VIII. κ-pic and Not Adding Reals

§0. Introduction

In the first section we show that we can iterate $\aleph_2$-complete forcing, and $\aleph_1$-complete forcing which satisfy the $\aleph_2$-c.c. in a strong sense.

In the second section we deal with a strong version of the $\aleph_2$-c.c. called $\aleph_2$-pic. It is useful for proving that for CS iteration of length ω_2 of proper forcing notions, the limit still satisfies the $\aleph_2$-c.c. This in turn will be used in order to get universes with $2^{\aleph_1} > 2^{\aleph_0} = \aleph_2$.

In the third section we deal again with the axioms; starting with a model of ZFC (not assuming the existence of large cardinals) we phrase the axioms we can get. There are four cases according to whether $2^{\aleph_0}$ is $\aleph_1$ or $\aleph_2$, and $2^{\aleph_1}$ is $\aleph_2$ or larger [our knowledge on the case $2^{\aleph_0} \geq \aleph_3$ is slim].

In the fourth section we return to the problem of when a CS iteration of proper forcing preserves "not adding reals". We weaken "each $\underset{\sim}{Q}_i$ (a P_i-name) is $\mathbb{D}$-complete for some $\mathbb{D}$ a $(\lambda, 1, \kappa)$-system", by replacing "each $\mathbb{D}_x$ is an $\aleph_1$-complete filter" or even just "each $\mathbb{D}_x$ is a filter" by "each $\mathbb{D}_x$ is a family of sets, the intersection of e.g. any two is nonempty". So we can deduce ZFC+CH $\nvdash \Phi^3_{\aleph_1}$. We also try to formulate the property preserved by iteration weaker than this completeness. See references in the relevant sections.

§1. Mixed Iteration – $\aleph_2$-c.c., $\aleph_2$-Complete

1.1 Lemma. Suppose P_α $(\alpha \leq \alpha_0)$ are forcing notions in V and $V \models$ CH and $\underset{\sim}{Q}_\alpha$ are such that:

1) $\underset{\sim}{Q}_\alpha$ is a P_α-name of a forcing in V^{P_α},
2) $P_\alpha = \{f : |\{2\beta : 2\beta < \alpha \text{ and } 2\beta \in \mathrm{Dom}(f)\}| \leq \aleph_0$ and $|\{2\beta+1 : 2\beta+1 < \alpha$ and $2\beta+1 \in \mathrm{Dom}(f)\}| \leq \aleph_1$ and $\emptyset \Vdash_{P_i}$ "$f(i) \in \underset{\sim}{Q}_i$" for all $i \in \mathrm{Dom}(f)\}$,
3) $\underset{\sim}{Q}_{2\beta+1}$ is $\aleph_2$-complete (i.e. in $V^{P_{2\beta+1}}$, if $q_i \in \underset{\sim}{Q}_{2\beta+1} (i < \delta < \omega_2)$ are increasing, then $(\exists q_\delta \in \underset{\sim}{Q}_{2\beta+1}) \bigwedge_{i<\delta} q_i \leq q_\delta$),
4) for $\alpha = 2\beta$, in V^{P_α} there is an $\underset{\sim}{h}_\alpha : \underset{\sim}{Q}_\alpha \to \omega_1$, such that: $\underset{\sim}{h}_\alpha(p) = \underset{\sim}{h}_\alpha(q) \Rightarrow$ (p, q has a least upper bound $p \wedge q$),
5) $\underset{\sim}{Q}_{2\beta}$ is $\aleph_1$-complete,
6) $V \models CH$.
7) The order on P_α is as usual: $P_\alpha \models$ "$p \leq q$" *iff* for every $\beta \in \mathrm{Dom}(p)$ we have $q \in \mathrm{Dom}(p)$ and $q{\restriction}\beta \Vdash_{P_\beta}$ "$p(\beta) \leq q(\beta)$"

Then: P_{α_0} is $\aleph_1$-complete and does not collapse $\aleph_2$.

1.1A Remark.

1) Condition 4) was introduced by Baumgartner for getting a weak MA for a $\aleph_1$-complete forcing.

2) For simplicity, we assume that $\emptyset \in Q_\beta$ is minimal, $h_{2\beta}(\emptyset) = 0$, and adopt the convention: if $f \in P_\alpha$ and $f(\beta)$ is not defined otherwise, then $f(\beta) = \emptyset$.

3) Of course the decision to use odd and even ordinals for the two different cases is arbitrary, since any other iteration along two disjoint sets of ordinals for the two different cases can be translated into such an iteration.

4) For a better theorem — see Chapter XIV.

5) We can replace $\aleph_1$ by κ if κ is regular, $\kappa = \kappa^{<\kappa}$ (so "countable" is replaced by "of cardinality $< \kappa$".)

Proof. Let λ be large enough, $N \prec (H(\lambda), \in)$, $P_{\alpha_0} \in N$, $||N|| = \aleph_1$, and every countable subset of N belongs to N. Let $p \in P_{\alpha_0} \cap N$, and $\langle \mathcal{I}_\alpha : \alpha < \omega_1 \rangle$ be a list of all maximal antichains of P_{α_0} which belong to N (so $\langle \mathcal{I}_\alpha : \alpha < \omega_1 \rangle \notin N$ but for each $\alpha^* < \omega_1$ we have that $\langle \mathcal{I}_\alpha : \alpha < \alpha^* \rangle \in N$, by the choice of N). It is trivial that P_{α_0} is $\aleph_1$-complete. It then suffices to prove the existence of a $p^* \in P_{\alpha_0}$, $p \leq p^*$ such that p^* is (N, P_{α_0})-generic, so proving that P_{α_0} is "somewhat proper".

By CH, we can let $\{\langle \alpha^\xi, A^\xi, \langle \gamma^\xi_{\beta,n} : \beta \in A^\xi, n < \omega \rangle \rangle : \xi < \omega_1\}$ be a list of all triples of the form $\langle \alpha, A, \langle \gamma_{\beta,n} : \beta \in A, n < \omega \rangle \rangle$ such that $\alpha < \omega_1$, $A \subseteq \{2\beta : 2\beta \in \alpha_0 \cap N\}$, $|A| \leq \aleph_0$ and $\gamma_{\beta,n} < \omega_1$.

We now inductively define conditions $p_\xi \in N \cap P_{\alpha_0}$ (for $\xi < \omega_1$) which are increasing, with $p_0 = p$, such that:

A) $\mathrm{Dom}(p_\xi) \cap \{2\beta : 2\beta < \alpha_0\} \subseteq \mathrm{Dom}(p)$

B) $2\beta \in \mathrm{Dom}(p_\xi) \Rightarrow p_\xi(2\beta) = p(2\beta)$

C) *if* there are $p^n \in P_{\alpha_0} \cap N$ (for $n < \omega$) such that

(i) $p_\xi \leq p^0 \leq p^1 \leq p^2 \ldots$,

(ii) $2\beta \in A^\xi \Rightarrow (p^{n+1} \restriction 2\beta) \Vdash_{P_{2\beta}}$ "$\underset{\sim}{h}_{2\beta}(p^n(2\beta)) = \gamma^\xi_{2\beta,n}$", and

(iii) for some $q \in \mathcal{I}_{\alpha^\xi}$ we have $q \leq p^0$

then there are such $p^0, p^1 \ldots$ such that:

$[2\beta + 1 \in \bigcup_{n<\omega} \mathrm{Dom}(p^n)] \Rightarrow \bigwedge_{n<\omega} [\Vdash_{P_{2\beta+1}}$ " if $p^0(2\beta+1) \leq p^1(2\beta+1) \leq \ldots \leq p^n(2\beta+1)$(in $\underset{\sim}{Q}_{2\beta+1}$) then $p^n(2\beta+1) \leq p_{\xi+1}(2\beta+1)$"$]$.

We let $p^n_\xi = p^n$ and $q_\xi = q$.

There is no problem in the definition; we can assume w.l.o.g. that

$$\Vdash_{P_{2\beta+1}} \text{"}p^n_\xi(2\beta+1) \leq p^{n+1}_\xi(2\beta+1)\text{"}$$

for every n, as we can replace p^n_ξ by r^n where $r^n(2\beta) = p^n_\xi(2\beta)$ and $r^n(2\beta+1) = p^\ell_\xi(2\beta+1)$ for the maximal $\ell \leq n$ such that $\langle p^m_\xi(2\beta+1) : m \leq \ell \rangle$ is increasing (this is of course a $P_{2\beta+1}$-name).

Now we define p^* : if $2\beta \in \mathrm{Dom}(p)$, $p^*(2\beta) = p(2\beta)$, and if $2\beta+1 \in \bigcup_{\xi<\omega_1} \mathrm{Dom}(p_\xi)$ then $p^*(2\beta+1)$ is (a $P_{2\beta+1}$-name of) an upper bound (in $Q_{2\beta+1}$) of $\{p_\xi(2\beta+1) : \xi < \omega_1\}$ if it exists, and $p(2\beta+1)$ otherwise. So $p = p_0 \leq p^* \in P_{\alpha_0}$. So if $(2\beta+1) \in N \cap \alpha_0$ then $p^* \restriction (2\beta+1) \Vdash_{P_{2\beta+1}}$ "$p^*(2\beta+1)$ is an upper bound of $\{p_\zeta(2\beta+1) : \zeta < \omega_1$ (and $2\beta+1 \in \mathrm{Dom}(p_\zeta))\}$".

Now we prove that for each $\alpha < \omega_1$, $\mathcal{I}_\alpha \cap N$ is pre-dense above p^*. Clearly, there are p^0_* and $q_* \in P_{\alpha_0}$ such that $p^* \leq p^0_*$, $q_* \leq p^0_*$, $q_* \in \mathcal{I}_\alpha$.

Let $A_0 = \mathrm{Dom}(p^0_*) \cap \{2\beta : 2\beta < \alpha_0\} \cap N$. Now, by the $\aleph_1$-completeness of P_{α_0}, there is a $p^1_* \in P_{\aleph_0}$, $p^0_* \leq p^1_*$, such that for every $2\beta \in A_0$,

$$(p^1_* \restriction 2\beta) \Vdash_{P_{2\beta}} \text{"}\underset{\sim}{h}_{2\beta}(p^0_*(2\beta)) = \gamma_{2\beta,0}\text{"}$$

for some $\gamma_{2\beta,0}$. We continue to define $p^{n+1}_* \geq p^n_*$ such that

$$(p^{n+1}_* \restriction 2\beta) \Vdash_{P_{2\beta}} \text{"}\underset{\sim}{h}_{2\beta}(p^n_*(2\beta)) = \gamma_{2\beta,n}\text{"}$$

for every $2\beta \in A_n \stackrel{\mathrm{def}}{=} \mathrm{Dom}(p^n_*) \cap \{2\beta : 2\beta < \alpha_0\} \cap N$. We now define $A \stackrel{\mathrm{def}}{=} (\bigcup_{n<\omega} A_n)$, $\gamma_{2\beta,n} = 0$ for $2\beta \in A \setminus A_n$. So, for some ξ we have $\alpha_\xi = \alpha$, $A^\xi = A$, $\langle \gamma_{2\beta,n} : 2\beta \in A, n < \omega\rangle = \langle \gamma^\xi_{2\beta,n} : 2\beta \in A, n < \omega\rangle$. As these objects are countable subsets of N, by the choice of N, they belong to N.

As $N \prec (H(\lambda), \in)$, for this ξ there are q, p^n as mentioned in clause (C) above, $p^0 \geq q \in \mathcal{I}_\alpha$. Again, without loss of generality, $p^n \in P_{\alpha_0} \cap N$ and $q \in \mathcal{I}_\alpha \cap N$, so $q_\xi, \langle p^n_\xi : n < \omega\rangle$, as in (C), are well defined. Now by the properties of $h_{2\beta}$, we can prove, by induction on $\gamma \leq \alpha_0$, that

($*$) *for every* $\zeta < \gamma$, and $r \in P_\alpha$ such that $p^n_* \restriction \zeta \leq r$ and $p^n_\xi \restriction \zeta \leq r$ for $n < \omega$, there is an $r^* \in P_\gamma$ such that $p^n_* \restriction \gamma \leq r^*$, $p^n_\xi \restriction \gamma \leq r^*$ for $n < \omega$, and $r^* \restriction \zeta = r$, and $\mathrm{Dom}(r^*) \cap [\zeta, \gamma) \subseteq \bigcup_{n<\omega} \mathrm{Dom}(p^n_\xi) \cup \bigcup_{n<\omega} \mathrm{Dom}(p^n_*)$.

For γ a limit there are no problems: use the induction hypothesis and the "bound" on the domain of r^*. For $\gamma = 2\beta+2$, w.l.o.g. $\zeta = 2\beta+1$ (by the induction hypothesis for "$\gamma^\dagger = 2\beta+1$"). By clause (C) and the induction

hypothesis we know:

$$r\restriction(2\beta+1)\Vdash_{P_{2\beta+1}}\text{ ``}p^n_*(2\beta+1)\geq p^*(2\beta+1)\geq p_{\xi+1}(2\beta+1)\geq p^n_\xi(2\beta+1)\text{''},$$

so by the $\aleph_1$-completeness of $\underset{\sim}{Q}_{2\beta+1}$, r exists. Lastly, for $\gamma=2\beta+1$, the nontrivial case is that 2β belongs to $\bigcup_{n<\omega}\mathrm{Dom}(p^n_\xi)$ and also to $\bigcup_{n<\omega}\mathrm{Dom}(p^n_*)$ hence $2\beta\in N\cap\alpha_0$. Again w.l.o.g. $\zeta=2\beta$, and by the hypothesis on r and $p^n,p^n_*(n<\omega)$:

$$r\restriction(2\beta)\Vdash_{P_{2\beta}}\text{ ``}\underset{\sim}{h}_{2\beta}(p^n_\xi(2\beta))=\underset{\sim}{h}_{2\beta}(p^n_*(2\beta))\text{''},$$

$$r\restriction(2\beta)\Vdash_{P_{2\beta}}\text{ ``}p^n_\xi(2\beta)\leq p^{n+1}_\xi(2\beta)\text{''},$$

$$r\restriction(2\beta)\Vdash_{P_{2\beta}}\text{ ``}p^n_*(2\beta)\leq p^{n+1}_*(2\beta)\text{''}.$$

So r forces that $p^n_\xi(2\beta),p^n_*(2\beta)$ have a least upper bound $p^n_\xi(2\beta)\wedge p^n_*(2\beta)$; and as $p^n_\xi(2\beta)\leq p^{n+1}_\xi(2\beta)$, and $p^n_*(2\beta)\leq p^{n+1}_*(2\beta)$ (i.e. r forces this), also $p^n_\xi(2\beta)\wedge p^n_*(2\beta)\leq p^{n+1}_\xi(2\beta)\wedge p^{n+1}_*(2\beta)$, hence by the $\aleph_1$-completeness of $\underset{\sim}{Q}_{2\beta}$, there is a $q(2\beta)$ such that $r\restriction(2\beta)\Vdash_{P_{2\beta}}\text{ ``}\bigwedge_{n<\omega}(p^n_\xi(2\beta)\wedge p^n_*(2\beta))\leq r^*(2\beta)\text{''}$ and $\emptyset\Vdash_{P_{2\beta}}\text{ ``}r^*(2\beta)\in\underset{\sim}{Q}_{2\beta}\text{''}$, so we are done.

Taking $\zeta=0,\gamma=\alpha_0$ in $(*)$, we see that the set $\{p^n_\xi,\ p^n_*:n<\omega\}$ has an upper bound which necessarily is a common upper bound to q_ξ, q_*, so as $\mathcal{I}_\alpha$ is an antichain, $q_*=q_\xi\in N$, so $\mathcal{I}_\alpha\cap N$ is pre-dense above p^*, and we finish. $\square_{1.1}$

1.2 Remark. The reason for including this is as follows. It was a consequence of the work on proper forcing that we can iterate $\aleph_1$-complete and $\aleph_1$-c.c. forcings together. So it was natural to ask the parallel for $\aleph_2$-complete and $\aleph_1$-complete with the $\aleph_2$-c.c. But we do not know how to iterate the second kind alone (and in general this is impossible since $\aleph_2$ will collapse). So it is reasonable to replace $\aleph_2$-c.c. by something stronger (here – clause (4) of the lemma). (Remember $p^n_*(2\beta)$ is $\emptyset$ when $2\beta\notin\mathrm{Dom}(p^n_*)$, and $h_{2\beta}(\emptyset)=0$). Of course, much better would be to find one condition unifying the two conditions - see Chapter XIV.

However as the interaction has no applications now, we shall not discuss it further (there are other tries at $\aleph_2$-c.c., see [Sh 80]).

Note also that the analogous lemma for $\aleph_1$-complete, $\aleph_1$-c.c. forcing holds, but now it has no application.

1.3 Claim. If $\diamondsuit_{\aleph_1}$ holds, then in 1.1 we can change the iteration to the usual $(< \aleph_2)$-support iteration and the conclusion still holds.

Proof. We let $N = \bigcup_{\xi<\omega_1} N_\xi$, $N_\xi \prec N$ where N_ξ are countable, increasing and continuous. By $\diamondsuit_{\aleph_1}$ there are for $\xi < \omega_1$, 2-place functions f_ξ from

$$Y_\xi \stackrel{\text{def}}{=} \{\langle \alpha, \beta\rangle : \beta, \alpha \in N_\xi, \alpha, \beta \text{ ordinals}\}$$

into ω_1, such that for every 2-place $f : \bigcup_{\xi<\omega_1} Y_\xi \to \omega_1$ the set $\{\xi : f \restriction Y_\xi = f_\xi\}$ is stationary. Repeat the proof of 1.1 but in the definition of the p_ξ's, we replace A), B), C) by:

A) if $\xi < \zeta < \omega_1$ and $2\beta \in N_\xi \cap \alpha_0$, then $p_\zeta(2\beta) = p_\xi(2\beta)$

B) $p_\xi \leq p_\zeta \in N \cap P_{\alpha_0}$ for $\xi \leq \zeta < \omega_1$

C) *If* $\xi < \omega_1$, and there are q and $p^i (i < \omega)$ such that:

(i) $q \leq p^0$, $q \in \mathcal{I}_\alpha$, and $p_\xi \leq p^0$

(ii) for $i < j < \omega, p^i \leq p^j$, moreover $\Vdash_{P_\gamma}$ "$p^i(\gamma) \leq p^j(\gamma)$" for each $\gamma \in \mathrm{Dom}(p^i)$

(iii) $p^{i+1}_\xi \restriction (2\beta) \Vdash_{P_{2\beta}}$ "$\underline{h}_{2\beta}(p^i(2\beta)) = f_\xi(i, 2\beta)$" for $2\beta \in N_\xi \cap \alpha_0$,

then there are q, $p^i_\xi (i < \omega)$ satisfying (i), (ii), (iii), such that:

for $i < \omega$ and $\gamma \in \mathrm{Dom}(p^i_\xi) \setminus \{2\beta : \beta \in N_\xi \text{ and } 2\beta < \alpha_0\}$, we have $\Vdash_{P_\gamma}$ "$p^i_\xi(\gamma) \leq p_{\xi+1}(\gamma)$"

In the end we define p^*, $\mathrm{Dom}(p^*) = \alpha_0 \cap N, p^*(\gamma)$ is $p_\xi(\gamma)$ for any even γ, and for any ξ such that $\gamma \in N_\xi$, and it is any upper bound of $\{p_\xi(\gamma) : \xi < \omega_1\}$ for γ odd. $\square_{1.3}$

1.4 Remark. See more in [Sh:186], [Sh:587].

§2. Chain Conditions Revisited

We here deal again with problems like those of §1 from VII, but allowing the continuum to increase somewhat. Here, κ is a fixed cardinal.

2.1 Definition. P satisfies the κ-p.i.c. (κ-properness isomorphism condition) provided the following holds, for λ large enough:

Suppose $i < j < \kappa$, $\kappa \in N_i \prec (H(\lambda), \in, <_\lambda)$, ($<_\lambda$ is a well ordering of $H(\lambda)$) and $\kappa \in N_j \prec (H(\lambda), \in, <_\lambda)$, $||N_i|| = ||N_j|| = \aleph_0$, $P \in N_i \cap N_j, i \in N_i, j \in N_j$, $N_i \cap \kappa \subseteq j, N_i \cap i = N_j \cap j$, $p \in P \cap N_i$, h an isomorphism from N_i onto N_j, $h{\restriction}(N_i \cap N_j) =$ the identity and $h(i) = j$.

Then there is a $q \in P$, such that:

(a) $p, h(p) \leq q$, and for every maximal antichain $\mathcal{I} \subseteq P$, $\mathcal{I} \in N_i$ we have that $\mathcal{I} \cap N_i$ is pre-dense above q, and similarly for $\mathcal{I} \in N_j$ (but clause (b) below implies that this follows from the rest of (a))

(b) for every $r \in N_i \cap P$ and $q^\dagger$ such that $q \leq q^\dagger \in P$ there is a $q'', q^\dagger \leq q'' \in P$ such that $[r \leq q''$ iff $h(r) \leq q'']$; equivalently;

(a'+b') letting $\underset{\sim}{G}$ be the P-name of the generic set

$$q \Vdash_P \text{“}(\forall r \in N_i \cap P)(r \in \underset{\sim}{G} \text{ iff } h(r) \in \underset{\sim}{G})\text{”},$$

$$q \Vdash_P \text{“}p \in \underset{\sim}{G}\text{”},$$

and q is $(N_i, P) -$ generic.

2.2 Claim.

1) If Definition 2.1 holds for P, $H(\lambda), <_\lambda$, *then* it holds for any $\lambda_1 > 2^\lambda$ and well ordering $<_1$ of $H(\lambda_1)$ (in fact, we can omit the well ordering).

2) If Definition 2.1 holds for $P, H(\lambda), <_\lambda$ *then* it holds for some $\lambda_1, <_1$ such that $\lambda_1 \leq (\mu + |P|)^+$, where μ is the number of maximal antichains of P (w.l.o.g. $P \in H(|P|^+)$).

Proof. Similar to the proof in III §1,2. $\square_{2.2}$

2.3 Lemma. Suppose $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$ where κ is regular and P satisfies the κ-p.i.c. *Then* P satisfies the κ-chain condition.

Proof. Let $p_i \in P$ for $i < \kappa$ be given. Let $(H(\lambda), \in, <_\lambda)$ be as in Definition 2.1. Find, for $i < \kappa$, models N_i such that $i, p_i \in N_i \prec (H(\lambda), \in, <_\lambda)$, $||N_i|| = \aleph_0$. Define $f(i) \stackrel{\text{def}}{=} \text{Sup}(N_i \cap i)$, so $\text{cf}(i) > \aleph_0 \Rightarrow f(i) < i$.
By Fodor's Lemma, for some γ the set $\{i : f(i) = \gamma\}$ is stationary. As $(\forall \mu < \kappa)$ $\mu^{\aleph_0} < \kappa$ and κ is regular, for some $A \subseteq \gamma$, $S = \{i : N_i \cap i = A\}$ is stationary. Similarly, we can assume that for some B, $i \neq j \in S \Rightarrow N_i \cap N_j = B$ (see the proof of V1.5A). Also $C = \{\delta < \kappa : (\forall i < \delta)(N_i \cap \kappa \subseteq \delta)\}$ is closed unbounded, so $S_1 = S \cap C$ is stationary. Now there are κ models $(N_i, p_i, i, a)_{a \in B}$ where p_i, i, a $(a \in B)$ are individual constants and $\kappa > 2^{\aleph_0}$, and the number of isomorphism types of such models is $2^{\aleph_0}$, so for some $i < j$ there is an isomorphism $h : N_i \to N_j$ (onto), $h(p_i) = p_j, h{\restriction}B =$ the identity. Now apply Definition 2.1 for N_i, N_j, h, p_i. $\square_{2.3}$

2.4 Lemma. Suppose $\overline{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha_0, j < \alpha_0 \rangle$ is an iteration with countable support. Suppose further
(*) Q_α satisfies the κ-pic (for each $\alpha < \alpha_0$) and κ is regular.
Then 1) If $\alpha_0 < \kappa$, P_{α_0} satisfies the κ-p.i.c.
2) If $\alpha_0 \leq \kappa$, P_{α_0} satisfies the κ-chain condition, provided that

$$(\forall \mu < \kappa)(\mu^{\aleph_0} < \kappa).$$

Proof. 1)Let $(H(\lambda), \in, <_\lambda)$, h, i, j be as required for Definition 2.1. Let $\underset{\sim}{G}_\beta$ be the P_β-name of the generic subset of P_β. Without loss of generality $\overline{Q} \in N_i \cap N_j$ (because $P_{\alpha_0} \in N_i \cap N_j$), hence $\alpha_0 \in N_i \cap N_j$; as $\alpha_0 < \kappa$, $N_i \cap \alpha_0 = N_j \cap \alpha_0$, (read Definition 2.1) and the proof is now similar to the proof of properness.

We prove by induction on $\xi \leq \alpha_0$ that:

(**) *for every* $\zeta < \xi$, $\zeta \in N_i \cap \alpha_0$, $\xi \in N_i \cap \alpha_0$ and $p \in N_i \cap P_{\alpha_0}$ or even just a P_ξ-name $\underset{\sim}{p}$ of such a condition, $q_\zeta \in P_\zeta$, $q_\zeta \geq p{\restriction}\zeta$, $q_\zeta \geq h(p){\restriction}\zeta$, such that q_ζ is (N_i, P_ζ)-generic and (N_j, P_ζ)-generic and $q_\zeta \Vdash_{P_\zeta}$ "$(\forall r \in N_i \cap P_\zeta)$ $[r \in \underset{\sim}{G}_\zeta$ iff

$h(r) \in \underset{\sim}{G}_\zeta]$", *there is* a $q_\xi \in P_\xi$ such that $q_\xi \restriction \zeta = q_\zeta$, $q_\xi \geq p \restriction \xi$, $q_\xi \geq h(p) \restriction \xi$; q_ξ is (N_i, P_ξ)-generic and (N_j, P_ξ)-generic and $q_\xi \Vdash_{P_\xi}$ "$(\forall r \in N_i \cap P_\xi)$ $[r \in \underset{\sim}{G}_\xi$ iff $h(r) \in \underset{\sim}{G}_\xi]$".

Note that $(**)$ with p an element implies the apparently more general version with p being $\underset{\sim}{p}$, a P_ξ-name of a member of $N \cap P_{\alpha_0}$, such that $q_\xi \Vdash_{P_\xi}$ " for some $p \in N_i \cap P_{\alpha_0}$, $\underset{\sim}{p}[\underset{\sim}{G}_{P_\xi}] = p$, and $p \restriction \zeta \leq q$ and $h(p) \restriction \zeta \leq q$". (Used in the inductive proof for ξ limit.)

For ξ a successor, we first, by the induction hypothesis, define $q_{\xi-1}$ as required (necessarily $\xi - 1 \in N_i \cap N_j$); then notice that, by the induction hypothesis, if we force with $P_{\xi-1}$ and get a generic $G_{\xi-1} \subseteq P_{\xi-1}$ and $q_{\xi-1}$ is in this generic set, then h is still an isomorphism etc, so we can use the κ-p.i.c. on $\underset{\sim}{Q}_{\xi-1}[G_{\xi-1}]$.

For $\mathrm{cf}(\xi) = \aleph_0$, we work as in the proof of properness (III 3.2), using the induction hypothesis. Noticing that $q_\xi \Vdash_{P_\xi}$ "$(\forall r \in P_\xi)$ $(r \in \underset{\sim}{G}_\xi$ iff $h(r) \in \underset{\sim}{G}_\xi)$" makes no problem in the limit, we do not have to take special care. For cf $(\xi) \geq \aleph_1$ the proof is similar but easier.

2) Trivial by 1) and 2.3 and the proof of III 4.1. $\square_{2.4}$

2.5 Lemma. If P is proper and $\kappa > |P|$ *then* P satisfies the κ-p.i.c.

Proof. We start with $i, j, N_i, N_j, (H(\lambda), \in, <_\lambda)$, p and κ as in 2.1. Remember that $P \in N_i \cap N_j$. In $(H(\lambda), \in, <_\lambda)$ there is a $<_\lambda$-first one-to-one function g from $|P|$ onto P, so $g \in N_i \cap N_j$. Also as $|P| \in N_i \cap N_j$, by the assumption on N_i, N_j, κ from Definition 2.1, $|P| < i, j$ and hence $N_i \cap |P| = N_j \cap |P|$. Hence (using the function g), $N_i \cap P = N_j \cap P$ and so h is the identity on $P \cap N_i$. Now it may well be that there is an $\mathcal{I} \in N_i$, a pre-dense subset of P, which does not belong to N_j. But if q is (N_j, P)-generic then for any $\mathcal{I} \in N_i$ a pre-dense subset of P, $h(\mathcal{I}) \in N_j$ is a pre-dense subset of P and $\mathcal{I} \cap N_i = \mathcal{I} \cap (P \cap N_i) = h(\mathcal{I}) \cap (P \cap N_j)$, hence $\mathcal{I} \cap N_i = h(\mathcal{I}) \cap N_j$ is pre-dense above q. So q is (N_i, P)-generic too and we can use the properness of P to define q as required (with clause (b) of Definition 2.1 being trivial). $\square_{2.5}$

2.6 Definition. The κ-p.i.c* is defined similarly to κ-p.i.c, but we add one assumption:
for any $a \in N_i$ there is a sequence $\langle a_\alpha : \alpha < \kappa \rangle$ in $N_i \cap N_j$ such that $a_i = a$ (this implies the corresponding condition on N_j); equivalently N_i is the Skolem hull of $(N_i \cap N_j) \bigcup \{i\}$, and N_j is the Skolem Hull of $(N_i \cap N_j) \bigcup \{j\}$.

2.7 Lemma.

1) The κ-p.i.c. implies the κ-p.i.c*.

2) Lemmas 2.2–2.5 hold for κ-p.i.c*, (and we call them 2.2*,..., respectively).

3) P satisfies the κ-p.i.c* if P satisfies the conditions from [Sh:80] which are:
   a) P is $\aleph_1$-complete.
   b) for any $p_i \in P$ $(i < \kappa)$ there are $p_i^\dagger \in P, p_i \leq p_i^\dagger$ and pressing down functions $F_n : \kappa \setminus \{0\} \to \kappa$, (i.e., $F_n(\alpha) < \alpha$) for $n < \omega$, such that: if $i < j$ and $\bigwedge_n F_n(i) = F_n(j)$, *then* $p_i^\dagger, p_j^\dagger$ have a least upper bound in P, called $p_i^\dagger \wedge p_j^\dagger$.

2.7A Remark. So 2.7(2), (3), 2.4*, 2.4* give an alternative proof of [Sh:80], for the case $\alpha_0 \leq \kappa$. In fact, 2.4* holds for α_0 not necessarily $< \kappa$, when each Q_i is $\aleph_1$-complete, and this gives an alternative axiom for [Sh:80].

Proof. 1) Trivial.

2) The least trivial part is 2.3. Here the extra assumption is the least obvious. So, by induction, we define $N_i^k (k < \omega)$. For $k = 0$ we choose N_i^0 such that: $\{p_i, i\} \in N_i^0 \prec (H(\lambda), \in, <_\lambda), ||N_i^0|| = \aleph_0$. Suppose N_i^k (for each i) has been defined and let $\{a_{i,e}^k : e < \omega\}$ enumerate the members of N_i^k. We choose N_i^{k+1} such that $N_i^k \in N_i^{k+1} \prec (H(\lambda), \in, <_\lambda)$, $||N_i^{k+1}|| = \aleph_0$, and $\langle \langle a_{j,e}^k : j < \kappa \rangle : e < \omega \rangle \in N_i^{k+1}$.

Now let $N_i = \bigcup_{k<\omega} N_i^k$ and proceed as in 2.3.

3) Let N_i, N_j, h be as in the definition of κ-p.i.c*, $p \in N_i$. Let $\langle \mathcal{I}_n : n < \omega \rangle$ be a list of all maximal antichains of P which belong to N_i. For every $a \in N_i$ let $\mathrm{seq}_a \in N_i \cap N_j$ be a sequence of length κ such that $a = \mathrm{seq}_a(i)$. We define,

by induction, conditions p_n:

$p_0 = p$,

if p_{2n} is defined, choose $p_{2n+1} \geq p_{2n}$, such that $p_{2n+1} \in P \cap N_i$, $p_{2n+1} \geq$ (some $q_n \in \mathcal{I}_n$),

if p_{2n+1} is defined, consider $\mathrm{seq}_{p_{n+1}} = \langle r_{\alpha,n} : \alpha < \kappa \rangle$. We can assume w.l.o.g. $(\forall \alpha < \kappa)\ r_{\alpha,n} \in P$. So there are $\langle F_n : n < \omega \rangle$, $\langle r^\dagger_{\alpha,n} : \alpha < \kappa \rangle$ as mentioned in 2.7(3)(b). As $\langle r_{\alpha,n} : \alpha < \kappa \rangle \in N_i \cap N_j$ and $<_\lambda$ is a well ordering, we can assume that $\langle F_m : m < \omega \rangle$, $\langle r^\dagger_{\alpha,n} : \alpha < \kappa \rangle \in N_i \cap N_j$. Let $p_{2n+2} = r^\dagger_{i,n}$ (remember $p_{2n+1} = r_{i,n} \leq r^\dagger_{i,n}$). Notice that $h(r_{i,n}) = r_{j,n}$, and by the choice of N_i, N_j we have $\bigwedge_m F_m(i) = F_m(j)$. So $r^\dagger_{i,n} = p_{2n+2}$ and $r^\dagger_{j,n} = h(p_{2n+2})$ have a least upper bound $q_{2n+2} \stackrel{\mathrm{def}}{=} p_{2n+2} \wedge h(p_{2n+2})$. In the end:

$$p_2 \leq p_4 \leq p_6 \leq \dots$$

$$h(p_2) \leq h(p_4) \leq h(p_6) \leq \dots$$

Now $q_{2n+2} \leq q_{2n+4}$, as they are *least* upper bounds. So by $\aleph_1$-completeness there is a q, $\bigwedge_n q \geq q_n$. Now q is as required. $\square_{2.7}$

2.8 Lemma. 1) All forcings used in VII §3 (= applications of Axiom II) satisfy the $\aleph_2$-p.i.c*, (but of course Levy$(\aleph_1, < \kappa)$ if $\kappa > \aleph_2$)

2) Moreover for each application we can find a forcing notion doing all the assigments of this kind present in the current universe and satisfies the $\aleph_2$-p.i.c*, and in fact all are $(< \omega_1)$-proper and $\mathbb{D}$-complete for a simple $\aleph_0$-completeness system $\mathbb{D}$.

Proof. We elaborate two of them leaving rest to the reader.

application F: Let $\overline{T} = \langle (T^\alpha, f^*) : \alpha < \alpha^* \rangle$ be a sequence of pairs (T, f^*), T an Aronszajn $\aleph_1$-tree, $f^* : T \to \omega$ satisfies the antecendent of $\otimes$ in VII 3.8 (in the main case: listing all of such pairs). We define a forcing notion $P_{\overline{T}}$. A member p of $P_{\overline{T}}$ has the form $p = (i, w, \overline{g}, \bar{C}, B)$, where:

(i) $w \subseteq \alpha^*$ is countable,

(ii) $i < \omega_1$, $\bar{C} = \langle C_\alpha : \alpha \in w \rangle$, C_α the characteristic function of a closed subset of $i+1$ to which i belongs

(iii) $\bar{g} = \langle g_\alpha : \alpha \in w \rangle$, g_α a function from $T^\alpha_{\leq i} = (T^\alpha)_{\leq i}$ to ω such that (g_α, C_α) is as in VII 3.11.

Notation: For finite $u \subseteq \alpha^*$ we letting T^u be the disjoint union of $\{T^\alpha : \alpha \in u\}$ (i.e. make them disjoint).

(iv) B is a countable family, for each member I for some finite $u = u(I) \subseteq \omega$, $(g^{[u]}, \bigcap\limits_{\alpha \in u} C_\alpha, \{I\}) \in P_{(T^u, f^*_u)}$

application C: Use product with countable aupport. $\square_{2.8}$

2.8A Remark. We do not investigate the connection between κ-p.i.c, and κ-e.c.c. However, κ-e.c.c. was introduced to deal with the case in which we iterate forcings which are $\mathbb{D}$-complete for some $\mathbb{D}$. We introduce the κ-p.i.c. to deal with the case in which we want to get $V \models$ "$\aleph_2 = 2^{\aleph_0} < 2^{\aleph_1}$". So we use an iteration of length ω_2 where each iterand does not add reals. On the other hand, κ-p.i.c.* seems to replace κ-p.i.c. totally.

Note that the property of being κ-p.i.c. *essentially* (but seemingly not formally) implies properness.

2.9 Claim. If $\alpha_0 < \kappa, \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \alpha_0 \rangle$ is a CS iteration, $\alpha_0 < \kappa$, each Q_α satisfies the κ-p.i.c* and $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$; *then*, in $V^{P_{\alpha_0}}$ we have $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$ and $2^{\aleph_0} < \kappa$.

Proof. Trivial.

§3. The Axioms Revisited

3.1 Thesis. Proper forcing is efficient for getting models in which $2^{\aleph_0} \leq \aleph_2$ and important things it gets are such universes of set theory which in addition satisfy conditions of the form "for every $A \subseteq \omega_1 \ldots$". The reason is that we, at present, can iterate only ω_2 times without collapsing $\aleph_2$ (of course, if we are

interested in c.c.c. forcing, we can increase this to $2^{\aleph_0}$). So we have a division to four main cases we can reasonably handle:

I) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$,

II) $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$,

III) $2^{\aleph_0} = \aleph_2 < 2^{\aleph_1}$,

IV) $2^{\aleph_0} = \aleph_1, \aleph_2 < 2^{\aleph_1}$.

3.2 Discussion. We have dealt in VII §2 with I), II), and have the appropriate axioms. Also in previous works we dealt with mainly I), II); sometimes we get more for free: e.g. in Laver [L1] (consistency of Borel conjecture), the value of $2^{\aleph_1}$ was immaterial.

For getting such models with some extra properties we iterate ω_2 times. At some stages we increase $2^{\aleph_1}$, or add "a few" reals (and preserve CH meanwhile (see 2.9) if $\aleph_2$ in the end is a given inaccessible κ, "few" can be interpreted as $< \kappa$) (according to the case – for I, III each time we add a few reals, for III, IV we start by adding many $A \subseteq \omega_1$). In other stages (for I and II) we consider $A \subseteq \omega_1$ and force "for" "it" some B. If we want III or IV, we consider all $A \subseteq \omega_1$ of a certain kind and simultaneously add for each such A an appropriate B. Sometimes we want the forcing to preserve something (e.g. ${}^{\omega}\omega$-boundedness, or a Ramsey ultrafilter etc.) but we shall not deal with those things here, for the number of axioms arising is not bounded.

For other possibilities of $2^{\aleph_0}, 2^{\aleph_1}$ the situation is not clear. On some consistency results see Abraham and Shelah [AbSh:114]. We get there results with $2^{\aleph_0} > \aleph_2$ but the results are on $A \subseteq \omega_1$. Resolving the problematic cases, first of all $2^{\aleph_0} \geq \aleph_3$ seems to me a major problem; we shall discuss this later.
Note: in case IV for example, we are restricted by our iterations being of length ω_2.

Generally, for getting the consistency of stronger axioms we have to assume the consistency of ZFC+ some large cardinal. Here we concentrate on assuming the consistency of ZFC only.

3.3 Notation. φ, ψ are first order sentences (in the language with $=, \in$ and one predicate P).

M is a model with universe ω_1 and language of cardinality $\leq \aleph_0$, $M = (|M|, \ldots, R_i \ldots)_{i<i_0\leq\omega}$. Let N denote an expansion of M, again with $\leq \aleph_0$ relations and φ a first order sentence in N's language.

3.4 Lemma. If ZFC is consistent, then so are ZFC + each one of the following axiom schema (separately):

1) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2+$

   Axiom Schema. I_b: For each (ψ, φ), for every M with universe $\subseteq H(\aleph_1)$ such that $(H(\aleph_1), \in, M) \models \psi$, there is an expansion N of M such that $N \models \varphi$, provided that

$(*)_{I_b}$: the following is provable from ZFC + G.C.H.: *if* $(H(\aleph_1), \in, M) \models \psi$ *then* for some proper forcing notion P satisfying the $\aleph_2$-p.i.c.*, $|P| \leq \aleph_2$, $\Vdash_P$ "there is an expansion N of M, N satisfying φ and $2^{\aleph_0} = \aleph_1$".

2) $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$ (+ G.C.H. if you want) +

   Axiom Schema. II_b: For each (ψ, φ), for every M with universe $\subseteq H(\aleph_1)$ such that $(H(\aleph_1), \in, M) \models \psi$, there is an expansion N of M such that $N \models \varphi$, provided that

$(*)_{II_b}$ the following is provable from ZFC + G.C.H.: *if* $(H(\aleph_1), \in, M) \models \psi$ *then* for some $(< \omega_1)$-proper P, $\mathbb{D}$-complete for some simple $\aleph_0$-completeness system $\mathbb{D}$, satisfying the $\aleph_2$-p.i.c.*, $|P| \leq \aleph_2$ and $\Vdash_P$ " there is an expansion N of M satisfying φ". We can use here $H(\aleph_2)$ instead of $H(\aleph_1)$.

3) $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1}$ = any cardinality of cofinality $\geq \aleph_2+$

   Axiom Schema. IV_b: For each (ψ, φ), for every M with universe $\subseteq H(\aleph_1)$ such that $(H(\aleph_1), \in, M) \models \psi$, there is an expansion N of M such that $N \models \varphi$, provided that

$(*)_{IV_b}$ the following is provable in ZFC + CH: there is a $(< \omega_1)$-proper forcing notion P, $\mathbb{D}$-complete for some simple $\aleph_0$-completeness system $\mathbb{D}$, satisfying the

$\aleph_2$-p.i.c.*, $|P| \leq 2^{\aleph_1}$, such that $\Vdash_P$ "for every $M \in V$, if $(H(\aleph_1)^V, \in, M) \vDash \psi$, then there is (in V^P) an expansion N of M satisfying φ".

4) $2^{\aleph_0} = \aleph_2 < 2^{\aleph_1}$ = anything of cofinality $\geq \aleph_2+$

Axiom Schema. III_b: For each (ψ, φ), for every M with universe $\subseteq H(\aleph_1)$ such that $(H(\aleph_1), \in, M) \vDash \psi$, there is an expansion N of M satisfying φ, provided that

$(*)_{III_b}$: the following is provable from ZFC + CH: there is a proper forcing P satisfying the $\aleph_2$-p.i.c*., $|P| \leq 2^{\aleph_1}$ such that:

$\Vdash_P$ "CH and for every $M \in V$ with universe $\subseteq H(\aleph_1)$,
if $(H(\aleph_1)^V, \in, M) \vDash \psi$, then
there is (in V^P) an expansion N of M satisfying φ".

Proof. Straightforward by now, when we use the relevant theorems on forcing. $\square_{3.4}$

3.4A Remarks on 3.4(3).

A) Notice that we use CH instead of G.C.H, and we here put first $\exists P$ and then $\forall M$. We can also assume that P is an iteration with countable support satisfying the above conditions.

B) If $2^{\aleph_1}$ is such that $(\forall \mu < 2^{\aleph_1})\ [\mu^{\aleph_0} < 2^{\aleph_1}]$, we can replace $H(\aleph_1)$, by $H(2^{\aleph_1})$.

Of course, as we use "larger" cardinals κ to be collapsed to $\aleph_2$, we can get stronger axioms:

3.5 Lemma. If "ZFC + $\exists$ an inaccessible cardinal" is consistent, then so are "ZFC + each of the following" (separately).

1) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2+$

Axiom Schema. I_a: like I_b, but we replace (in $(*)_{I_b}$ the demand) "... P, satisfying the $\aleph_2$-p.i.c.*", by "... P, $|P|$ < first strongly inaccessible".

2) G.C.H. +

Axiom Schema. II_a: like II_b, but we replace "$\ldots, P$ satisfies the $\aleph_2$-p.i.c.*, $|P| \leq \aleph_2$" by "$\ldots, |P| <$ first strongly inaccessible".

3) $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} =$ first inaccessible +

Axiom Schema III_a. Like III_b but we replace "$\ldots P$, P satisfies the $\aleph_2$-p.i.c.*, $|P| \leq 2^{\aleph_1}$" by "$\ldots P$, $|P| <$ first inaccessible".

4) $2^{\aleph_0} = \aleph_2 = 2^{\aleph_1} =$ first inaccessible +

Axiom Schema IV_a. like IV_b but we replace "$\ldots, P$ satisfying the $\aleph_2$-p.i.c.*" by "$\ldots P, |P| <$ first inaccessible".

3.5A Remark. 1) Our use here of I_a is not the same as in VII §2, but we can take their union as I_a.

2) We can replace in this section "$\mathbb{D}$ simple $\aleph_0$-completeness system" by $\mathbb{D} \in V^{P_0}$ is a 2-completeness system (see §4).

3) We can replace in 3.5 "$|P| < \kappa$" by "$P \vDash \kappa$-p.i.c.*".

Proof. Again easy.

Remark. We can of course try more axioms, but those mentioned above seem to suffice.

§4. More on Forcing Not Adding ω-Sequences and on the Diagonal Argument

4.1 Discussion. We have proved in VII §3, application D, that CH $\not\to \Phi^{\aleph_0}_{\aleph_1}$ whereas in Devlin and Shelah [DvSh:65] (or see somewhat more [Appendix §1]) it is shown that CH $\to \Phi^2_{\aleph_1}$. But Axiom II does not prove the consistency of "not $\Phi^3_{\aleph_1}$" with CH. More generally, we can ask whether we can make the condition on the Q_α (in V 7.1, 7.2) weaker.

We saw no point in trying to weaken the assumption "α-proper for every $\alpha < \omega_1$" to e.g. ω-proper, as it seemed to us that every natural example of forc-

ing will satisfy it (truly, sometimes we want to destroy some stationary subsets of ω_1 and under reasonable conditions we can succeed, for example, the proof the consistency of "the closed unbounded filter on ω_1 is precipitous" (see Jech, Magidor, Mitchell and Prikry [JMMP]), but we can amalgamate such a proof with our constructions). The hard part seems to be the $\mathbb{D}$-completeness, where $\mathbb{D} \in V$ or $\mathbb{D}$ is simple, again, do not seem to be a serious obstacle to anything; but the requirement that any finitely many possibilities are compatible (i.e. $\mathbb{D}_{\langle N,P,p\rangle}$ generates a filter) seemed to be an obstacle – e.g. to the natural forcing for making $\Phi^3_{\aleph_1}$ false. Remember, $\mathbb{D}_{\langle N,\ldots\rangle}$ was a family of subsets of $\mathcal{P}(N)$ with the finite intersection property. We shall try to replace this requirement by the requirement that the intersection of any two is nonempty. As an application we get consistency with CH of variants of $\Phi^3_{\aleph_1}$ (we hope there will be more). Note that we replace here $\mathbb{D}_{\langle N,P,p\rangle}$ by another equivalent formulation.

Note another drawback, which at present is only aesthetical, the $\mathbb{D}$-completeness is not preserved; i.e. we have not stated a natural condition, preserved by CS iterations, that implying that no ω-sequence of ordinals is added. Note that we do not use the full generality of Definition 4.2. We treat it in 4.14—4.22. A minor difference with Chapter V is that we use countable subsets of some λ instead of $H(\lambda)$, but this is just a matter of presentation.

For another point of view and more results see [Sh:177] or better yet XVIII §1, §2 .

4.2 Definition. 1) Let λ be a cardinal. $\mathbb{D}$ is called a $(\lambda, 1, k)$-system (or completeness system) if:

(i) $\mathbb{D}$ is a function (where $\mathbb{D}(x)$ may be written as $\mathbb{D}_x$),

(ii) $\mathbb{D}_{\langle a,<^*,x,p\rangle}$ is well defined iff $a \in \mathcal{S}_{\aleph_0}(\lambda)$, $<^*$ is a partial order of a, $x \subseteq a \times a$ and $p \in a$,

(iii) $\mathbb{D}_{\langle a,<^*,x,p\rangle}$ is a family of subsets of $\mathcal{P}(a)$, the intersection of any i of them is nonempty when $i < 1 + k$.

2) We say that $\mathbb{D}$ is a k-completeness system if for some λ it is $(\lambda, 1, k)$-completeness system.

4.3 Definition. A forcing notion P is called $(\mathcal{D}, \mathbb{D})$-complete, where $\mathcal{D}$ is a filter over $\mathcal{S}_{\aleph_0}(\lambda)$ and $\mathbb{D}$ a $(\lambda, 1, k)$-system, if $\lambda \geq 2^{|P|}$, P is isomorphic to $P^* = (P^*, \leq^*)$, $P^* \subseteq \lambda$, and for every $p \in P^*$ and $\mathcal{I}_\alpha = \{p_i^\alpha : i < i_\alpha \leq \lambda\}$ pre-dense subsets of P^* (for $\alpha < \lambda$), for some $x \subseteq \lambda \times \lambda$, the family of all $a \in \mathcal{S}_{\aleph_0}(\lambda)$ that satisfy the following, is in $\mathcal{D}$:

$(*)$ the following contains, as a subset, a member of $\mathbb{D}_{\langle a, <^* \restriction a,\, x \restriction a \times a,\, p \rangle}$:

$\{G \subseteq a$: 1) for every $\alpha \in a$, for some $i \in a \cap i_\alpha$, $p_i^\alpha \in G$
2) $(\exists q \in P^*)(\forall r \in G)(r \leq^* q)$
3) $p \in G\}$.

If $\mathcal{D} = \mathcal{D}_{<\aleph_1}(\lambda)$ (see V §2) we may omit it and write $\mathbb{D}$ instead of $(\mathcal{D}, \mathbb{D})$. If $\mathcal{D} = \mathcal{D}_{<\aleph_1}(\lambda) + S$, where $S \subseteq \mathcal{S}_{<\aleph_1}(\lambda)$ is stationary, then we may write S instead of $\mathcal{D}$.

4.3A Remark.

1) In Definitions 4.2, 4.3 we can replace λ by any set of this cardinality or by any larger cardinality.
2) We omit $\mathcal{D}$ in 4.5 below from laziness only.
3) In 4.4 and 4.3 of [Sh:b] we use $\bar{a} \in \mathcal{S}^\alpha_{\aleph_0}(\lambda)$ for some fixed α, but we use only the case $\alpha = 1$ in Theorem 4.5; to help the reader we delay this generality to a later part of this section.
4) If $\mathcal{D}^1_{\lambda_0}$ is a fine normal filter on $\mathcal{S}_{<\aleph_1}(\lambda_0)$, and $\lambda_1 \geq \lambda_0$, we let $\mathcal{D}^1_{\lambda_1}$ be the fine normal filter on $\mathcal{S}_{<\aleph_1}(\lambda_1)$ generated by

$$\left\{\{a \in \mathcal{S}_{<\aleph_1}(\lambda_1) : a \cap \lambda_0 \in X\} : X \in \mathcal{D}^0_{\lambda_0}\right\}.$$

5) If $S \subseteq \mathcal{S}_{<\aleph_1}(\lambda)$ is stationary we may write S instead of $\mathcal{D}_{<\aleph_1}(\lambda) + S$.
6) Instead of $\mathcal{I}_\alpha$ for $\alpha < \lambda$ we can use $\mathcal{I}_\alpha$ for $\alpha < \alpha^*$, $\alpha^* \leq \lambda$.

4.3B Fact. 1) In Definition 4.3, any choice of $(P^*, \leq^*)$ gives an equivalent definition. Also we can increase λ in a natural way.

2) Definitions 4.3 and 4.4 (simplicity) are compatible with the definition of completeness systems from V 5.2, 5.3, 5.5.

Specifically:

(A) For a forcing notion P, $k \leq \aleph_1$ and a family $\mathcal{E}$ of subsets of $\mathcal{S}_{<\aleph_1}(\mu)$, the following are equivalent:

(i) For some $\lambda \geq \mu$, P is $(\mathcal{D}_{<\aleph_1}(\lambda) + \mathcal{E}, \mathbb{D})$-complete for some simple $(\lambda, k, 1)$-completeness system $\mathbb{D}$ in the sense of 4.3, 4.4 below, where $\mathcal{D}_{<\aleph_1}(\lambda) + \mathcal{E}$ is the fine normal filter on $\mathcal{S}_{<\aleph_1}(\lambda)$ generated by

$$\{\{a \in \mathcal{S}_{<\aleph_1}(\lambda) : a \cap \mu \in X\} : X \in \mathcal{E}\}.$$

(ii) P is $(\mathcal{E}, \mathbb{D})$-complete for some simple k-completeness system in the sense of V §5.

(B) For a forcing notion P, $k \leq \aleph_1$, $\mathcal{E}$ a family of subsets of $\mathcal{S}_{<\aleph_1}(\mu)$, and a subuniverse V_0 (V_0 a transitive sub-class of V containing all ordinals and being a model of ZFC) such that $\mathcal{S} \in V_0$, the following are equivalent:

(i) For some $\lambda \geq \mu$ and $(\lambda, k, 1)$-completeness system $\mathbb{D} \in V_0$, the forcing notion P is $(\mathcal{D}_{<\aleph_1}(\lambda) + \mathcal{E}, \mathbb{D})$-complete in the sense of V §5.

(ii) For some $\lambda \geq \mu$ and k-complete system $\mathbb{D}$ which is almost simple over V_0, the forcing notion P is $(\mathcal{E}, \mathbb{D})$-complete in the sense of V §5.

Proof. Straightforward. $\square_{4.3B}$

4.4 Definition. We call $\mathbb{D}$ simple if for some first order formula $\psi(v, u)$, $\mathbb{D}_{\langle a, <^*, x, p\rangle} = \{\{G \subseteq a : p \in G, \langle a \bigcup \mathcal{P}(a \times a), \in, \lceil a, <^*, x, p\rangle \models \psi(G, u)\} : u \subseteq a\}$, ($\psi$ can have a countable sequence of ordinals as a parameter).

4.5 Theorem. If $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \alpha_0\rangle$ is an iteration with countable support and each $\underset{\sim}{Q}_\alpha$ is β-proper, for every $\beta < \omega_1$ and $\mathbb{D}^\alpha$-complete for some $(\lambda_\alpha, 1, 2)$-system, $\mathbb{D}^\alpha \in V$ (possibly $\mathbb{D}^\alpha, \lambda_\alpha$ are actually P_α-names $\underset{\sim}{\mathbb{D}}^\alpha, \underset{\sim}{\lambda}_\alpha$ but it does not matter), *then* forcing with P_{α_0} does not add any new ω-sequences of ordinals.

Note:

4.6 Claim. Any simple system in V^{P_α} (where P_α is a forcing notion adding no new ω-sequences) is in V.

Note:

4.6A Remark. Every Q_α is ${}^\omega\omega$-bounding because it is $\mathbb{D}^\alpha$-complete, which implies it does not add (to V^{P_α}) reals and even does not add ω-sequences of ordinals.

Proof of 4.5. We prove some claims and then the theorem becomes obvious (Claim 4.10 is the heart of the matter).

4.7 Definition. Let $\overline{A} = \langle A_i : i \leq \beta \rangle$, β countable, each A_i is a countable set of ordinals, $A_i (i \leq \beta)$ is (strictly) increasing and continuous. For $\xi \leq \zeta, \xi \in A_0, \zeta \in A_0, \overline{A}$ as above, we define when $\overline{A}$ *is long for* (ξ, ζ), by induction on ζ:

Case (i). $\zeta = \xi$: $\overline{A}$ is long for (ξ, ζ) (under the assumptions above) if $\beta > 0$.

Case (ii). ζ a successor, $\zeta > \xi$: $\overline{A}$ is long for (ξ, ζ) if for some $\beta^\dagger < \beta$ we have that $\langle A_i : i \leq \beta^\dagger \rangle$ is long for $(\xi, \zeta - 1)$.

Case (iii). ζ a limit: $\overline{A}$ is long for (ξ, ζ) if there are $\beta_i (i \leq \omega^2)$ (the ordinal square of ω) such that: $i < j \leq \omega^2 \Rightarrow \beta_i < \beta_j$; $\beta_{\omega^2} + \omega + 1 < \beta$; and for every i and (ξ_1, ζ_1) we have: $\xi_1 \in A_{\beta_i}, \zeta_1 \in A_{\beta_i}$, $\xi \leq \xi_1 \leq \zeta_1 < \zeta$ and $i < \omega^2$, implies that $\langle A_j : \beta_i + 2 \leq j \leq \beta_{i+1} \rangle$ is long for (ξ_1, ζ_1).

4.8 Claim.

1) If $\xi < \zeta \in A_0$, $\langle A_i : i \leq \beta \rangle$ is as in the assumptions of Definition 4.7, $\beta_0 < \beta_1 \leq \beta$, and $\langle A_i : \beta_0 \leq i \leq \beta_1 \rangle$ is long for (ξ, ζ) *then* $\langle A_i : i < \beta \rangle$ is long for (ξ, ζ).
2) $\langle A_i : i \leq \beta \rangle$ is long for (ξ, ζ) iff $\langle A_i \cap (\zeta + 1) \setminus \xi : i \leq \beta \rangle$ is long for (ξ, ζ).

Proof. 1) By induction on ζ, and there are no problems.

2) Easy. $\square_{4.8}$

4.9 Claim. Let $\lambda \geq \omega_1$, $\beta < \omega_1$ and for $i < \beta$ we have $N_i \prec (H(\lambda), \in)$, and N_i are countable increasing continuous and $\langle N_j : j \leq i \rangle \in N_{i+1}$ and $\xi < \zeta \in N_0$. *Then* we can find an α such that $\beta \leq \alpha < \omega_1$, and a countable N_i for $\beta \leq i \leq \alpha$ such that $N_i \prec (H(\lambda), \in)$, $\langle N_j : j \leq i \rangle \in N_{i+1}$ for $i < \alpha, N_i$ are countable increasing continuous in i and $\langle N_i \cap \lambda : i < \alpha \rangle$ is long for (ξ, ζ).

Proof. Again by induction on ζ.

4.10 Claim. Suppose $\underset{\sim}{Q}_i, P_i, \alpha_0$ are as in Theorem 4.5, λ is large enough and for $i \leq \beta (< \omega_1)$ $N_i \prec (H(\lambda), \in, <^*_\lambda)$ is countable, increasing, continuous and

$$\langle N_j : j \leq i \rangle \in N_{i+1},\ \xi \leq \zeta \in N_0 \cap (\alpha_0 + 1),\ \langle P_i, \underset{\sim}{Q}_i : i < \alpha_0 \rangle \in N_0.$$

Suppose further that $\langle N_i \cap \alpha_0 : i \leq \beta \rangle$ is long for (ξ, ζ), $G_e (e = 0, 1)$ are directed subsets of $P_\xi \cap N_\beta$, $r_e \in P_\xi$ (for $e = 0, 1$), $(\forall q \in G_e) q \leq r_e$, $G_0 \cap N_0 = G_1 \cap N_0$ and $i \leq \beta \Rightarrow \mathcal{I} \cap G_e \cap N_i \neq \emptyset$ for every pre-dense $\mathcal{I} \subseteq P_\xi$ with $\mathcal{I} \in N_i$ and $e = 0, 1$. Suppose also $p \in P_\zeta \cap N_0$ and $p \restriction \xi \in G_0$.

Then there is a directed $G^* \subseteq P_\zeta \cap N_0$ such that $G_0 \cap N_0 \subseteq G^*$, G^* not disjoint to any pre-dense $\mathcal{I} \subseteq P_\zeta, \mathcal{I} \in N_0, p \in G^*$, and $r_e \Vdash_{P_\xi}$ "$\{q \restriction [\xi, \zeta) : q \in G^*\}$ has an upper bound in P_ζ / P_ξ" for $e = 0, 1$.

Proof. By induction on ζ (for all $\xi \leq \zeta$).

Note that the assertion (for $\xi = 0$) implies that forcing by P_ζ does not add ω-sequences of ordinals.

Also note if $p \leq q$ are in $P_\xi \cap N_0$, $q \in G_0 \cap N_0$ then $p \in G_0 \cap N_0$ (as $\{r \in P_\xi : r \geq p$ or r, p incompatible in $P_\xi\}$ is pre-dense in P_ξ and belongs to N_0); similarly for N_β instead N_0 and/or for G_1 instead of G_0. Also, for $e = 0, 1$ and $i < \beta$, we have $G_e \cap N_i \in N_{i+1}$. Why?

This is easy, as the set $\mathcal{I} \subseteq P_\xi$, defined below, is pre-dense and belongs to N_{i+1}, hence is not disjoint from $G_e \cap N_{i+1}$. So there is an $r^\dagger \in \mathcal{I} \cap N_{i+1}$ with

$r^\dagger \in G_e \cap N_{i+1}$; but by the assumption $(\forall q)[q \in G_e \cap N_i \to q \leq r_e]$ and $r^\dagger \leq r_e$. Hence, by the definition of $\mathcal{I}^0$ below, we know that $r^\dagger \notin \mathcal{I}^0$, so $r^\dagger \in \mathcal{I}^1$ (see below), and necessarily $\{q \in N_i \cap P_\xi : q \leq r^\dagger\} = G_e \cap N_i$. So $G_e \cap N_i \in N_{i+1}$ as it is defined from parameters $(N_i, P_\xi, r^\dagger)$ in it. Here is the definition of $\mathcal{I}$:

$$\mathcal{I} = \mathcal{I}^0 \bigcup \mathcal{I}^1, \text{ where}$$

$$\begin{aligned}\mathcal{I}^0 = \{q \in P_\xi : &\text{ there are no } r \in P_\xi, r \geq q \text{ and } G \subseteq N_i \cap P_\xi \text{ such that}\\ &(\forall p' \in G) \quad p' \leq r \text{ and}\\ &(\forall \mathcal{J} \in N_0)\ [\mathcal{J} \text{ pre-dense in } P_\xi \to \mathcal{J} \cap G \neq \emptyset]\}\end{aligned}$$

$$\begin{aligned}\mathcal{I}^1 = \{q \in P_\xi : &\text{ there is a } G \subseteq N_0 \cap P_\xi \text{ such that } (\forall p' \in G) \quad p' \leq q \text{ and}\\ &(\forall \mathcal{J} \in N_0)[\mathcal{J} \text{ pre-dense in } P_\xi \to \mathcal{J} \cap G \neq \emptyset]\}.\end{aligned}$$

Case (i): $\zeta = \xi$.

Trivial. Just let $G^* = G_0 \cap N_0$.

Case (ii): ζ a successor ordinal $> \xi$.

So by Definition 4.7, for some $\gamma < \beta$, $\langle N_i \cap \lambda : i \leq \gamma\rangle$ is long for $(\xi, \zeta - 1)$. For $e = 0, 1$, we can find $r_e^a \in G_e \cap N_{\gamma+1}$ such that r_e^a is above every member of $G_e \cap N_\gamma$ (see the proof above). Hence, by the induction hypothesis, there is a $G^* \subseteq N_0 \cap P_{\zeta-1}$ such that:

(a) for every pre-dense $\mathcal{I} \subseteq P_{\zeta-1}, \mathcal{I} \in N_0$, the intersection $G^* \cap \mathcal{I}$ is not empty,

(b) $r_e^a \Vdash_{P_\xi}$ "G^* has an upper bound in $P_{\zeta-1}/P_\xi$",

(c) $G_e \cap N_0 = G^* \cap P_\xi$.

Now without loss of generality G^* is definable (in $(H(\lambda), \in, <^*_\lambda)$) from the parameters $\overline{Q}$, $\langle N_i : i \leq \gamma\rangle$, r_0^a, r_1^a, ζ, ξ, hence

(d) $G^* \in N_{\gamma+1}$.

Now we want to define $G^{**} \subseteq P_\zeta \cap N_0$ as in the conclusion of the claim. G^* determines $\mathbb{D}^{\zeta-1} \in V$, as $\zeta \in N_0$ (i.e. some member of G^* forces $(\Vdash_{P_{\zeta-1}})$ $\underset{\sim}{\mathbb{D}}^{\zeta-1}$ to be some $\mathbb{D}^{\zeta-1}$). Let $G^* \subseteq G^f_{\zeta-1} \subseteq P_{\zeta-1}$ where $G^f_{\zeta-1}$ is generic (over V). Clearly, some members of $G^f_{\zeta-1}$ force $N_0 \cap \lambda_{\zeta-1}$ ($\lambda_{\zeta-1}$ is from the definition

of $\mathbb{D}^{\zeta-1}$) to be in the appropriate closed unbounded subset of $\mathcal{S}_{\aleph_0}(\lambda_{\zeta-1})$, so we "know" $\mathbb{D} = \mathbb{D}^{\zeta-1}$ and it belongs to $N_{\gamma+1}[G^f_{\zeta-1}]$ (and to V; why not to N_1? we need G^*; "know" means independently of the particular choice of $G^f_{\zeta-1}$). In $V^{P_{\zeta-1}}$, let $(Q^*_\zeta, <^*_\zeta) \stackrel{\text{h}}{\cong} (Q_\zeta, <)$, $Q^*_\zeta \subseteq \lambda$, $x \subseteq \lambda \times \lambda$ code a list $\langle \mathcal{J}_\alpha = \langle p^\alpha_i : i < i_\alpha\rangle : \alpha < \alpha^*\rangle$ of the pre-dense subsets of $(Q^*_{\zeta-1}, <^*_{\zeta-1})$ from Definition 4.3 and x is as in Definition 4.3. They may be $P_{\zeta-1}$-names, but without loss of generality $\langle \underset{\sim}{\mathcal{J}}_\alpha = \langle \underset{\sim}{p}^\alpha_i : i < \underset{\sim}{i}_\alpha\rangle : \alpha < \underset{\sim}{\alpha}^*\rangle$, $(\underset{\sim}{Q}^*_{\zeta-1}, \underset{\sim}{\leq}^*_{\zeta-1}), \underset{\sim}{h}$ and $\underset{\sim}{x}$ belong to $N_0[G^f_{\zeta-1}]$. [Why? By III 2.13.] And we can compute $\underset{\sim}{x}{\restriction}(N_0 \cap \lambda_{\zeta-1})$, $(\underset{\sim}{Q}^*_\zeta, \underset{\sim}{\leq}^*_{\zeta-1}){\restriction}(N_0 \cap \lambda_{\zeta-1})$ so that $N_{\gamma+1}[G^f_{\zeta-1}] \prec (H(\lambda), \in, <^*_\lambda)$. Finally, let $y = \langle N_0 \cap \lambda_{\zeta-1}, \leq^*_{\zeta-1}{\restriction}(N_0 \cap \lambda_{\zeta-1}), \underset{\sim}{x}{\restriction}((N_0 \cap \lambda_{\zeta-1}) \times (N_0 \cap \lambda_{\zeta-1})), \underset{\sim}{h}(p)\rangle$. There is a set $A \in \mathbb{D}_y$ such that (in $V[G^f_{\zeta-1}]$) for any $G \in A$, we have that $G \subseteq N_0 \cap \underset{\sim}{Q}^*_{\zeta-1}[G^f_{\zeta-1}]$ is not disjoint to any pre-dense $\mathcal{I} \subseteq Q^*_{\zeta-1}$ which belongs to $N_0[G^f_{\zeta-1}]$ and G has an upper bound in $\underset{\sim}{Q}^*_{\zeta-1}[G^f_{\zeta-1}]$. Note that $G^f_{\zeta-1} \subseteq P_{\zeta-1}$ is an arbitrary generic set which includes G^*, so there is a $P_{\zeta-1}$-name $\underset{\sim}{\mathcal{I}}$ such that $\Vdash_{P_{\zeta-1}}$ " if there is an A as above then $\underset{\sim}{\mathcal{I}}$ names it ". Again without loss of generality $\underset{\sim}{\mathcal{I}} \in N_{\gamma+1}[G^f_{\zeta-1}]$.

Now $\mathcal{I}^* = \{q \in P_{\zeta-1} : q$ forces $\underset{\sim}{\mathcal{I}}$ to be some specific $A \in \mathbb{D}_y\}$ is pre-dense in $P_{\zeta-1}$ and belongs to $N_{\gamma+1} \subseteq N_\beta$. So, by the assumptions on G_e, there are r^*_e for $e = 0, 1$, such that $r^*_e \in P_{\zeta-1} \cap N_\beta$, $r^*_e{\restriction}\xi \in G_e$, $(\forall q \in G^*)(q \leq_{P_{\zeta-1}} r^*_e)$ and $r^*_e \Vdash_{P_{\zeta-1}}$ "$\underset{\sim}{\mathcal{I}} = A_e$", where $A_e \in \mathbb{D}_y, A_e \in N_\beta$. By the hypothesis on $\mathbb{D}^{\zeta-1}$, $A_0 \cap A_1 \neq \emptyset$ so as A_0, $A_1 \in N_\beta$ there is a $G^\otimes \in A_0 \cap A_1 \cap N_\beta$.

Then

$$G^{**} = \{q : q \in N_0 \cap P_\zeta \text{ and } q{\restriction}(\zeta-1) \in G^* \text{ and for some } r \in G^\otimes,$$
$$r \Vdash_{P_{\zeta-1}} \text{"}h(q(\zeta-1)) = r\text{"}\},$$

is as required (it is well defined as though $\underset{\sim}{h}$ is a $P_{\zeta-1}$-name, it belongs to N_0 hence $\underset{\sim}{h}{\restriction}N_0$ can be computed from G^*).

Case (iii). ζ a limit.

So there are $\beta_i (i \leq \omega^2)$ such that $i < j \leq \omega^2 \to \beta_i < \beta_j$, $\beta_{\omega^2} + \omega + 1 < \beta$ and for any (ξ_1, ζ_1) in N_{β_i} if $\xi \leq \xi_1 \leq \zeta_1 < \zeta$ then $\langle N_j \cap \zeta : \beta_i + 2 \leq j \leq \beta_{i+1}\rangle$ is

long for (ξ_1, ζ_1) and $\langle \beta_i : i \leq \omega^2 \rangle$ is increasing continuous (we can assume this by 4.8(1)).

Let $\beta^* = \beta_{\omega^2}$ and w.l.o.g. we can assume $\langle \beta_i : i \leq \omega^2 \rangle \in N_{\beta^*+1}$ (as there is such a sequence in N_{β^*+1}, because it exists in V). Similarly $i < \omega^2 \Rightarrow \langle \beta_j : j \leq i \rangle \in N_{\beta_i+1}$. Choose $\zeta_n \in N_0 \cap \zeta$ such that $\xi = \zeta_0 < \zeta_1 < \ldots \zeta_n < \zeta_{n+1} \ldots < \zeta$ and $[\gamma \in N_0 \,\&\, \gamma < \zeta \Rightarrow \bigvee_n \gamma < \zeta_n]$.

Now we define by induction on $n < \omega$, $k_n = k(n) < \omega$, G^n_η (for $\eta \in {}^{(k(n))}2$), r^*_n, $\underset{\sim}{\eta}_n$ such that:

1) r^*_n is a member of $P_{\zeta_n} \cap N_{\beta^*+\omega+1}$ with domain $\zeta_n \cap N_{\beta^*+\omega}$, $r^*_0 \leq r_0, r^*_0 \leq r_1$ (see below for formal problems or let $r^*_0 = \emptyset$ (so $\mathrm{Dom}(r^*_0) = \emptyset$) but then we use $r^*_n \cup r_e$ to force anything),
2) $r^*_{n+1} \restriction \zeta_n = r^*_n$, (or you can say that r^*_n is a P_ζ-name of such a condition with r_0, r_1 deciding the value but see (8) or the beginning of the proof below)
3) if $n + 2 \leq m < \omega$, r^*_n is $(N_{\beta^*+m}, P_{\zeta_n})$-generic,
4) if $\mathcal{I} \subseteq P_{\zeta_n}$ is a maximal antichain, $\mathcal{I} \in N_{\beta^*+1}$ *then* for some finite $\mathcal{J} \subseteq \mathcal{I} \cap N_{\beta^*+1}$, $\mathcal{J}$ is pre-dense above r^*_n (note that, as described in the beginning of the proof of 4.10, this implies that the function giving $\mathcal{J}$ (from $\mathcal{I}$) belongs to N_{β^*+n+2}),
5) $k_0 = 1$, $G^0_{\langle e \rangle} = G_e \cap N_{\beta^*}$,
6) $G^n_\eta \subseteq P_{\zeta_n} \cap N_{\beta^*}, G^n_\eta \in N_{\beta^*+1}$ for $\eta \in {}^{k(n)}2$,
7) if $m < n$ and $\eta \in {}^{k(n)}2$, then $G^m_{\eta \restriction k(m)} = G^n_\eta \cap P_{\zeta_m}$,
8) $\underset{\sim}{\eta}_n$ is a P_{ζ_n}-name which belongs to N_{β^*+1},

$$r^*_n \Vdash \text{“}\underset{\sim}{\eta}_n \in {}^{k(n)}2\text{”},$$
$$r^*_{n+1} \Vdash \text{“}\underset{\sim}{\eta}_{n+1} \restriction k_n = \underset{\sim}{\eta}_n\text{”},$$
$$r^*_n \Vdash \text{“}G^n_{\underset{\sim}{\eta}_n} \text{ is included in the generic subset of } P_{\zeta_n}\text{”},$$

9) if $j \leq \beta^*$, *but* for no $k \leq k(n)$ is $\beta_{\omega k} + 1 < j \leq \beta_{\omega(k+1)}$ *then* for every pre-dense $\mathcal{I} \subseteq P_{\zeta_n}$, $\mathcal{I} \in N_j$, and $\eta \in {}^{k(n)}2$ we have $N_j \cap \mathcal{I} \cap G^n_\eta \neq \emptyset$,
10) $\eta, \nu \in {}^{k(n)}2$, $\eta \restriction k = \nu \restriction k$, $k < k(n)$ implies $G^n_\eta \cap N_{\beta_{\omega k}+1} = G^n_\nu \cap N_{\beta_{\omega k}+1}$ and we denote both by $G^n_{\eta \restriction k}$,
11) $(\forall q \in N_{\beta^*} \cap P_{\zeta_n})\ [(\exists q^\dagger \in G^n_\eta)(q \leq q^\dagger) \to q \in G^n_\eta]$ for $\eta \in {}^{k(n)}2$.

There is no problem for $n = 0$ (how do we define r_0^*? we can assume that $\underset{\sim}{Q}_\gamma$ is closed under disjunction. Let $\varepsilon \in N_\beta \cap (\zeta + 1)$ be maximal such that $(\forall r \in N_{\beta+\omega} \cap P_\varepsilon)(r \leq r_0 \equiv r \leq r_1)$, by the definition of CS iteration it is well defined. As before we can find $r'_e \in N_{\beta^*+\omega+1}$ which is below r_e and $(\forall r \in N_{\beta+\omega} \cap P_\zeta)(r < r'_e \equiv r < r_e)$, $r'_0 \restriction \varepsilon = r'_1 \restriction \varepsilon$, $\varepsilon < \zeta \Rightarrow r'_0 \restriction \varepsilon \Vdash_{P_\varepsilon}$ "$r'_0(\varepsilon), r'_1(\varepsilon)$ are incompatible in $\underset{\sim}{Q}_\varepsilon$" and define r_0^* as follows: $\mathrm{Dom}(r_0^*) = \mathrm{Dom}(r'_0) \cup \mathrm{Dom}(r'_1)$, and

$$r_0^*(\gamma) = r'_0(\gamma) \text{ if } \gamma < \varepsilon,$$

$$r_0^*(\gamma) = r'_0(\gamma) \vee r'_1(\varepsilon) \text{ if } \gamma = \varepsilon,$$

$$r_0^*(\gamma) = r'_1(\gamma) \text{ if } r'_1 \restriction (\varepsilon + 1) \in \underset{\sim}{G}_{P_{\varepsilon+1}},$$

$$r_0^*(\gamma) = r'_0(\gamma) \text{ otherwise.})$$

So assume we have defined for n and we shall define for $n + 1$.

First we define, by induction on $\ell \leq k(n)$, for every $\eta \in {}^\ell 2$, the sets G_η^{n+1} (see (10) above), and we have to satisfy (9),(7), and r_n^* should force G_η^{n+1} is (bounded by) a condition, if $\eta = \underset{\sim}{\eta}_n \restriction \ell$.

This makes no problem, using the induction hypothesis on ζ and $(< \omega_1)$-properness.

Second we want to define $k(n+1), G_\eta^{n+1}$ (for $\eta \in {}^{k(n+1)}2$). Let $G_{\zeta_n} \subseteq P_{\zeta_n}$ be generic, $r_n^* \in G_{\zeta_n}$ and work for a while in $V[G_{\zeta_n}]$.

For each $p' \in N_{\beta^*}[G_{\zeta_n}] \cap (P_{\zeta_{n+1}}/G_{\zeta_n})$, there is a $G \subseteq P_{\zeta_{n+1}}/G_{\zeta_n}$, $p' \in G$, G has an upper bound, and *if* $p' \in N_j, j \leq \beta^*$ is as in (9) then $G \cap N_j$ is generic for $(N_j, P_{\zeta_{n+1}}/G_{\zeta_n})$ [equivalently, if $\mathcal{I} \subseteq P_{\zeta_{n+1}}$ is pre-dense, $\mathcal{I} \in N_j$, *then* the set $\mathcal{I} \cap N_j \cap G$ is not empty]. So there is a function F giving such a G with $\mathrm{Dom}(F) = (P_{\zeta_{n+1}}/G_{\zeta_n}) \cap N_{\beta^*}[G_{\zeta_n}]$. So, in V, we have a P_{ζ_n}-name $\underset{\sim}{F}$ for it. As its domain is countable, $\Vdash_{P_{\zeta_n}}$ "$\underset{\sim}{F} \in V$" (the domain is essentially $\subseteq N_{\beta^*}$).
Also it is clear that, without loss of generality, $\underset{\sim}{F} \in N_{\beta^*+1}$ as $\langle N_i : i \leq \beta^* \rangle \in N_{\beta^*+1}$, $r_n^* \in G_{\zeta_n}$ and condition (9), (4).

So, by condition (4), there are $F_1, \ldots, F_m \in N_{\beta^*+1}$ such that $r_n^* \Vdash$ "$\underset{\sim}{F} \in \{F_1, \ldots, F_n\}$", $F_1 \ldots \in V$ (note that so their domain is computed by $G_{\underset{\sim}{\eta}_n} \ldots$).

By renaming, choose $k(n+1)$, F_η (for $\eta \in {}^{k(n+1)}2$) such that for every $\eta \in {}^{k(n)}2$ we have $\{F_1, \ldots, F_n\} = \{F_\nu : \nu \in {}^{k(n+1)}2, \nu \restriction k(n) = \eta\}$. Now for any

$\eta \in {}^{k(n+1)}2$ we can first define $G^{n+1}_\eta \cap N_{\omega k(n+1)+2}$ so that it depends on $\eta \restriction k(n)$ and not η, and then let $G_\eta = F_\eta(\bigwedge(G^{n+1}_\eta \cap N_{\beta_{\omega k_{n+1}+2}}))$.

Third we define r^*_{n+1} by V 4.5. $\square_{4.10}$

Proof of 4.5. Immediate by 4.10. $\square_{4.5}$

4.10A Remark.

1) So now everywhere we can use 4.5 instead of V 7.1 and strengthen Axiom II, II_a, II_b etc.
2) We could use shorter sequences, e.g. $\beta = \beta^* + 2$ is o.k. (see implicitly VI §1, and explicitly XVIII 2.10).
3) By easy manipulations, it does not matter in Theorem 4.3, whether $\mathbb{D}^\alpha$ is a P_α-name of a member of V, or simply a member of V (i.e. the function $\alpha \mapsto \mathbb{D}^\alpha$ is in V).

4.11 Conclusion. It is consistent with ZFC + G.C.H. that:

$(*)$ If $k < \omega$ and η_δ is an ω-sequence converging to δ for any limit $\delta < \omega_1$ and $\langle A_\delta : \delta < \omega_1 \rangle$ is such that $A_\delta \subseteq k$, $|A_\delta| < k/2$
then there is an $h : \omega_1 \to k$ such that for every limit $\delta < \omega_1$, $\langle h(\eta_\delta(n)) : n < \omega \rangle$ is eventually constant, and its constant value $\notin A_\delta$.

4.12 Conclusion. G.C.H. $\not\Rightarrow \Phi^3_{\aleph_1}$.

4.13 Lemma.

1) The demand on each $\underset{\sim}{Q}_\alpha$ in 4.5 follows from: $\underset{\sim}{Q}_\alpha$ is γ-proper for every $\gamma < \omega_1$, and $\mathbb{D}_\alpha$-complete for some simple 2-completeness system $\mathbb{D}_\alpha$.
2) We can demand in 4.5 that each $\underset{\sim}{Q}_\alpha$ is γ-proper for every $\gamma < \omega_1$, and $\mathbb{D}_\alpha$-complete for some almost simple 2-completeness system over V.

Remark. See Definition V 5.5.

Proof. Straightforward. $\square_{4.13}$

* * *

Finally, we indicate how to rephrase the proof in the form of a condition which is preserved by iteration.

4.14 Definition. We[†] call $E \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$ *stationary* if for every $\chi > \lambda$ (equivalently, some $\chi > \lambda^{\aleph_0}$) and for every $x \in H(\chi)$ there is a sequence $\overline{N} = \langle N_i : i \leq \alpha \rangle$ such that

(a) $\overline{N} = \langle N_i : i \leq \alpha \rangle$ is an increasing continuous sequence of countable elementary submodels[††] of $(H(\chi), \in, <^*_\chi)$, $\alpha < \omega_1$ and $\overline{N}{\restriction}(i+1) \in N_{i+1}$ for $i < \ell g(\overline{N})$,

(b) $x \in N_0$,

(c) $\langle N_i \cap \lambda : i \leq \alpha \rangle \in E$.

4.15 Definition. Let λ be a cardinal, $E \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$ stationary, and κ a cardinal (may be finite). We call $\mathbb{D}$ a (λ, E, κ)-system if:

(A) $\mathbb{D}$ is a function (written $\mathbb{D}_x$).

(B) $\mathbb{D}_{\langle \overline{a}, <^*, x, p \rangle}$ is defined iff $\overline{a} = \langle a_i : i \leq \alpha \rangle \in E$, $<^*$ a partial order of a_α, x is a binary relation on a_α and $p \in a_0$. If $\mathbb{D}_{\langle \overline{a}, <^*, x, p \rangle}$ is well defined then it is a family of subsets of $\mathcal{P}(a_0)$.

(C) *If* $i < 1 + \kappa$ and $\langle \overline{a}, \leq^*_j, x_j, p \rangle \in \mathrm{Dom}(\mathbb{D})$ for $j < i$ (note: same $\overline{a}$ and p and possibly distinct $\leq^*_j, x_j$) and $\leq^*_j {\restriction} a_0 = \leq^*_0 {\restriction} a_0$, $x_j {\restriction} a_0 = x_i {\restriction} a_0$ and $A_j \in \mathbb{D}_{\langle \overline{a}, \leq^*, x_j, p \rangle}$ for $j < i$, *then* $\bigcap_{j<i} A_j \neq \emptyset$.

4.16 Definition. A forcing notion P is called $(\mathcal{D}, \mathbb{D})$-complete, where $\mathcal{D}$ is a (fine normal) filter on $\mathcal{S}_{\leq \aleph_0}(\lambda)$ and $\mathbb{D}$ is a (λ, E, κ)-system (so $E \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$ is stationary) *if*:

(A) $\lambda > 2^{|P|}$, P is isomorphic to $(P^*, <^*)$, $P^* \subseteq \lambda$,

(B) for any $p \in P^*$ and pre-dense subsets $\mathcal{I}_\alpha$ of P^*, $\mathcal{I}_\alpha = \{p^\alpha_i : i < i_\alpha \leq \lambda\}$ (for $\alpha < \lambda$), for some $x \subseteq \lambda \times \lambda$ and a nonstationary $Y \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$, we have:

[†] Remember $\mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda) \stackrel{\mathrm{def}}{=} \{\bar{A} : \bar{A}$ an increasing continuous sequence of countable subsets of λ of length $< \omega_1\}$.

[††] Of course we can omit $<^*$ and still get an equivalent formulation.

(*) If $\overline{a} = \langle a_i : i \leq \alpha \rangle \in E \setminus Y$ for $i \leq \alpha$, *then* the following set includes a member of $\mathbb{D}_{\langle \overline{a}, <^* \restriction a_0, x \restriction a_0, p \rangle}$:

$$\Big\{ G \subseteq a_0 \text{: } 1) \text{ for every } \alpha \in a_0 \text{ for some } i \in a_0,\ p_i^\alpha \in G.$$
$$2)(\exists q \in P^*)(\forall r \in G)[r \leq q]$$
$$3) p \in G \Big\},$$

(C) for every $Y \in \mathcal{D}$ we have $\{\overline{a} \in E : a_0 \notin Y\}$ is not stationary (as a subset of $\mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$).

(D) if $Y \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$, $Y \neq \emptyset \bmod \mathcal{D}$ then $\{\overline{a} \in E : a_0 \in Y\}$ is stationary.

4.17 Convention. If $\mathcal{D} = \mathcal{D}_{\leq \aleph_0}(\lambda) + F_0$, $F_0 \neq \emptyset \bmod \mathcal{D}_{\leq \aleph_0}(\lambda)$ *then* we shall write F_0 instead of $\mathcal{D}$. If $\mathcal{D} = \mathcal{D}_{\leq \aleph_0}(\lambda)$ we shall write $\mathbb{D}$ instead of $(\mathcal{D}, \mathbb{D})$. We do not always distinguish between $\mathcal{D}$, a fine normal filter on $\mathcal{S}_{\leq \aleph_0}(\lambda)$ in V, and the fine normal filter it generates in a generic extension of V.

4.18 Claim. (1) If P is $(\mathcal{D}, \mathbb{D})$-complete, *then* forcing with P adds no new ω-sequence of ordinals (in particular, no reals).

(2) Suppose P is $(\mathcal{D}, \mathbb{D})$-complete for a (λ, E, κ)-completeness system $\mathbb{D}$, and $\mu \geq \lambda$.

Then for some $\mathbb{D}'$, $E', \mathcal{D}'$ we have :

(a) $E' \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\mu)$ is stationary and for $\mathbb{D}'$

a (μ, E', κ)-completeness system P is $(\mathcal{D}', \mathbb{D}')$-complete,

(b) $\mathcal{D}'$ is the normal fine filter on $\mathcal{S}_{\leq \aleph_0}(\mu)$ generated by

$$\Big\{ \{a : a \cap \lambda \in y\} : y \in \mathcal{D} \Big\}.$$

(c) $E' = \{\overline{a} : \langle a_i \cap \lambda : i \leq \ell g(\overline{a}) \rangle \in E$ and $\overline{a} \in S^{<\omega_1}_{\leq \aleph_0}(\mu)$ and $a_0 \cap \lambda \in \mathcal{D}\}$.

(d) $\mathbb{D}'$ is defined naturally.

(3) If $V \models 2^\lambda \leq \mu$ then (in the first part) $\mathcal{D}'$ has the form $\mathcal{D}_{\leq \aleph_0}(\lambda) + F$ for some stationary $F \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$.

4.19 Claim. 1) Assume $E \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$ is stationary and $\mathcal{D}, E$ satisfy condition (C) of 4.16. If $Y \in \mathcal{D}$ and H is a function from Y to λ, *then* $\{\overline{a} \in E : H(a_0) \notin a_1\}$ is not stationary.

2) In definition 4.14 the value of χ is immaterial and, if $\chi > 2^\lambda$, we can omit "$x \in N_0$".

3) Assume $\bar{Q} = \langle P_\xi, \underset{\sim}{Q}_\zeta : \xi \leq \zeta^* \text{ and } \zeta < \zeta^* \rangle$ is a CS iteration of proper forcings. Assume also that $\mathcal{D}$ is a (fine) normal filter on $\mathcal{S}_{\leq \aleph_0}(\lambda)$, and for $\zeta < \xi \leq \zeta^* < \lambda$, $E_{\zeta,\xi} \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$ is stationary and each quadruple $(\mathcal{D}, E, P_\zeta/P_\xi)$ satisfies in V^{P_ζ} conditions (C), (D) of Definition 4.16. *Then*

$E \overset{\text{def}}{=} \{\overline{a} : \overline{a} \in \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$ and there is $\langle \varepsilon_\zeta : \zeta \in a_0 \cap (\zeta^* + 1) \rangle$ increasing continuous, $0 < \varepsilon_0$, $\varepsilon_{\zeta^*} < \ell g(\overline{a})$, for each $\zeta \in a_0 \cap (\zeta^* + 1)$ we have $\langle a_0 \rangle \hat{} \langle a_\varepsilon : \varepsilon_\zeta + 2 \leq \epsilon \leq \varepsilon_{\zeta+1} \rangle \in E_{\varepsilon_\zeta, \varepsilon_{\zeta+1}}$ and $a_0, a_{\ell g(\overline{a})-1}, a_{\varepsilon_\zeta+1}$ belong to $\{b_0 : \overline{b} \in E_0\}\}$

is stationary and satisfies clauses (C), (D) of Definition 4.16.

(4) If $F \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$ is stationary and in (3) we add $[\overline{a} \in E_\zeta \& \zeta < \zeta^* \Rightarrow \bigwedge_{i \leq \ell g(\overline{a})} a_i \in F]$ *then* we can replace E by $E' = \{\overline{a} \in E : \bigwedge_{i \leq \ell g(\overline{a})} a_i \in F\}$.

Proof. Straightforward. $\square_{4.19}$

4.19A Remark. If $\lambda < \chi, N_i \prec (H(\chi), \in, <^*_\chi)$ for $i \leq \alpha$ is countable, increasing, continuous (in i) and $\langle N_j : j \leq i \rangle \in N_{i+1}$ *then* we can find a limit $\beta > \alpha$ and N_i (for $i \in (\alpha, \beta]$) such that:

(a) $\langle N_i : i \leq \beta \rangle$ is increasing and continuous, $N_i \prec (H(\chi), \in, <^*_\chi)$ is countable, and $\langle N_j : j \leq i \rangle \in N_{i+1}$.

(b) $\lambda \in N_{\alpha+1}$ and *if* $E \in N_\beta$, $E \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$ is stationary *then* for some i, j, we have $\alpha < i < j < \beta$, $E \in N_i$ and $\langle N_\gamma \cap \lambda : i \leq \gamma \leq j \rangle \in E$.

4.20 Theorem. Assume

(a) $\overline{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \alpha^* \rangle$ is a CS iteration,

(b) each $\underset{\sim}{Q}_\alpha$ is $(< \omega_1)$-proper (in V^{P_α}),

(c) $\kappa \geq 2, \lambda \geq \omega_1$ a cardinal, $\alpha^* \leq \lambda, \lambda = \lambda^{\aleph_0}$, and P_{α^*} has a dense subset of cardinality $\leq \lambda$,

(d) λ is a cardinal, $\mathcal{D}$ is a normal filter on $\mathcal{S}_{\leq\aleph_0}(\lambda)$ (or $\mathcal{D} = \mathcal{D}_{\leq\aleph_0}(\lambda) + F$),

(e) for every α, $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ is $(\mathcal{D}, \underset{\sim}{\mathbb{D}}^\alpha)$-complete", $\underset{\sim}{\mathbb{D}}^\alpha$ a P_α-name of a $(\lambda_\alpha, \underset{\sim}{E}_\alpha, \kappa)$-completeness system from V, $\underset{\sim}{E}_\alpha$ is a P_α -name of a member of V which is a stationary subset of $\mathcal{S}^{<\omega_1}_{\leq\aleph_0}(\lambda)$ (in V) satisfying (C), (D) of 4.18.

Then:

(1) P_{α^*} is $(\mathcal{D}, \mathbb{D})$-complete for some (λ, E, κ)-completeness system $\mathbb{D} \in V$, for some stationary $E \subseteq \mathcal{S}^{<\omega_1}_{\leq\aleph_0}(\lambda)$, satisfying:

$$\bigcup_{\overline{a}\in E} \mathrm{Rang}(\overline{a}) \subseteq \{a_i : \nVdash_{P_\alpha} \text{"}\overline{a} \notin E^\alpha \text{ for some } \alpha < \alpha^*\text{"} \text{ and } i \leq \ell g(\overline{a})\},$$

(2) moreover, if $\alpha < \beta \leq \alpha^*$, then in V^{P_α}, P_β/P_α is $(\mathcal{D}, \mathbb{D})$-complete for some $(\lambda, E^{\alpha,\beta}, \kappa)$-completeness system $\mathbb{D} \in V$, for some $\underset{\sim}{E}^{\alpha,\beta} \in V$ as above.

4.20A Remark. Why can we assume that λ is constant? see 4.18(2).

Proof. This is proved by induction on α^*. Fix a one to one function H^* from $\mathcal{S}^{<\omega_1}_{\leq\aleph_0}(\lambda)$ onto λ and let $\langle(\mathbb{D}^\zeta, E^\zeta) : \zeta < \lambda\rangle$ list, in V, all pairs $(\mathbb{D}, E)$, $\mathbb{D}$ a (λ, E, κ)-completeness system such that for some $\alpha < \beta < \alpha^*$, $\Vdash_{P_\alpha}$ "$(\mathbb{D}, E)$ is $<^*_{\lambda^+}$-first for which P_β/P_α is $(\mathcal{D}, \mathbb{D})$-complete, $\mathbb{D}$ is a (λ, E, κ)-completeness system and $\mathbb{D}, E \in V$". There is such a list of length λ, as $\alpha^* \leq \lambda$ and P_{α^*} has a dense subset cardinality $\leq \lambda$ (see assumption (a) of 4.20).

By the following subclaim, without loss of generality $(\underset{\sim}{\mathbb{D}}^\alpha, \underset{\sim}{E}^\alpha)$, for $\alpha < \alpha^*$, are really $(\mathbb{D}^\alpha, E^\alpha)$ (i.e. members of V rather than names of such members) and $E^\alpha \cap \mathcal{S}^{\leq 2}_{\leq\aleph_0}(\lambda) = \emptyset$ and similarly $(\mathbb{D}^{\alpha,\beta}, E^{\alpha,\beta})$, for $\alpha < \beta < \alpha^*$. (We can alternatively redefine the iteration, inserting many trivial forcings, i.e. replace Q_i by $Q'_{\lambda\times i+\zeta}(\zeta < \lambda)$ which is $(\mathcal{D}, \mathbb{D}^\zeta)$-complete (one of them is Q_i, the others are trivial).)

4.21 Subclaim. Under the assumption in 4.20, for each $\alpha < \alpha^*$, for some $(\mathbb{D}, E) \in V$, $\mathbb{D}$ is a (λ, E, κ)-completeness system in V, $E \in \mathcal{S}^{<\omega_1}_{\leq\aleph_0}(\lambda)$ is stationary and $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ is $(\mathcal{D}, \mathbb{D})$-complete".

Proof. Let

$$E^*_\alpha = \{\bar{a} : \bar{a} \in \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda), \ell g(\bar{a}) > 2 \text{ and for some } \bar{\varepsilon} = \langle (\varepsilon^\zeta_0, \varepsilon^\zeta_1) : \zeta \in a_0 \cap \alpha \rangle$$
$$\text{for every } \zeta \in a_0, \varepsilon^\zeta_0 < \varepsilon^\zeta_1 < \ell g(\bar{a}) \text{ and } \bar{\varepsilon} \restriction \zeta \in N_{\varepsilon^\zeta_0} \text{ and}$$
$$\langle a_i : i = 0 \text{ or } \varepsilon^\zeta_0 \leq i \leq \varepsilon^\zeta_1 \rangle \in E^\zeta\}.$$

By 4.19(3) we can show that E^* is of the right kind. $\square_{4.21}$

Continuation of the proof of 4.20: By the associativity law for CS iterations of proper forcing, the following cases suffice. Also, clause (1) of the conclusion is a special case of clause (2) and if $\beta < \alpha^*$ then the statement has already been proved (by the induction hypothesis).

First Case: $\alpha = 0$, $\beta = \alpha^* = 1$.

Trivial.

Second Case: $\alpha = 1$, $\beta = \alpha^* = 3$.

Let $E^{1,3} = \{\bar{a} : \bar{a} \in \mathcal{S}^{\omega_1}_{\leq \aleph_0}(\lambda), \bar{a} = \langle a_i : i \leq i^* \rangle$ and for some j^*_1, j^*_2 we have :
$$0 < j^*_1 < j^*_2 < i^*, j^*_2 + 5 \leq i^* \text{ and } \bar{a} \restriction (j^*_1 + 1) \in E^1$$
$$\text{and } \langle a_0 \rangle \hat{} \bar{a} \restriction (j^*_1 + 3, j^*_2) \in E^2\}$$

Third Case: $\alpha = 0, \beta = \alpha^* = 2$.

Similar to the second case, but easier.

Fourth Case: $\alpha = 1, \beta = \alpha^* = \omega$.
$E^{\alpha,\beta} = \{\bar{a} :$ for some $\langle \beta_j : j \leq \omega^2 \rangle$ increasing and continuous, $\beta_0 = 0$, $\ell g(\bar{a}) = \beta_{\omega^2} + \omega + 1$, and for each n, $m < \omega$, $\langle a_{\beta_{\omega n}} \rangle \hat{} \langle a_\gamma : \beta_{\omega m+n} + 2 \leq \gamma \leq \beta_{\omega m+n+1} \rangle$ belongs to $E^n\}$.
If $\kappa \geq \aleph_0$, $\langle \beta_j : j \leq \omega \rangle$ suffices.

Fifth Case: $\alpha = 1$, $\beta = \alpha^* = \mathrm{cf}(\alpha^*) > \aleph_0$.
$E^{\alpha,\beta} = \{\bar{a} \in \mathcal{S}^{<\omega_1}_{\leq \aleph_o}(\lambda) : \beta \in a_0$ and for some $\langle \beta_j : j \leq \omega^2 \rangle$ increasing continuous $\beta_0 = 0, \ell g(\bar{a}) = \beta_{\omega^2} + \omega + 1$, and we can choose $\langle \alpha(n) : n < \omega \rangle$ such that: $\alpha(n) \in a_0 \cap \beta, \alpha(n) < \alpha(n+1), \alpha(0) = \alpha(= 1)$ and $\sup(\beta \cap a_0) = \bigcup_{n<\omega} \alpha(n)$,

and for each $n, m < \omega$ we have $\langle a_{\beta_{\omega n}} \rangle \hat{} \langle a_\gamma : \beta_{\omega m+n} + 2 \leq \gamma \leq \beta_{\omega m+n+1} \rangle$ belongs to $E^{\alpha(n),\alpha(n+1)}\}$. $\square_{4.20}$

4.24 Concluding Remarks. 1) There is not much difference between using $(\mathcal{D}, \mathbb{D})$-completeness and $(F, \mathbb{D})$-completeness (where F is a stationary subset of $\mathcal{S}_{\leq \aleph_0}(\lambda)$ for some λ), as long as we do not mind increasing λ (see 4.18).

2) In Theorem 4.20 $(< \omega_1)$-proper can be weakened by restricting ourselves to e.g., E^α_W-proper for every α, where $E^\alpha_W = \{\langle a_i : i \leq \alpha \rangle \in \mathcal{S}^\alpha_{\leq \aleph_0}(\lambda) : a_i \cap \omega_1 \in W\}$ for a stationary subset $W \subseteq \omega_1$; demanding that each E_β is a subset of $\bigcup_{\alpha < \omega_1} E^\alpha_W$, then also in the definition of "long " (4.17) we restrict ourselves to the case $\bigwedge_i N_i \cap \omega_1 \in W$.

3) We could have replaced E^α by $\mathrm{Dom}(\mathbb{D}^\alpha)$, etc.

4) We may note the following generalization. Call $E \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$ unambiguous if for no $\bar{a}, \bar{b} \in E$ is $\bar{a}$ a proper initial segment of $\bar{b}$. Then for $\bar{c} \in \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$, we let $\zeta_E(\bar{c})$ be the unique $\zeta \leq \ell g(\bar{c})$ such that $\bar{c}{\restriction}\zeta \in E$ (maybe $\zeta_E(\bar{c})$ is not defined.)

Now in Definition 4.15 we will have also $E^* \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$ stationary such that $E \subseteq \mathrm{Dom}(\zeta_{E^*})$ and now we call $\mathbb{D}$ a $(\lambda, E, E^*, \kappa)$-system and note that only now (in (B)) $\mathbb{D}_{\langle \bar{a}, <^*, \lambda, p \rangle}$ is a family of subsets of $\mathcal{P}(a_{\zeta_{E^*}(\bar{a})})$. Now, in Definition in 4.16, the family is:

$\{G$: 1) for every $\varepsilon \leq \zeta(\bar{a}), \alpha \in a_\varepsilon$ for some $i \in a_\varepsilon$ we have $p^i_\alpha \in G$

2) $(\exists q \in P^*)(\forall r \in G)(r \leq q)$ (i.e. G has an upper bound in $(P^*, <^*)$)

3) $p \in G\}$.

(This generalization adds some indices to the proofs but no essential changes. This was the point of the original version of 4.2, 4.3.)

5) We can view (4) as a particular case of, more generally, putting an induction hypothesis on $G \cap N_0$.

6) See more in X §7 and XVIII §1, §2.

4.22 Definition. We say $E \subseteq \mathcal{S}^{<\omega_1}_{\leq \aleph_0}(\lambda)$ is simple if letting $H(\aleph_1, \lambda)$ be the closure of λ under taking countable subsets, E is first order definable in $(H(\aleph_1, \lambda), \in, <^*)$.

4.23 Claim. In 4.20 condition (e), instead of "$\mathbb{D}, E$ are from V", it is enough to demand that they are simple.