in e_\eta$.

(c) $\eta \in T_\delta \setminus A$ implies $\mathrm{Suc}_{T_\delta}(\eta) \notin \mathbf{I}_\eta$.

(d) for every $\eta \in T_\delta$: $H(\eta) < \delta$.

Proof. For each $\zeta < \theta$ we define a game $\eth_\zeta$. The game lasts ω moves, in the nth move $\eta_n \in T_0$ of length n is chosen.

For $n = 0$ necessarily $\eta_0 = \langle\,\rangle$.

For $n = m + 1$: If $|\mathrm{Suc}_{T_0}(\eta_m)| = \theta$, then the *first* player chooses $\eta_{m+1} \in \mathrm{Suc}_{T_0}(\eta_n)$, $\eta_{m+1}(m) < \zeta$.

If $|\mathrm{Suc}_{T_0}(\eta_m)| < \theta$, then the *first* player chooses any $\eta_{m+1} \in \mathrm{Suc}_{T_0}(\eta_m)$.

If $\eta_m \notin A$, then the first player chooses $A_m \in \mathbf{I}_{\eta_m}$, and then the *second* player chooses $\eta_{m+1} \in \mathrm{Suc}_{T_0}(\eta_m) \setminus A_m$.

At the end, the second player wins if for all n, $H(\eta_n) < \zeta$ and $|\mathrm{Suc}_{T_0}(\eta_n)| = \theta \Rightarrow \zeta \in e_{\eta_n}$. Now clearly

$(*)$ if for a club of $\zeta < \theta$ the second player has a winning strategy for the game $\eth_\zeta$, *then* there are trees T_δ (as required).

Let $S = \{\delta < \theta :$ second player does not have a winning strategy for the game $\eth_\delta\}$; we assume that S is stationary, and get a contradiction.

Let for $\delta \in S, F_\delta$ be a winning strategy for first player in $\eth_\delta$ (he has a winning strategy as the game is determined being closed for the second player). So F_δ gives for the first $(n-1)$-moves of the second player, the nth move of the first player.

Let χ be regular large enough, and let $(N_0, \in) \prec (H(\chi), \in)$ be such that $\theta + 1 \subseteq N_0$, $|N_0| = \theta$, $(T_0, \mathbf{I}) \in N_0$, $\bar{e} \in N_0$, and $\langle F_\delta : \delta \in S\rangle \in N_0$. We can find $N_1 \prec N_0$ such that $|N_1| < \theta$, $N_1 \cap \theta$ is an ordinal and $(T_0, \mathbf{I}) \in N_1$, $\langle F_\delta : \delta \in S\rangle \in N_1$ and $\bar{e} \in N_1$. Let $\delta \stackrel{\text{def}}{=} N_1 \cap \theta$. Since S was assumed to be stationary, we may assume $\delta \in S$.

Now we shall define by induction on $n, \eta_n \in T_0 \cap N_1$ of length n, such that $\langle \eta_\ell : \ell \leq n\rangle$ is an initial segment of a play of the game $\eth_\delta$ in which the first player uses his winning strategy F_δ.

Case 1. $n = 0$:
We let $\eta_0 = \langle\rangle$. (The $A_\ell \in \mathbf{I}_{\eta_\ell}$ are not mentioned as they are not arguments of F_δ).

Case 2. For $n = m+1$, $\eta_m \in A$: the first player has a winning strategy F_δ for the game $\eth_\delta$. So F_δ gives us η_n. Now if $|\mathrm{Suc}_{T_0}(\eta_m)| < \theta$ then $\mathrm{Suc}_{T_0}(\eta_m) \subseteq N_1$ (because T_0, η_m belongs and $N_1 \cap \theta$ is an ordinal), hence $\eta_n \in N_1$ as required. If $|\mathrm{Suc}_{T_0}(\eta_m)| = \theta$ then necessarily $\mathrm{Suc}_{T_0}(\eta_m) \subseteq \{\eta_m{}^\frown\langle i\rangle : i < \theta\}$, $\eta_n = \eta_m{}^\frown\langle i\rangle, i < \delta$ (as the play is of the game $\eth_\delta$), so necessarily $i \in N_1$ hence (as $\eta_m \in N_1$) also $\eta_n \in N_1$.

Case 3. Lastly if $n = m+1, \eta_m \notin A$: so F_δ gives us $A^\delta_m \in \mathbf{I}_{\eta_m}$ which is not necessarily in N_1, however let $A^* = \bigcup\{A^\zeta_m : \zeta \in S$, and there is a play of $\eth_\zeta$ in which $\langle \eta_\ell : \ell \le m\rangle$ were played and the first player plays according to F_ζ (this play is unique) and the strategy F_ζ dictates to the first player to choose $A^\zeta_m\}$.

Now A^* is in N_1 (as $\bar{F} \in N_1$) and as the union of $\le \theta$ members of $\mathbf{I}_{\eta_m}$ it belongs to $\mathbf{I}_{\eta_m}$ hence $A^* \cap \mathrm{Suc}_{T_0}(\eta_m)$ is a proper subset of $\mathrm{Suc}_{T_0}(\eta_m)$, so there is $\eta_m{}^\frown\langle i\rangle \in \mathrm{Suc}_{T_0}(\eta_m) \setminus A^*$, so there is such $i \in N_1$ (so necessarily $i < \delta$). Let the second player choose $\eta_n = \eta_m{}^\frown\langle i\rangle$.

So we have played a sequence $\langle \eta_n : n \in \omega\rangle$ of elements of N_1, always obeying F_δ so this sequence was produced by a play of $\eth_\delta$ in which the first player plays according to the strategy F_δ. But then for all n : $\eta_n \in N_1 \Rightarrow H(\eta_n) \in N_1$, so $H(\eta_n) < \delta$, and

$$\eta_n \in N_1 \Rightarrow e_{\eta_n} \in N_1 \Rightarrow \delta = \sup(e_{\eta_n} \cap \delta) \Rightarrow \delta \in e_{\eta_n};$$

hence second player wins in this play. So F_δ cannot be a winning strategy. Contradiction, so S is not stationary. $\square_{2.6}$

2.6A Lemma. Suppose $(T, \mathbf{I})$ is an $\mathbb{I}$-tree, θ regular uncountable, $\langle A_\eta : \eta \in T\rangle$ is such that: A_η is a set of ordinals, $[\eta \triangleleft \nu \Rightarrow A_\eta \subseteq A_\nu]$ and

(∗) (a) $\mathbf{S} \subseteq$ RUCar,

(b) $\mathbb{I}' \stackrel{\text{def}}{=} \mathbb{I} \setminus \{I : |\mathrm{Dom}(I)| < \mu\}$ is μ^+-complete or at least strongly μ-indecomposable for every μ such that $\mu \in \mathbf{S}$ or $\mu \in \mathrm{pcf}\,(\mathbf{S} \cap A_\eta)$ for some $\eta \in T$ and

(c) $\mathbb{I}$ is θ-complete and $|\,\mathrm{pcf}\,(\mathbf{S} \cap A_\eta)| < \theta$ for $\eta \in T$ and $\theta \le \min(\mathbf{S})$

(d) $|A_\eta| < \min(\mathbf{S})$ for $\eta \in T$

Then there is $T^\dagger$, $(T, \mathbf{I}) \le^* (T^\dagger, \mathbf{I})$, such that:

if $\lambda \in A_\nu \cap \mathbf{S}$ and $\nu \in T^\dagger$ then for some $\alpha_\nu(\lambda) < \lambda$ for every ρ such that $\nu \triangleleft \rho \in \lim T^\dagger$ we have $\alpha_\nu(\lambda) \ge \sup(\lambda \cap \bigcup_{n<\omega} A_{\rho\restriction n})$.

Proof. It is enough to prove the existence of a $T^\dagger$ as required just for $\nu = \langle\ \rangle$ (as we can repeat the proof going up in the tree). This can be proved by induction on max(pcf ($\mathbf{S} \cap A_{\langle\ \rangle}$)) (exist see [Sh:g, I 1.9]). Let $\alpha_\lambda(\eta) = \sup(A_\eta \cap \lambda)$.

As this lemma (2.6A) is not used in this book we assume knowledge of [Sh:g].

Let $\mathfrak{a} \stackrel{\text{def}}{=} \mathbf{S} \cap A_{\langle\rangle}$ (if $\mathfrak{a}$ is empty we have nothing to do), $\mu = \max\mathrm{pcf}\,(\mathfrak{a})$, and $\langle f_\alpha : \alpha < \mu\rangle$ be $<_{J_{<\mu}[\mathfrak{a}]}$-increasing cofinal. Let $\{\mathfrak{b}_\varepsilon : \varepsilon < \varepsilon(*)\}$ be cofinal in $J_{<\mu}[\mathfrak{a}]$ e.g. this set is $\{\bigcup_{\theta\in\mathfrak{c}} b_\theta[\mathfrak{a}] : \mathfrak{c} \subseteq \mathrm{pcf}\,\mathfrak{a} \setminus \{\mu\}$ finite$\}$ so $\varepsilon(*) < \theta$ hence by an assumption $\mathbb{I}'$ is $|\varepsilon(*)|^+$-complete.

For $\varepsilon < \varepsilon(*)$, $\zeta < \mu$ we define:

$(*)^\varepsilon_\zeta$ there is a subtree T' of T, $(T, \mathbf{I}) \leq^* (T', \mathbf{I})$ such that for every $\eta \in \lim(T')$ and $\lambda_n \in \mathfrak{a} \setminus \mathfrak{b}_\varepsilon$ we have $\alpha_{\lambda_n}(\eta{\restriction}n) \leq f_\zeta(\lambda_n)$.

It suffices to find such T' (for some ε, ζ) as: $\max\mathrm{pcf}\,(\mathfrak{b}_\varepsilon) < \max\mathrm{pcf}\,(\mathfrak{a})$, so we can apply the induction hypothesis on T'.

In V define for $\zeta < \mu$ and $\varepsilon < \varepsilon(*)$.

$$B_\zeta \stackrel{\text{def}}{=} \{\eta \in \lim(T) : \text{ for some } \varepsilon < \varepsilon(*) \text{ for every } \lambda \in \mathfrak{a} \setminus \mathfrak{b}_\varepsilon \text{ we have } n < \omega \Rightarrow \alpha_\lambda(\eta{\restriction}n) \leq f_\zeta(\lambda)\}$$

$$B_{\zeta,\varepsilon} \stackrel{\text{def}}{=} \{\eta \in \lim(T) : \text{ for every } \lambda \in \mathfrak{a} \setminus \mathfrak{b}_\varepsilon, n < \omega \Rightarrow \alpha_\lambda(\eta{\restriction}n) \leq f_\zeta(\lambda)\}$$

Clearly $B_{\zeta,\varepsilon}$ is closed and $B_\zeta = \bigcup\limits_{\varepsilon<\varepsilon(*)} B_{\zeta,\varepsilon}$. Now $\zeta < \xi < \mu \Rightarrow B_\zeta \subseteq B_\xi$, (as $f_\zeta <_{J_{<\mu}[\mathfrak{a}]} f_\xi$) and $\lim(T) = \bigcup_{\zeta<\mu} B_\zeta$ (as $\langle f_\zeta : \zeta < \mu\rangle$ is cofinal in $\prod_{n<\omega}\lambda_n$), hence using 2.6B(3) below (with μ, $\varepsilon(*)$ here standing for θ, ε_i there) for some $\zeta(*) < \mu$ and $\varepsilon < \varepsilon(*)$ and T' we have $(T, \mathbf{I}) \leq^* (T', \mathbf{I})$ and $\lim(T') \subseteq B_{\zeta,\varepsilon}$. So $(*)^\varepsilon_\zeta$ holds, but as said above this suffices. $\square_{2.6A}$

Question. If $\mathbb{I} \in H(\chi)$ is there a countable $N \prec (H(\chi), \in, <^*_\chi)$ such that: $\mathbb{I} \in N$ and for every $\lambda \in \mathrm{RCar} \cap N$, letting $\mathbb{I}^{[\lambda]} \stackrel{\text{def}}{=} \{J \in \mathbb{I} : J \text{ is } \lambda^+\text{-complete}\}$, there is $\langle N^\lambda_\eta : \eta \in (T, \mathbf{I})\rangle$ an $\mathbb{I}^{[\lambda]}$-suitable tree (see Definition 2.10) such that $N <_\lambda N^\lambda_{\langle\rangle}$? (Or replace RCar by a thinner set.)

2.6B Lemma. Let $(T, \mathbf{I})$ be an $\mathbb{I}$-tree, $\mathbb{I}$ a family of ideals,

1) If $H : T \to \mu$ and $\mu^{\aleph_0} < \lambda$ and $\mathbb{I}$ is λ - complete *then* there is T' such that

$$(T, \mathbf{I}) \leq^* (T', \mathbf{I})$$
$$\eta, \nu \in T' \ \& \ \ell g(\eta) = \ell g(\nu) \Rightarrow H(\eta) = H(\nu)$$

2) If $\lim(T) = \bigcup_{i<\theta} B_i, \theta < \lambda$, $\mathbb{I}$ is λ - complete and each B_i is a Borel set *then* there is T' such that

$$(T, \mathbf{I}) \leq^* (T', \mathbf{I})$$
$$\text{for some } i : [\lim T \subseteq B_i]$$

3) If θ is regular uncountable, $\lim(T) = \bigcup\limits_{i<\theta} B_i$, and B_i is a Borel subset of $\lim(T)$, increasing with i, and $(*)$ below holds then

(a) for some $i < \theta$ and T' we have $(T, \mathbf{I}) \leq^* (T', \mathbf{I})$ and $\lim(T') \subseteq B_i$

(b) if in addition $\eta \in T \setminus \text{split}((T, \mathbf{I})) \Rightarrow |\text{Suc}_T(\eta)| < \theta$ then in (a) we can demand $(T, \mathbf{I}) \leq^{\otimes} (T', \mathbf{I})$

where

$(*)$ every $I \in \mathbb{I}$ is θ^+-complete or at least strongly θ-indecomposable (see 2.5(7)).

4) Assume $\lim(T) = \bigcup_{i<\theta} \bigcup_{\varepsilon<\varepsilon_i} B_{i,\varepsilon}$, each $B_{i,\varepsilon}$ is a Borel set, $[i < \theta \Rightarrow \varepsilon_i < \sigma]$, $\mathbb{I}$ is σ-complete, and each $I \in \mathbb{I}$ is strongly θ-indecomposable, and $B_i \stackrel{\text{def}}{=} \bigcup_{\varepsilon<\varepsilon_i} B_{i,\varepsilon}$ is increasing in i and

$$[\eta \in T \setminus \text{split}(T, \mathbf{I}) \Rightarrow |\text{Suc}_T(\eta)| < \sigma].$$

Then for some $i < \theta$ and $\varepsilon < \varepsilon_i$ and T' we have $(T, \mathbf{I}) \leq^{\otimes} (T', \mathbf{I})$, and $\lim(T') \subseteq B_{i,\varepsilon}$.

2.6C Remark. 1) We can combine 2.6B(3), (4) with 2.6A.

2) To what can we weaken "strongly θ-indecomposable"? A sufficient condition is the existence of a precipitous normal filter E on θ such that for every $I \in \mathbb{I}$ and $A_i \in I$ for $i < \theta$ and $A^* \in I^+$ there are $x_i \in A^*$ for $i < \theta$ such that $\{i \in A^* : \{x_j : j < i\} \not\subseteq A_i\} \neq \emptyset \mod E$

3) We can elaborate 2.6B(4). We can have $t \subseteq {}^{\omega>}\text{Ord}$ be a tree with no ω-branch, $\langle B_\eta : \eta \in t \rangle$ a sequence of subsets of $\lim(T)$ such that:

(α) if $\eta \in \max(t)$ then B_η is Borel

(β) if $\eta \in t$ is not maximal, then (a) or (b)

(a) $\mathbb{I}$ is $|\mathrm{Suc}_t(\eta)|^+$-complete, $B_\eta = \bigcup\{B_\nu : \nu \in \mathrm{Suc}_t(\eta)\}$

(b) $\langle B_{\eta\hat{}\langle i\rangle} : \eta\hat{}\langle i\rangle \in \mathrm{Suc}_t(\eta)\rangle$ is increasing and letting $\theta_\eta = \mathrm{cf}(\mathrm{otp}(\{i : \eta\hat{}\langle i\rangle \in t\}))$, $\mathbb{I}$ is strongly θ_η-indecomposable.

Now we prove by downward induction on $\eta \in t$ that

$(*)_\eta$ there are $(T', \mathbf{I})$ and ν such that: $\eta \trianglelefteq \nu \in \max(t)$, $(T, \mathbf{I}) \leq^* (T', \mathbf{I})$ and $\lim(T') \subseteq B_i$ *or* in the game corresponding to $\bigcup\{B_\rho : \eta \trianglelefteq \rho \in \max(t)\}$ the first player wins

4) We can combine 2.6C(3) with 2.6B(3).

Proof. 1), 2) By [RuSh:117] or see here XI 3.5, 3.5A.

3) Similar to the proof of 2.6. First we prove clause (a). Without loss of generality $(T, \mathbf{I})$ is standard, so for notational simplicity it is full (see Definition 2.4(6), (7)). For each $\zeta < \theta$ let ∂_ζ be the following game with ω moves, letting $\eta_0 = \langle\rangle$ and in the n'th move $\eta_n \in T$ is chosen; the first player chooses $A_n \in \mathbf{I}_{\eta_n}$ and the second player $\eta_{n+1} \in \mathrm{Suc}_T(\eta_n) \setminus A_n$. In the end $\bigcup_{n<\omega} \eta_n \in \lim(T)$, and the second player wins the play if $\bigcup_n \eta_n \in B_\zeta$. It suffices to prove for some $\zeta < \theta$, the second player has a winning strategy. So otherwise for each ζ the first player has a winning strategy F_ζ. Let χ be large enough, $N_1 \prec (H(\chi), \in, <^*_\chi)$, $\|N_1\| < \theta$, $\delta \overset{\mathrm{def}}{=} N_1 \cap \theta < \theta$ such that $(T, \mathbf{I}), \langle B_\zeta : \zeta < \theta\rangle$ and $\langle F_\zeta : \zeta < \theta\rangle$ belongs to N_1. We shall simulate a play $\langle A_m, \eta_{m+1} : m < \omega\rangle$ of ∂_δ such that $\eta_{m+1} \in N_0$. Assume $\langle A_\ell, \eta_{\ell+1} : \ell < m\rangle$ is already defined. Let $S'_m = \{\zeta < \theta$: there is an initial segment of a play of ∂_ζ in which the first player uses the strategy F_ζ and the second player plays $\langle \eta_\ell : \ell \leq m\rangle\}$, note that such initial segment is unique, for a given ζ. For $\zeta \in S'_m$ let A^ζ_m be the $(m+1)$'th move of the first player, for such a play with the second player using the strategy F_ζ, so $\langle A^\zeta_m : \zeta \in S'_m\rangle \in N_1$, also clearly $\delta \in S'_m$, hence $|S'_m| = \theta$ and by the assumption $(*)$ for some $B \in N_1$, $B \subseteq \mathrm{Suc}_T(\eta_m)$, $|B| < \theta$ and $\bigwedge_{\zeta \in S'_m} B \not\subseteq A^\zeta_m$. As $B \in N_1$ and $N_1 \cap \theta = \delta$, clearly $B \subseteq N_1$ and choose $i \in B \setminus A^\delta_m$ and let $\eta_{n+1} = \eta_m\hat{}\langle i\rangle$.

For proving clause (b), defining $\Game_\zeta$ if $|\mathrm{Suc}_T(\eta_n)| < \theta$ we change the rule and let player I choose $\eta_{n+1} \in \mathrm{Suc}_T(\eta_n)$.

4) We define, for $\zeta < \theta$ and $\varepsilon < \varepsilon_\zeta$ a game $\Game_{\zeta,\varepsilon}$ as in the proof of 2.6B(3) clause (a) for $B_{\zeta,\varepsilon}$. If for some $\zeta < \theta$, $\varepsilon < \varepsilon_\zeta$ the second player wins then we get the desired conclusion. Otherwise as each such game is determind (as $B_{\zeta,\varepsilon}$ is a Borel set) there is a winning strategy $F_{\zeta,\varepsilon}$ for the first player. As $\mathbf{I}$ is $|\varepsilon_\zeta|^+$-complete, there is one strategy F_ζ good in all the games $F_{\zeta,\varepsilon}(\varepsilon < \varepsilon_\zeta)$ simultaniously (take the union of the sets suggested by all those strategies). So F_ζ is a winning strategy in $\Game_\zeta$, and we can proceed as in the proof of 2.6B(3).

$\square_{2.6B}$

2.7 Definition. Let $\mathbb{I}$ be a set of $\aleph_2$-complete ideals, $\mathbf{S}$ a set of regular cardinals, $\aleph_1 =$ Min $(\mathbf{S})$ and P a forcing notion.

1) We say that $(T, \mathbf{I}, \bar{\lambda}, \bar{\underset{\sim}{\xi}}, \bar{\zeta})$ is a $(\mathbb{I}, P, \mathbf{S})$-tree if:

a) $(T, \mathbf{I})$ is a $\mathbb{I}$-tree (see Definition 2.4(2))

b) $\bar{\lambda}$ is a function from T to $\mathbf{S}$

c) $\bar{\underset{\sim}{\xi}}$ is a function with domain T such that for every $\eta \in T, \bar{\underset{\sim}{\xi}}(\eta)$ is a P-name of an ordinal $< \bar{\lambda}(\eta)$

d) $\bar{\zeta}$ is a function from $T \setminus \{\langle\rangle\}$ such that each $\bar{\zeta}(\eta)$ is an ordinal.

2) We say that the $(\mathbb{I}, P, \mathbf{S})$-tree $(T, \mathbf{I}, \bar{\lambda}, \bar{\underset{\sim}{\xi}}, \bar{\zeta})$ *obeys* a function F if there are fronts $J_n \subseteq T$ for $n < \omega$ (see Definition 2.5 (2)) such that every member of J_{n+1} has a strict initial segment in J_n and $\eta \in J_n$ implies

$$\Big\langle \mathrm{Suc}_T(\eta), \mathbf{I}_\eta, \langle \bar{\zeta}(\nu) : \nu \in \ \mathrm{Suc}_T(\eta)\rangle \Big\rangle =$$
$$F\Big\langle \eta, w[\eta], \langle \langle \bar{\lambda}(\eta{\restriction}\ell), \bar{\underset{\sim}{\xi}}(\eta{\restriction}\ell),\ \bar{\zeta}(\eta{\restriction}\ell)\rangle\ : \ell \leq \ell g(\eta)\rangle \Big\rangle$$

where $w[\eta]$ is $\{k : \eta{\restriction}k \in \bigcup_{\ell<\omega} J_\ell\}$.

2.7A Definition. We say that the forcing notion P satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ (the "universal property"), where $\mathbf{W} \subseteq \omega_1$ is stationary, $\underset{\sim}{\mathbf{S}}$ a P-name of a set of uncountable regular cardinals (in V) which contains $\aleph_1^V$, provided that: letting $\mathbf{S}^* = \mathbf{S}^*[\underset{\sim}{\mathbf{S}}] = \{\kappa : \kappa \text{ regular} \leq |P|, \nVdash \text{“}\kappa \notin \underset{\sim}{\mathbf{S}}\text{”}\}$, for every $p \in P$ there is a function F_p (with domain and range as implied implicitly in (2)) such that: for any $(\mathbb{I}, P, \mathbf{S}^*)$-tree $(T, \mathbf{I}, \bar{\lambda}, \bar{\xi}, \bar{\zeta})$ obeying F_p and any $T^\dagger$, $(T, \mathbf{I}) \le^* (T^\dagger, \mathbf{I})$ there is $q \in P$, $p \le_{pr} q$ such that:

$q \Vdash_P$ "there is $\eta \in \lim T^\dagger$ such that: *if* $\sup\{\bar{\zeta}(\eta{\restriction}\ell) : \ell < \omega$ and $\bar{\zeta}(\eta{\restriction}\ell) < \omega_1\}$ is not obtained and belongs to $\mathbf{W}$ *then*: for every $m < \omega$ satisfying $\bar{\lambda}(\eta{\restriction}m) \in \underset{\sim}{\mathbf{S}}$, for some $\ell < \omega$ we have $\underset{\sim}{\xi}(\eta{\restriction}m)[\underset{\sim}{G}_P] < \bar{\zeta}(\eta{\restriction}\ell) < \bar{\lambda}(\eta{\restriction}m)$"

2.7B Notation. 1) If $\mathbb{I}$ is the set $\{J^{\mathrm{bd}}_\lambda : \lambda \ge \aleph_2, \lambda$ is regular$\}$ (where $J^{\mathrm{bd}}_A = \{B \subseteq A : \sup(B) < \sup(A)\}$) then we may omit it. We let $\bar{\lambda}_\eta \overset{\mathrm{def}}{=} \bar{\lambda}(\eta)$, $\underset{\sim}{\bar{\xi}}_\eta \overset{\mathrm{def}}{=} \underset{\sim}{\bar{\xi}}(\eta)$, $\bar{\zeta}_\eta \overset{\mathrm{def}}{=} \bar{\zeta}(\eta)$. If $\underset{\sim}{\mathbf{S}} = \{\aleph_1\}$ we may omit it and omit $\bar{\lambda}$. If $\underset{\sim}{\mathbf{S}} = \mathrm{RUCar}^V$ we may write $*$ instead of $\mathbf{S}$. If $\mathbf{W} = \omega_1$ we may omit it (note: no object can serve as two among $\mathbb{I}, \mathbf{S}$ and $\mathbf{W}$, so no confusion should arise).

It is always understood that the trivial I is in $\mathbb{I}$ (even if we write $\mathbb{I} = \emptyset$), a trivial I is the empty set with domain a singleton.

2) If not said otherwise, we shall ignore the non-$\aleph_2$-complete members of $\mathbb{I}$, i.e. $UP(\mathbb{I}, \mathbf{S}, \mathbf{W})$ means $UP(\mathbb{I}', \mathbf{S}, \mathbf{W})$ where $\mathbb{I}' = \{I \in \mathbb{I} : I$ is $\aleph_2$-complete$\}$.

2.7C Remark. 1)Why do we use $\mathbf{S}^*$ and why can we require $\mathbf{S}^* \subseteq |P|^+$?

(a) $\underset{\sim}{\mathbf{S}}$ is only a name (if $\mathbf{S}$ was a set $\in V$, $\mathbf{S}^* = \mathbf{S}$ is o.k.) and

(b) P-names of an ordinal $< \lambda$, $\lambda = \mathrm{cf}\lambda \ge |P|^+$ have an apriori bound.

2) A reader may use $\mathbf{S} = \{\aleph_1\}$ all the time.

2.7D Claim. 1) In Definition 2.7, if $\underset{\sim}{\mathbf{S}} = \mathrm{RUCar}^V$ we can replace in 2.7(1)(c) "a P-name of an ordinal $< \bar{\lambda}(\eta)$" by "a P-name of a member of V", in 2.7A demand $\underset{\sim}{\xi}(\eta{\restriction}m) = \bar{\zeta}(\eta{\restriction}(\ell))$ and omit $\bar{\lambda}$ and get an equivalent definition (we can also replace $\le |P|$ by $< \mathrm{Min}\{\kappa : P$ satisfies the κ-c.c.$\}$).

2) The forcing notion P satisfies $UP(\mathbb{I}, \mathbf{S}, \mathbf{W})$ iff its completion (to a complete Boolean algebra) satisfies it (assuming $\le_{\mathrm{pr}}=\le$).

3) If Q satisfies $UP(\mathbb{I}, *, \mathbf{W})$ (i.e. as in part (1)) and $\mathbb{I}$ is μ^+-complete (e.g. $\mathbb{I} = \emptyset$) *then* any "new" countable set of ordinals $< \mu$ is included in an "old" countable set of ordinals i.e. one from V.

4) Q satisfies $UP(\emptyset, *)$ iff Q is proper

5) Q satisfies $UP(\emptyset, \{\aleph_1\})$ iff Q is semiproper.

6) If Q satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ and $\mathbb{I} \subseteq \mathbb{I}_1, \underset{\sim}{\mathbf{S}}_1 \subseteq \underset{\sim}{\mathbf{S}}$ and $\mathbf{W}_1 \subseteq \mathbf{W}$ *then* Q satisfies $\mathrm{UP}(\mathbb{I}_1, \underset{\sim}{\mathbf{S}}_1, \mathbf{W}_1)$.

7) In Def. 2.7A, we can replace $\underset{\sim}{\mathbf{S}}$ by any set $\underset{\sim}{\mathbf{S}}'$ of uncountable regular cardinals of V, such that $\Vdash_P$ "$\underset{\sim}{\mathbf{S}} \cap |P|^+ = \underset{\sim}{\mathbf{S}}' \cap |P|^+$".

Proof. (sketch) (1) is easy.

(2) Note that F is defined on sequences of names, and it is well known that P-names can be canonically translated to Q-names, if P is a dense subset of Q.

(3) Use 2.6B(2) repeatedly.

(4), (5): If $\mathbb{I} = \emptyset$, then each branch of an $\mathbb{I}$-tree is itself an $\mathbb{I}$-tree, so a strategy from XII 1.1 (or 1.7(3)) easily yields a function F.

(6) Easy.

(7) By 2.7C(1)(b). $\square_{2.7D}$

2.7E Convention. 1) We write $F_w(\eta, \langle \lambda_\ell, \underset{\sim}{\xi}_\ell, \zeta_\ell : \ \ell \leq lg(\eta)\rangle)$ for

$$F\Big(\eta, w, \Big\langle \langle \lambda_\ell, \underset{\sim}{\xi}_\ell, \zeta_\ell\rangle : \ \ell \leq lg(\eta)\Big\rangle\Big);$$

we omit λ_ℓ when $\underset{\sim}{\mathbf{S}} = \{\aleph_1\}$.

In Definition 2.7, the value F gives to $\mathrm{Suc}_T(\eta)$ is w.l.o.g. $\{\eta\hat{\ }\langle\alpha\rangle : \alpha < \lambda\}$ for some λ, and we do not strictly distinguish between λ and $\mathrm{Suc}_T(\eta)$.

2.8 Definition. 1) For an ideal collection $\mathbb{I}$, a set $\mathbf{S}$ of uncountable regular cardinals, (where $\aleph_1 = \min(\mathbf{S})$, and $\mathbb{I}$ is $\aleph_2$-complete) and χ regular large enough, we say a countable model $N \prec (H(\chi), \in, <^*_\chi)$ is strictly $(\mathbb{I}, \mathbf{S}, \mathbf{W})$-suitable for χ *if*: $N \cap \omega_1 \in \mathbf{W}$ and in the following game the second player has a winning strategy (letting $N_0 = N$).

in the nth move: the first player chooses $I_n \in \mathbb{I} \cap N_n$ and set A_n (not necessarily in N_n), $A_n \subseteq \mathrm{Dom}(I_n), A_n \in I_n$,

then the second player chooses $x_n \in (\mathrm{Dom}(I)) \setminus A_1$ and let $N_{n+1} \supseteq$ Skolem

Hull of $N_n \cup \{x_n\}$ such that for each $\lambda \in \mathbf{S} \cap N_n$:

$$\sup(N_{n+1} \cap \lambda) = \sup(N_n \cap \lambda)$$

2) If $\mathbf{W}$ is omitted we mean $\mathbf{W} = \omega_1$, if $\mathbf{S}$ is omitted we mean $\{\aleph_1\}$, if both are omitted we write strictly $\mathbb{I}$-suitable.

2.9 Claim. A model $N \prec (H(\chi), \in, <^*_\chi)$ is strictly $(\mathbb{I}, \mathbf{S}, \mathbf{W})$-suitable for χ *iff* there is an $\mathbb{I}$-tagged tree $(T, \mathbf{I})$ and $\langle N_\eta : \eta \in T\rangle$ such that:

a) $N = N_{\langle\,\rangle}, \{\mathbb{I}, \mathbf{S}, \mathbf{W}\} \in N$

b) $N_\eta \prec (H(\chi), \in, <^*_\chi)$ is countable

c) $N_{\eta \restriction k} \prec N_\eta$

d) for $\lambda \in \mathbf{S} \cap N_{\eta\restriction k}$, $k < lg(\eta)$ we have: $\sup(N_\eta \cap \lambda) = \sup(N_{\eta\restriction k} \cap \lambda)$

e) for every $\eta \in T$ and $I \in \mathbb{I} \cap N_\eta$

$\{\nu : \eta \trianglelefteq \nu, \nu$ a splitting of $(T, \mathbf{I})$ and $\mathbf{I}_\nu = I\}$ contains a front of $T^{[\eta]}$

f) $\eta \in N_\eta$.

g) $N \cap \omega_1 \in \mathbf{W}$.

Proof Easy: from a winning strategy we can build a tree, and for any such tree $\langle N_\eta : \eta \in T\rangle$ a winning strategy of player II is to choose some $\eta_{n+1} \in T, \eta_n \trianglelefteq \eta_{n+1}$ preserving $\bigcup_{\ell \le n} N_\ell \cup N \subseteq N_n = N_{\eta_n}$. $\square_{2.9}$

2.10 Definition. Fix $\mathbb{I}, \mathbf{S}, \mathbf{W}$.

1) An $\mathbb{I}$-tagged tree of models is an $\mathbb{I}$-tagged tree $(T, \mathbf{I})$ whose nodes η are used to label countable models N_η (we write this as $\bar{N} = \langle N_\eta : \eta \in (T^*, \mathbf{I})\rangle$) satisfying the following:

(a) for $\eta \in T$ we have $N_\eta \prec (H(\chi), \in, <^*_\chi)$ is a countable model.

(b) $N_{\langle\rangle}$ contains all necessary information, in particular $\mathbb{I}, \mathbf{S}, \mathbf{W}$.

(c) $\eta \triangleleft \nu \in T$ implies $N_\eta \prec N_\nu$

(d) for $\eta \in T$ we have $\eta \in N_\eta$ and $\mathbf{I}_\eta \in N_\eta$.

Whenever we have such an $\mathbb{I}$-tagged tree $\bar{N}$ of models, we write $N_\eta = \bigcup_{k<\omega} N_{\eta\restriction k}$ for all $\eta \in \lim(T)$.

2) We call such a tree $\mathbb{I}$-*suitable* if

(e) $\forall \eta \in T \forall I \in \mathbb{I} \cap N_\eta \{\nu \in T^{[\eta]} : \nu \in \mathrm{split}(T), \mathbf{I}_\nu = I$ (or just they are isomorphic)$\}$ contains a front of $T^{[\eta]}$.

3) We call $\bar{N}$ *suitable** if instead of (e) we only have

(e)* $\forall \eta \in T \forall I \in \mathbb{I} \cap N \{\nu \in T^{[\nu]} : \nu \in \mathrm{split}(T), \mathbf{I}_\nu \leq_{\mathrm{RK}} I\}$ (see 2.10A below) contains a front of T.

4) We call $\bar{N}$ $\aleph_1$-strictly $(\mathbb{I}, \mathbf{S}, \mathbf{W})$-suitable if $\bar{N}$ is suitable and in addition

(f) for some $\delta \in \mathbf{W}$, for all $\eta \in T$ we have: $N_\eta \cap \omega_1 = \delta$

5) we call $\bar{N}$ strictly $(\mathbb{I}, \mathbf{S}, \mathbf{W})$-suitable, if in addition to clauses (a) – (e) we have:

(g) for all $\nu \in T$, $\lambda \in \mathbf{S} \cap N_\delta$ there is $\delta_\lambda < \lambda$ such that $\forall \eta \in T$ $[\nu \trianglelefteq \eta \Rightarrow \sup(N_\eta \cap \lambda) = \delta_\lambda]$.

6) We call $\bar{N}$ uniformly suitable or $\aleph_1$-uniformly suitable if (g) or (f) respectively hold only for all $\eta \in \lim(T)$.

Remark. Note: for suitable trees, $\mathbf{S}$ is essentially redundant so we may omit it or allow names. Similarly so for $\mathbf{W}$. In 2.9 and 2.10 we omit $\mathbf{W}$ when it is ω_1, and omit $\mathbf{S}$ when $\mathbf{S} = \{\aleph_1\}$, so $\mathbb{I}$-suitable means $(\mathbb{I}, \{\aleph_1\}, \omega_1)$-suitable. Let $\eta \in (T, \mathbf{I})$ means $\eta \in T$ and we write T when $\mathbf{I}$ is clear.

2.10A Definition. 1) For ideals J_1, J_2 we say

$$J_1 \leq_{RK} J_2$$

if there is a function h witnessing it, i.e. $h : \mathrm{Dom}(J_2) \to \mathrm{Dom}(J_1)$ is such that

$$\text{for every } A \subseteq \mathrm{Dom}(J_2) : A \neq \emptyset \bmod J_2 \Rightarrow h''(A) \neq \emptyset \bmod J_1$$

or equivalently, $J_2 \supseteq \{h^{-1}(A) : A \in J_1\}$.

2) For families $\mathbb{I}_1, \mathbb{I}_2$ of ideals we say $\mathbb{I}_1 \leq_{RK} \mathbb{I}_2$ if there is a function H witnessing it i.e.

(i) H is a function from $\mathbb{I}_1$ into $\mathbb{I}_2$

(ii) for every $J \in \mathbb{I}_1$ we have $J \leq_{RK} H(J)$

3) For families $\mathbb{I}_1, \mathbb{I}_2$ of ideals, $\mathbb{I}_1 \equiv_{RK} \mathbb{I}_2$ *if* $\mathbb{I}_1 \leq_{RK} \mathbb{I}_2 \& \mathbb{I}_2 \leq_{RK} \mathbb{I}_1$.

2.10B Fact. Assume $\mathbb{I} \leq_{RK} \mathbb{I}'$, where $\mathbb{I}, \mathbb{I}'$ are families of ideals.

1) If $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is a $\mathbb{I}'$-suitable* tree and $\mathbb{I} \in N_{\langle\rangle}$, *then* $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is also $\mathbb{I}$-suitable*.
2) If $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is $\mathbb{I}$-suitable*, *then* there is a tree $(T', \mathbf{I}')$ satisfying the following:
   (a) $T' \subseteq T$ (but in general not $T \leq T'$, as the function $\mathbf{I}'$ will be different)
   (b) $\mathrm{split}(T', \mathbf{I}') = T' \cap \mathrm{split}(T, \mathbf{I})$
   (c) $\langle N_\eta : \eta \in (T', \mathbf{I}')\rangle$ is $\mathbb{I}$-suitable.

Proof. (1) Should be clear, as $\leq_{\mathrm{RK}}$ is transitive (as a relation among ideals and also among families of ideals).

(2) For each $\eta \in \mathrm{split}(T, \mathbf{I})$ pick an ideal $\mathbf{I}'_\eta \in \mathbb{I} \cap N_\eta$, $\mathbf{I}'_\eta \leq_{\mathrm{RK}} \mathbf{I}_\eta$ such that: for all $\nu \in T$, for all $I' \in \mathbb{I} \cap N_\nu : \{\eta \in T^{[\nu]} : I' = \mathbf{I}'_\eta\}$ contains a front of $T^{[\nu]}$. This can be done using a bookkeeping argument.

Now define T' as follows: If $\eta \in T' \setminus \mathrm{split}(T, \mathbf{I})$, then $\mathrm{Suc}_{T'}(\eta) = \mathrm{Suc}_T(\eta)$. If $\eta \in T' \cap \mathrm{split}(T, \mathbf{I})$, then $\mathbf{I}'_\eta$ is already defined and it belongs to N_η. Let g_η be a witness for $\mathbf{I}'_\eta \leq_{\mathrm{RK}} \mathbf{I}_\eta$, so g_η introduces an equivalence relation on $\mathrm{Suc}_T(\eta)$. Let A_η be a selector set for this equivalence relation, i.e. $g_\eta \restriction A_\eta$ is 1-1 and has the same range as g_η. Note that we can choose g_η and A_η in N_η. So without loss of generality we may assume that $g_\eta \restriction A_\eta$ is the identity, and let $\mathrm{Suc}_T(\eta) = A_\eta$.

$\square_{2.10B}$

2.11 Claim. Assume $\mathbb{I}$ is a restriction closed family of ideals, $\underset{\sim}{\mathbf{S}}$ a P-name of a set of regular uncountable cardinals, P a forcing notion, $\mathbb{I}$ is $\aleph_2$-complete and $\mathbf{W} \subseteq \omega_1$. Then TFAE:

(A) P satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$.

(B) for large enough regular χ, *if* $\mathbf{S}^* = \{\lambda : \nVdash_P$ "$\lambda \notin \mathbf{S}$" and $\lambda \leq |P|\}$ and $\overline{N} = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is a $(\mathbb{I}, \mathbf{S}^*, \mathbf{W})$-suitable tree of models for χ (see Definition 2.10(2)) and $p \in N_{\langle\,\rangle} \cap P$, *then* there is a $q \in P$, $p \leq_{pr} q$, such that:

$$q \Vdash_P \text{" for some } \eta \in \lim T \text{ (in } (V^P)) \bigcup_{k<\omega} N_{\eta\restriction k} \cap \omega_1 \notin \mathbf{W} \text{ or :}$$

$$\text{for every } k<\omega, \text{ and } \lambda \in \underset{\sim}{\mathbf{S}} \cap N_{\eta\restriction k} \text{ and}$$

$$\underset{\sim}{\alpha} \in N_{\eta\restriction k}, \text{ a } P\text{--name of an ordinal } < \lambda$$

$$\text{we have } \underset{\sim}{\alpha}[\underset{\sim}{G}_P] < \sup[\bigcup_n (N_{\eta\restriction n} \cap \lambda)]\text{"}$$

i.e. $q \Vdash_P$ "for some $\eta \in \lim(T)$: if $\sup\big(\bigcup_{k<\omega} N_{\eta\restriction k} \cap \omega_1\big) \in \mathbf{W}$ then $\sup(\bigcup_k N_{\eta\restriction k}[G_P] \cap \lambda) = \sup(\bigcup_k N_{\eta\restriction k} \cap \lambda)$ for $\lambda \in \underset{\sim}{\mathbf{S}} \cap (\bigcup_k N_{\eta\restriction k})$"

(B)* Like (B) replacing suitable by suitable*.

2.11A Remark. We can use, in (B), "$\lambda \in \underset{\sim}{\mathbf{S}} \cap N_{\eta\restriction k}[\underset{\sim}{G}_P]$" instead of "$\lambda \in \underset{\sim}{\mathbf{S}} \cap N_{\eta\restriction k}$" if in 2.7(1) we change all $\bar{\lambda}(\eta)$ to be P-names. Such a change would not hurt the rest of this chapter.

Proof. $\underline{(A) \Rightarrow (B)}$

So let $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ be $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable tree of models for χ and $p \in N_{\langle\rangle} \cap P$. We should find q as in (B). There are $F \in N_{\langle\,\rangle}$ witnessing $\mathrm{UP}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ for p and $\chi_0 \in N_{\langle\,\rangle}$ (such that $<^*_{\chi_0} \in N_{\langle\rangle}$) where $\{F, P, 2^{|P|}\} \in N_{\langle\,\rangle} \cap H(\chi_0)$.

Now we form an $(\mathbb{I}, P, \underset{\sim}{\mathbf{S}})$-tree $(T^\dagger, \mathbf{I}^\dagger, \bar{\lambda}, \bar{\underset{\sim}{\xi}}, \bar{\zeta})$ which obey F, and a function $h : T^\dagger \to T$ satisfying $[\eta \trianglelefteq \nu \Rightarrow h(\eta) \trianglelefteq h(\nu)]$ and $[\eta \in T^\dagger \Rightarrow \{\eta, \mathbf{I}^\dagger, \bar{\lambda}(\eta), \bar{\underset{\sim}{\xi}}(\eta), \bar{\zeta}(\eta)\} \in N_{h(\eta)}]$, and:

$(*)_1$ for every $\eta \in T^\dagger, \lambda \in N_\eta \cap \mathbf{S}^*, I \in \mathbb{I} \cap N_\eta$ and $\underset{\sim}{\xi} \in N_\eta$ a P-name of an ordinal $< \lambda$, for some front J of $T^{[\eta]}$ consisting of splitting nodes of $(T^\dagger, \mathbf{I})$ above η,

$$[\nu \in J \Rightarrow \langle \bar{\lambda}(\nu), \bar{\underset{\sim}{\xi}}(\nu)\rangle = \langle \lambda, \underset{\sim}{\xi}\rangle]$$

$$[\nu \in J \Rightarrow \mathbf{I}^\dagger_\nu = I]$$

Note that as $F \in N_{\langle\rangle} \prec N_\nu$ necessarily

$$[\nu \in J \;\&\; \rho \in \mathrm{Suc}_T(\nu) \Rightarrow \zeta_\rho \in N_\rho].$$

Now apply Def 2.7A to $(T^\dagger, \mathbf{I}^\dagger, \bar\lambda, \bar\xi, \bar\zeta)$ so in V^P we get $q \in P$ and $\underset{\sim}{\eta} \in \lim(T^\dagger)$ (a P-name) as required there (i.e. forced by q to be so), now there is a P-name $\underset{\sim}{\nu} \in \lim(T)$ such that $\bigwedge_{k<\omega} h(\underset{\sim}{\eta}\restriction k) \lhd \underset{\sim}{\nu}$; so q, $\underset{\sim}{\nu}$ are as required.

$\underline{(B) \Rightarrow (A)}$

Easy. Choose χ large enough, and let us define a function F which will exemplify (A). Let $\langle A_n : n < \omega\rangle$ be pairwise disjoint infinite subsets of ω, with $\mathrm{Min}(A_n) \geq n$ and $\omega = \bigcup_{n<\omega} A_n$.

Now

$$F(\eta, w, \langle\langle\lambda_l, \underset{\sim}{\xi}_l, \zeta_l\rangle : l \leq \ell g(\eta)\rangle) = \Big\langle Y^*, I^*, \langle x(\nu) : \nu \in Y^*\rangle\Big\rangle$$

is defined as follows: let n be the unique $n < \omega$ such that $|w| \in A_n$, so $n \leq \ell g(\eta)$, and let $\nu = \nu_\eta = \eta\restriction n$, we let $N_\eta \overset{\text{def}}{=}$ the Skolem Hull of $\{\mathbf{S}, \mathbb{I}, \eta\} \cup \langle\langle\lambda_l, \underset{\sim}{\xi}_l, \zeta_l\rangle : l \leq n\rangle$ in $(H(\chi), \in, <^*_\chi)$; and let $\langle(I^\nu_m, \lambda^\nu_m, \underset{\sim}{\xi}^\nu_m, \zeta^\nu_m) : m \in A_n\rangle$ be the $<^*_\chi$-first list of this form of all tuples $(I,\ \lambda,\ \underset{\sim}{\xi}, \zeta)$ such that $I \in N_\nu \cap \mathbb{I}$, $\lambda \in \mathrm{RUCar} \cap N_\nu$, $\underset{\sim}{\xi} \in N_\nu$ a P-name of an ordinal $< \lambda$ and $\zeta \in N_\nu$ an ordinal.

Lastly,

$$\begin{aligned}
Y^* &= \{\eta\,\hat{}\,\langle x\rangle : x \in \mathrm{Dom}(I^\nu_{|w|})\} \\
I^* &= \{\{\eta\,\hat{}\,\langle x\rangle : x \in B\} : B \in I^\nu_{|w|}\} \\
\lambda(\eta\,\hat{}\,\langle x\rangle) &= \lambda^\nu_{|w|} \\
\underset{\sim}{\xi}(\eta\,\hat{}\,\langle x\rangle) &= \underset{\sim}{\xi}^x_{|w|} \\
\zeta(\eta\,\hat{}\,\langle x\rangle) &= \sup(\lambda_{|w|} \cap N_\nu)
\end{aligned}$$

So let $\langle T, \mathbf{I}, \bar\lambda, \bar\xi, \bar\zeta\rangle$ be an $(\mathbb{I}, P, \mathbf{S})$-tree obeying F.

Now apply (B) to $\langle N_\nu : \nu \in T\rangle$ and get $q, \underset{\sim}{\eta}$ as required in (A), i.e. they are as required in 2.7(3).

$\underline{(B)^* \Rightarrow (B)}$ Easy as a suitable tree is a suitable* tree.

$\underline{(B) \Rightarrow (B)^*}$ By 2.10B(2). $\square_{2.11}$

2.12 Claim. If $\underset{\sim}{\mathbf{S}}, P, \mathbb{I}, \mathbf{W}, x \in H(\chi)$ and $\mathbf{S}^*$ is as in 2.7A (i.e. $\mathbf{S}^* = \{\theta : \theta = \mathrm{cf}(\theta) \le |P| \text{ and } \nVdash_P \text{“}\theta \notin \underset{\sim}{\mathbf{S}}\text{”}\}$, e.g. $\mathbf{S} = \{\aleph_1\}$), $\mathbb{I}$ is $\aleph_2$-complete *or* for each $I \in \mathbb{I}$, $\kappa \in \mathbf{S}$ we have I is κ-indecomposable, *then* there is a $\aleph_1$-strictly, uniformly $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable tree $\bar{N}$ with $x \in N_{\langle\rangle}$.

Proof. We will construct this tree in three steps: first we find a suitable tree, then we thin it out to be a uniformly suitable tree, then we blow up the models to make it $\aleph_1$-strict. For notational simplicity let $\underset{\sim}{\mathbf{S}} = \{\aleph_1\}$ so $\mathbf{S}^* = \{\aleph_1\}$.
First Step: An easy bookkeeping argument (to ensure 2.10(e)) yields an $(\mathbb{I} \cup \{J^{\mathrm{bd}}_{\omega_1}\})$-suitable tree $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$; so for $\eta \in \lim(T)$ we let $N_\eta = \bigcup\limits_{\ell<\omega} N_{\eta\restriction\ell}$.
Hence we get that for all $\eta \in \lim(T)$, for all $I \in (\mathbb{I} \cap N_\eta) \cup \{J^{\mathrm{bd}}_{\omega_1}\}$, there are infinitely may k such that $\eta\restriction k \in \mathrm{split}(T, \mathbf{I})$ and $\mathrm{Suc}_T(\eta\restriction k) = \{\eta\,\hat{}\,\langle x\rangle : x \in \mathrm{Dom}(I)\}$.
Second Step: Define $H : T \to \omega_1$ by $H(\eta) = \sup(N_\eta \cap \omega_1) < \omega_1$. Apply 2.6 to get a subtree T', and a limit ordinal $\delta \in \mathbf{W} \subseteq \omega_1$ such that clauses (a) – (d) of 2.6 hold. By clause (d) of 2.1, for all $\eta \in T'$, $N_\eta \cap \omega_1 \subseteq \delta$. Let $\delta_0 < \delta_1 < \ldots$, $\bigcup_n \delta_n = \delta$, and let

$$T_2 \stackrel{\mathrm{def}}{=} \{\eta \in T' : \forall k < \ell g(\eta), \text{ if } \mathrm{Suc}_T(\eta\restriction k) = \{\eta\restriction k\,\hat{}\,\langle\alpha\rangle : \alpha < \omega_1\} \\ (\text{so } \mathrm{Suc}_{T'}(\eta\restriction k) = \{\eta\restriction k\,\hat{}\,\langle\alpha\rangle :\ \alpha < \delta\}) \text{ then } \eta(k) = \delta_k\}.$$

Clearly T_2 will be $\aleph_1$-uniformly suitable.
Third Step: For $\eta \in T_2$, let N'_η =the Skolem hull of $N_\eta \cup \delta$. So $N'_\eta \cap \omega_1 \supseteq \delta$. Conversely, let $\nu \in \lim(T_2)$, $\eta \lhd \nu$, then $N_\eta \cup \delta \subseteq N_\nu$, so $N'_\eta \subseteq N_\nu$ hence $N'_\eta \cap \omega \subseteq \delta$. So $N'_\eta \cap \omega_1 = \delta$, i.e. $\langle N'_\eta : \eta \in T_2\rangle$ is an $\aleph_1$-strictly by $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-tree of models (see Definition 2.10(4)).

We claim that this tree is still suitable. Indeed, let $\eta \in T_2$, $\nu \in \lim(T_2)$, $\eta \trianglelefteq \nu$ and $I \in \mathbb{I} \cap N'_\eta$. Then for some $\alpha < \delta$, I is in the Skolem hull of $N_\eta \cup \alpha$. Let $k < \omega$ be such that $\alpha \in N_{\nu\restriction k} \cap \omega_1$, $k \le \ell g(\eta)$. Then since $\langle N_\eta : \eta \in T_2\rangle$ was suitable, there is $\ell \ge k$ such that $\mathbf{I}_{\nu\restriction\ell} = I$. So $\langle N'_\eta : \eta \in T_2\rangle$ is also suitable.
$\square_{2.12}$

2.12A Conclusion. If P satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ and $\underset{\sim}{\mathbf{S}}$ is as in 2.7A (or $\underset{\sim}{\mathbf{S}} = \{\aleph_1\}$) (recall that this notation implies $\mathbb{I}$ is $\aleph_2$-complete, $\aleph_1 \in \underset{\sim}{\mathbf{S}}$, $\mathbf{W} \subseteq \omega_1$ stationary) then $\Vdash_P$ "$\mathbf{W}$ is stationary". Moreover, if $\mathbf{W}' \subseteq \mathbf{W}$ is stationary then also $\Vdash_P$ "$\mathbf{W}$ is a stationary subset of ω_1".

Proof. The "moreover" fact is by 2.7D(6) (i.e. monotonicity in $\mathbf{W}$).

Assume that $p \Vdash$ "$\underset{\sim}{C}$ is a club of ω_1 and $\underset{\sim}{C} \cap \mathbf{W} = \emptyset$". By 2.12 we can find an $\aleph_1$-strictly $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable tree of models $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ with $\underset{\sim}{C}, p \in N_{\langle\rangle}$. Let $\delta = N \cap \omega_1$, so $\delta \in \mathbf{W}$. By $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ we can find a condition q as in 2.11(B) in particular $p \leq_{\rm pr} q$. Clearly $q \Vdash$ "$N_{\langle\rangle}[G] \cap \omega_1 = \delta$" and, trivially $p \Vdash_P$ "$\underset{\sim}{C}$ is unbounded in $N_{\langle\rangle}[G] \cap \omega_1$" hence $p \Vdash$ "$N_{\langle\rangle}[G] \cap \omega_1 \in C$". So $q \Vdash$ "$\delta \in \underset{\sim}{C} \cap \mathbf{W}$". $\square_{2.12A}$

2.12B Remark. From now we shall use 2.11+2.12 freely. Usually we assume $\mathbb{I}, \mathbf{S}$ satisfies 2.6A$(*)$(a)+(b), $\mathbf{S} = \{\aleph_1\}$ is the main case. We could have started with 2.11(B) as a definition of UP but did not as the definition 2.7 was closer to Chapter XI.

2.13 Remark. From the proof of 2.12 we can conclude that in 2.11; in clause (B) we can replace "$(\mathbb{I}, \mathbf{S}, \mathbf{W})$-suitable" by "$\aleph_1$-strictly $(\mathbb{I}, \mathbf{S}, \mathbf{W})$-suitable, $N_\eta \cap \omega_1 = \delta \in \mathbf{W}$", and then the condition q will be $N_{\langle\rangle}$-semi generic.

2.14 Conclusion. 1) If P satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$, $\underset{\sim}{Q}$ a P-name of a purely proper forcing *then* $P * \underset{\sim}{Q}$ satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$.

2) If $\underset{\sim}{\mathbf{S}} = \{\aleph_1\}$, $\underset{\sim}{Q}$ purely semiproper is enough.

3) Generally $\underset{\sim}{Q}$ is purely $(\underset{\sim}{\mathbf{S}}, \mathbf{W})$-semiproper is enough where:

Q is $(\underset{\sim}{\mathbf{S}}, \mathbf{W})$-semiproper *when*: if χ regular large enough, $Q \in N \prec (H(\chi), \in, <^*_\chi)$, $\|N\| = \aleph_0$, $p \in Q \cap N$ and $N \cap \omega_1 \in \mathbf{W}$ *then* there is q, $p \leq_{\rm pr} q \in Q$, such that:

$q \Vdash$ "for every $\lambda \in N \cap \underset{\sim}{\mathbf{S}}$, if $\underset{\sim}{\alpha} \in N$ is a Q-name of an ordinal $< \lambda$ then $\underset{\sim}{\alpha}[\underset{\sim}{G}_Q] <$ sup $(N \cap \lambda)$".

(Note that Q is $(\underset{\sim}{\mathbf{S}}, \mathbf{W})$-semiproper iff Q satisfies the $UP(\emptyset, \underset{\sim}{\mathbf{S}}, \mathbf{W})$).

4) Suppose Q_0 is a proper forcing (in V), $\lambda \geq |Q_0|^{\aleph_0}$, (of course $\lambda = \lambda^{\aleph_0} \geq$ density of Q_0 suffices), $\mathbb{I} \in V$ is a λ^+-complete family of ideals which is λ^+-directed under $\leq_{\mathrm{RK}}$ and $\Vdash_{Q_0}$ "$\underset{\sim}{Q}_1$ is a forcing notion satisfying $UP(\mathbb{I})$".

Then $Q_0 * \underset{\sim}{Q}_1$ satisfies $UP(\mathbb{I})$ (in V).

5) In 4) we can add $\mathbf{W}$.

6) If Q satisfies the λ-c.c., satisfaction of "Q satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$" depend on $\underset{\sim}{\mathbf{S}} \cap \lambda$ only so we shall ignore $\underset{\sim}{\mathbf{S}} \setminus \lambda$. For notational convenience we will demand $\mathrm{URCard} \setminus \lambda \subseteq \underset{\sim}{\mathbf{S}}$.

Proof. 1), 2), 3), 5), 6). Left to the reader.

4) Let χ be regular large enough and let $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ be an $\mathbb{I}$-suitable tree of models for χ, $(p_0, \underset{\sim}{p}_1) \in Q_0 * \underset{\sim}{Q}_1$ and $\{(p_0, \underset{\sim}{p}_1), Q_0 * \underset{\sim}{Q}_1, \mathbb{I}\} \in N_{\langle\rangle}$. For $\eta \in \lim(T)$ we let $N_\eta \overset{\text{def}}{=} \bigcup_{k<\omega} N_{\eta \restriction k}$. As $\lambda \geq |Q_0|^{\aleph_0}$, as $\mathbb{I}$ is λ^+-complete, by 2.6B(1) w.l.o.g. for $\eta \in T$ we have: $N_\eta \cap Q_0$ depends only on $\ell g(\eta)$ and hence $N_\eta \cap Q_0$ is the same for all branches $\eta \in \lim(T)$. Now for each $\eta \in \lim(T)$, in V N_η is a countable elementary submodel of $(H(\chi), \in, <^*_\chi)$ hence there is $q^\circ \in Q_0, p_0 \leq_{\mathrm{pr}} q^\circ$, and q° is (N_η, Q_0)-generic.

Now for each $q, p_0 \leq_{\mathrm{pr}} q \in Q_0$, let

$$B_q = \{\eta \in \lim(T) : q \text{ is } (N_\eta, Q_0)\text{-generic}\}.$$

So $\lim(T) = \cup\{B_q : p_0 \leq_{\mathrm{pr}} q \in Q\}$

Note

(*) for $\eta \in \lim(T)$, $p_0 \leq_{\mathrm{pr}} q \in Q_0$, we have: $\eta \in B_q$ *iff* for any maximal antichain $\mathcal{J} \in N_\eta$ of Q_0, we have: $[r \in \mathcal{J} \setminus N_\eta \Rightarrow r, q$ incompatible$]$.

Hence, B_q is a closed subset of $\lim(T)$, (as if $\eta \in \lim(T) \setminus B_q$ then for some $\mathcal{J} \in MAC(Q_0) \cap N_\eta$ and $r \in \mathcal{J} \setminus N_\eta$ we have r, q are compatible; then for some $m < \omega$, $\mathcal{J} \in N_{\eta \restriction m}$, and $\eta \restriction m \lhd \nu \in \lim(T)$ still implies $r \in \mathcal{J} \setminus N_\nu$ (because $N_\eta \cap Q_o = N_\nu \cap Q_0$) but r, q compatible. So $\lim(\underset{\sim}{T}) \setminus B_q$ contains the neighborhood determined by $\eta \restriction m$).

So by 2.6B(2) if $\lambda \geq |Q_0|^{\aleph_0}$, for some $q \in Q_0$ and T' we have: $p^0 \leq_{\rm pr} q$, $(T, \mathbf{I}) \leq^* (T', \mathbf{I})$ and $\lim(\underset{\sim}{T}') \subseteq B_q$. So $q \Vdash_{Q_0}$ "$N_\eta[\underset{\sim}{G}_{Q_0}] \cap \omega_1 = N_\eta \cap \omega_1$ for every $\eta \in T'$" and clearly $q \Vdash$ "$\langle N_\eta[\underset{\sim}{G}_{Q_0}] : \eta \in T' \rangle$ is an $\mathbb{I}$ -suitable* tree of models for χ". Why the suitable* not suitable? There may be $\eta \in T'$, $I \in \mathbb{I} \cap N_\eta[\underset{\sim}{G}_{Q_0}] \setminus N_\eta$; we get the $\mathbb{I}$-suitable* by 2.14A below.

So we can finish easily. $\square_{2.14}$

2.14A. Assume that $\langle N_\eta : \eta \in (T, \mathbf{I}) \rangle$ is a $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable* tree, Q is a forcing notion satisfying κ-c.c. and $(\mathbb{I}, \leq_{RK})$ is κ-directed. *Then* $\Vdash_Q$ "$\langle N_\eta[G] : \eta \in (T, \mathbf{I}) \rangle$ is $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable*".

Proof. First we claim that for each name $\underset{\sim}{I}$, if $p \Vdash$ "$\underset{\sim}{I} \in \mathbb{I}$" then there is $J \in \mathbb{I}$ such that $p \Vdash$ "$\underset{\sim}{I} \leq_{RK} J$". Indeed, since Q satisfies the κ-c.c. we can find a set $Y \subseteq \mathbb{I}$, $|Y| < \kappa$ such that $p \Vdash$ "$\underset{\sim}{I} \in Y$". Now let J be a $\leq_{RK}$-upper bound for Y. So for all $I' \in Y$ we have $I' \leq_{RK} J$. The function witnessing this relation will also witness it in V^Q, hence $p \Vdash$ "$\underset{\sim}{I} \leq_{RK} J$".

Now work in $V[G]$. Let $I \in N_\eta[G] \cap \mathbb{I}$. Applying the claim we have just proved, in N_η we can find $J \in N_\eta \cap \mathbb{I}$ such that $I \leq_{RK} J$. In $V[G]$ the set $\{x \in T^{[\eta]} : \eta \trianglelefteq \nu, J \leq_{RK} \mathbf{I}_\nu\}$ contains a front F of $T^{[\eta]}$. F is also a front in $V[G]$, so by transitivity of $\leq_{RK}$ we are done. $\square_{2.14A}$

2.15 Theorem. Suppose

a) Q_0 is a forcing notion, satisfying $UP(\mathbb{I}_0, \underset{\sim}{\mathbf{S}}_0, \mathbf{W})$

b) $\Vdash_{Q_0}$ "$\underset{\sim}{Q}_1$ is a forcing notion satisfying $UP(\underset{\sim}{\mathbb{I}}_1, \underset{\sim}{\mathbf{S}}_1, \mathbf{W})$". So: $\underset{\sim}{\mathbf{S}}_0, \underset{\sim}{\mathbb{I}}_1$ are $\underset{\sim}{Q}_0$-names and $\underset{\sim}{\mathbf{S}}_1$ is a $Q_0 * \underset{\sim}{Q}_1$-name.

c) $\lambda = \lambda^{\aleph_0} \geq |MAC(Q_0)|$, and $[I \in \mathbb{I}_0 \Rightarrow \lambda^{|\mathrm{Dom}(I)|} = \lambda]$

d) $\underset{\sim}{\mathbb{I}}_1$ is λ^+-complete. (i.e. $\Vdash_{Q_0}$ "each $I \in \underset{\sim}{\mathbb{I}}_1$ is λ^+-complete ").

e) $\{\aleph_1\} \subseteq \underset{\sim}{\mathbf{S}}_0 \subseteq \{\mu : \aleph_1 \leq \mu = \mathrm{cf}(\mu) \leq \lambda\}$,

f) $\mathbb{I}_0 \subseteq \mathbb{I}$ and $\mathbb{I} \setminus \mathbb{I}_0$ is λ^+-complete and $(\mathbb{I} \setminus \mathbb{I}_0, <_{\rm RK})$ is λ^+-directed (or just κ-directed where Q_0 satisfies the κ-c.c).

g) $\Vdash_{Q_0}$ "for every $I \in \underset{\sim}{\mathbb{I}}_1$ for some $I' \in \mathbb{I}, I \leq_{\rm RK} I'$" ($\leq_{RK}$ - Rudin Keisler order, see 2.10A), moreover $I' \in \mathbb{I} \setminus \mathbb{I}_0$ (hard to fail this addition).

h) $\mathbf{S} = \mathbf{S}_0 \cap \mathbf{S}_1$ i.e. $\mathbf{S}_1 \cap \left(\mathbf{S}_0 \cup (|Q_0|, |Q_0 * \underset{\sim}{Q}_1|]\right)$.

Then $Q_0 * \underset{\sim}{Q}_1$ satisfies $UP(\mathbb{I}, \mathbf{S}, \mathbf{W})$

2.15A Remark. 1) Comparing with Ch XI, 5.1 we lose a little: we demand $\lambda \geq |MAC(Q_0)|$ instead demanding $\lambda \geq |Q_0|$ but this seems marginal, (see (*) of 1.2A(2)).
2) More on $\leq_{RK}$ is this context see §4.

2.15B Example. Let $Q_0 = \mathrm{Nm}$, $\underset{\sim}{Q}_1 = \mathrm{Levy}(\aleph_1, \lambda_1)$ (for some large enough λ_1 (in V^{Q_0})) $Q_2 = \mathrm{Nm}$ (in $V^{Q_0 \times Q_1}$), $Q_3 = \mathrm{Levy}(\aleph_1, \lambda_3)$ for some even larger λ_3, etc., then Q_0, $Q_0 * (\underset{\sim}{Q}_1 * \underset{\sim}{Q}_2)$, $(Q_0 * (\underset{\sim}{Q}_1 * \underset{\sim}{Q}_2)) * (\underset{\sim}{Q}_3 * \underset{\sim}{Q}_4), \dots$ satisfy $UP(\mathbb{I})$ for appropriate $\mathbb{I}$, by 2.15.

Before we prove 2.15 we will remind the reader of a definition and a combinatorial lemma.

2.16 Definition. For a subset A of (an ω-tree) T we define by induction on the length of a sequence η, $\mathrm{res}_T(\eta, A)$ for each $\eta \in T$. Let $\mathrm{res}_T(\langle\,\rangle, A) = \langle\,\rangle$. Assume $\mathrm{res}_T(\eta, A)$ is already defined and we define $\mathrm{res}_T(\eta\char`^\langle\alpha\rangle, A)$ for all members $\eta\char`^\langle\alpha\rangle$ of $\mathrm{Suc}_T(\eta)$. If $\eta \in A$ then $\mathrm{res}_T(\eta\char`^\langle\alpha\rangle, A) = \mathrm{res}_T(\eta, A)\char`^\langle\alpha\rangle$, and if $\eta \notin A$ then $\mathrm{res}_T(\eta\char`^\langle\alpha\rangle, A) = \mathrm{res}_T(\eta, A)\char`^\langle 0\rangle$. If $\eta \in \lim(T)$, we let $\mathrm{res}(\eta, A) = \bigcup_{k\in\omega} \mathrm{res}(\eta{\restriction}k, A)$.

Explanation. Thus $\mathrm{res}(T, A) \stackrel{\mathrm{def}}{=} \{\mathrm{res}_T(\eta, A) : \eta \in T\}$ is a tree obtained by projecting, i.e., gluing together all members of $\mathrm{Suc}_T(\nu)$ whenever $\nu \notin A$.

We state now (see Chapter XI, 5.3):

2.17 Lemma. Let λ, μ be uncountable cardinals satisfying $\lambda^{<\mu} = \lambda$ and let $(T, \mathbf{I})$ be a tagged tree in which for each $\eta \in T$ either $|\mathrm{Suc}_T(\eta)| < \mu$ or $\mathbf{I}(\eta)$ is λ^+-complete. *Then* for every function $H : T \to \lambda$ there exist T' satisfying $(T, \mathbf{I}) \leq^* (T', \mathbf{I})$ such that for $\eta^1, \eta^2 \in T'$ we have: (letting $A = \{\mu \in T : |\mathrm{Suc}_T(\mu)| < \mu\}$):
$\mathrm{res}_T(\eta^1, A) = \mathrm{res}_T(\eta^2, A)$ implies:

$H(\eta^1) = H(\eta^2)$ and $\eta^1 \in A \Leftrightarrow \eta^2 \in A$, and: if $\eta \in T' \cap A$, then $\mathrm{Suc}_T(\eta) = \mathrm{Suc}_{T'}(\eta)$

Proof of Theorem 2.15. Let χ be large enough. Let $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ be an $\aleph_1$-strictly $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable tree of models for χ such that

$$\{Q_0, \underset{\sim}{Q}_1, \underset{\sim}{\mathbf{S}}_0, \underset{\sim}{\mathbf{S}}_1, \mathbb{I}_0, \underset{\sim}{\mathbb{I}}_1, \mathbf{W}\} \in N_{\langle\,\rangle}, \text{ and } (p_0, \underset{\sim}{p}_1) \in (Q_0 * \underset{\sim}{Q}_1) \cap N_{\langle\,\rangle},$$

let $\mu = \mathrm{Min}\{\mu : \lambda^\mu > \lambda\}$, so $\mu > \aleph_0, \mu > |\mathrm{Dom}(I)|$ for $I \in \mathbb{I}_0$, $\mu = \mathrm{cf}(\mu)$, and $\lambda = \lambda^{<\mu}$. Let us define a function H with domain T: $H(\eta)$ is the pair

$$\Big(N_\eta \cap MAC(Q_0), \text{ isomorphism type of } (N_\eta, N_{\eta\restriction 0} \dots, N_{\eta\restriction(lg(\eta)-1)}, \eta, c)_{c \in N_{\langle\rangle}}\Big)$$

so $|\mathrm{Rang}(H)| \leq \lambda$. By the lemma above there is T^1 satisfying $(T, \mathbf{I}) \leq^* (T^1, \mathbf{I})$ such that for $\eta, \nu \in T^1$:

$$\mathrm{res}_T(\eta, A) = \mathrm{res}_T(\nu, A) \Rightarrow H(\eta) = H(\nu) \;\&\; [\eta \in A \Longleftrightarrow \nu \in A]$$
$$\text{where } A = \{\eta \in T : \; |\mathrm{Suc}_T(\eta)| < \mu\}$$

let $T^* = \{\mathrm{res}_T(\eta, A) : \eta \in T^1\}$.

We can find T^2 satisfying $(T^1, \mathbf{I}) \leq (T^2, \mathbf{I})$ such that the mapping $\eta \mapsto \mathrm{res}_T(\eta, A)$ on T^2, is one to one onto T^*. By 2.6A (for $\mathbf{S} = \{\aleph_1\}$) without loss of generality for some $\delta < \omega_1$, $\eta \in \lim(T^1) \Rightarrow \delta = \bigcup_{\ell<\omega} N_{\eta\restriction\ell} \cap \omega_1$, by the proof of 2.12 without loss of generality $\eta \in T^1 \Rightarrow N_\eta \cap \omega_1 = \delta$; and looking at the definition without loss of generality $\delta \in \mathbf{W}$. Let $N'_{\mathrm{res}_T(\eta,A)} = N_\eta$ for $\eta \in T^2$.

By assumption (f) we have $\langle N'_\nu : \nu \in T^*\rangle$ is an $(\mathbb{I}_0, \underset{\sim}{\mathbf{S}}_0, \mathbf{W})$-suitable tree for χ, $p_0 \in Q_0 \cap N'_{\langle\,\rangle}$. So there are $q_0, p_0 \leq_{\mathrm{pr}} q_0 \in Q_0$, and Q_0-name $\underset{\sim}{\nu} \in \lim(T^*)$ such that $q_0 \Vdash_{Q_0}$ "$\bigcup_k N'_{\underset{\sim}{\nu}\restriction k}[\underset{\sim}{G}_{Q_0}] \cap \lambda$ and $\bigcup_k N'_{\underset{\sim}{\nu}\restriction k} \cap \lambda$ has the same supremum for $\lambda \in \underset{\sim}{\mathbf{S}}_0 \cap N_{\underset{\sim}{\nu}\restriction k}$".

Let $\underset{\sim}{T}^+ = \{\eta \in T^1 : \; \mathrm{res}_T(\eta, A) = \underset{\sim}{\nu}\restriction\ell\mathrm{g}(\eta)\}$, this is a Q_0-name.

Let $q_0 \in G \subseteq Q_0, G$ generic over V, and let $\nu = \underset{\sim}{\nu}[G]$.

Now we need:

2.18 Fact. 1) $\langle N_\eta[G] : \eta \in (\underset{\sim}{T}^+[G], \mathbf{I})\rangle$ is an $\underset{\sim}{\mathbb{I}}_1[G]$- suitable* tree for χ.

2) For $\eta \in \lim(\underset{\sim}{T}^+[G])$ and $\kappa \in (\mathbf{S}_1^* \cup \underset{\sim}{\mathbf{S}}_0[G]) \cap \bigcup_{k<\omega} N_{\eta\restriction k}$ we have:

$$\sup(\bigcup_k N_{\eta\restriction k}[G] \cap \kappa) = \sup(\bigcup_k N_{\eta\restriction k} \cap \kappa).$$

3) Moreover if in (2) we choose η in some further generic extension, it still holds.

Proof of 2.18. 1) One point is $N_\eta[G] \cap \omega_1 = N_\eta \cap \omega_1 (\in \omega_1)$ which follows by the choice of q_0, ν as $\aleph_1 \in \mathbf{S}_0$ (in fact $N_\eta[G] \cap \omega_1 = \delta$, for every $\eta \in T$, as $\bar{N}$ was $\aleph_1$-strict). Another point is that for $I \in \underset{\sim}{\mathbb{I}}_1[G] \cap N_\eta[G]$, $\eta \in \underset{\sim}{T}^+[G]$ we have

$$\{\nu : \eta \trianglelefteq \nu \in \underset{\sim}{T}^+[G] \text{ and } I = \mathbf{I}_\eta \text{ or at least } I \leq_{RK} \mathbf{I}_\eta\}$$

is a front of $(\underset{\sim}{T}^+[G])^{[\eta]}$, this follows as : if $I \in N_\eta[G] \cap \underset{\sim}{\mathbb{I}}_1[G]$ then there is $Y \in N_\eta$, $|Y| \leq |Q_0|$ (even $|Y| < \kappa$ if Q_0 satisfies the κ-c.c) such that

$$(\exists I')(I \leq_{RK} I' \in Y \;\&\; I' \in \mathbb{I} \setminus \mathbb{I}_0),$$

note $I \notin \mathbb{I}_0$. Now $Y \cap \mathbb{I} \setminus \mathbb{I}_0$ has a $\leq_{\mathrm{RK}}$-upper bound in $\mathbb{I}$ hence in $\mathbb{I} \cap N_\eta$ by the assumption on $\mathbb{I} \setminus \mathbb{I}_0$ being λ^+-directed, $\lambda \geq |Q_0|$ (or κ-directed, Q_0 satisfying the κ-c.c.).

2) If $\kappa > \lambda$ then $\kappa > |Q_0|$ hence this is immediate; so assume $\kappa \leq \lambda$. Let $\nu = \underset{\sim}{\nu}[G]$ be the branch we obtained by applying $UP(\mathbb{I}, \mathbf{S}, \mathbf{W})$ to Q_0, and let $\eta \in \lim(T^+)$ be any branch. Now there is an isomorphism $g = g_\eta$ from $\bigcup_k N_{\eta\restriction k}$ onto $\bigcup_k N'_{\nu\restriction k}$ such that $g(\eta\restriction l) = \nu\restriction l$ for $\ell < \omega$, $g\restriction N_{\langle\,\rangle}$ = the identity, $g''(N_{\eta\restriction\ell}) = N'_{\nu\restriction\ell}$ for $\ell < \omega$, and necessarily $g\restriction(MAC(Q_0) \cap N_{\eta\restriction k}) =$ the identity (as $N_\eta \cap MAC(Q_0) = N'_\nu \cap MAC(Q_0)$). Now for every $\underset{\sim}{\alpha} \in \bigcup_k N'_{\nu\restriction k}$ a Q_0-name of an ordinal $< \kappa, \underset{\sim}{\alpha}$ is just a maximal antichain of Q_0 with a function from it to ordinals. So $g(\underset{\sim}{\alpha}) = \underset{\sim}{\alpha}$ and of course $g(\kappa) = \kappa$. So as the isomorphism g is onto $\bigcup_k N_{\nu\restriction k}$ we see that $q_0 \Vdash_{Q_0}$ "if $\eta \in \lim(\underset{\sim}{T}^+[\underset{\sim}{G}_{Q_0}])$ then $\underset{\sim}{\alpha} = g_\eta(\underset{\sim}{\alpha}) < \sup(\bigcup_k N_{\eta\restriction k} \cap \kappa)$" as required.

3) Same proof as g can still be defined. $\square_{2.18}$

Continuation of the proof of 2.15: By the fact 2.18 in $V[G]$ there are $\underset{\sim}{\nu}_1$ and q_1 such that $\underset{\sim}{Q}_1[G] \models [p_1 \leq_{\rm pr} q_1;\quad q_1 \Vdash$ "$\underset{\sim}{\nu}_1$ a branch of $\underset{\sim}{T}^+[G]$, and for $\kappa \in \underset{\sim}{\mathbf{S}}_1[G] \cap N_{\nu_1}$ we have $\sup(\bigcup_{\ell<\omega} N_{\nu_1\restriction\ell} \cap \kappa) = \sup \bigcup_{\ell<\omega}(N_{\nu_1\restriction\ell}[G] \cap \kappa)$"$]$.

Now, back in V, there are $\underset{\sim}{\nu}_1, \underset{\sim}{q}_1$ and $q_0' \in G$ such that $q_0' \Vdash_{Q_0}$ "$\underset{\sim}{q}_1, \underset{\sim}{\nu}_1$ are as above" (actually $\underset{\sim}{\nu}_1$ is a Q_0-name of a $\underset{\sim}{Q}_1$-name); w.l.o.g. $q_0 = q_0' (\in Q_0)$ as the existence proof works for any G such that $q_0 \in G$. Note $(q_0', \underset{\sim}{q}_1) \in Q$ and $\underset{\sim}{\nu}_1 \in \lim(\underset{\sim}{T}^+) \subseteq \lim(T)$ are as required (remembering 2.15A(3)).

$\square_{2.15}$

Now clearly

2.19 Claim. 1) If (a forcing notion) P satisfies the $(\mathbb{I}, \mathbf{W})$ - condition (see Ch XI) *then* P satisfies $UP(\mathbb{I}, \mathbf{W})$ [look at Definition 2.7, 2.7A]

2) if $P = \mathrm{Nm}'(D)$ (see chapter X), D an $\aleph_2$ - complete filter, $\mathbb{I} = \{I :$ for some $A \subseteq \mathrm{Dom}(D)$ satisfying $A \neq \emptyset \mod D$ we have $I = \{X \subseteq A : X = \emptyset \mod D\}\}$ *then* P satisfies $UP(\mathbb{I})$

3) Let $(T^*, \mathbf{I}^*)$ be an $\mathbb{I}$-tagged full tree, and

$$P = \{(T, \mathbf{I}) : (T^*, \mathbf{I}^*) \leq (T, \mathbf{I}),\ \text{and for every } \eta \in \lim(T) \text{ we have } (\exists^\infty n)[\eta\restriction n \text{ is a splitting point of } (T, \mathbf{I})\}$$

ordered by inverse inclusion.

$$P' = \{(T, \mathbf{I}) : (T^*, \mathbf{I}^*)^{[\eta]} \leq^* (T, \mathbf{I}) \text{ for some } \eta \in T^*\}$$

ordered by inverse inclusion.

Then

(a) P, P' satisfies $\mathrm{UP}(\mathbb{I})$

(b) if for λ regular

$$\forall \eta \in \lim(T^*)\ \exists^\infty n\ \forall A \in (\mathbf{I}^*_{\eta\restriction n})^+ [\mathbf{I}_\eta \restriction A \text{ is not } \lambda\text{-indecomposable}]$$

then $\Vdash_{P'}$ "$\mathrm{cf}(\lambda) = \aleph_0$".

(c) if $(\forall \eta \in \lim(T^*)) \exists n \bigwedge_{m \geq n} \forall A \in (\mathbf{I}^*_{\eta\restriction n})^+$ $[\mathbf{I}_\eta \restriction A$ is not λ-indecomposable$]$ then $\Vdash_P$ "$\mathrm{cf}(\lambda) = \aleph_0$".

4) Let $\lambda = \mathrm{cf}(\lambda) > \aleph_1$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ stationary, $\mathrm{club}(S) = \{h :$ h an increasing continuous function from some $i+1 < \omega_1$ into $S\}$ ordered by inclusion. *Then* $\mathrm{club}(S)$ satisfies $\mathrm{UP}(\{I\})$ if I is a uniform filter on λ.

Proof. We will only give a sketch of (2), leaving the other claims to the reader. We will use the following fact about $\mathrm{Nm}'(D)$:

(*) If $p \in \mathrm{Nm}'(D)$, $\underset{\sim}{\alpha}$ is a $\mathrm{Nm}'(D)$-name of an ordinal, then there is q, $p \leq^* q$ such that the set $\{\eta \in q :$ for some β we have $q^{[\eta]} \Vdash$ "$\underset{\sim}{\alpha} = \beta$"$\}$ contains a front.

This fact follows easily from 2.6B(2) (let $H : P \to \{0,1\}$ be defined by $H(\eta) = 1$ iff $p^{[\eta]}$ decides $\underset{\sim}{\alpha}$, define $H(\eta) = \lim_{n\in\omega}(H(\eta\restriction n))$ for $\eta \in \lim(p)$, and find q such that H is constant on $\lim(q)$). Let $\mathbb{I}$ be such that the ideal dual to D is in it.

Now let $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ be an $\aleph_1$-strictly $\mathbb{I}$-suitable tree, $\{p, D\} \in N_{\langle\rangle}$ a condition. We can now find a condition q, $p \leq^* q$, an index set $\langle p_\eta : \eta \in p\rangle$ of conditions and a function $f : q \to T$ satisfying the following:

1. If $\eta \lhd \nu$ in q, then $f(\eta) \lhd f(\nu)$
2. For all η in q, $\mathrm{Suc}_T(f(\eta)) \neq 0 \bmod D$ and $\mathbf{I}_\eta$ is the ideal dual to D
3. For all η in q, $\mathrm{Suc}_q(\eta) \subseteq \mathrm{Suc}_T(f(\eta))$
4. For all η in q, $p_\eta \in N_{f(\eta)}, \mathrm{tr}(p_\eta) = \eta, p^{[\eta]} \leq^* p_\eta$.
5. For all η in q, $p_\eta \leq q^{[\eta]}$.
6. For all η in q, all names $\underset{\sim}{\alpha}$ in $N_{f(\eta)}$, the set $\{\nu \in q : p_\nu$ decides $\alpha\}$ contains a front of p_η.

We can do this as follows: by induction on $n < \omega$ we choose $q \cap {}^n(\mathrm{Dom}(D))$ and $\langle(f(\eta), p_\eta, \mathrm{Suc}_q(\eta)) : \eta \in q \cap {}^n(\mathrm{Dom}(D))\rangle$ satisfying the relevant demands. If $n \leq \ell g(\mathrm{tr}(p))$ this is trivial. If we have defined for n, for each $\nu \in q \cap {}^n(\mathrm{Dom}(D))$ and $\eta \in \mathrm{Suc}_q(\nu)$, we do the following. We can find $f(\eta)$ satisfying (1)+(2) because $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is $\mathbb{I}$-suitable. We choose p_η using a bookkeeping argument to take care of (4)+(6), using (*). Then we choose $\mathrm{Suc}_q(\eta)$ such that (3) and (5) are satisfied.

Now let G be $\mathrm{Nm}'(D)$-generic, $q \in G$. Now G defines a generic branch η through q. This induces a branch ν through T by: $\nu = \bigcup_{n\in\omega} f(\eta\restriction n)$. Let $\underset{\sim}{\alpha} \in N_{\nu\restriction k}$, then there is ℓ such that $p_{\eta\restriction\ell} \Vdash$ "$\underset{\sim}{\alpha} = \beta$ and $\beta \in N_{f(\eta\restriction\ell)} \subseteq N_\nu$".

$\square_{2.19}$

2.19A Remark. 1) Note: 2.19(1) tells us that various specific forcing notions satisfy $UP(\mathbb{I}, \mathbf{W})$ via Chapter XI 4.4, 4.4A, 4.5, 4.6.
2) We leave to the reader to compute the natural $\mathbf{S}$'s.

§3. Preservation of the $UP(\mathbb{I}, \mathbf{S}, \mathbf{W})$ by Iteration

3.1 Definition. We say that $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ satisfies $\langle\, \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j}, \underset{\sim}{\mathbf{S}}_{i,j} : \langle i,j\rangle \in W^*\rangle$ for $\mathbf{W}$ provided that the following hold:

(0) $W^* \subseteq \{\langle i,j\rangle :\ i < j \le \alpha, i$ is not strongly inaccessible$\}$, $W^* \supseteq \{\langle i+1, j\rangle : i < j < \alpha\}$ (we can use some variants, but there is no need),

(1) $\bar{Q}$ is a GRCS iteration.

(2) $P_{i,j} = P_j/P_i$ satisfies $UP(\mathbb{I}_{i,j}, \underset{\sim}{\mathbf{S}}_{i,j}, \mathbf{W})$ for $\langle i,j\rangle \in W^*$ (in V^{P_i}).

(3) for every $I \in \mathbb{I}_{i,j}$, the set $\mathrm{Dom}(I)$ is a cardinal, I is $\lambda^+_{i,j}$-complete (in V), $\lambda_{i,j} < |\mathrm{Dom}(I)| < \mu_{i,j}$ and $|MAC(P_i)| \le \lambda_{i,j}$, and $\lambda_{i,j} \ge \aleph_2$ and $(\mathbb{I}_{i,j}, \le_{RK})$ is $\lambda^+_{i,j}$ - directed (note that $\mathbb{I}_{i,j}$ is from V and not V^{P_i}, and $i \le \lambda_{i,j} < \mu_{i,j}$).

(4) if $i(0) < i(1) < i(2) < \alpha, \langle i(0), i(1)\rangle \in W^*, \langle i(1), i(2)\rangle \in W^*$ *then* $(\lambda_{i(1),i(2)})^{<\mu_{i(0),i(1)}} = \lambda_{i(1),i(2)}$. (Hence $\lambda_{i(0),i(1)} < \mu_{i(0),i(1)} \le \lambda_{i(1),i(2)}$.)

(5) for every $I \in \mathbb{I}_{i(2),i(3)}$ and $i(0) < i(1) \le i(2) \le i(3)$ such that $\langle i(0), i(1)\rangle \in W^*$ and $\langle i(2), i(3)\rangle \in W^*$ we have: I is $\lambda^+_{i(0),i(1)}$-complete.

(6) if $i(0) < i(1) < i(2)$,

$$\langle i(0), i(1)\rangle \in W^*,\ \langle i(1), i(2)\rangle \in W^*,\ \langle i(0), i(2)\rangle \in W^*$$

and then $\underset{\sim}{\mathbf{S}}_{i(0),i(2)}$ is $\underset{\sim}{\mathbf{S}}_{i(0),i(1)} \cap \underset{\sim}{\mathbf{S}}_{i(1),i(2)}$ [this holds if (always) $\mathbf{S}_{i,j} = \{\aleph_1\}$, and essentially if (always) $\mathbf{S}_{i,j} = \text{RUCar}$]. (Remember - by 2.13(6) every $\chi \in$ RCar $\setminus |P_{i(1)}|^+ \subseteq$ RCar $\setminus \lambda_{i(1),i(2)}$ is considered to be in $\mathbf{S}_{i(0),i(1)}$.)

Note:

3.1A Remark. If $\mathbb{I}$ is λ^+ - complete, $\kappa \leq \lambda^+$,

$$\bigwedge_{I \in \mathbb{I}} |\text{Dom}(I)| < \mu, \quad [\bigwedge_{i<\alpha<\kappa} \mu_i < \mu \Rightarrow \prod_{i<\alpha} \mu_i < \mu] \quad \text{and}$$

$$\mathbb{I}^{[\kappa]} \stackrel{\text{def}}{=} \mathbb{I} \cup \{\Pi_{i<\alpha} I_i : \quad \alpha < \kappa, \quad \kappa < \mu, I_i \in \mathbb{I}\}$$

(product $\prod_{i<\alpha} I_i$ defined in 4.11(1)) *then* $\mathbb{I}^{[\kappa]}$ is λ^+ - complete, $I \in \mathbb{I}^{[\kappa]} \Rightarrow |\text{Dom}(I)| < \mu$ and $(\mathbb{I}^{[\kappa]}, \leq_{\text{RK}})$ is κ - directed (see more in 4.11).

We shall use 3.1A freely.

3.2 Lemma. If $\bar{Q} = \langle P_n, \underset{\sim}{Q}_n : n < \omega \rangle$ satisfies $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j}, \underset{\sim}{\mathbf{S}}_{i,j} : i < j < \omega \rangle$ for $\mathbf{W}$ and $\mathbb{I} = \bigcup_{n<\omega} \mathbb{I}_{n,n+1}$ *then* $P_\omega = \text{Rlim}\bar{Q}$ satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ where $\underset{\sim}{\mathbf{S}} = \{\lambda : \lambda \text{ is regular} > \aleph_0 \text{ and for every } n, \text{cf}(\lambda) \in \underset{\sim}{\mathbf{S}}_n\}$ (a P_ω-name.)

3.2A Remark. For the case $\leq_{pr} \neq \leq$ use VI 1.10.

Proof. Let $\mathbb{I}_n = \mathbb{I}_{n,n+1}$ and $\lambda_i = \lambda_{i,i+1}$ and $\mu_i = \mu_{i,i+1}$, note that $P_{i,i+1} = \underset{\sim}{Q}_i$ [see 3.1(2)], so $\underset{\sim}{Q}_n$ satisfies the $\mathbb{I}_n$-condition, $|P_i| \leq \lambda_i$, and $\mu_i \leq \lambda_{i+1} = (\lambda_{i+1})^{<\mu_i} < \mu_{i+1}$.

Let $\langle N_\eta : \eta \in (T, \mathbf{I}) \rangle$ be an $\aleph_1$-strict $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable tree of models for $\chi, N_{\langle\rangle} \cap \omega_1 \in \mathbf{W}, p \in N_{\langle\rangle} \cap P_\omega$ and $\{\mathbf{W},\ \bar{Q},\ \langle\langle \mathbb{I}_n, \lambda_n, \mu_n \rangle : n < \omega \rangle\} \in N_{\langle\rangle}$.

The proof will combine the proof of Ch XI 6.2 and the argument in preservation of (semi) properness.

We now define by induction (similarly to Ch XI§1) on $n < \omega$, a tree T_n such that (letting $A_n = \{\eta \in T : |Suc_T(\eta)| < \mu_n\}$):

(i) $T_0 = T$

(ii) $(T_n, \mathbf{I})$ is an $\mathbb{I}$-tree

(iii) $(T_n, \mathbf{I}) \leq^* (T_{n+1}, \mathbf{I})$

(iv) if $\eta, \nu \in T_{n+1}, \mathrm{res}(\eta, A_n) = \mathrm{res}(\nu, A_n)$ *then*

(a) $N_\eta \cap \mathrm{MAC}(P_{n+1}) = N_\nu \cap \mathrm{MAC}(P_{n+1})$

(b) the structures

$$\langle N_\eta, N_{\eta\restriction(lg(\eta)-1)}, \ldots, N_{\langle\rangle}, \eta, c\rangle_{c\in N_{\langle\rangle}} \text{ and}$$

$$\langle N_\nu, N_{\nu\restriction lg(\nu)-1)}, \ldots, N_{\langle\rangle}, \nu, c\rangle_{c\in N_{\langle\rangle}}$$

are isomorphic

(v) if $\eta \in T_{n+1}, |\mathrm{Suc}_{T_n}(\eta)| < \mu_n$ *then* $\mathrm{Suc}_{T_{n+1}}(\eta) = \mathrm{Suc}_{T_n}(\eta)$.

This is done by applying to $(T_n, \mathbf{I})$ Lemma 2.17 (for the function H implicit in (iv), and (λ, μ) there correspond to (λ_{n+1}, μ_n) here).

In the end let $T^* \stackrel{\text{def}}{=} \bigcap_{n<\omega} T_n$; now for every n we have $(T_n, \mathbf{I}) \leq^* (T^*, \mathbf{I})$; why? if $\eta \in T^*$, then for some n $(\forall k \leq \ell g(\eta))[\mathbf{I}_{\eta\restriction k} \in \bigcup_{\ell\leq n} \mathbb{I}_\ell]$, hence for $k > n$, $\mathrm{Suc}_{T_k}(\eta) = \mathrm{Suc}_{T_{k+1}}(\eta)$.

We let $T_n^- \stackrel{\text{def}}{=} \{\mathrm{res}(\nu, A_n) : \nu \in T^*\}$.

We now define by induction on $n < \omega$, p_n, q_n, η_n and P_n-name $\underset{\sim}{\nu}_n$ such that:

(a) $q_n \in P_n$

(b) $q_{n+1}\restriction n = q_n$

(c) $p\restriction n \leq_{pr} q_n$

(d) $q_n \Vdash$ "$\underset{\sim}{\nu}_n \in \lim(T_n^-)$" (lim is taken in V^{P_n})

(e) $q_{n+1} \Vdash$ "$\mathrm{res}(\underset{\sim}{\nu}_{n+1}, A_n) = \underset{\sim}{\nu}_n$" (more exactly $\mathrm{res}(\underset{\sim}{\nu}_{n+1}, \{\mathrm{res}(\rho, A_{n+1}) : \rho \in A_n\}) = \underset{\sim}{\nu}_n)$

(f) $q_{n+1} \Vdash$ "*if* $\rho \in \lim T^*, \mathrm{res}(\rho, A_n) = \underset{\sim}{\nu}_n$ *then*

$$(\bigcup_{n<\omega} N_{\rho\restriction n})[G_{P_n}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$$

moreover for every $\kappa \in \bigcap_{\ell<n}(\underset{\sim}{\mathbf{S}}_{\ell,\ell+1} \cup [|P_\ell|^+, |P_\omega|])$ which belongs to $\bigcup_{n<\omega} N_{\rho\restriction n}$ we have

$$\sup\left[(\bigcup_{n<\omega} N_{\rho\restriction n})[G_{P_n}] \cap \kappa\right] = \sup\left[(\bigcup_{n<\omega} N_{\eta\restriction n} \cap \kappa\right] \text{''}$$

(g) $\eta_n \in T^*, \eta_n \lhd \eta_{n+1}, lg(\eta_n) = n$

(h) $\text{res}(\eta_n, A_n) = \underset{\sim}{\nu}_n\restriction n$

(i) $p_0 = p, p_n \leq_{pr} p_{n+1} \in N_{\eta_{n+1}}$ and $p_n\restriction n = p_{n+1}\restriction n$ (we can get this from chapter VI).

(j) $p_n\restriction n \leq_{pr} q_n$

(k) if $\kappa \in N_{\eta_n}$ a regular cardinal, $\underset{\sim}{\alpha} \in N_{\eta_n}$ is a P_ω-name of an ordinal $< \kappa$ *then* for some m, and $\underset{\sim}{\beta} \in N_{\eta_m}$ we have $\underset{\sim}{\beta}$ is a P_m-name of an ordinal $< \kappa$ and $p_{m+1} \Vdash$ "$\underset{\sim}{\alpha} \leq \underset{\sim}{\beta}$ or $\kappa \notin \underset{\sim}{S}$".

The induction step is done as in the proof of 2.15 (remember that by 3.1(3) $(\mathbb{I}_{i,j}, \leq_{\text{RK}})$ is $\lambda^+_{i,j}$-directed), (plus bookkeeping for (k) if $\leq_{\text{pr}} \neq \leq$ we use VI 1.10). $\square_{3.2}$

3.3 Lemma.

1) If $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \omega_1\rangle$ satisfies $\langle \mathbb{I}_{\alpha,\beta}, \lambda_{\alpha,\beta}, \mu_{\alpha,\beta}, \underset{\sim}{\mathbf{S}}_{\alpha,\beta} : \alpha < \beta < \omega_1$ & α non-limit $\rangle$ for $\mathbf{W}$, and $\mathbb{I} = \bigcup\{\mathbb{I}_{i,j} : i < j < \omega_1, i \text{ non-limit }\}$ *then* $P = P_{\omega_1} = \text{Rlim}\bar{Q}$ satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ where

$$\underset{\sim}{\mathbf{S}} = \bigcap_{\langle\alpha,\beta\rangle} \left(\underset{\sim}{\mathbf{S}}_{\alpha,\beta} \cup [|P_\beta|^+, ||P_{\omega_1}|]\right).$$

2) For $P = R\lim \bar{Q}$ as above, $\bigcup_{\alpha<\omega_1} P_\alpha$ is a dense subset of P, moreover for every $p \in P$ there is q such that $p \leq_{pr} q \in \bigcup_{\alpha<\omega_1} P_\alpha$.

3) We can replace ω_1 by a δ such that $\text{cf}^V\delta < \lambda_{i,j}$ and $\Vdash_{P_i}$ "$\text{cf}\delta \in \underset{\sim}{\mathbf{S}}_{i,j}$" for any $\langle i,j\rangle \in W^*$.

Proof. 1) Let $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ be an $\aleph_1$-strict $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable tree of models for $\chi, N_{\langle\rangle} \cap \omega_1 \in \mathbf{W}$, $\{\mathbf{W}, \bar{Q}, \langle\langle \mathbb{I}_{\alpha,\beta}, \lambda_{\alpha,\beta}, \mu_{\alpha,\beta}, \underset{\sim}{\mathbf{S}}_{\alpha,\beta}\rangle : \alpha < \beta <$

ω_1, α non-limit $\rangle\}$ belongs to $N_{\langle\,\rangle}$ and $p \in P_{\omega_1} \cap N_{\langle\,\rangle}$. So $\mathbb{I} \in N_{\langle\,\rangle}$, hence (as $N_\eta \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$):

(*) for every $\eta \in T, \mathbf{I}_\eta \in \mathbb{I}_{i,j}$ for some $i < j < N_{\langle\,\rangle} \cap \omega_1$.

Let $0 = \gamma_0 < \gamma_1 < \gamma_2 < \dots$, $\bigcup_{n<\omega} \gamma_n = \delta = N_{\langle\,\rangle} \cap \omega_1$ and each γ_{1+n} is a successor ordinal. We repeat the proof of 3.2 (again remembering 3.1(3)) using P_{γ_n} instead P_n, and get $q = q_\omega$, $\underset{\sim}{\nu}_\omega$ as there.

The new point is why $p \leq_{pr} q$, and not only $p{\restriction}\delta = p{\restriction}(\bigcup_{n<\omega} \gamma_n) \leq_{pr} q$. The answer (as in Chapter XI, proof of 2.6) is: Let $\underset{\sim}{\zeta}^n(p_i)$ be prompt names as in XI 1.9. By (k) above q forces $\underset{\sim}{\zeta}^n(p_i)$ to be bounded by $\delta = N_{\langle\,\rangle} \cap \omega_1$, so we can finish.

2) We have proved this: for every $p \in P_{\omega_1}$ by 2.12 we can find $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ and $q \in P_{\omega_1}, p \leq_{pr} q$, as above; q is as required.

3) Almost the same proof. $\square_{3.3}$

3.4 Conclusion. 1) For $\bar{Q}$ an iteration as in 3.1, and limit $\delta \leq \ell g(\bar{Q})$ such that $\mathrm{cf}(\delta) = \omega_1$ and $[i < j < \delta,\ i$ non-limit $\Rightarrow \langle i, j\rangle \in W^*]$ *then* $\bigcup_{i<\delta} P_i$ is a dense subset of P_δ.

2) Instead $\mathrm{cf}(\delta) = \omega_1$, it is enough that for some $i < \delta, \Vdash_{P_i}$ "$\mathrm{cf}(\delta) = \omega_1$",

3) Also if δ is strongly inaccessible, $|P_i| < \delta$ for $i < \delta$ *then*

(a) conclusion of (1) holds.

(b) P_κ satisfies the κ-c.c. (in a strong sense: Δ-system lemma!)

4) In (1) we can weaken the demand on W^*, it is enough: for some unbounded $A \subseteq \delta$ we have $[i < j \;\&\; i \in A \;\&\; j \in A \Rightarrow \langle i, j\rangle \in W^*]$.

5) Moreover in (4) we can replace A by a set of strictly increasing sequences of ordinals $< \delta$, such that $[\eta \in t \Rightarrow \eta(0) = 0], [m < k < \ell g \eta \;\&\; \eta \in t \Rightarrow \langle \eta(n), \eta(k)\rangle \in W^*]$ and $[\eta \in t \;\&\; \alpha < \delta \Rightarrow \bigvee_{\beta \in (\alpha,\delta)} \eta\hat{\ }\langle\beta\rangle \in t]$. Of course this is because we can use a sequence from t as $\langle \gamma_0, \gamma_1, \dots\rangle$ in 3.3. Similar claims holds for 3.5, 3.6.

Proof. Easy. $\square_{3.4}$

3.5 Lemma. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$ satisfies $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j}, \underset{\sim}{\mathbf{S}}_{i,j} :\ i < j < \kappa,\ i$ non-limit $\rangle$ for $\mathbf{W}, \kappa$ is strongly inaccessible $|P_i| + \lambda_{i,j} + \mu_{i,j} + |\mathrm{Dom}(I)| < \kappa$

for every $\langle i, j\rangle \in W^*, I \in \mathbb{I}_{i,j}$ and $\mathbb{I} = \bigcup_{i,j} \mathbb{I}_{i,j}$. *Then* $P_\kappa = \lim(\bar{Q})$ satisfies the condition $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ where $\underset{\sim}{\mathbf{S}} = \{\lambda : \text{for every } (i,j) \in W^*, \text{ in } V^{P_i}, \operatorname{cf}(\lambda) \in \underset{\sim}{\mathbf{S}}_i$ (or $\lambda = \operatorname{cf}\lambda > |P_j|)\}$.

Proof. Let $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ be an $\aleph_1$-strict $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable tree of models for χ. Choose for each $\eta \in T$ a strictly increasing sequence $\langle \gamma^n_\eta : n < \omega\rangle$ of non-limit ordinals from $N_\eta \cap \kappa$ such that $0 = \gamma^0_\eta$, $\sup(N_\eta \cap \kappa) = \bigcup_{n<\omega} \gamma^n_\eta$, and $\gamma^n_{\eta\restriction k} < \gamma^n_\eta$ for $k < \ell g(\eta)$. Let $A_{\eta,n} = \{\rho \in T : |\mathrm{Suc}_T(\eta)| < \mu_{\gamma^n_\eta, \gamma^{n+1}_\eta}\}$.

We define by induction on n, T_n such that

(i) $T_0 = T$

(ii) $(T_n, \mathbf{I})$ is an $\mathbb{I}$-tree

(iii) $(T_n, \mathbf{I}) \le^* (T_{n+1}, \mathbf{I})$

(iv) if $\eta \in T_n$, and $\ell g(\eta) \le n$ then $\eta \in T_{n+1}$

(v) if $\eta \in T_n$, $\ell g(\eta) = n$, $\eta \trianglelefteq \nu_1 \in T_{n+1}, \eta \trianglelefteq \nu_2 \in T_{n+1}$ and $\operatorname{res}(\nu_1, A_{\eta,n}) = \operatorname{res}(\nu_2, A_{\eta,n})$ then

(a) $N_{\nu_1} \cap MAC(P_{\gamma^n_\eta}) = N_{\nu_2} \cap MAC(P_{\gamma^n_\eta})$.

(b) the structures

$$\langle N_{\nu_1}, N_{\nu_1\restriction(\ell g(\nu_1)-1)}, \ldots, N_{\langle\,\rangle}, \nu_1, c\rangle_{c \in N_\eta} \quad \text{and}$$

$$\langle N_{\nu_2}, N_{\nu_2\restriction(\ell g(\nu_2)-1)}, \ldots, N_\eta \ldots, N_{\langle\,\rangle}, \nu_2, c\rangle_{c \in N_\eta}$$

are isomorphic.

There is no problem in this: Let $T^* = \bigcap_{n<\omega} T_n$, easily $(T_n, \mathbf{I}) \le^* (T^*, \mathbf{I})$ for every n (by (iii) and (iv)). The rest is like the proof of 3.3. The only difference is that instead of actual ordinals γ_n we will have prompt names: $\underset{\sim}{\gamma}_n = \gamma^n_{\underset{\sim}{\eta}_n}$. We can use XI 1.10 to get the conditions $p_n \in P_{\underset{\sim}{\gamma}_n}$. Also remember that every $\bar{Q}$-named ordinal $\underset{\sim}{\zeta}(< \kappa)$ is bounded below κ (as for $\delta < \kappa$ of cofinality $\aleph_1, \bigcup_{\alpha<\delta} P_\alpha$ is a dense subset of P_δ). $\square_{3.5}$

3.6 Theorem. Suppose

(a) κ is strongly inaccessible,

(b) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i < \kappa \rangle$ satisfies $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j}, \underset{\sim}{\mathbf{S}}_{i,j} : i < j < \kappa, i$ non-limit$\rangle$ for $\chi, \mathbf{W}$, $|P_i| < \kappa$ for $i < \kappa$, $P_\kappa = \bigcup_{i<\kappa} P_i$.

(c) $\Vdash_{P_\kappa}$ "$\underset{\sim}{Q}_\kappa$ satisfies $UP(\mathbb{I}_\kappa, \underset{\sim}{\mathbf{S}}_\kappa, \mathbf{W})$, $\mathbb{I}_\kappa$ is κ-complete" and $(\mathbb{I}_\kappa, \leq_{RK})$ is $(< \kappa)$ - directed.

(d) $\mathbb{I} = \mathbb{I}_\kappa \cup \bigcup \{\mathbb{I}_{i,j} : \langle i, j \rangle \in W^*\}$.

(e) $\underset{\sim}{\mathbf{S}} = \bigcap_{\langle i,j \rangle \in W^*} \underset{\sim}{\mathbf{S}}_{i,j} \cap \underset{\sim}{\mathbf{S}}_\kappa$.

Then $P_\kappa * \underset{\sim}{Q}_\kappa$ satisfies $UP(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$.

Remark. This generalizes Gitik, Shelah [GiSh:191] which improves the relevant theorem in XI §6.

Proof. Let $\langle N_\eta : \eta \in (T, \mathbf{I}) \rangle$ be an $\aleph_1$-strict $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable tree of models for $\chi, N_{\langle\,\rangle} \cap \omega_1 \in \mathbf{W}, (p_a, p_b) \in (P_\kappa * \underset{\sim}{Q}_\kappa) \cap N_{\langle\rangle}$. You may assume, for simplicity that $\underset{\sim}{\mathbf{S}}_{i,j} = \{\aleph_1\}, \underset{\sim}{\mathbf{S}}_\kappa = \{\aleph_1\} = \underset{\sim}{\mathbf{S}}$. Let T^* be as in the proof of the previous theorem. Let $G^* \subseteq \text{Levy}(\aleph_0, 2^\chi)$ be generic over V. Let $\kappa = \bigcup_{n<\omega} \alpha_n$, each α_{n+1} a successor, $\alpha_0 = 0$, $\alpha_n < \alpha_{n+1}$, $\bigcup_{n<\omega} \alpha_n = \kappa$ (in $V[G^*]$!).

We choose by induction on $n, \beta_n, G_n \in V[G^*], T_n, \nu_n$ such that

(a) $G_n \subseteq P_{\beta_n}$, G_n generic over V, $G_{n+1} \cap P_{\beta_n} = G_n$, $\alpha_n \leq \beta_n < \beta_{n+1} < \kappa$,

(b) $T^0 = T^*$, $T^{n+1} \subseteq T^n$,

(c) $\langle N_\eta[G_n] : \eta \in (T^n, \mathbf{I}) \rangle$ is an $\mathbb{I}_n$-suitable* tree of models for χ, $\mathbb{I}_n = \bigcup \{\mathbb{I}_{i,j} : \langle i, j \rangle \in W^*, i \geq \beta_n\} \cup \mathbb{I}_\kappa$,

(d) $N_\eta[G_n] \cap \omega_1 = N_{\langle\,\rangle} \cap \omega_1$ for all $\eta \in T_n$

(e) T^n has a unique member of length k_n, ν_n,

(f) if H is a function (from V), $\text{Dom}(H) = T$, $H(\eta) \in \mathbf{I}_\eta$, for $n < \omega$, J_n is a front of T and $[\eta \in J_{n+1} \Rightarrow \bigvee_{\ell<\ell g \eta} \eta \restriction \ell \in J_n]$ and $\eta \in J_n \Rightarrow \mathbf{I}_\eta \in \mathbb{I}_\kappa$ *then* for infinitely many n, there exists m_n, $k_n \geq n$ such that $\nu_n \restriction k_n \in J_{m_n}$ and $\nu_{n+1}(k_n) \in H(\nu_n \restriction k_n)$.

Now $G = \bigcup\limits_{n<\omega} G_n$ is a generic subset of P_κ over V (as P_κ satisfies the κ-c.c., every maximal antichain in P_κ is contained in some P_{α_n}, hence meets some G_n).

We define $T^a = \{\eta \in T : N_\eta[G] \cap \omega_1 = N_{\langle\,\rangle} \cap \omega_1\}$.

We define a depth function on T^a:

$Dp(\eta) \geq 0$ iff $\eta \in T^a$.

$Dp(\eta) \geq \alpha (> 0)$ iff for every $\beta < \alpha$

for every $I \in \mathbb{I}_\kappa \cap N_\eta[G]$, there is ν, $I \leq_{\mathrm{RK}} I_\nu$, $\eta \triangleleft \nu \in T^a$ such that $\{i : \nu\char`^\langle i \rangle \in T^a, Dp(\nu\char`^\langle i \rangle) \geq \beta\} \neq \emptyset \bmod I_\nu$.

Easily $Dp \in V[G]$ and its definition is absolute. $[\eta \triangleleft \nu \in T^a \Rightarrow Dp(\nu) \leq Dp(\eta))]$ and $Dp(\langle\ \rangle) = \infty$ as $\bigcup_{n<\omega} \nu_n$ witnesses (in $V[G^*]$).

So in $V[G]$, $T^b \stackrel{\mathrm{def}}{=} \{\nu \in T^a : \ Dp(\nu) = \infty\}$ is the desired tree (i.e. we can continue as in 2.15 with $P_\kappa, \underset{\sim}{Q}_\kappa$ here corresponding to Q_0, $\underset{\sim}{Q}_1$ there). $\square_{3.6}$

3.7 Lemma. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ satisfies $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j}, \underset{\sim}{\mathbf{S}}_{i,j} : i < j \leq \alpha, i \text{ non-limit}\rangle$ for $\mathbf{W}$; $i(*) < \alpha$ is a non-limit, $G_{i(*)} \subseteq P_{i(*)}$ generic over V, and $\langle i_\zeta : \zeta \leq \beta \rangle$ is an increasing continuous sequence of ordinals in $V[G_{i(*)}]$, $i_0 = i(*)$, $i_\beta = \alpha$, each $\alpha_{\zeta+1}$ a successor ordinal.

In $V[G_{i(*)}]$, we define $P'_\zeta = P_{i_\zeta}/G_{i(*)}$, $\underset{\sim}{Q}'_\zeta = \underset{\sim}{Q}_{\alpha_\zeta}/G_{i(*)}$, $\bar{Q}' = \langle P'_\zeta, \underset{\sim}{Q}'_\xi : \zeta \leq \beta, \xi < \beta \rangle$, *then,* in $V[G_{i(*)}]$, $\bar{Q}'$ satisfies $\langle \mathbb{I}_{\alpha_\zeta, i_\xi}, \lambda_{i_\zeta, i_\xi}, \mu_{i_\zeta, i_\xi}, \mathbf{S}_{i_\zeta, i_\xi} : \zeta < \zeta \leq \beta, \zeta$ a non-limit$\rangle$ for $\mathbf{W}$.

Proof. Straightforward. $\square_{3.7}$

3.8 Conclusion. For every function F, stationary $\mathbf{W} \subseteq \omega_1$ and ordinal α^* there are $\alpha \leq \alpha^*$ and a GRCS iteration $\bar{Q}$ of length α satisfying $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j}, \underset{\sim}{\mathbf{S}}_{i,j}; i < j \leq \alpha,\ i$ non-limit $\rangle$ for $\mathbf{W}$ with $(\underset{\sim}{Q}_i, \mathbb{I}_{i,i+1}, \underset{\sim}{\mathbf{S}}_{i,i+1}) = F(\bar{Q} \restriction i)$ and $\alpha = \alpha^*$ *or* $\alpha < \alpha^*$, and $F(\bar{Q})$ does not satisfy $(*)$ below or there is β, $\beta + \omega \leq \alpha$ and $\bigwedge_n \nVdash_{P_{\beta+n}}$ "$|\operatorname{MAC}(P_\beta)| = \aleph_1$"

$(*) F(\bar{Q})$ has the form $(\underset{\sim}{Q}, \mathbb{I}, \underset{\sim}{\mathbf{S}})$, $\underset{\sim}{Q}$ a P_α-name of a forcing notion satisfying UP$(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$, and (a) or (b) below holds:

(a) $\Vdash_{P_\alpha}$ "$\alpha \neq \aleph_2$" (i.e. $\Vdash_\alpha$ "$|\alpha| < \aleph_2$") and for some λ the family $\mathbb{I}$ is λ^+-complete, where $\lambda = \lambda^{|\mathrm{Dom}(I)|}$ whenever $I \in \mathbb{I}_{i,j}, i < j \leq \alpha$, and $|MAC(P_\alpha)| \leq \lambda$ and $(\mathbb{I}, \leq_{RK})$-is λ^+-directed.

(b) $\Vdash_{P_\alpha}$ "$\alpha = \aleph_2$" (i.e. α is strongly inaccessible, $|P_i| < \alpha$ for $i < \alpha$), $\mathbb{I}$ is α-complete and $I \in \mathbb{I}_{i,j} \& i < j < \alpha \Rightarrow |\mathrm{Dom}(I)| < \alpha$ and $(\mathbb{I}, \leq_{RK})$ is α-directed.

Proof. Straightforward. $\square_{3.8}$

§4. Families of Ideals and Families of Partial Orders

4.1 Definition. 1) We call an ideal J fine if $\{x\} \in J$ for every $x \in \mathrm{Dom}(J)$.
2) We call the ideal with domain $\{0\}$, which is $\{\emptyset\}$, the trivial ideal.

4.2 Claim. 1) If an ideal J is not fine then $J \leq_{RK}$ "the trivial ideal". (See 2.10A for the definition of $\leq_{RK}$).
2) In 2.10B we can weaken the hypothesis to $\mathbb{I}_1 \leq_{RK} \mathbb{I}'_2$ where $\mathbb{I}'_2 \stackrel{\text{def}}{=} \mathbb{I}_2 \cup$ {the trivial ideal}. The same holds in similar situations.
3) $\leq_{RK}$ is a partial quasiorder (among ideals and also among families of ideals).

Proof. Easy. $\square_{4.2}$

4.3 Definition.

1) For an (upward) directed partial or just quasi order $L = (B, \leq)$ we define an ideal id_L:

$$\mathrm{id}_L = \{A \subseteq B : \text{ for some } y \in L \quad \text{we have} \quad A \subseteq \{x \in B : \neg y \leq x\}\}\,.$$

(Equivalently the dual filter fil_L is generated by the "cones" $L_y \stackrel{\text{def}}{=} \{x \in L : y \leq x\}$.) We call such an ideal a partial order ideal or a quasi order ideal. We let $\mathrm{Dom}(L) = \mathrm{Dom}(\mathrm{id}_L)(= B)$, but we may use L instead of $\mathrm{Dom}(L)$ (like $\forall x \in L$) abusing notation as usual.

2) For a partial order L let $\mathrm{dens}(L) = \mathrm{Min}\{|\Theta| : \Theta \subseteq \mathrm{Dom}(L)$ is dense i.e. $(\forall a \in \mathrm{Dom}(L))(\exists b \in \Theta)[a \leq b])\}$ (this applies also to ideals considered as the quasi order $(I, \subseteq)$).

3) For a family $\mathcal{L}$ of directed quasi orders let $\mathrm{id}_{\mathcal{L}} = \{\mathrm{id}_L : L \in \mathcal{L}\}$.

4.4 Fact.

1) id_L is λ-complete *iff* L is λ-directed.

2) $\mathrm{dens}(L) = \mathrm{dens}(\mathrm{id}_{(L,<)}, \subseteq)$

3) If $h : L_1 \to L_2$ preserves order (i.e. $\forall x, y \in L$, $(x \leq y \Rightarrow h(x) \leq h(y))$) and has cofinal range (i.e. $\forall x \in L_2 \exists y \in L_1 (x \leq h(y))$) *then* $\mathrm{id}_{L_2} \leq_{RK} \mathrm{id}_{L_1}$.

4) $h : L_1 \to L_2$ exemplifies $\mathrm{id}_{L_2} \leq_{RK} \mathrm{id}_{L_1}$ *iff* for every $x_2 \in L_2$ there is $x_1 \in L_1$ such that: $y \in L_1$ & $x_1 \leq_{L_1} y \Rightarrow x_2 \leq_{L_2} h(y)$ (i.e. for $y \in L_1 : \neg x_2 \leq_{L_2} h(y) \Rightarrow \neg x_1 \leq_{L_1} y$ but h is not necessarily order preserving).

5) the ideal $\mathrm{id}_{(L,<)}$ is fine iff $(L, <)$ has no maximal element.

Proof. Straight. E.g.

4) Note: h exemplifies $\mathrm{id}_{L_2} \leq \mathrm{id}_{L_1}$ *iff*

$$(\forall A \subseteq L_1)(A \neq \emptyset \mod \mathrm{id}_{L_1} \to (\forall x_2 \in L_2)[h''(A) \cap \{y \in L_2 : x_2 \leq_{L_2} y\} \neq \emptyset])$$

iff

$$(\forall x_2 \in L_2)(\forall A \subseteq L_1)[A \neq \emptyset \mod \mathrm{id}_{L_1} \to h''(A) \cap \{y \in L_2 : x_2 \leq_{L_2} y\} \neq \emptyset]$$

iff

$$(\forall x_2 \in L_2)[\{y \in L_1 : \neg x_2 \leq_{L_2} h(y)\} = \emptyset \mod \mathrm{id}_{L_1}]$$

iff

$$(\forall x_2 \in L_2)(\exists x_1 \in L_1)(\forall y \in L_1)(\neg x_2 \leq_{L_2} h(y) \to \neg x_1 \leq_{L_1} y)$$

iff

$$(\forall x_2 \in L_2)(\exists x_1 \in L_1)(\forall y \in L_1)(x_1 \leq_{L_1} y \to x_2 \leq_{L_2} h(y))$$

$\square_{4.4}$

4.5 Fact. 1) For every ideal J (such that $(\mathrm{Dom}(J)) \notin J$), let $J_1 = \mathrm{id}_{(J,\subseteq)}$, *then*

(i) J_1 is a partial order ideal

(ii) $|\mathrm{Dom}(J_1)| = |J| \leq 2^{|\mathrm{Dom}(J)|}$

(iii) $J \leq_{RK} J_1$

(iv) if J is λ-complete *then* $(J, \subseteq)$ is λ-directed hence J_1 is λ-complete

(v) $\mathrm{dens}(J, \subseteq) = \mathrm{dens}(J_1, \subseteq)$

2) For every dense $\Theta \subseteq J$ we can use $\mathrm{id}_{(\Theta,\subseteq)}$ and get the same conclusions.

3) For every ideal J there is a directed order L such that:

$$J \leq_{RK} \mathrm{id}_L, \quad \mathrm{dens}(J) = \mathrm{dens}(L) \text{ and:}$$

$$\text{for every } \lambda \text{ if } J \text{ is } \lambda\text{-complete then so is } \mathrm{id}_L.$$

Proof. Least trivial is (1)(iii), let $h : J \to \mathrm{Dom}(J)$ be such that $h(A) \in (\mathrm{Dom}(J)) \setminus A$ (exists as $(\mathrm{Dom}(J)) \notin J$). Let $J_1 = \mathrm{id}_{(J,\subseteq)}$.

If $X \subseteq \mathrm{Dom}(J_1) = J, X \notin J_1$ and $A \stackrel{\mathrm{def}}{=} h''(X)$ belongs to J, then $\{B \in J : \neg A \subseteq B\} \in \mathrm{id}_{(J,\subseteq)} = J_1$ (by the definition of $\mathrm{id}_{(J,\subseteq)}$) hence (as $X \notin J_1$) for some $B \in X, A \subseteq B$, so $h(B) \in h''(X) = A$ contradicting the choice of $h(B)$ (as $A \subseteq B$). $\square_{4.5}$

4.5A Remark. So we can replace the ideals by partial orders without changing much the relevant invariants such as completeness or density.

4.6 Conclusion. For any family of ideals $\mathbb{I}$ there is a family of $\mathcal{L}$ of directed partial order such that

(i) $\mathbb{I} \leq_{RK} \{\mathrm{id}_{(L,<)} : (L,<) \in \mathcal{L}\}$

(ii) $|\mathcal{L}| \leq |\mathbb{I}|$

(iii) $\sup\{|L| : (L,<) \in \mathcal{L}\} \leq \sup\{|J| : J \in \mathbb{I}\}) \quad \leq (\sup\{2^{|\mathrm{Dom}(J)|} : J \in \mathbb{I}\})$

(iv) $\sup\{\mathrm{dens}(L,<) : (L,<) \in \mathcal{L}\} = \sup\{\mathrm{dens}(J,\subseteq) : J \in \mathbb{I}\}$

(v) if $\mathbb{I}$ is λ-complete then every $(L,<) \in \mathcal{L}$ is λ-directed.

Proof. Easy. $\square_{4.6}$

4.7 Definition. For a forcing notion Q, satisfying the κ-c.c., a Q-name $\underset{\sim}{L}$ of a directed partial (or just quasi) order with (for notational simplicity) $\mathrm{Dom}(\underset{\sim}{L}) \in V$; let $L^* = ap_\kappa(\underset{\sim}{L})$ be the following partial order

$$\mathrm{Dom}(L^*) = \{a : \ a \subseteq \mathrm{Dom}(\underset{\sim}{L}) \text{ and } |a| < \kappa\}$$

$$a \leq^* b \text{ iff } \Vdash_Q \text{"}(\forall y \in a)(\exists x \in b)[\underset{\sim}{L} \models y < x]\text{"}$$

(this is a quasi order only, e.g. maybe $a \leq^* b \leq^* a$ but $a \neq b$).

4.8 Claim. For a forcing notion Q satisfying the κ-c.c. and a Q-name $\underset{\sim}{L}$ of a λ-directed partial order (with $\mathrm{Dom}(\underset{\sim}{L}) \in V$ for simplicity) such that $\lambda \geq \kappa$ we have:

(i) $ap_\kappa(\underset{\sim}{L})$ is λ-directed partial order (in V and also in V^Q).

(ii) $|ap_\kappa(\underset{\sim}{L})| \leq |\mathrm{Dom}(\underset{\sim}{L})|^{<\kappa}$

(iii) $\Vdash_Q$ "$\mathrm{id}_{\underset{\sim}{L}[G]} \leq_{RK} \mathrm{id}_{ap_\kappa(\underset{\sim}{L})}$"

Proof. We leave (i), (ii) to the reader. We check (iii). Let $G \subseteq Q$ be generic over V, and in $V[G]$ we define a function h from $ap_\kappa(\underset{\sim}{L})$ to $\mathrm{Dom}(\underset{\sim}{L}[G])$:

$h(a)$ will be an element of $\mathrm{Dom}(\underset{\sim}{L}[G])$ such that

$$(\forall x \in a)\underset{\sim}{L}[G] \models \text{"}x < h(a)\text{"}.$$

We can now easily verify the condition in 4.4(4). $\square_{4.8}$

4.9 Conclusion. 1) Suppose Q is a forcing notion satisfying the κ-c.c., $\underset{\sim}{\mathbb{I}}_1$ a Q-name of a family of λ-complete filters and $\lambda \geq \kappa$. *Then* there is, (in V), a family $\mathbb{I}_2$ of λ-complete filters such that:

(i) $\Vdash_Q$ "$\underset{\sim}{\mathbb{I}}_1 \leq_{RK} \mathbb{I}_2$"

(ii) $|\mathbb{I}_2| = |\mathbb{I}_1|$

(iii) $\sup\{|\mathrm{Dom}(J)| : J \in \mathbb{I}_2\} = \sup\{(2^\mu)^{<\kappa}$: some $q \in Q$ forces that some $J \in \underset{\sim}{\mathbb{I}}_1$ has domain of power $\mu\}$.

2) If $\mathbb{I}_1$ has the form $\{\mathrm{id}_{(L,<)} : (L,<) \in \mathcal{L}\}$ then in (iii) we can have

(iii)$'$ $\sup\{|\mathrm{Dom}(J)| : J \in \mathbb{I}_2\} = \sup\{\mu^{<\kappa}$: some $q \in Q$ force some $(L,<) \in \mathcal{L}$ has power $\mu\}$.

Proof. Easy. $\square_{4.9}$

4.9A Remark. The aim of 4.8, 4.9 is the following: We will consider iterations $\langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ where $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ satisfies $UP(\underset{\sim}{\mathbb{I}}_i)$", but $\underset{\sim}{\mathbb{I}}_i$ may not be a subset of the ground model V. Now 4.9 gives us a good $\leq_{RK}$-bound $\mathbb{I}_i$ in V, and we can prove (under suitable assumptions) that P_α will satisfy the $UP(\bigcup_{i<\alpha} \mathbb{I}_i)$.

4.10 Definition. 1) We say a family $\mathbb{I}$ of ideals is κ-closed *if: for every* $\alpha < \kappa$ and $J_i \in \mathbb{I}$ for $i < \alpha$ *there is* $J \in \mathbb{I}$, $\bigwedge_{i<\alpha} J_i \leq_{RK} J$. It is strongly κ-closed it is κ-closed, and it is closed under restriction.
2) We say a family $\mathcal{L}$ of partial orders is κ-closed if $\{\mathrm{id}_L : L \in \mathcal{L}\}$ is.

4.11 Fact. 1) Let $\langle J_i : i < \alpha \rangle$ be a sequence of ideals; we define $J = \prod_{i<\alpha} J_i$ as the ideal on $\prod_{i<\alpha}(\mathrm{Dom}(J_i))$ generated by $\{\bigcup_{j<\alpha} \prod_{i<\alpha} A_i^j$: for $i, j < \alpha$ we have $A_i^j \subseteq \mathrm{Dom}(J_i)$ and for each $i < \alpha$ we have $A_i^i \in J_i\}$, *then*

(i) J is an ideal

(ii) $|\mathrm{Dom}(J)| = \prod_{i<\alpha} |\mathrm{Dom}(J_i)|$

(iii) $\mathrm{dens}(J) \leq \prod_{i<\alpha} \mathrm{dens}(J_i)$

(iv) if each J_i is λ-complete then J is λ-complete

(v) $J_i \leq_{RK} J$ for each $i < \alpha$

(vi) if for each i, $(\mathrm{Dom}(J_i)) \notin J_i$ then $(\mathrm{Dom}(J)) \notin J$

(vii) if $J_i = \mathrm{id}_{(L_i, <_i)}$ then J is naturally isomorphic to $\mathrm{id}_{(L,<)}$ where $(L, <) = \prod_{i<\alpha}(L_i, <_i)$

2) This product is associative.

4.12 Definition. 1) For κ a regular cardinal the κ-closure of a family $\mathbb{I}$ of ideals is

$$\mathbb{I} \cup \{\prod_{i<\alpha} J_i : \alpha < \kappa, J_i \in \mathbb{I}\}$$

2) Similarly for a family of partial orders

4.13 Fact. For a family $\mathbb{I}$ of ideals let $\mathbb{I}'$ be the κ-closure of $\mathbb{I}$, then:

(i) $|\mathbb{I}'| \leq |\mathbb{I}|^{<\kappa}$

(ii) $\mathbb{I}'$ is κ-closed

(iii) $\sup_{J\in\mathbb{I}'} |\mathrm{Dom}(J)| \leq (\sup_{J\in\mathbb{I}}(|\mathrm{Dom}(J)|)^{<\kappa}$

(iv) if $\mathbb{I}$ is λ-complete so is $\mathbb{I}'$

(v) $\sup_{J\in\mathbb{I}}(\operatorname{dens}(J)) \le (\sup_{J\in\mathbb{I}}\operatorname{dens}(J))^{<\kappa}$

(vi) if $\mathbb{I} = \{\mathrm{id}_L : L \in \mathcal{L}\}$ then $\mathbb{I}' \equiv_{RK} \mathrm{id}_{\mathcal{L}'}$ where $\mathcal{L}'$ is the κ-closure of $\mathcal{L}$ (in fact, $\mathbb{I}'$, $\mathrm{id}_{\mathcal{L}}$, are isomorphic)

Proof: Easy. $\square_{4.13}$

The following claim gives better cardinality restrictions in §3 (and 2.17) and not having to use "not too large $\mathbb{I}$ for P_i in the iteration for the sake of Q_i" (also alternative proofs). Here $\underset{\sim}{\mathbf{S}}$ is just $\{\aleph_1\}$.

4.14 Claim. Suppose Q satisfies $UP(\mathbb{I},\mathbf{W})$, Q satisfies the κ-c.c. and $\langle N_\eta : \eta \in (T,\mathbf{I})\rangle$ is an $\aleph_1$-strict $(\mathbb{I},\mathbf{W})$ - suitable tree of models (for χ). Let $\mathbb{I}' = \{I \in \mathbb{I} : I$ is κ-complete $\}$ and assume $\mathbb{I}'$ is κ-closed, $N_\eta \cap \omega_1 = \delta \in \mathbf{W}$.
Then for every $p \in N_{\langle\rangle} \cap Q$ there is an $(N_{\langle\rangle}, Q)$ - semi generic $q, p \le_{\mathrm{pr}} q \in Q$ such that

$$q \Vdash_Q \text{" there is } T' \subseteq T \text{ such that } \langle N_\eta[\underset{\sim}{G}_Q] : \eta \in (T',\mathbf{I})\rangle \text{ is a } \aleph_1\text{-strictly } (\mathbb{I}',\mathbf{W}) \text{ - suitable* tree of models"}$$

Proof. Let $G \subseteq Q$ be generic over V. Let $\delta = N_{\langle\rangle} \cap \omega_1$. By 2.14A we know that $\langle N_\eta[G] : \eta \in (T,\mathbf{I})\rangle$ is $(\mathbb{I}',\mathbf{W})$-suitable*, but it is not necessarily $\aleph_1$-strict. So let (in $V[G]$) :

$$T^* = \underset{\sim}{T}^*[G] \stackrel{\text{def}}{=} \{\eta \in T : N_\eta[G] \cap \omega_1 = \delta\}.$$

$UP(\mathbb{I},\mathbf{W})$ implies that we can find q such that $p \le_{\mathrm{pr}} q$ and q forces that $\underset{\sim}{T}^*$ contains a branch, but we want $\underset{\sim}{T}^*$ to contain even an $(\mathbb{I}',\mathbf{W})$-suitable* tree.

Define (in $V[G]$)) a depth function Dp_T as follows:

$$\mathrm{Dp}_T(\eta) \ge \alpha \text{ iff} : \eta \in T^* \text{ and } \forall \beta < \alpha \, \forall I \in \mathbb{I}' \cap N_\eta[G] \, \exists \nu_\eta \in T^*$$
$$[\eta \trianglelefteq \nu_\eta \;\&\; I \le_{RK} \mathbf{I}_{\nu_\eta} \;\&\; \{\rho : \rho \in \mathrm{Suc}_{T^*}(\nu_\eta),\ \mathrm{Dp}_T(\rho) \ge \beta\} \notin \mathbf{I}_{\nu_\eta}].$$

Clearly $\mathrm{Dp}_T : T^* \to \mathrm{Ord} \cup \{\infty\}$ is well-defined, and if $\eta \trianglelefteq \nu$, then $\mathrm{Dp}_T(\eta) \ge \mathrm{Dp}_T(\nu)$.

For each $\eta \in T$, define A_η as follows:

$$\text{if } \eta \notin \mathrm{split}(T^*,\mathbf{I}) \text{ or } \mathbf{I}_\eta \notin \mathbb{I}', \text{ then } A_\eta = \emptyset$$
$$\text{otherwise } A_\eta = \{\rho \in \mathrm{Suc}_{T^*}(\eta) : \mathrm{Dp}_T(\eta) = \mathrm{Dp}_T(\rho)\}.$$

If $A_\eta \in \mathbf{I}_\eta$ let $B_\eta = A_\eta$, otherwise let $B_\eta = \emptyset$. Now we return to V. So for each η we have a name $\underset{\sim}{B}_\eta$ such that $\Vdash_Q$ "$\underset{\sim}{B}_\eta \in \mathbf{I}_\eta \in \mathbb{I}'$". As $\mathbb{I}'$ is κ-complete and Q satisfies the κ-c.c., there is $B^*_\eta \in \mathbf{I}_\eta$ such that $\Vdash_Q$ "$\underset{\sim}{B}_\eta \subseteq B^*_\eta$". Now define T^0 as follows:

$$T^0 = \{\eta : \text{ for all } \ell < \ell g(\eta), \text{ if } \mathbf{I}_{\eta\restriction\ell} \in \mathbb{I}' \text{ and } \eta\restriction\ell \in \text{split}(T, \mathbf{I}), \text{ then } \eta(\ell) \notin B^*_\eta\}.$$

So we have $(T, \mathbf{I}) \leq^* (T^0, \mathbf{I})$, and $\langle N_\eta : \eta \in (T^0, \mathbf{I})\rangle$ is still an $\aleph_1$-strictly $(\mathbb{I}, \mathbf{W})$-suitable* tree of models. So we can find a condition q and a name $\underset{\sim}{\eta}$ such that $p \leq_{\text{pr}} q$ and $q \Vdash$ "$\underset{\sim}{\eta} \in \lim(T^0)$ and for all $\ell < \omega$: $N_{\eta\restriction\ell}[G] \cap \omega_1 = \delta$". We now claim

$(*)$ $q \Vdash$ " for all $\ell < \omega$, $\text{Dp}_T(\underset{\sim}{\eta}\restriction\ell) = \infty$".

So work in $V[G]$, where $q \in G$. Clearly $\eta\restriction\ell \in T^*$ for $\ell < \omega$ and assume toward condition $\bigvee_\ell \text{Dp}_T(\eta\restriction\ell) < \infty$. As $\eta \trianglelefteq \nu (\in T^*[G]) \Rightarrow \text{Dp}(\eta) \geq Dp(\nu)$ for some $\ell_0 < \omega$, $(\forall \ell \geq \ell_0)[\text{Dp}_T(\eta\restriction\ell) = \alpha_0 < \infty]$. Let $\eta_0 = \eta\restriction\ell_0$. By definition of Dp_T, there are $I \in N_\eta[G] \cap \mathbb{I}'$ and $\beta < \alpha_0 + 1$ such that for all $\nu \in T^*$: if $\eta \trianglelefteq \nu$, and $I \leq_{RK} \mathbf{I}_\nu$ and $\nu \in \text{split}(T, \mathbf{I})$ then $\{\rho \in \text{Suc}_{T^*}(\nu) : \text{Dp}_T(\rho) \geq \beta\} \in \mathbf{I}_\eta$. W.l.o.g. $\beta = \alpha_0$. Since $\langle N_\nu[G] : \nu \in T^0\rangle$ is suitable*, and $\underset{\sim}{\eta}$ is a branch, we can find $\ell_1 > \ell_0$, such that (letting $\eta_1 = \eta\restriction\ell_1$): $I \leq_{RK} \mathbf{I}_{\eta_1}$ and $\text{Suc}_T(\eta_1) \notin \mathbf{I}_{\eta_1}$; now as $\ell_1 > \ell_0$ clearly $\eta \trianglelefteq \eta_1$ and (by the choice of ℓ_0) $\text{Dp}_T(\eta_1) = \alpha_0$; by those things and by the previous sentence $\{\rho \in \text{Suc}_{T^*}(\eta_1) : \text{Dp}_T(\rho) = \alpha_0\} \in \mathbf{I}_{\eta_1}$. But then we must have $\eta\restriction(\ell_1 + 1) \in \{\rho \in \text{Suc}_{T^*}(\eta_1)\text{: } \text{Dp}_T(\rho) = \alpha_0\} \subseteq A_{\eta_1} = B_{\eta_1} \subseteq B_\eta$. This is impossible as $\eta\restriction(\ell_1 + 1) \in T^0$. So we have proved $(*)$. Now it is easy to see that $\underset{\sim}{T}'[G] = \{\eta \in \underset{\sim}{T}^*[G] : \text{Dp}_T(\eta) = \infty\}$ satisfies all requirements. $\square_{4.14}$

We can conclude (and it should be easy for a reader who has arrived here):

4.15 Iteration Lemma. Suppose:

(a) $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$ is an *RCS* iteration

(b) for every i for some n we have $\Vdash_{P_{i+n}}$ "$|P_i| \leq \aleph_1$"

(c) $\mathbf{W} \subseteq \omega_1$ stationary

(d) for each i for some P_i-name of regular cardinal $\underset{\sim}{\kappa}_i \geq \aleph_1$ (in V) and P_i-name $\underset{\sim}{\mathbb{I}}_i$:

(α) P_i satisfies the $\underset{\sim}{\kappa}_i$-c.c. (i.e. if $p \Vdash_{P_i}$ "$\underset{\sim}{\kappa}_i = \kappa$" then $P_i\restriction\{q \in P_i, q \geq p\}$ satisfies the κ-c.c.) and

(β) $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ satisfies $UP(\underset{\sim}{\mathbb{I}}_i, \mathbf{W})$ and $\underset{\sim}{\mathbb{I}}_i$ is $\underset{\sim}{\kappa}_i$ - complete."

Then

(1) P_α satisfies $UP(\mathbb{I}, \mathbf{W})$ for some $(\mathrm{Min}_{i<\alpha}\underset{\sim}{\kappa}_i)$-complete $\mathbb{I}(\in V)$ (i.e. $\mathbb{I}$ is κ-complete where $\kappa \stackrel{\text{def}}{=} \mathrm{Min}\{\kappa :$ for some i and $p \in P_i, p \Vdash_{P_i} \kappa_i = \kappa)$.

(2) $\bigcup_{i<\delta} P_i$ is a dense subset of P_δ (δ limit ordinal $\leq \alpha$) if: $\mathrm{cf}(\delta) = \aleph_1$ or $\Vdash_{P_\delta}$ "$\mathrm{cf}(\delta) = \aleph_1$" or δ strongly inaccessible and $\bigwedge_{i<\delta} |P_i| < \delta$.

(3) also the existence lemma holds, (like 3.8).

Proof. Should be clear. $\square_{4.15}$

We note:

4.15A Claim. 1) In 4.15, we can use the "strong preservation" version (and it works).

4.16 Lemma. The following property, $UP_{con}(\mathbb{I}, \mathbf{W})$, is preserved (even strongly preserved) by iterations as in 4.15, and implies that forcing by Q add no real, where:

$UP_{con}(\mathbb{I}, \mathbf{W})$ is satisfied by the forcing notion Q, if: *for any* $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ an $\aleph_1$-strict $(\mathbb{I}, \mathbf{W})$-suitable tree of models for χ, such that for every $\eta, \nu \in T$, of the same length $h_{\eta,\nu}$ is an isomorphism from N_η onto $N_\nu, h_{\eta,\nu}(Q) = Q$, $h_{\eta\restriction\ell,\nu\restriction\ell} \subseteq h_{\eta,\nu}$ and: *if* $\eta^* \in \lim(T)$ and G_{η^*} is a directed subset of $\bigcup_{\ell<\omega} N_{\eta^*\restriction\ell} \cap Q$, not disjoint to any dense subset of $\bigcup_{\ell<\omega} N_{\eta^*\restriction\ell} \cap Q$ defined in $(\bigcup_{m<\omega} N_{\eta^*\restriction m}, \quad N_{\eta^*\restriction\ell}, Q, \mathbf{I}_{\eta^*\restriction\ell})_{\ell<\omega}$ *then there is* $q \in Q$ such that $q \Vdash_Q$ "there is $\nu \in \lim(T)$ (in V^Q) such that $\bigcup_{\ell<\omega} h_{\eta^*\restriction\ell,\nu\restriction\ell}(G \cap N_{\nu\restriction\ell})$ is a subset of $\underset{\sim}{G}_Q$".

4.16A Remark. 1) This property relates to the $\mathrm{UP}(\mathbb{I}, \mathbf{W})$ as E-complete relate to E-proper (see V §1).

2) Who satisfies this condition? $\mathbf{W}$-complete forcing notions, $\mathrm{Nm}'(D)$, $\mathrm{Nm}(D)$

(D is $\aleph_2$-complete) $\mathrm{Nm}^{(')}(T, \mathfrak{D})$ ($\mathfrak{D}$ is $\aleph_2$-complete), and shooting a club through a stationary subset of some $\lambda = \mathrm{cf}(\lambda) > \aleph_1$ consisting of ordinals of cofinality ω (and generally those satisfying the $\mathbb{I}$-condition from Chapter XI).

Proof: Should be clear (and will be elaborated elsewhere, see [Sh:311]). $\square_{4.16}$