XII. Improper Forcing

§0. Introduction

In Chapter X we proved general theorems on semiproper forcing notions, and iterations. We apply them to iterations of several forcings. One of them, and an important one, is Namba forcing. But to show Namba forcing is semiproper, we need essentially that $\aleph_2$ was a large cardinal which has been collapsed to $\aleph_2$ (more exactly – a consequence of this on Galvin games). In XI we took great trouble to use a notion considerably more complicated than semiproperness which is satisfied by Namba forcing. However it was not clear whether all this is necessary as we do not exclude the possibility that Namba forcing is always semiproper, or at least some other forcing, fulfilling the main function of Namba forcing (i.e., changing the cofinality of $\aleph_2$ to ω without collapsing $\aleph_1$). But we prove in 2.2 here, that: there is such semiproper forcing, iff Namba forcing is semiproper, iff player II wins in an appropriate game $\Game(\{\aleph_1\}, \omega, \aleph_2)$ (a game similar to the game of choosing a decreasing sequence of positive sets (modulo appropriate filter, see X 4.10 (towards the end) and the divide and choose game, X 4.9, Galvin games) and, in 2.5, that this implies Chang's conjecture. In our game player I divide, played II choose but here it continue to choose more possibilities later. Now it is well known that Chang's conjecture implies $0^{\#}$ exists, so e.g., in ZFC we cannot prove the existence of such semiproper forcing. An amusing consequence is that if we collapse a measurable cardinal to κ by Levy-collapsing P, then Chang's conjecture holds as by X 6.13 in V^P player II wins the game mentioned above (Silver's original proof of the consistency of Chang conjecture uses a Ramsey cardinal, but he has first to force MA + $2^{\aleph_0} > \aleph_1$ and then use a more complicated collapsing).

In Sect. 1 we give the various variants of properness equivalent formations using games 1.1, 1.7; we also show how the preservation theorems work in this setting, thus getting alternative proofs of the preservation theorems for semiproper and proper, (1.8, 1.8A, 1.9). This alternative proof for properness was later and independently discovered by Gray in [Gr]. Related games have been investigated by Jech (which are like Galvin games, but using complete Boolean algebras) but his interests were different. Compared to [Sh:b] the order of the sections is inverted.

§1. Games and Properness

1.1 Theorem. A forcing notion P is proper *iff* player II has a winning strategy in the game $\mathrm{P}\eth^{\omega}(p, P)$, for every p, where:

1.2 Definition. In a play of the game $\mathrm{P}\eth^{\alpha}(p, P)$ (α a limit ordinal, $p \in P$) in the β-th move player I chooses a P-name $\underset{\sim}{\xi}_\beta$ of an ordinal, and player II chooses an ordinal ζ_β.

In the end after α moves player II wins if there is q such that $p \leq q \in P$ and $q \Vdash$ " for every $\beta < \alpha$ we have $\underset{\sim}{\xi}_\beta \in \{\zeta_{\beta+n} : n < \omega\}$", and player I wins otherwise.

1.3 Remarks.

1) Note that we can allow player II to choose countably many ordinals $\zeta_{\beta,\ell}$ ($\ell < \omega$) and demand $q \Vdash$ "$\underset{\sim}{\zeta}_\beta \in \{\xi_{\beta+n,\ell} : n, \ell < \omega\}$". Similarly player I can choose countably many P-names, and nothing is changed, i.e., the four variants of the definition, together satisfy (or do not satisfy) "player I (II) has a winning strategy".

2) Similarly for $\mathrm{P}\partial^{\omega}(p, P, \lambda)$ (see Definition 1.4(1)).

Proof of 1.1.

The "if" part.

Let λ be big enough, $N \prec (H(\lambda), \in)$ is countable, $\{P, p\} \in N, p \in P$. Then a winning strategy of $\mathrm{P}\partial^{\omega}(p, P)$ belongs to N. So there is a play of this game $\underset{\sim}{\xi}_n, \zeta_n (n < \omega)$ in which player II uses his winning strategy in choosing $\zeta_n \in N$ and every P-name $\underset{\sim}{\xi} \in N$ of an ordinal appears in $\{\underset{\sim}{\xi}_n : n < \omega\}$ and each $\underset{\sim}{\xi}_n$ belongs to N. So clearly $\zeta_n \in N$ for every n.

So there is q, witnessing the victory of II, i.e., $p \le q \in P$, $q \Vdash$ "$\underset{\sim}{\xi}_n \in \{\zeta_m : m < \omega\}$", (for every n), but $\zeta_m \in N$, so $q \Vdash$ "$\underset{\sim}{\xi}_n \in N$ for every n". As $\{\underset{\sim}{\xi}_n : n < \omega\}$ lists all P-names of ordinals which belong to N, q is (N, P)-generic; and $q \ge p$, so we finish.

The "only" if part.

For λ big enough, expand $(H(\lambda), \in)$ by Skolem functions and get a model M^* and we shall describe a strategy for II: If player I has chosen up to now $\underset{\sim}{\xi}_0, \dots, \underset{\sim}{\xi}_n$, let N_n be the Skolem hull of $\{p, P\} \bigcup \{\underset{\sim}{\xi}_0, \dots, \underset{\sim}{\xi}_n\}$ in M^*, and $\{\zeta_{n,\ell} : \ell < \omega\}$ will be the set of ordinals which belong to N_n (remember Remark 1.3).

Suppose $\underset{\sim}{\xi}_0, \zeta_{0,\ell}(\ell < \omega), \underset{\sim}{\xi}_1, \zeta_{1,\ell}(\ell < \omega), \dots$ is a play in which player II uses his strategy (see remark 1.3(1)). Why does he win? Clearly $N = \bigcup_n N_n$, which is the Skolem hull in M^* of $\{p, P\} \bigcup \{\underset{\sim}{\xi}_\ell : \ell < \omega\}$, is an elementary submodel of M^*, (similarly for the reducts) so there is $q \ge p$ which is (N, P)-generic, so as $\underset{\sim}{\xi}_n \in N$ we have $q \Vdash$ "$\underset{\sim}{\xi}_n \in N$" but the set of ordinals of N is $\{\zeta_{n,\ell} : n, \ell < \omega\}$. So we finish. $\square_{1.1}$

1.4 Definition.

(1) $\mathrm{P}\partial^{\alpha}(p, P, \lambda)$ is defined similarly, but $\underset{\sim}{\xi}_\beta$ are P-names of ordinals $< \lambda$. For $\lambda = \infty$ (or just $\lambda > |P|$) we get $\mathrm{P}\partial^{\alpha}(p, P)$.

(2) For a set S of cardinals, the game $\mathrm{P}\partial^{\alpha}(p, P, S)$ is defined as follows: in the β-th move player I chooses $\lambda_\beta \in S$, and a P-name $\underset{\sim}{\xi}_p$, and player II

chooses for $\gamma \leq \beta$ subsets $A_{\beta,\gamma}$ of λ_γ, of power $< \lambda_\gamma (A_{\beta,\gamma} \in V)$ (for λ_γ regular these are w.l.o.g. initial segments of λ_γ).

In the end player II wins if there is $q \in P$, $q \geq p$ such that $q \Vdash$ "$\underset{\sim}{\xi}_\gamma \in \bigcup_{\gamma \leq \beta < \gamma+\omega} A_{\beta,\gamma}$ for $\gamma < \alpha$". The definition for a P-name $\underset{\sim}{S}$ is similar, but player I chooses λ_β (not a P-name), and in the end, q forces only that: "if $\lambda_\beta \in \underset{\sim}{S}$ then ..."

If not said otherwise we restrict ourselves to sets of regular cardinals in V.

1.4A Remark. Note that 1.4(1) is in fact a special case of 1.4(2) when we identify λ with $S = \{\mu : \aleph_1 \leq \mu \leq \lambda, \mu \text{ regular}\}$, i.e. a winning strategy for $\mathrm{P}\eth^\alpha(p, P, \lambda)$ can, in a canonical way, be translated to a winning strategy in $\mathrm{P}\eth^\alpha(p, P, S)$, and conversely.

1.5 Definition.

(1) $\mathrm{P}\eth^\alpha_\ell(p, P, \lambda)$ is defined similarly for $\ell = 0, 1, 2$, but:

for $\ell = 0$ it is exactly $\mathrm{P}\eth^\alpha(p, P, \lambda)$;

for $\ell = 1$ player I chooses $\underset{\sim}{\xi}_\beta$, a P-name of an ordinal $< \lambda$. Player II chooses countably many ordinals ($\zeta_{\beta,n}$ for $n < \omega$) and player II wins if $q \Vdash$ "$\underset{\sim}{\xi}_\beta \in \{\zeta_{\beta,n} : n < \omega\}$ for each $\beta < \alpha$" for some $q \geq p$

for $\ell = 2$ player II chooses $\aleph_0$ P-names ($\underset{\sim}{\xi}_{\beta,n}$ for $n < \omega$) of ordinals $< \lambda$ and player II chooses $\aleph_0$ ordinals $\zeta_{\beta,n}$ for $n < \omega$; and player II wins if $q \Vdash$ "$\underset{\sim}{\xi}_{\beta,n} \in \{\zeta_{\beta,\ell} : \ell < \omega\}$ for every β, n" for some $q \geq p$.

(2) The games $\mathrm{P}\eth^\alpha_\ell(p, P, S)$ ($\ell = 0, 1, 2$, S a set of cardinals of V) are similar to $\mathrm{P}\eth^\alpha(p, P, S)$;

$$\mathrm{P}\eth^\alpha_0(p, P, S) = \mathrm{P}\eth^\alpha(p, P, S)$$

In $\mathrm{P}\eth^\alpha_1, (p, P, S)$, player I chooses (in move β) a cardinal $\lambda_\beta \in S$ and a name of an ordinal $\underset{\sim}{\xi}_\beta < \lambda_\beta$. Player II plays a set $A_\beta \subseteq \lambda_\beta, |A_\beta| < \lambda_\beta,\ A_\beta \in V$ (if $V \models$ "λ regular" w.l.o.g. $A_\beta = \xi_\beta$). After α moves II wins if he can find a condition $q \geq p$ such that $q \Vdash$ "$(\forall \beta < \alpha) \underset{\sim}{\xi}_\beta \in A_\beta$".

In $\mathrm{P}\partial_2^\alpha(p,P,S)$, player I chooses $\lambda_\beta \in S$ and $\aleph_0$ names $\underset{\sim}{\xi}_{\beta,n}$ of ordinals $< \lambda_\beta$, player II chooses $A_\beta \subseteq \lambda_\beta$, as above and in the end player II has to find a condition $q \geq p$ such that $q \Vdash$ "$\forall\beta < \alpha\, \forall n < \omega\, (\underset{\sim}{\xi}_{\beta,n} \in A_\beta)$".

Similarly, we can define $\mathrm{P}\partial_\ell^\alpha(p,P,\underset{\sim}{S})$, where $\underset{\sim}{S}$ is a name of a set of regular cardinals of V.

1.6 Claim.

(1) Player II wins $\mathrm{P}\partial_1^\alpha(p,P,\lambda)$ iff he wins $\mathrm{P}\partial_2^\alpha(p,P,\lambda)$ provided that $\lambda = \aleph_1$ or $\lambda^{\aleph_0} = \lambda$ or at least there is in V a family $\{A_i : i < \lambda\}$ of countable subsets of λ such that in V $(\forall A \subseteq \lambda)(\exists i)$ $(|A| \leq \aleph_0 \to A \subseteq A_i)$.

(2) Player II wins $\mathrm{P}\partial_1^\alpha(p,P,\underset{\sim}{S})$ iff he wins $\mathrm{P}\partial_2^\alpha(p,P,\underset{\sim}{S})$ (when $\underset{\sim}{S}$ is a P-name of a set of regular cardinals $> \aleph_0$).

1.6A Remark. Note that if player II wins one of the games $\mathrm{P}\partial_\ell^\alpha(p,P,\lambda)$ then p forces $(\forall A \in V^P)(\exists B \in V)$ $[A \subseteq \lambda \wedge |A| \leq \aleph_0 \Rightarrow A \subseteq B \wedge |B| = \aleph_0]$.

Proof. (2) The "if" part is trivial.

For the "only if" part: Note that by the assumption $\Vdash_P$ "$\lambda \in \underset{\sim}{S} \to \lambda = \mathrm{cf}^V(\lambda) > \aleph_0$" and that as player II wins $\mathrm{P}\partial_1^\alpha(p,P,S)$ we have: for any λ, $\Vdash_P$ "$\lambda \in \underset{\sim}{S} \to \mathrm{cf}(\lambda) > \aleph_0$"). Now when player I chooses λ_β and $\{\underset{\sim}{\xi}_{\beta,n} : n < \omega\}$, player II "pretends" player I has chosen $\mathrm{Sup}\{\underset{\sim}{\xi}_{\beta,n} : n < \omega\}$, and chooses a suitable initial segment of λ_β. So he translates a play of $\mathrm{P}\partial_2^\alpha(p,P,s)$ to one of "$\mathrm{P}\partial_1^\alpha(p,P,s)$".

(1) Really the same as the proof of (2): the "if" part is trivial and for the "only if" part, player II let $\underset{\sim}{i}_\beta = \mathrm{Min}\{i : \{\underset{\sim}{\xi}_{\beta,n} : n < \omega\} \subseteq A_i\}$ where $\{A_i : i < \lambda\}$ is as mentioned in 1) (such a family exists in V if $\lambda = \aleph_1$ $(A_i = i)$ or $\lambda^{\aleph_0} = \lambda$ $(\{A_i : i < \lambda\} = \{A \subseteq \lambda : |A| \leq \aleph_0\})$ and as in the proof of part (2) above, the forcing preserves this property).

Now because player II wins $\mathrm{P}\partial_1^\alpha(p,P,\lambda)$, he can find in the β-th move (of a play of $\mathrm{P}\partial_2^\alpha(p,P,\lambda)$) ordinals $i_{\beta,n} (n < \omega)$, such that at the end of the game $q \Vdash$"$\exists n(\underset{\sim}{i}_\beta = i_{\beta,n})$", and so his move in $\mathrm{P}\partial_2^\alpha(p,P,\lambda)$ will be the countable set $\bigcup_n A_{i_{\beta,n}}$. (More formally, fixing a winning strategy F_1 for player

II in $(\mathrm{P}\eth_2^\alpha(p,P,\lambda)$, we shall describe a winning strategy F_2 of player II in $\mathrm{P}\eth_2^\alpha(p,P,\lambda)$. During the play he simulate a play of $\mathrm{P}\eth_1^\alpha(p,P,\lambda)$ by playing in the β-th move $\underset{\sim}{i}_\beta$.) $\square_{1.6}$

1.7 Theorem.

(1) P is α-proper *iff* player II wins $\mathrm{P}\eth^{\omega(1+\alpha)}(p,P;\infty)$ for every p.

(2) P is $(\alpha,1)$-proper *iff* player II wins $\mathrm{P}\eth_2^\alpha(p,P;\infty)$ for every p.

(3) A forcing notion P is semiproper iff player II has a winning strategy in $\mathrm{P}\eth_0^\omega(p,P,\aleph_1)$ for every $p\in P$.

(4) A forcing notion P is $\underset{\sim}{S}$-semiproper *iff* player II has a winning strategy in $\mathrm{P}\eth_0^\omega(p,P,\underset{\sim}{S})$ for every $p\in P$.

Proof. Similar to 2.1, for (2) use V 2.5. $\square_{1.7}$

1.7A Remark.

(1) Call a forcing notion P "$\underset{\sim}{S}$-semi-α-proper" (for $\underset{\sim}{S}$ a P-name of a set of cardinals of V), if for all large enough cardinals λ, and all $\langle N_i : i\leq\alpha\rangle\in SQS_\alpha^0(\lambda)$ (see V2.1) for which $P,\underset{\sim}{S}\in N_0$:

for all $p\in P\cap N_0$ there is $q\geq p$, such that for every $i\leq\alpha$ we have $q\Vdash_P$ " if $\mu\in\underset{\sim}{S}$ then $\sup(\mu\cap N_i[\underset{\sim}{G}])=\sup(\mu\cap N_i)$".

Then we have: P is $\underset{\sim}{S}$-semi-α-proper iff player II wins $\mathrm{P}\eth^{\omega(1+\alpha)}(p,P,\underset{\sim}{S})$.

(2) For countable ordinals to demand $\underset{\sim}{\xi}_\beta\in\{\zeta_{\beta,n}:n<\omega\}$" is equivalent to demanding $\underset{\sim}{\xi}_{\beta,n}<\mathrm{Sup}\{\zeta_{\beta,n}:n<\omega\}$, etc.

1.8 Theorem. The property "player II wins $\mathrm{P}\eth_\ell^\alpha(p,P,\lambda)$ for every $p\in P$" is preserved by CS iterations $\langle P_i,\underset{\sim}{Q}_i : i<\alpha(*)\rangle$, for $\ell=0,2$, $\alpha<\omega_1$, if $|\alpha(*)|\leq\lambda$.

1.8A Remark.

(1) So we get here an alternative proof of the preservation of properness and $(\alpha,1)$-properness: we could give such proofs for other theorems as well.

(2) The situation is similar with $\underset{\sim}{S}$ instead of λ, and easier for RCS iteration by X 2.5 (stated in 1.9).

Proof. Let $\langle P_i, \underset{\sim}{Q}_i : i < \alpha(*)\rangle$ be a countable support iteration. Let us consider a game $\mathrm{P}\eth_0^\omega(p, P_{\alpha(*)}, \lambda)$ (the others are similar). By the hypothesis for each $i < \alpha(*)$, player II has, for every $\underset{\sim}{q} \in \underset{\sim}{Q}_i$, a winning strategy $\underset{\sim}{st}_i = \underset{\sim}{st}_i(\underset{\sim}{q})$ in the game $\mathrm{P}\eth_0^\omega(\underset{\sim}{q}, \underset{\sim}{Q}_i, \lambda)$ where $\underset{\sim}{q}$, $\underset{\sim}{Q}_i, \underset{\sim}{st}_i$ are P_i-names.

Without loss of generality for $\underset{\sim}{Q}_i$ we use the version of the games in which player I plays a countable set (of names of countable ordinals) at each stage, and player II answers with a single ordinal. But for $\mathrm{P}\eth_0^\omega(p, P_{\alpha(*)}, \lambda)$ we use the version where both players play singletons, this is legitimate by remark 1.3(1). (The remark at the end of the proof will explain why it is more convenient to let player I choose a countable set of names in the games $\mathrm{P}\eth_0^\omega(q, \underset{\sim}{Q}_i, \lambda)$).

Now player II plays as follows: in the n-th move he will define $w_n, p_n, t_i^n (i \in w_n)$ such that:

(1) $p_n \in P_{\alpha(*)}, p \leq p_0$, $p_n \leq p_{n+1}$,

(2) w_n is a finite subset of $\mathrm{Dom}(p_n), w_n \subseteq w_{n+1}$, and if $\mathrm{Dom}(p_n) = \{i_{n,k} : k < \omega\}$ and w.l.o.g. $i_{n,0} = 0$ then $w_n = \{i_{m,k} : m < n, k < n\}$, (so eventually $\bigcup_{n<\omega} w_n = \bigcup_n \mathrm{Dom}(p_n)$ and w_n depends just on $\langle p_\ell : \ell < n\rangle$)

(3) $p_{n-1}\restriction w_n = p_n \restriction w_n$ for $n > 0$.

(4) For $i \in w_n$, let $n(i) = \mathrm{Min}\{m : i \in w_m\}$, and $t_n^i = \langle\langle\Gamma_k^i, \underset{\sim}{\zeta}_k^i\rangle : n(i) \leq k \leq n\rangle$ is such that Γ_k^i is a countable set of P_i-names of $\underset{\sim}{Q}_i$-names of ordinals $< \lambda$ and $\underset{\sim}{\zeta}_k^i$ a P_i-name of an ordinal, and $p_n\restriction i \Vdash_{P_i}$ "t_n^i is an initial segment of a play of $\mathrm{P}\eth_\ell^\alpha(p_{n(i)}(i), \underset{\sim}{Q}_i, \lambda)$, in which player II uses the strategy $\underset{\sim}{st}_i = \underset{\sim}{st}_i(p_{n(i)}(i))$".

(5) In the zero move (as $w_0 = \emptyset$), player I chooses a $P_{\alpha(*)}$-name of an ordinal $\underset{\sim}{\xi}_0$, and player II chooses $p_0 \geq p$, $p_0 \Vdash_{P_{\alpha(*)}}$ "$\underset{\sim}{\xi}_0 = \zeta_0$", and play the ordinal ζ_0.

(6) In the n-th move, let player I play $\underset{\sim}{\xi}_n$, a $P_{\alpha(*)}$-name of an ordinal $< \lambda$. Let $\langle j_n(m) : m < l_n \rangle$ enumerate w_n in increasing order (so $j_n(0) = 0$) as $i_{n,0} = 0$. Let $j_n(l_n) = \alpha(*)$, and let $\underset{\sim}{\xi}_n^{\alpha(*)} = \underset{\sim}{\xi}_n^{j_n(l_n)} = \underset{\sim}{\xi}_n$.

By downward induction on m $(l_n > m \geq 0)$, player II will define $p_{n,m}, \Gamma_n^{j_n(m)}$, $\underset{\sim}{\zeta}_n^{j_n(m)}$ as follows:

A) Given $\underset{\sim}{\zeta}_n^{j_n(m+1)}$ (for $m = l_n - 1$ this is $\underset{\sim}{\xi}_n$), a $P_{j_n(m+1)}$-name of an ordinal, he can find a condition $p_{n,m}$ (but see below) of the forcing notion $P_{j_n(m+1)}/P_{j_n(m)+1}$ (in the universe $V^{P_{j_n(m)+1}}$); $p_n \restriction [j_n(m)+1, j(m+1)) \leq p_{n,m}$, such that $p_{n,m}$ decides $\underset{\sim}{\zeta}_n^{j_n(m+1)}$ up to a $P_{j_n(m)+1}$-name. So we have the freedom to choose a countable set $\Gamma_n^{j_n(m)}$ of $P_{j_n(m)+1}$-names (of ordinals $< \lambda$, but see below) such that

$$p_{n,m} \Vdash_{P_{j_n(m+1)}/P_{j_n(m+1)}} \text{“}\underset{\sim}{\zeta}_\ell^{j_n(m)} \in \Gamma_n^{j_n}(m)\text{”}.$$

B) Once $\Gamma_n^{j_n(m)}$ is defined, demand (4) yields a $P_{j_n(m)}$-name $\underset{\sim}{\zeta}_n^{j_n(m)}$.

Finally, player II plays $\zeta_n = \zeta_n^0 = \zeta_n^{j_n(0)}$ (which is a P_0-name, hence a real ordinal) and define p_{n+1} by: for $j \in w_n$, $p_{n+1}(j) = p_n(j)$, and $p_{n+1} \restriction [j_n(m) + 1, j(m+1)) = p_{n,m}$.

This is easily done and in the end, as for each $i \in \bigcup_{n<\omega} w_n$, player II has simulated a play of $\mathrm{P}\eth_0^\omega(p_{n(i)}(i), \underset{\sim}{Q}_i, \lambda)$ in which the second player uses his winning strategy st_i (the union of $t_n^i (n < \omega)$), there is a P_i-name $\underset{\sim}{q}(i)$, such that $\Vdash_{P_i}$ “$p_{n(i)}(i) \leq \underset{\sim}{q}(i)$”, and $\Vdash_{P_i} [\underset{\sim}{q}(i) \Vdash_{Q_i}$ “$\bigcup\{\Gamma_m : n(i) \leq n < \omega\} \subseteq \{\underset{\sim}{\xi}_m^i : m < \omega\}$”$]$.

It is clear that q (i.e. $\mathrm{Dom}(q) = \bigcup_{n<\omega} w_n, q(i)$ defined above) belongs to $P_{\alpha(*)}$. We have to prove $q \Vdash$ “$\underset{\sim}{\xi}_\ell \in \{\zeta_n^0 : n < \omega\}$”. Let $r \geq q$ be such that $r \Vdash$ “$\underset{\sim}{\xi}_\ell = \xi^*$”.

Note first that $q \geq p_n$ for every $n < \omega$.

We prove by induction on $i \in \bigcup_{n<\omega} w_n$ that if $\underset{\sim}{\xi} \in \cup\{\Gamma_k^j : j < i, n(j) \leq k < \omega\}$ then $q \restriction i \Vdash_{P_i}$ “$\underset{\sim}{\xi} \in \{\zeta_n^0 : n < \omega\}$”. This suffices as for each n we have $q \Vdash \underset{\sim}{\xi}_n \in \bigcup_{i \in w_n} \Gamma_i^n$. For i limit-trivial; for i successor - use the choice of $p_{n,m}$ and the winning of the second player in $\mathrm{P}\eth_0((p_{n(i)}, \underset{\sim}{Q}_i, \lambda)$.

At first glance we get it too cheaply.

But there is a delicate point – the choice of $p_{n,m}$; it is really not a condition of $P_{j(m+1)}$ or $P_{\alpha(*)}$ but a $P_{j(m)+1}$-name for it. The delicate point is that its domain is a name, whereas it is required to be a set. However, it is not fatal as long as at least q would be a real condition, for which it is not necessary to really know $\mathrm{Dom}(p_{n,m})$, just to find a countable set including it. So into the list of names player II is manipulating he has to add names of the members of $\mathrm{Dom}(p_{n,m})$. So it is enough to ask that $\bigcup_{n<\omega} \Gamma^0_n \cap \alpha^* \subseteq \bigcup_{n<\omega} w_n$ and $\mathrm{Dom}(p_{n,m}) \subseteq \Gamma^{j_n(m)}_n$ i.e., there are $P_{j_n(m)}$-names $\underset{\sim}{\varepsilon}^{j_n(m)}_{n,\ell}$ $(\ell < \omega)$ such that $\Vdash_{P_{j(m,n)+1}}$ "$\mathrm{Dom}(p_{n,m}) = \{\underset{\sim}{\varepsilon}^{j_n(m)}_{n,\ell} : \ell < \omega\}$" and we let $\underset{\sim}{\varepsilon}^{j_n(m)}_{n,\ell} \in \Gamma^{j_n(m)}_n$. But Player II can manipulate only names of ordinals $< \lambda$; this is why $|\alpha(*)| = \lambda$ was required so in the end we know $\mathrm{Dom}(p_{n,m}) \subseteq \{\xi_k : k < \omega\}$, and there are no problems.

* * *

The cases $\ell = 0$, $\alpha > \omega$ are proved by induction on α.

For $\ell = 2$, we already know from what we prove that player II wins $\mathrm{P}\eth^\omega_0(p, P_{\alpha(*)}; \lambda)$ for every $p \in P_{\alpha(*)}$, and the proof is simple. $\square_{1.8}$

We leave to the reader the easier theorem (by now):

1.9 Theorem.

1) RCS iterations preserves the following property of forcing notions:

for every $p \in P$, Player II wins $PG^\omega_0(p, P, \aleph_1)$.

2) Similarly for $\mathrm{P}\eth^\alpha_0(p, P, \aleph_1)$, and variants with $\underset{\sim}{S}$.

§2. When Is Namba Forcing Semiproper, Chang's Conjecture and Games

2.1 Definition. For a filter D on a set I, and a set S of regular cardinals and ordinal α we define a game $\eth(S, \alpha, D)$: the game lasts α moves, in the β-th move player I chooses a cardinal $\lambda_\beta \in S$ and a function F_β from I to λ_β.

Then player II has to choose an ordinal $i_\beta < \lambda_\beta$. In the end of the game, player II wins if

$\{t \in I$: for every $\lambda \in S$, if for some β, $\lambda = \lambda_\beta$ then $\mathrm{Sup}\{F_\beta(t) : \lambda_\beta = \lambda\}$ is $\leq \mathrm{Sup}\{i_\beta : \lambda_\beta = \lambda\}\} \not\equiv \emptyset \bmod D$; also player I has to choose each λ_β infinitely often otherwise player II wins (though not every $\lambda \in S$ appears).

If $I = \lambda$, $D = D^{cb}_{\lambda}$ (the filter of cobounded subsets of λ) we replace D by λ.

Remark.

(1) This is close to the games in Definition X 4.9, X 4.10 (toward end) and reference there but here we do not choose immediately the bound, we increase it later.

(2) We do not list the obvious monotonicity properties.

2.2 Theorem.

Namba forcing is $\{\aleph_1\}$-semiproper *iff* player II wins $\Game(\{\aleph_1\}, \omega, \aleph_2)$ *iff* there is a $\{\aleph_1\}$-semiproper forcing P changing the cofinality of $\aleph_2$ to $\aleph_0$.

2.2A Remark.

It does not matter whether we use Nm or Nm' (see XI 4.1, X 4.4), so we deal with the somewhat harder case: Nm'.

Proof. We use the criterion for semiproperness from 1.7(3)

third condition implies second condition

We assume P is $\{\aleph_1\}$-semiproper; and so (by 1.7(3)) we can choose a winning strategy for player II in $\mathrm{P}\Game = \mathrm{P}\Game^{\omega}(p_0, P, \{\aleph_1\})$ (in the variant where both players choose countable sets) and we shall call its players $I_{\mathrm{P}\Game}$,$II_{\mathrm{P}\Game}$ for clarity. Now we describe a winning strategy for player II in $\Game(\{\aleph_1\}, \omega, \aleph_2)$. Player II will simulate for this a play of $\mathrm{P}\Game$, in which $II_{\mathrm{P}\Game}$ uses his winning strategy and $I_{\mathrm{P}\Game}$ is played by him (player II in $\Game(\{\aleph_1\}, \omega, \aleph_2)$). Let $\underset{\sim}{\xi}_n$ be the P-name of the n-th element in an ω-sequence which is unbounded in $\aleph_2$.

In the n-th move let player I choose $F_n : \aleph_2 \to \aleph_1$, player II will play for $I_{\mathrm{P}\Game}$ the P-names $F_n(\underset{\sim}{\xi}_\ell)$ $(\ell < \omega)$, (see 1.3(2)), so he knows the ordinals $\alpha_{n,\ell} < \omega_1$ (for $\ell < \omega$) which $II_{\mathrm{P}\Game}$ plays (according to his strategy). Now player II returns to his play and makes the move $\alpha_n = \mathrm{Sup}\{\alpha_{n,\ell} + 1 : \ell < \omega\} < \omega_1$.

In the end there is a condition $q \in P$, $q \geq p_0$ such that $q \Vdash$ "$F_n(\underset{\sim}{\xi}_\ell) < \bigcup_m \alpha_m$" for n, $\ell < \omega$. Let for each $\ell < \omega$:

$$A_\ell = \{\alpha < \omega_2 : q \text{ does not force } \underset{\sim}{\xi}_\ell \neq \alpha, \text{ i.e.,}$$
$$\text{there is } r \in P \text{ such that } r \geq q \text{ and } r \Vdash \text{"}\underset{\sim}{\xi}_\ell = \alpha\text{"}\}$$

By our choice of $\underset{\sim}{\xi}_\ell(\ell < \omega)$, for some ℓ, A_ℓ is an unbounded subset of ω_2 (otherwise $\bigcup_{\ell<\omega} A_\ell$ would be bounded, as in V we have $\mathrm{cf}(\omega_2) = \aleph_2 > \aleph_0$ so $q \Vdash$ "$\{\underset{\sim}{\xi}_\ell : \ell < \omega\} \subseteq \bigcup\limits_{\ell<\omega} A_\ell$" give a contradiction to the choice of $\langle \underset{\sim}{\xi}_\ell : \ell < \omega \rangle$). Now clearly $\alpha \in A_\ell$ implies $F_n(\alpha) < \bigcup_m \alpha_m$; hence A_ℓ is as required.

First implies third condition

Trivial

Second condition implies first condition

The proof is similar to that of X 4.12(3), it is similar for Nm and Nm', and we present the one for Nm'.

We again use the games and prove player II wins $\mathrm{P}\eth_0^\omega(p, P, \aleph_1\}$ for every $p \in P$, $P = Nm' = Nm'(D^{cb}_{\aleph_2})$ (see X Definition 4.4(4) (we can in fact replace $\aleph_1$ by $\{\lambda : \lambda \text{ a regular cardinal} \neq \aleph_2\}$)). For notational simplicity assume p's trunk is $\langle\rangle$ i.e. $(\forall \eta \in p)(\exists^{\aleph_2} i)(\eta\,\hat{}\,\langle i \rangle \in p)$.

During the play, in the n-th move player I chooses a P-name of a countable ordinal $\underset{\sim}{\xi}_n$, and player II will choose $\xi_n < \omega_1$. On the side, player II in stage n also chooses a condition $p_n \in P$ and a function F_n from $p_n \cap ({}^{n\geq}(\omega_2))$ to ω_1 such that:

(a) $p \leq p_0, p_n \leq p_{n+1}, p_n \cap ({}^{n\geq}(\omega_2)) = p_{n+1} \cap ({}^{n\geq}(\omega_2))$, and the trunk of each p_n is $\langle\rangle$.

(b) For $\eta \in p_{n+1} \cap ({}^{n}(\omega_2))$ and $\ell < n$, either $(p_{n+1})_{[\eta]} \Vdash_P$ "$\underset{\sim}{\xi}_\ell \leq F_n(\eta)$" *or* there is no q, $(p_n)_{[\eta]} \leq q$, with trunk η and $\zeta < \omega_1$, such that $q \Vdash_P$ "$\underset{\sim}{\xi}_\ell \leq \zeta$" (remember $q_{[\eta]} = \{\nu \in q : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$.

(c) For $\eta \in p_n \cap ({}^{m}(\omega_2))$ and $m < n$ let $F_{n,\eta} : \omega_2 \to \omega_1$ be defined by $F_{n,\eta}(i) = F_n(\eta\,\hat{}\,\langle i \rangle)$; and we demand that $F_n(\eta)$ is defined such that

$\langle\langle F_{m+\ell,\eta}, F_{m+\ell}(\eta)\rangle : \ell < n-m\rangle$ is an initial segment of a play of $\partial(\{\aleph_2\},\omega,\aleph_1)$ in which the second player uses his winning strategy (i.e. one we choose a priori).

(d) At last player II actually plays $\zeta_n = F_n(\langle\rangle)$.

There is no problem for player II to use this strategy: Stage 0, is trivial, *stage* $n+1$: for each $\eta \in p_n \cap ({}^n\omega_2)$ define $q^\ell_{\eta,n}$ by induction on $\ell \le n+1 : q^0_{\eta,n} = (p_n)_{[\eta]}$, $q^{\ell+1}_{\eta,n} \ge q^\ell_{\eta,n}$, has trunk η and either it forces $\underset{\sim}{\xi}_\ell \le \zeta$ for some $\zeta < \omega_1$ which we call $\zeta^\ell_{\eta,n}$ or there is no such q (satisfying the two previous conditions). Then

$$p_{n+1} = \bigcup\{q^{n+1}_{\eta,n} : \eta \in p_n \cap ({}^n(\omega_2))\}.$$

and for $\eta \in p_n \cap ({}^n(\omega_2))$.

$$F_{n+1}(\eta) = \mathrm{Max}\{\zeta^\ell_{\eta,n} : \ell \le n+1, \zeta^\ell_{\eta,n} \text{ well defined}\} \bigcup \{0\}.$$

Now he defines $F_{n+1}(\eta)$ for $\eta \in p_n \cap ({}^{n>}(\omega_2)) = p_{n+1} \cap ({}^{n>}(\omega_2))$ by downward induction on $lg(\eta)$, using (c) (i.e., the winning strategy of the second player in $\partial(\{\aleph_2\},\omega,\aleph_1)$.)

In the end player II has to provide the suitable condition $q \ge p$; we define $q \cap {}^n\omega_2$ by induction on n:

$\langle\rangle \in q$, and if we have decided that $\eta \in q$ then: $\eta\hat{\ }\langle i\rangle \in q$ *iff* $\mathrm{Sup}\{F_n(\eta\hat{\ }\langle i\rangle) : n < \omega, n \ge lg(\eta\hat{\ }\langle i\rangle)\} \le \mathrm{Sup}\{F_n(\eta) : n < \omega,\ n \ge lg(\eta)\}$ (and $\eta\hat{\ }\langle i\rangle \in p_{\ell g(\eta)+1}$ of course).

By (c), q will be a condition; and by (b) it forces $\underset{\sim}{\xi}_n < \bigcup_{n<\omega}\zeta_n$: otherwise there is $r \ge q$, $r \Vdash_P$ "$\underset{\sim}{\xi}_n \ge \bigcup_{n<\omega}\zeta_n$". Let η be the trunk of r, and then using (b) we get a contradiction. $\square_{2.2}$

2.3 Definition. More generally, if we have a family $\mathbb{D}$ of filters, a set S of regular cardinals, and a relation $K \subseteq S \times \mathbb{D}$, we define the game $\partial(K,\alpha)$ as follows: In the β^{th} move ($\beta < \alpha$), player I chooses a pair $(\lambda_\beta, D_\beta) \in K$ and a function F'_β from $\cup D_\beta$ to λ_β. Player II chooses an ordinal $i_\beta < \lambda_\beta$. In the end, player II wins if for each $D \in \mathbb{D}$ the set $\{t \in \cup D :$ for every λ with $(\lambda, D) \in K$,

if for some β, $(\lambda, D) = (\lambda_\beta, D_\beta)$, then Sup $\{F_\beta(t) : (\lambda, D) = (\lambda_\beta, D_\beta)\} \le$ Sup $\{i_\beta : (\lambda, D) = (\lambda_\beta, D_\beta)\}\}$ is $\neq \emptyset$ mod D. (Also, player I has to choose each

λ_β infinitely often, otherwise II wins but some $\lambda \in S$ may never be chosen). Again, we may write κ for D^{cb}_κ.

2.4 Theorem. For a countable set S of regular cardinals, and a countable set $\mathbb{D}$ of $\aleph_1$-complete filters, player II wins $\Game(S \times \mathbb{D}, \omega)$ iff there is an S-semiproper forcing notion P, such that for all $D \in \mathbb{D}$

$$\Vdash_P \text{“}\exists w \subseteq \cup D[w \text{ countable, } w \neq \emptyset \bmod D, \text{ that is, it is not disjoint to any } A \in D (\text{so } A \in V)]\text{”}.$$

(If $D = D^{cb}_\lambda$, then the above condition is clearly equivalent to $\Vdash_P$ “ $\mathrm{cf}(\lambda) = \aleph_0$.”)

Proof. The proof is similar to the proof of 2.2. To show the “only if” part, let $\mathbb{D} = \{D_n : n < \omega\}$, where each D_n occurs infinitely often. Let $P = Nm'(T, \mathfrak{D})$, where $T = \{\eta : \eta \text{ a finite sequence, } (\forall k)[k \leq \ell g\, \eta \Rightarrow \eta{\restriction}k \in \mathrm{Dom}(D_k)]\}$, and for each $\eta \in T$, let $\mathfrak{D}_\eta = \big\{\{\eta\char`^\langle i\rangle : i \in A\} : A \in D_{\ell g(\eta)}\big\}$. $\square_{2.4}$

2.5 Theorem.

(1) If player II wins $\Game = \Game(\{\aleph_1\}, \omega_1, \aleph_2)$, *then* Chang's conjecture holds.

(2) Moreover if e.g. $\chi > 2^{\aleph_2}$, M^* an expansion of $((H(\chi), \in))$ by Skolem functions, $N \prec M^*$ is countable, *then* for arbitrarily large $\alpha < \aleph_2$, there is $N_\alpha, N \prec N_\alpha \prec M^*$, $\alpha \in N_\alpha$, and N_α, N have the same countable ordinals.

(3) In (2) we can find N', $N \prec N' \prec M^*$, $N' \cap \omega_1 = N \cap \omega_1$ and $|N \cap \omega_2| = \aleph_1$.

Proof. (1) Follows easily from (2). We can easily build a strictly increasing elementary chain of countable models, of length ω_1, all having the same countable ordinals.

(2) Clearly some winning strategy for player II in $\Game$ belongs to N. So we can construct a play of $\Game$, $F_0, \alpha_0, F_1, \alpha_1, \ldots$ such that player II uses his strategy, each F_n belongs to N, and every function from ω_2 to ω_1 which belong to N appears in $\{F_n : n < \omega\}$.

As the strategy and $F_0, \dots, F_n$ belong to N, also α_n belongs to N. In the end for arbitrarily large $\alpha < \aleph_2$, $F_n(\alpha) < \bigcup_m \alpha_m$ for every n. But clearly $N \cap \omega_1$ is an initial segment of ω_1, hence $\bigcup_m \alpha_m \subseteq N$, so $F_n(\alpha) \in N$ for every α satisfying $\bigwedge_n F_n(\alpha) < \bigcup_m \alpha_m$. Now we can take as N_α the Skolem hull of $N \bigcup \{\alpha\}$.

We have to show that N and N_α have the same countable ordinals. Every $\gamma \in N_\alpha \cap \omega_1$ can be written as $\gamma = \tau(\alpha, a_1, \dots, a_\alpha)$, where $a_1, \dots a_\kappa \in N$, and τ is a Skolem term.

In N, we can define a function $f : \omega_2 \to \omega_1$, by

$$f(i) = \begin{cases} \tau(i, a_1, \dots, a_k), & \text{if this is } < \omega_1 \\ 0 & \text{otherwise.} \end{cases}$$

So $f = F_n$ for some n, but clearly $\gamma = \tau(\alpha, a_n, \dots, a_n) = f(\alpha) = F_n(\alpha) \in N$.

(3) By the proof of (1). $\square_{2.5}$

We can prove similar theorems for S not necessarily $\{\aleph_1\}$, e.g.

2.6 Theorem. If player II wins $\eth(\{\lambda\}, \omega, \mu)$, $\lambda < \mu$, M a model with universe $H((2^\mu)^+)$ and countably many relations and functions, including $\in$ and Skolem functions, $N \prec M$ countable, *then* there is $N^\dagger, N \prec N^\dagger \prec M$, $N^\dagger \neq N$, $\mathrm{Sup}(N^\dagger \cap \lambda) = \mathrm{Sup}(N \cap \lambda)$, $\mathrm{Sup}(N^\dagger \cap \mu) > \mathrm{Sup}(N \cap \mu)$, and $|N^\dagger \cap \mu| = \aleph_1$.

Proof. Similar. $\square_{2.6}$

2.7 Conclusion. (1) For some regular λ, player II wins in $\eth(\{\aleph_1\}, \omega, \lambda)$ iff there is an $\{\aleph_1\}$-semiproper forcing P not preserving "$\mathrm{cf}(\alpha) > \aleph_0$".

(2) So if e.g. $0^\#$ does not exist, a forcing notion is proper *iff* it is $\underset{\sim}{S}$-semiproper, $\underset{\sim}{S} = \{\aleph_1^V\} \cup \{\kappa : \mathrm{cf}(\kappa)^{V^P} > \aleph_0 \ \kappa \text{ regular (in } V) \text{ and } \kappa > \aleph_1\}$.

Proof. (1) By 2.4, $\mathbb{D} = \{D_\lambda^{\mathrm{cb}}\}$.

(2) If P is $\underset{\sim}{S}$-semiproper and preserves "$\mathrm{cf}(\alpha) > \aleph_0$" then by Claim X 2.3(1) P is proper (see Definition X 2.2, $\underset{\sim}{S}$ is essentialy equal to the S from Claim X 2.3(1) because P preserves "$\mathrm{cf}(\alpha) > \aleph_0$").

If P is $\underset{\sim}{S}$-semiproper not preserving "cf$(\alpha) > \aleph_0$" then for some $p \in P$, and regular $\lambda > \aleph_1$ (in V) $Q = P\restriction\{q \in P : p \leq q\}$ is $\underset{\sim}{S}$-semiproper hence $\{\aleph_1\}$-semiproper and $\emptyset \Vdash$ "cf$(\lambda) = \aleph_0$". But then by 2.7(1) player II wins the game $\partial(\{\aleph_1\}, \omega_1, \lambda)$ hence the conclusion of Theorem 2.6 holds and by well known theorems such variants of Chang conjecture imply $0^\# \in V$. $\square_{2.7}$

Note that

2.8 Conclusion. If κ is measurable in V, $P = \text{Levy}(\aleph_1, < \lambda)$ (so elements of P are countable partial functions), then in V^P, Chang's conjecture holds.

Proof. By 2.2 and X 4.11.