II. Iteration of Forcing

§0. Introduction

Suppose $V_{\ell+1}$ is a generic extension of V_ℓ, for $\ell = 0, 1$. Is V_2 a generic extension of V_0? In §1 we present the possible answer, in fact if $V_{\ell+1} = V_\ell[G_\ell]$, G_0 is a subset of P generic over V_0, G_1 is a subset of $\underset{\sim}{Q}[G_0]$ generic over V_1, we can get V_2 by some subset G of $P * \underset{\sim}{Q}$ generic over V_0, and there are natural mappings between the family of possible pairs $[G_0, G_1]$ and the family of possible G's. In §2 we deal with iterations $\langle P_\ell, \underset{\sim}{Q}_\ell : \ell < \alpha \rangle$ of length an ordinal α.

This seems suitable to deal with proving the consistency of "for every x there is y such that ..." each Q_α producing a y_α for some $x_\alpha \in V^{P_\alpha}$. However V^{P_α} is not $\bigcup_{i<\alpha} V^{P_i}$, still if we speak of, say, $x \in H(\lambda)$ and $\mathrm{cf}(\alpha) \geq \lambda$ and $P_\alpha = \bigcup_{i<\alpha} P_i$, and P_α satisfies the c.c.c. (or less), then no "new" x appear in V^{P_α}, so we can "catch our tail."

An important point is what we do for limit ordinals δ. We choose $P_\delta = \bigcup_{i<\delta} P_i$ (direct limit), this is the meaning of FS (finite support iteration). An important property is (see 2.8): if each $\underset{\sim}{Q}_i$ satisfies the c.c.c. then so does P_δ.

In §3 we present MA (Martin's axiom) and prove its consistency. The axiom says inside the universe, for any c.c.c. forcing notion P we can find directed $G \subseteq P$ which are "quite generic", say not disjoint to $\mathcal{I}_i$ for $i < i^*$ if $\mathcal{I}_i \subseteq P$ is dense and $i^* < 2^{\aleph_0}$. The proof of its consistency (3.4) is by iterations as in §2 of c.c.c. forcing notions, the point being the right bookkeeping and the "catching of your tail." We then give some applications; if $A_i \subseteq \omega$ $(i < \lambda < 2^{\aleph_0})$ are infinite almost disjoint, $S \subseteq \lambda$ then for some $f : \omega \to \{0,1\}$ $f{\restriction}A_i =_{ae} 1_{A_i}$ iff $i \in S$ (this is 3.7), and we can omit "$A_i \subseteq \omega$" (by 3.5). Can we, for $\langle f_i \in {}^{A_i}\{0,1\} : i < \lambda\rangle$, $\lambda > \aleph_0$ find $f : {}^{\omega}\{0,1\}$ such that $f{\restriction}A_i =_{ae} f_i$? (we say $\langle A_i : i < \lambda\rangle$ has uniformization). If the A_i's are like branches of trees and the f_i are constant then yes (by 3.9).

In §4 we continue with this question. *If* $A_i \subseteq \omega$ $(i < \lambda)$ and the A_i has splitting ≤ 2 (if not branches of a tree this mean: if we know the first n members of A_i, there are ≤ 2 possibilities for the nth member) *then* uniformization fail (see 4.2 and 4.4). Moreover if MA holds, then $\langle A_i : i < \lambda\rangle$ has a subsequence as above so the answer is no. Still a positive answer is consistent (see 4.6) where we fix $\langle A_i : i < \lambda\rangle$ and preserve a strong negation of: for no uncountable $S \subseteq \lambda$, $\langle A_i : i \in S\rangle$ has spliting ≤ 2. For this we demand each Q_i to satisfy a strong version of the c.c.c.

Lastly §5 deals with the existence of mad (=maximal almost disjoint) families of subsets of ω, showing the consistency of the existence.

Also this chapter presents old material. The material is mostly from Solovay, Tennenbaum [ST] (consistency of Souslin hypothesis using FS iteration of c.c.c. forcing notions) and Martin, Solovay [MS] (on applications of MA), but note that [ST] use Boolean algebras. But the iteration like here was shortly later known, and MA was formulated and proved consistent independently by Martin and Rowbottom.

The material of §4 is from [Sh:98, §4, §1 (mainly 1.1(3), 1.2)] (where we phrase the iteration theorem more generally). See more (for higher cardinals) in Mekler and Shelah [MkSh:274].

§1. The Composition of Two Forcing Notions

1.1 Discussion. We shall now ask the following question. Suppose we start with V, extend it once by means of the forcing notion P to $V[G]$, where G is a generic subset of P, then we take a forcing notion Q in $V[G]$ and extend

$V[G]$ to $V[G][H]$, where H is a subset of Q generic over $V[G]$; can $V[G][H]$ be obtained from V by a single forcing extension? The answer is positive as we shall now see.

Since $Q \in V[G], Q$ does not necessarily belong to V but it has a P-name $\underset{\sim}{Q}$ in V. Since Q is a forcing notion we have, by the Forcing Theorem that $p \Vdash_P$ "$\underset{\sim}{Q}$ is a forcing notion (i.e. a partial order)", for some $p \in G$. We shall, however, make the stronger assumption, which suffices for all our needs, that $\Vdash_P$ "$\underset{\sim}{Q}$ is a forcing notion." If this stronger assumption would fail to hold we can define

$$\underset{\sim}{Q}^\dagger = \begin{cases} \underset{\sim}{Q} & \text{if } p \in G \\ \{0\} & \text{otherwise} \end{cases}$$

and then $\Vdash_P$ "$\underset{\sim}{Q}^\dagger$ is a P-name for Q in $V[G]$". To prove that "forcing twice" is like forcing once we want to define a forcing notion $P * \underset{\sim}{Q}$, i.e. the set and the order.

1.2 Definition. 1) $P * \underset{\sim}{Q} = \{\langle p, \underset{\sim}{q}\rangle : p \in P, \underset{\sim}{q}$ is a canonical P-name of a potential member of $\underset{\sim}{Q}$, i.e., $\Vdash_P$ "$\underset{\sim}{q} \in \underset{\sim}{Q}$", $\underset{\sim}{q}$ a canonical P-name"$\}$.

[What are the canonical names of a potential member of $\underset{\sim}{Q}$ and why do we use this concept? A member of $V[G]$ may have a proper class of names in V so $P * \underset{\sim}{Q}$ would be a proper class if we would not restrict the names to a certain representative set. What is important is that every member of $\underset{\sim}{Q}[G]$ for every generic subset G of P has a name in this set, e.g., has a canonical name. By I 5.13 this is true. Also there is a class of P-names $\underset{\sim}{q}$ such that: for some $p \in P$, we have $p \Vdash_P$ "$\underset{\sim}{q} \in Q$"; again demanding $\Vdash$ "$\underset{\sim}{q} \in Q$" suffice as for every P-name $\underset{\sim}{q}^0$ there is a P-name $\underset{\sim}{q}^1$ such that $\Vdash_P$ "$\underset{\sim}{q}^1 \in Q$" and $\Vdash_P$ "if $\underset{\sim}{q}^0 \in Q$ then $\underset{\sim}{q}^0 = \underset{\sim}{q}^1$" (why? by I 1.19, definition by cases as we assume $\underset{\sim}{Q} \neq \emptyset$).]

2) We define on $P * \underset{\sim}{Q}$ a (pre) partial order as follows:

$\langle p_1, \underset{\sim}{q}_1\rangle \leq \langle p_2, \underset{\sim}{q}_2\rangle$ *if* $p_1 \leq p_2$ (i.e. $P \models p_1 \leq p_2$) and $p_2 \Vdash_P$ "$\underset{\sim}{q}_1 \leq \underset{\sim}{q}_2$ in the partial order $\underset{\sim}{Q}$".

1.3 Claim. If $P \in V$ is a forcing notion, $\Vdash_P$ "$\underset{\sim}{Q}$ is a forcing notion" *then* $P * \underset{\sim}{Q}$ is a forcing notion.

Proof. First $\langle p, \underset{\sim}{q} \rangle \leq \langle p, \underset{\sim}{q} \rangle$ since $p \Vdash_P$ "$\underset{\sim}{q} \leq \underset{\sim}{q}$" because $\langle p, \underset{\sim}{q} \rangle \in P * \underset{\sim}{Q}$ implies $p \Vdash_P$ "$\underset{\sim}{q} \in \underset{\sim}{Q}$" and since $\Vdash_P$ "$(\underset{\sim}{Q}, \leq)$ is a forcing notion". Now assume $\langle p_1, \underset{\sim}{q}_1 \rangle \leq \langle p_2, \underset{\sim}{q}_2 \rangle \leq \langle p_3, \underset{\sim}{q}_3 \rangle$ then, since $p_3 \geq p_2 \geq p_1$ we have $p_3 \geq p_1$ and $p_3 \Vdash_P$ "$\underset{\sim}{q}_1 \leq \underset{\sim}{q}_2 \,\&\, \underset{\sim}{q}_2 \leq \underset{\sim}{q}_3$". Since also $\Vdash_P$ "$(\underset{\sim}{Q}, \leq)$ is a forcing notion", clearly $p_3 \Vdash_P$ "$\underset{\sim}{q}_1 \leq \underset{\sim}{q}_3$" and so $\langle p_1, \underset{\sim}{q}_1 \rangle \leq \langle p_3, \underset{\sim}{q}_3 \rangle$. We shall use $P * \underset{\sim}{Q}$ as a forcing notion. $\square_{1.3}$

1.4 Theorem. This theorem asserts, essentially, that forcing by $P * \underset{\sim}{Q}$ is equivalent to forcing first by P and then by $\underset{\sim}{Q}[G]$.

(1) Let $G_{P*\underset{\sim}{Q}} \subseteq P * \underset{\sim}{Q}$ be generic over V then

a. $G_P \stackrel{\text{def}}{=} \{p \in P : (\exists p^\dagger)(\exists \underset{\sim}{q})(p \leq p^\dagger \,\&\, \langle p^\dagger, \underset{\sim}{q} \rangle \in G_{P*\underset{\sim}{Q}})\} \subseteq P$ is generic over V.

b. $G_{P*\underset{\sim}{Q}}/G_P \stackrel{\text{def}}{=} \{\underset{\sim}{q}[G_P] : (\exists p)[\langle p, \underset{\sim}{q} \rangle \in G_{P*\underset{\sim}{Q}}]\}$ is a generic subset of $\underset{\sim}{Q}[G_P]$ over $V[G_P]$.

(2) If G_P is a generic subset of P over V and $H \subseteq \underset{\sim}{Q}[G_P]$ is a generic subset of $\underset{\sim}{Q}[G_P]$ over $V[G]$ *then* $G_P * H \stackrel{\text{def}}{=} \{\langle p, \underset{\sim}{q} \rangle : \langle p, \underset{\sim}{q} \rangle \in P * \underset{\sim}{Q} \,\&\, p \in G_P \,\&\, \underset{\sim}{q}[G] \in H\} \subseteq P * \underset{\sim}{Q}$ is a generic subset of $P * \underset{\sim}{Q}$ over V.

(3) The operations in (1) and (2) are one inverse of the other.

Proof. (1) a. G_P is downward closed by its definition. G_P is directed since $G_{P*\underset{\sim}{Q}}$ is. Now let $\mathcal{I}$ be a dense subset of P. Define $\mathcal{I}^+ = \{\langle p, \underset{\sim}{q} \rangle \in P * \underset{\sim}{Q} : p \in \mathcal{I}\}$. We shall see that $\mathcal{I}^+$ is a dense subset of $P * \underset{\sim}{Q}$. Let $\langle p, \underset{\sim}{q} \rangle \in P * \underset{\sim}{Q}$, then there is a $p^\dagger \geq p$ such that $p^\dagger \in \mathcal{I}$. Now $\langle p^\dagger, \underset{\sim}{q} \rangle \in P * \underset{\sim}{Q}$. Also $\langle p^\dagger, \underset{\sim}{q} \rangle \geq \langle p, \underset{\sim}{q} \rangle$ (since $p^\dagger \Vdash$ "$\underset{\sim}{q} \leq \underset{\sim}{q}$", as $\Vdash_P$ "$(\underset{\sim}{Q}, \leq)$ is a forcing notion"). Since $\langle p^\dagger, \underset{\sim}{q} \rangle \in \mathcal{I}^+$ this shows that $\mathcal{I}^+$ is dense. Therefore $\mathcal{I}^+ \cap G_{P*\underset{\sim}{Q}} \neq \emptyset$, let $\langle p, \underset{\sim}{q} \rangle \in \mathcal{I}^+ \cap G_{P*\underset{\sim}{Q}}$, then $p \in \mathcal{I} \cap G_P$, hence $\mathcal{I} \cap G_P \neq \emptyset$. As we prove it for every $\mathcal{I}$, G_P is a generic subset of P over V.

(1)b. (i) We want first to prove that $G_{P*\underset{\sim}{Q}}/G_P$ is downward closed, so let $\underset{\sim}{Q}[G_P] \models$ "$q^\dagger \le \underset{\sim}{q}[G_P]$" in $V[G_P]$ and assume $\underset{\sim}{q}[G_P] \in G_{P*\underset{\sim}{Q}}/G_P$. Then by the Forcing Theorem (and I 5.13) there is a canonical name $\underset{\sim}{q}^\dagger$ such that $q^\dagger = \underset{\sim}{q}^\dagger[G_P]$, and for some $p^\dagger \in G_P$ we have $p^\dagger \Vdash$ "$\underset{\sim}{q}^\dagger \le \underset{\sim}{q}$ and $\underset{\sim}{q}^\dagger \in \underset{\sim}{Q}$". By the "definition by cases" w.l.o.g. $\Vdash_P$ "$\underset{\sim}{q}^\dagger \in \underset{\sim}{Q}$" (we will usually forget to mention this explicity). As we assume $\underset{\sim}{q}[G_P] \in G_{P*\underset{\sim}{Q}}/G_P$ clearly for some $p \in G_P$ we have $\langle p, \underset{\sim}{q}\rangle \in G_{P*\underset{\sim}{Q}}$. Since $p^\dagger \in G_P$ there are $p'', \underset{\sim}{q}''$ such that $\langle p'', \underset{\sim}{q}''\rangle \in G_{P*\underset{\sim}{Q}}$ and $p^\dagger \le p''$ (by the definition of G_P.) Since $G_{P*\underset{\sim}{Q}}$ is directed there is $\langle p^*, \underset{\sim}{q}^*\rangle \in G_{P*\underset{\sim}{Q}}$ such that $\langle p^*, \underset{\sim}{q}^*\rangle \ge \langle p, \underset{\sim}{q}\rangle$, $\langle p'', \underset{\sim}{q}''\rangle$. We claim that $\langle p^*, \underset{\sim}{q}^\dagger\rangle \le \langle p^*, \underset{\sim}{q}^*\rangle$. Since $\langle p^*, \underset{\sim}{q}^*\rangle \ge \langle p, \underset{\sim}{q}\rangle$, we have $p^* \Vdash$ "$\underset{\sim}{q}^* \ge \underset{\sim}{q}$". Since also $p^* \ge p'' \ge p^\dagger$ and $p^\dagger \Vdash$ "$\underset{\sim}{q}^\dagger \le \underset{\sim}{q}$" we have $p^* \Vdash$ "$\underset{\sim}{q}^\dagger \le \underset{\sim}{q}$", so together with the previous sentence $p^* \Vdash$ "$\underset{\sim}{q}^\dagger \le \underset{\sim}{q}^*$", hence $\langle p^*, \underset{\sim}{q}^*\rangle \ge \langle p^*, \underset{\sim}{q}^\dagger\rangle$. Since $\langle p^*, \underset{\sim}{q}^*\rangle \in G_{P*\underset{\sim}{Q}}$ also $\langle p^*, \underset{\sim}{q}^\dagger\rangle \in G_{P*\underset{\sim}{Q}}$ and $q^\dagger = \underset{\sim}{q}^\dagger[G_P] \in G_{P*\underset{\sim}{Q}}/G_P$. Thus $G_{P*\underset{\sim}{Q}}/G_P$ is downward closed.

To see that $G_{P*\underset{\sim}{Q}}/G_P$ is directed let $q, q^\dagger$ be in $G_{P*\underset{\sim}{Q}}/G_P$. Then $q = \underset{\sim}{q}[G_P], q^\dagger = \underset{\sim}{q}^\dagger[G_P]$ for some $\underset{\sim}{q}, \underset{\sim}{q}^\dagger$ such that there are $p, p^\dagger$ satisfying $\langle p, \underset{\sim}{q}\rangle$, $\langle p^\dagger, \underset{\sim}{q}^\dagger\rangle \in G_{P*\underset{\sim}{Q}}$. Since $G_{P*\underset{\sim}{Q}}$ is directed there is a $\langle p'', \underset{\sim}{q}''\rangle \in G_{P*\underset{\sim}{Q}}$ such that $\langle p'', \underset{\sim}{q}''\rangle \ge \langle p, \underset{\sim}{q}\rangle, \langle p^\dagger, \underset{\sim}{q}^\dagger\rangle$. We have $p'' \Vdash$ "$\underset{\sim}{q}, \underset{\sim}{q}^\dagger \le \underset{\sim}{q}''$" and since $p'' \in G_P$ we have $\underset{\sim}{q}[G_P], \underset{\sim}{q}^\dagger[G_P] \le \underset{\sim}{q}''[G_P]$ and obviously $q''[G_P] \in G_{P*\underset{\sim}{Q}}/G_P$.

(ii) Let $\mathcal{I}$ be a dense subset of $\underset{\sim}{Q}[G_P]$ in $V[G_P]$, and we shall show that $\mathcal{I} \cap G_{P*\underset{\sim}{Q}}/G_P \neq \emptyset$. Clearly $\mathcal{I}$ has a P-name $\underset{\sim}{\mathcal{I}}$ and for some $p_0 \in G_P$ we have $p_0 \Vdash_P$ "$\underset{\sim}{\mathcal{I}}$ is a dense subset of $\underset{\sim}{Q}$".

Let $\mathcal{I}^+ = \{\langle p, \underset{\sim}{q}\rangle \in P * \underset{\sim}{Q} : p \Vdash_P \text{"}\underset{\sim}{q} \in \underset{\sim}{\mathcal{I}} \,\&\, p \ge p_0\text{"}\}$. Since $p_0 \in G_P$ and, as we can replace p_0 by any p', $p_0 \le p' \in G_P$, w.l.o.g. there is a $\underset{\sim}{q}_0$ such that $\langle p_0, \underset{\sim}{q}_0\rangle \in G_{P*\underset{\sim}{Q}}$. We claim $\mathcal{I}^+$ is dense above $\langle p_0, \underset{\sim}{q}_0\rangle$ in $P * \underset{\sim}{Q}$. Let $\langle p^\dagger, \underset{\sim}{q}^\dagger\rangle \ge \langle p_0, \underset{\sim}{q}_0\rangle$ and let $G^\dagger$ be any generic subset of P such that $p^\dagger \in G^\dagger$. In $V[G^\dagger]$ we know that $\underset{\sim}{\mathcal{I}}[G^\dagger]$ is a dense subset of $\underset{\sim}{Q}[G^\dagger]$, hence (by I 3.1, I 5.13) there is a canonical name $\underset{\sim}{q}''$ of a member of $\underset{\sim}{Q}$ such that $\underset{\sim}{q}''[G^\dagger] \ge \underset{\sim}{q}^\dagger[G^\dagger]$ and $q''[G^\dagger] \in \underset{\sim}{\mathcal{I}}[G^\dagger]$. Let $p'' \in G^\dagger$ be such that $p'' \Vdash_P$ "$\underset{\sim}{q}'' \in \underset{\sim}{\mathcal{I}}$" and $p'' \Vdash_P$ "$\underset{\sim}{q}'' \ge \underset{\sim}{q}^\dagger$"

and (as $G^\dagger$ is directed) $p'' \geq p^\dagger$. Then clearly $\langle p'', \underset{\sim}{q}'' \rangle \geq \langle p^\dagger, \underset{\sim}{q}^\dagger \rangle$ and $\langle p'', \underset{\sim}{q}'' \rangle \in \mathcal{I}^+$. So we have proved that $\mathcal{I}^+$ is dense in $P * \underset{\sim}{Q}$, hence $\mathcal{I}^+ \cap G_{P*\underset{\sim}{Q}} \neq \emptyset$. Let $\langle p_1, \underset{\sim}{q}_1 \rangle \in \mathcal{I}^+ \cap G_{P*\underset{\sim}{Q}}$, then $\underset{\sim}{q}_1[G_P] \in \underset{\sim}{\mathcal{I}}[G_P] = \mathcal{I}$. So really in $V[G_P]$ we have $\mathcal{I} \cap (G_{P*\underset{\sim}{Q}}/G_P)$ is not empty, as required.

(2) (i) To see that $G_P * H$ is downward closed let $\langle p, \underset{\sim}{q} \rangle \in G_P * H$ and $\langle p^\dagger, \underset{\sim}{q}^\dagger \rangle \leq \langle p, \underset{\sim}{q} \rangle$. Then $p \in G_P, p^\dagger \leq p$ hence also $p^\dagger \in G_P$. Now $\underset{\sim}{q}[G_P] \in H$ and $q^\dagger[G_P] \leq q[G_P]$ (since $p \in G_P$) hence $\underset{\sim}{q}^\dagger[G_P] \in H$, therefore $\langle p^\dagger, \underset{\sim}{q}^\dagger \rangle \in G_P * H$; so we have proved that $G_P * H$ is downward closed.

To see that $G_P * H$ is directed let $\langle p, \underset{\sim}{q} \rangle, \langle p^\dagger, \underset{\sim}{q}^\dagger \rangle \in G_P * H$. Since $p, p^\dagger \in G_P$ there is a $p'' \geq p, p^\dagger$ in G_P. Since $\underset{\sim}{q}[G_P], \underset{\sim}{q}^\dagger[G_P] \in H$ there is a canonical name $\underset{\sim}{q}''$ of a potential member of $\underset{\sim}{Q}[G_P]$ such that $\underset{\sim}{q}''[G_P] \geq \underset{\sim}{q}[G_P]$, $q^\dagger[G_P]$ and $\underset{\sim}{q}''[G_P] \in H$. Since G_P is directed we can assume, without loss of generality that $p'' \Vdash_P$ "$\underset{\sim}{q}'' \in \underset{\sim}{Q} \,\&\, \underset{\sim}{q}'' \geq \underset{\sim}{q} \,\&\, \underset{\sim}{q}'' \geq \underset{\sim}{q}^\dagger$". Thus $\langle p'', \underset{\sim}{q}'' \rangle \in P * \underset{\sim}{Q}$ and $\langle p'', \underset{\sim}{q}'' \rangle \geq \langle p, \underset{\sim}{q} \rangle, \langle p^\dagger, \underset{\sim}{q}^\dagger \rangle$.

(ii) Let $\mathcal{I}$ be a dense open subset of $P * \underset{\sim}{Q}$. Let $\mathcal{I}/G_P = \{\underset{\sim}{q}[G_P] : (\exists p \in G_P) [\langle p, \underset{\sim}{q} \rangle \in \mathcal{I}]\}$. We shall see that $\mathcal{I}/G_P$ is a dense subset of $\underset{\sim}{Q}[G_P]$ in $V[G_P]$. Let $q_0 \in \underset{\sim}{Q}[G_P]$, then for some canonical name $\underset{\sim}{q}_0$ of a member of $\underset{\sim}{Q}$ we have $\underset{\sim}{q}_0[G_P] = q_0$, and let $p_0 \in G_P$, so $p_0 \Vdash_P$ "$\underset{\sim}{q}_0 \in \underset{\sim}{Q}$". Then $\langle p_0, \underset{\sim}{q}_0 \rangle \in P * \underset{\sim}{Q}$. Let $\mathcal{I}_{\underset{\sim}{q}_0} = \{p \in P : p \geq p_0 \,\&\, (\exists \underset{\sim}{q}^\dagger)[p \Vdash_P$ "$\underset{\sim}{q}^\dagger \geq \underset{\sim}{q}_0$" and $\langle p, \underset{\sim}{q}^\dagger \rangle \in \mathcal{I}]\} \in V$. We shall see that $\mathcal{I}_{\underset{\sim}{q}_0}$ is dense in P above p_0. Let $p_1 \geq p_0$, then $\langle p_1, \underset{\sim}{q}_0 \rangle \in P * \underset{\sim}{Q}$. Since $\mathcal{I}$ is dense in $P * \underset{\sim}{Q}$ there is a $\langle p^\dagger, \underset{\sim}{q}^\dagger \rangle \in \mathcal{I}$ such that $\langle p^\dagger, \underset{\sim}{q}^\dagger \rangle \geq \langle p_1, \underset{\sim}{q}_0 \rangle$. We have also $p^\dagger \Vdash_P$ "$\underset{\sim}{q}^\dagger \geq \underset{\sim}{q}_0$" hence $p^\dagger \in \mathcal{I}_{\underset{\sim}{q}_0}, p^\dagger \geq p_1$ and so $\mathcal{I}_{\underset{\sim}{q}_0}$ is dense in P above p_0. Since $p_0 \in G_P$ by I 1.18 there is p_2 such that $p_0 \leq p_2 \in G_P$ and $p_2 \in \mathcal{I}_{\underset{\sim}{q}_0}$, hence for some $\underset{\sim}{q}^\dagger$ we have $p_2 \Vdash_P$ "$\underset{\sim}{q}^\dagger \geq \underset{\sim}{q}_0$" and $\langle p_2, \underset{\sim}{q}^\dagger \rangle \in \mathcal{I}$. Since $p_2 \in G_P$ we have $\underset{\sim}{q}^\dagger[G_P] \geq \underset{\sim}{q}[G_P] = q$ and $q^\dagger[G_P] \in \mathcal{I}/G_P$ (as $\langle p_2, \underset{\sim}{q}^\dagger \rangle \in \mathcal{I}$). So really $\mathcal{I}/G_P$ is a dense subset of $\underset{\sim}{Q}[G_P]$ (in $V[G_P]$). Since H is a generic subset of $\underset{\sim}{Q}[G_P]$ over $V[G_P]$ we have $(\mathcal{I}/G_P) \cap H \neq \emptyset$, let $\underset{\sim}{q}[G_P] \in (\mathcal{I}/G_P) \cap H$. Since $\underset{\sim}{q}[G_P] \in \mathcal{I}/G_P$ there is a $p \in G_P$ such that $\langle p, \underset{\sim}{q} \rangle \in \mathcal{I}$. Therefore $\langle p, \underset{\sim}{q} \rangle \in \mathcal{I} \cap (G_P * H)$, which establishes the genericity of $G_P * H$.

(3) Left to the reader. $\square_{1.4}$

1.5 Lemma. For every $P * \underset{\sim}{Q}$ -name $\underset{\sim}{\tau}$ there is a P-name $\underset{\sim}{\tau}^*$ such that for every generic subset G_P of P, $\underset{\sim}{\tau}^*[G_P]$ is a $\underset{\sim}{Q}[G_P]$-name in $V[G_P]$ and for every subset G_Q of Q which is generic over $V[G_P]$ we have $(\underset{\sim}{\tau}^*[G_P])[G_Q] = \underset{\sim}{\tau}[G_P * G_Q]$. We use the notation $\underset{\sim}{\tau}^*[G_P] = \underset{\sim}{\tau}/G_P$.

Proof. By induction on the rank α of the $(P * \underset{\sim}{Q})$-name $\underset{\sim}{\tau}$. Suppose $\underset{\sim}{\tau} = \{\langle\langle p_i, \underset{\sim}{q}_i\rangle, \underset{\sim}{\tau}_i\rangle : i < i_0\}$ where $\langle p_i, \underset{\sim}{q}_i\rangle \in P * \underset{\sim}{Q}$ and $\underset{\sim}{\tau}_i$ is a $(P * \underset{\sim}{Q})$-name of rank $< \alpha$. By the induction hypothesis each $\underset{\sim}{\tau}_i$ has a translation $\underset{\sim}{\tau}_i^*$ to a P-name such that $\underset{\sim}{\tau}_i^*[G_P]$ is a $\underset{\sim}{Q}[G_P]$-name and $\underset{\sim}{\tau}_i^*[G_P][G_Q] = \underset{\sim}{\tau}_i[G_P * G_Q]$. Let $\underset{\sim}{\sigma}_i$ be the P-name of $\langle \underset{\sim}{q}_i, \underset{\sim}{\tau}_i^*\rangle$ and let $\underset{\sim}{\tau}^* = \{\langle p_i, \underset{\sim}{\sigma}_i\rangle; i < i_0\}$; this is clearly suitable. (See I 1.8). $\square_{1.5}$

1.6 Definition. If $P \lessdot Q$ (see Definition I 5.3(2)), and $G \subseteq P$ is generic over V, *then* let $Q/G \in V[G]$ be the following forcing notion:

(1) its set of elements is $\{q \in Q : q \text{ is compatible in } Q \text{ with every } p \in G\}$,

(2) its order is inherited from Q.

Sometimes we write Q/P instead $Q/\underset{\sim}{G}_P$, (so it is a P-name) and if h is a complete embedding of P into Q (or even to $RO(Q)$) we write $Q/(P, h)$.

1.7 Lemma. (1) $P \lessdot P * \underset{\sim}{Q}$ (when P a forcing notion, $\Vdash_P$ "$\underset{\sim}{Q}$ a forcing notion" and we identify $p \in P$ with $\langle p, \emptyset_{\underset{\sim}{Q}}\rangle$)

(2) The forcing notions $(P * \underset{\sim}{Q})/P$ and $\underset{\sim}{Q}$ are equivalent (i.e., this is forced by P). Moreover for any generic $G \subseteq P$, the function f, $f(\langle p, \underset{\sim}{q}\rangle/\approx) = \underset{\sim}{q}[G]/\approx$ (for $\langle p, \underset{\sim}{q}\rangle \in P * \underset{\sim}{Q}/G$, equivalently $p \in G$) is an isomorphism from $(P * \underset{\sim}{Q})[G]/\approx$, onto $\underset{\sim}{Q}[G]/\approx$ (where $\approx$ denotes the relation defined in I 5.5).

(3) If $P \lessdot Q$ (both forcing notions in V) *then* Q is equivalent to $P * (Q/P)$.

$\square_{1.7}$

It is not hard to see that

1.8 The Associative Law Lemma. If P is a forcing notion, $\Vdash_P$ "$\underset{\sim}{Q}$ a forcing notion", $\Vdash_{P*\underset{\sim}{Q}}$ "$\underset{\sim}{R}$ a forcing notion", then $(P * \underset{\sim}{Q}) * \underset{\sim}{R}$, and $P * (\underset{\sim}{Q} * \underset{\sim}{R})$ are equivalent. $\square_{1.8}$

§2. Iterated Forcing

2.1 Discussion. We saw already that two successive extensions of V by forcing are equivalent to an appropriate single extension. We want to ask now the same question about an infinite number of extensions by forcing. The need for this arose in the following case.

By a classical theorem, if a linearly ordered set is dense and complete, has no first or last member, and has a dense countable subset then it is order-isomorphic to the real numbers. Souslin raised the question whether one can replace the last requirement, that there is a dense countable subset, by the requirement that every set of pairwise disjoint intervals is at most countable. The statement that these two additional requirements are equivalent is called *Souslin's hypothesis*, and an ordered set which is a counterexample to Souslin's hypothesis is called a *Souslin's continuum.* Jech and Tennenbaum proved the independence of Souslin's hypothesis of the axioms of ZFC by using forcing to obtain a universe in which there is a Souslin continuum. Later Jensen proved that in the constructible universe there is a Souslin continuum. To prove the consistency of Souslin's hypothesis Solovay and Tennenbaum, [ST] proceeded as follows: Given V and a Souslin continuum C, one can construct a generic extension $V[G]$ of V in which C is no more a Souslin continuum by a generic introduction of an uncountable set of pairwise disjoint intervals in C. In this case we say that we have "destroyed" C. Using the same method we can go on destroying more and more Souslin continua. It can be shown that a Souslin's continuum has exactly $2^{\aleph_0}$ points, hence there are at most $2^{2^{\aleph_0}}$ Souslin continua (up to isomorphism). However, when we destroy one Souslin continuum, new ones will be created and we must be sure we have destroyed them all. Since we have infinite time at our disposal this may be possible if we can "catch out tail", but for this we need that in the end no new Souslin continua arise which look doubtful by the above estimate. However we can show that it is enough to deal with subsets of Souslin continuum of power $\aleph_1$, so there are only $2^{\aleph_1}$ such orders. We shall iterate λ times, so that in the new universe, $2^{\aleph_0} = 2^{\aleph_1} = \lambda$, so if the cofinality of the length of the iteration is $> \aleph_1$ we have a chance to catch our tail.

2.2. Definition. We shall call $\bar{Q} = \langle \underset{\sim}{Q}_j : j < \alpha \rangle$ (or $\bar{Q} = \langle P_j, \underset{\sim}{Q}_j : j < \alpha \rangle$ or $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : j < \alpha \text{ and } i \leq \alpha \rangle$), for some ordinal α, *a system of FS (finite support) iterated forcing* (or FS iteration) if each $\underset{\sim}{Q}_j, j < \alpha$ is a name, for the forcing notion P_j, of a forcing notion (=quasi-order), i.e., $\Vdash_{P_j}$ "$\underset{\sim}{Q}_j$ is a forcing notion", where P_j for $j \leq \alpha$ is defined by recursion as the set of all finite functions f with domain included in j such that for all $i \in \mathrm{Dom}(f)$ we have $f(i)$ is a canonical name for the forcing notion P_i of a potential member of $\underset{\sim}{Q}_i$ and we call P_α the direct limit of $\bar{Q}$ and denote it $\lim_{<\aleph_0}(\bar{Q})$. So $i < j \leq \alpha$, $p \in P_j \Rightarrow p{\restriction}i \in P_i$ and $P_i \subseteq P_j$. We use freely I 5.13.

This is called iterated forcing *with finite support* since the functions f we use in the P_j's are finite functions. The P_j's are sets since we restrict the choice of the $f(i)$'s to be canonical names of members of $\underset{\sim}{Q}_i$. We can replace "finite support" by CS ("countable support") or "$< \kappa$-support", see Chapter III.

The partial order on P_j for $j \leq \alpha$ is defined as follows: $f \leq g \Leftrightarrow \mathrm{Dom}(f) \subseteq \mathrm{Dom}(g)\,\&\,(\forall i \in \mathrm{Dom}(f))\ [g{\restriction}i \Vdash_{P_i} \text{"}f(i) \leq_{\underset{\sim}{Q}_i} g(i)\text{"}]$.

2.2A Fact. In Definition 2.2:

(1) If $i < j \leq \alpha$ then $P_i \subseteq P_j$ as sets and even as partial orders.

(2) If $i < j \leq \alpha$ and $p \in P_j$ then $p{\restriction}i \in P_i$; moreover $P_j \models$ "$p{\restriction}i \leq p$" and if $p{\restriction}i \leq q \in P_i$ then $r \overset{\text{def}}{=} q \cup p{\restriction}(j \setminus i)$ belong to P_j and is the least upper bound of q, p in P_j (actually as we are dealing with quasi order we should say a least upper bound).

(3) If $i < j \leq \alpha$ then $P_i \lessdot P_j$ and $q \in P_i, p \in P_j \Rightarrow P_j \models q \leq p \leftrightarrow P_i \models q \leq p{\restriction}i$.

(4) If $j \leq \alpha$ is a limit ordinal then $P_j = \bigcup\limits_{i<j} P_i$.

(5) The sequence $\langle \underset{\sim}{Q}_j : j < \alpha\rangle$ uniquely determines the sequence $\langle P_j, \underset{\sim}{Q}_j : j < \alpha\rangle$ and vice versa and similarly for $\langle P_i, \underset{\sim}{Q}_j : j < \alpha$, and $i \le \alpha\rangle$.

(6) If $\underset{\sim}{Q}'_i$ is a P_i-name, such that $\Vdash$ "$\underset{\sim}{Q}'_i$ is a dense subset of $\underset{\sim}{Q}_i$" *then* $P'_i = \{f \in P_i$: for every $j \in \mathrm{Dom}(f)$ we have: $\Vdash_{P_i}$ "$f(j) \in \underset{\sim}{Q}'_i$"$\}$ is a dense subset of P_i. Moreover we can define and prove by induction on $i \le \alpha$, that $P''_i = \{f \in P_i :$ for every $j \in \mathrm{Dom}(f)$ we have: $f(j)$ is a P''_i-canonical name of a member of $\underset{\sim}{Q}'_i\}$ is a dense subset of P_i and $\underset{\sim}{Q}''_i$ is a canonical P''_i-name satisfying $\Vdash_{P_i}$ "$\underset{\sim}{Q}''_i = \underset{\sim}{Q}'_i$" and $\langle P''_{j_0}, \underset{\sim}{Q}''_{j_1} : j_0 \le i, j_1 < i\rangle$ is a FS iteration.

(6A) Assume Q'_i is a set of canonical P_i-names of member of $\underset{\sim}{Q}_i$ such that for every P_i-name $\underset{\sim}{p}$ for some $\underset{\sim}{q} \in Q'_i$ we have $\Vdash_{P_i}$ "if $\underset{\sim}{p} \in \underset{\sim}{Q}_i$ then $\underset{\sim}{Q}_i \models$ "$\underset{\sim}{p} \le \underset{\sim}{q}$"". Then $P'_i = \{f \in P_i :$ for every $j \in \mathrm{Dom}(f)$ we have $f(j) \in Q'_i\}$ is a dense subset of P'_i and $\langle P'_i, \underset{\sim}{Q}_j : i \le \alpha, j < \alpha\rangle$ satisfies (1) — (4) above.

(7) If λ is regular uncountable, $\alpha < \lambda$ and $\Vdash_{P_i}$ " the density of $\underset{\sim}{Q}_i$ is $< \lambda$" *then* the density of P_α is $< \lambda$.

(8) If for $i < \alpha$, $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i \in H(\lambda_i)$", $\langle \lambda_i : i \le \alpha\rangle$ is an increasing sequence of regulars, $2^{\lambda_i} < \lambda_{i+1}$ and for limit $\delta \le \alpha$, $\sum_{i<\delta} \lambda_i < \lambda_\delta$, *then* $\bar{Q} \in H(\lambda_\alpha)$.

2.3 Definition. Let $\langle P_i, \underset{\sim}{Q}_j : i \le \alpha, j < \alpha\rangle$ be a FS iteration. For $\beta \le \gamma \le \alpha$ we define $P_{\beta,\gamma}$ and if $\gamma < \alpha$, $\underset{\sim}{Q}^{[\beta]}_\gamma$ by recursion on γ as follows: $P_{\beta,\gamma}$, $\underset{\sim}{Q}^{[\beta]}_\gamma$ are P_β-names, $P_{\beta,\gamma}$ is the set of all finite functions f from $\gamma \setminus \beta$ such that for $i \in \mathrm{Dom}(f)$, $f(i)$ is a canonical P_β-name for a canonical $P_{\beta,i}$-name of a potential member of $\underset{\sim}{Q}^{[\beta]}_i$. Now if $\gamma < \alpha$ then essentially $P_\gamma = P_\beta * \underset{\sim}{P}_{\beta,\gamma}$ (see 2.4(a)) and let $\underset{\sim}{Q}^{[\beta]}_\gamma$ be $\underset{\sim}{Q}_\gamma / \underset{\sim}{G}_{P_\beta}$ (see Definition 1.6).

The next theorem is given here without proof.

2.4 The General Associativity Theorem.

(a) $P_\gamma \approx P_\beta * \underset{\sim}{P}_{\beta,\gamma}$ for $\beta \le \gamma \le \alpha$, where $\underset{\sim}{P}_{\beta,\gamma}$ is a name in the forcing notion P_β for the forcing notion which is $P_{\beta,\gamma}$ in $V[G]$, where G is a generic subset of P_β, and where the $\underset{\sim}{Q}_j$'s, $\beta \leq j < \alpha$ are translated to names for the forcing notions $P_{\beta,j}$, $\approx$ means that one of the two forcing notions are isomorphic to a dense subset of the other, so that they represent essentialy the same forcing notion (see I §5).

(b) $P_{\beta,\beta+1} \approx Q_\beta$ over $V[G_\beta]$ where G_β is a generic subset of P_β.

(c) If $\langle \beta_i : i \leq \gamma \rangle$ is an increasing and continuous sequence such that $\beta_0 = 0$ and $\beta_\gamma = \alpha$, then $\langle P_{\beta_i,\beta_{i+1}} : i < \gamma \rangle$ is an iterated forcing equivalent (in the $\approx$ sense) to $\langle \underset{\sim}{Q}_i : i < \alpha \rangle$. This is, in some sense, a general associative rule. $\square_{2.4}$

2.5 The Definition by Induction Theorem. (One can construct $\underset{\sim}{Q}_i$'s by a given recursive recipe.) If F is a function and α is an ordinal then there is a unique FS iterated forcing $\langle \underset{\sim}{Q}_j : j < \alpha_0 \rangle$ such that for all $j < \alpha_0$, $\underset{\sim}{Q}_j = F(\langle \underset{\sim}{Q}_i : i < j \rangle)$ and either $\alpha_0 = \alpha$ or else $F(\langle \underset{\sim}{Q}_i : i < \alpha_0 \rangle)$ is not suitable for $\underset{\sim}{Q}_{\alpha_0}$, i.e., it is not a name of a forcing notion in the forcing notion P_{α_0}.

Proof. This theorem is an obvious consequence of the standard definition-by-recursion theorem. $\square_{2.5}$

2.6 Theorem. If P and $\underset{\sim}{Q}$ are as in the definition of $P * \underset{\sim}{Q}$ and P and $\underset{\sim}{Q}$ satisfy the c.c.c., where by "$\underset{\sim}{Q}$ satisfying the c.c.c.", we mean $\Vdash_P$ "$\underset{\sim}{Q}$ satisfies the c.c.c.", *then* $P * \underset{\sim}{Q}$ satisfies the c.c.c.

Proof. Let $\{\langle p_i, \underset{\sim}{q}_i \rangle : i < \aleph_1\}$ be a sequence of conditions in $P * \underset{\sim}{Q}$. We claim first that there is a $p \in P$ such that $p \Vdash_P$ "$|\{i : p_i \in \underset{\sim}{G}_P\}| = \aleph_1$", where G_P is the generic subset of P. Suppose this is not the case, then $\Vdash_P$ "$|\{i : p_i \in G_P\}| = \aleph_0$". Let $B = \{\zeta : (\exists r \in P)\ [r \Vdash$ "$\sup\{i : p_i \in \underset{\sim}{G}_P\} = \zeta$"$]\}$. Since each i for which p_i is defined is a countable ordinal and $\{i : p_i \in G_P\}$ is a countable set in $V[G_P]$ also each $\zeta \in B$ is a countable ordinal in $V[G_P]$ and hence in V (since P satisfies the c.c.c., by I 3.6 we know $\aleph_1^{V[G_P]} = \aleph_1^V$). Since P satisfies the c.c.c again by I 3.6 we know B is countable, so let $\xi \stackrel{\text{def}}{=} \sup(B)$, we have $\xi < \aleph_1$.

Obviously $p_{\xi+1} \Vdash$ "$\xi + 1 \in \{i : p_i \in G_P\}$" hence $p_{\xi+1} \Vdash$ "$\sup\{i : p_i \in G_P\} > \xi$" on the other hand we have $\Vdash_P$ "$\sup\{i : p_i \in G_P\} \leq \xi$", which is a contradiction.

Hence now we know that for some $p \in P$ we have $p \Vdash_P$ "$|\{i : p_i \in G_P\}| = \aleph_1$"; let G_P be a generic subset of P such that $p \in G_P$. Let $A = \{i : p_i \in G_P\}$, then the set $\{\underset{\sim}{q}_i[G_P] : i \in A\}$ is an uncountable subset of $\underset{\sim}{Q}[G_P]$, in $V[G_P]$ of course. Since $\underset{\sim}{Q}[G_P]$ satisfies the c.c.c. there are $i, j \in A$ such that $\underset{\sim}{q}_i[G_P]$ and $\underset{\sim}{q}_j[G_P]$ are compatible, hence there is a $\underset{\sim}{q}[G_P] \in \underset{\sim}{Q}[G_P]$ such that $\underset{\sim}{q}[G_P] \geq \underset{\sim}{q}_i[G_P], \underset{\sim}{q}_j[G_P]$. Since $p_i, p_j \in G_P$ and G_P contains conditions which force $\underset{\sim}{q} \geq \underset{\sim}{q}_i$ and $\underset{\sim}{q} \geq \underset{\sim}{q}_j$, because G_P is directed *there exists* a $p^* \in G_P$ such that $p^* \geq p_i, p^* \geq p_j$, $p^* \geq p$ and $p^* \Vdash$ "$\underset{\sim}{q} \geq \underset{\sim}{q}_i$" and $p^* \Vdash$ "$\underset{\sim}{q} \geq \underset{\sim}{q}_j$", thus $\langle p^*, \underset{\sim}{q}\rangle \in P * \underset{\sim}{Q}$ and $\langle p^*, \underset{\sim}{q}\rangle \geq \langle p_i, \underset{\sim}{q}_i\rangle$, $\langle p_j, \underset{\sim}{q}_j\rangle$ and so $\langle p_i, \underset{\sim}{q}_i\rangle$ $\langle p_j, \underset{\sim}{q}_j\rangle$ are compatible (in $P * \underset{\sim}{Q}$). $\square_{2.6}$

Within the proof of this theorem we have proved the following:

2.7 Observation. Let P be a forcing notion which satisfies the c.c.c. and let $\{p_i : i < \omega_1\}$ be a sequence of members of P. *Then* for some $p \in P$ we have $p \Vdash_P$ "$|\{i < \omega_1 : p_i \in \underset{\sim}{G}_P\}| = \aleph_1$", where $\underset{\sim}{G}_P$ denotes the generic subset of P (in fact for every $\xi < \omega_1$ large enough, $p = p_\xi$ satisfies the conclusion). $\square_{2.7}$

2.8 Theorem. If $\langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ is a system of FS iterated forcing and for each $i < \alpha$ the forcing $\underset{\sim}{Q}_i$ satisfies the c.c.c. (i.e., $\Vdash_{P_i}$ "$\underset{\sim}{Q}$ satisfies the c.c.c.") *then* P_α satisfies the c.c.c.

Proof. We proceed by induction on α. For $\alpha = 0$, P_0 consists of the null function only, hence it satisfies, trivially, the c.c.c. If α is a successor let $\alpha = \beta + 1$ then by 2.4 we know $P_\alpha = P_{\beta+1} \approx P_\beta * P_{\beta,\beta+1} \approx P_\beta * \underset{\sim}{Q}_\beta$. By the induction hypothesis P_β satisfies the c.c.c. and since also $\underset{\sim}{Q}_\beta$ satisfies the c.c.c., theorem 2.6 establishes that also $P_\beta * \underset{\sim}{Q}_\beta$ satisfies the c.c.c. The relation $\approx$ obviously preserves the c.c.c. (see I 5.15), hence P_α too satisfies the c.c.c.

So assume α is a limit ordinal. Let $\{p_i : i < \omega_1\} \subseteq P_\alpha$. Now $\{\mathrm{Dom}(p_i) : i < \omega_1\}$ is an uncountable family of finite subsets of α.

If $\mathrm{cf}(\alpha) > \aleph_1$ then for some $\xi < \alpha$ we have $\bigcup_{i<\omega_1} \mathrm{Dom}(p_i) \subseteq \xi$, so $p_i \in P_\xi$ for $i < \omega_1$ and we can apply the induction hypothesis.

If $\mathrm{cf}(\alpha) = \aleph_0$, let $\alpha = \bigcup_{n<\omega} \alpha_n$, $\alpha_n < \alpha_{n+1} < \alpha$. So for each $i < \omega_1$ for some $n(i) < \omega$ we have $\mathrm{Dom}(p_i) \subseteq \alpha_{n(i)}$. So for some $n(*) < \omega$ the set $A \stackrel{\mathrm{def}}{=} \{i < \omega_1 : n(i) = n^*\}$ is uncountable. So $\{p_i : i \in A\}$ is an uncountable subset of $P_{\alpha_{n(*)}}$ so by the induction hypothesis for some $i \neq j$ from A, p_i, p_j are compatible in $P_{\alpha_{n(*)}}$ hence in P_α as required.

Lastly, assume $\mathrm{cf}(\alpha) = \aleph_1$, so let $\langle \alpha_i : i < \aleph_1 \rangle$ be a (strictly) increasing continuous sequence of ordinals with limit α. Clearly for every $i < \omega_1$, $\mathrm{Dom}(p_i)$ being a finite subset of α is included in $\alpha_{g(i)}$, for some $g(i) < \omega_1$ and is disjoint to $[\alpha_{f(i)}, \alpha_i)$ for some countable ordinal $f(i) < i$ when i is limit ordinal (remember α_i is increasing continuous).

So clearly $E = \{i < \aleph_1$: i is a limit ordinal and for every $j < i$ we have $g(j) < i\}$ is a club of $\aleph_1$, and by Fodor lemma for some $j(*)$ the set $S = \{i < \aleph_1 : f(i) = j(*)\}$ is stationary and let $\xi = \alpha_{j(*)}$. Now $\{p_i \restriction \xi : i \in E \cap S\}$ is an uncountable subset of P_ξ, hence by induction hypothesis there are in it two compatible numbers $p_i \restriction \xi$, $p_j \restriction \xi$, i.e., there is $q \in P_\xi$ such that $q \geq p_i \restriction \xi$, $p_j \restriction \xi$, but then clearly p_i, p_j are compatible in P_α. e.g. $q \bigcup (p_i \restriction [\xi, \alpha) \bigcup (p_j \restriction [\xi, \alpha))$ is a common bound. $\square_{2.8}$

2.9 Lemma. Assume $\langle P_i, \underset{\sim}{Q} : i < \alpha \rangle$ is a FS iteration of c.c.c. forcing notions, $\Vdash_{P_i}$ "$|\underset{\sim}{Q}_i| \leq \lambda$" (forcing) and $\lambda^{\aleph_0} = \lambda$ and $|\alpha| \leq \lambda$.

1) If $\Vdash_{P_i}$ " the set of elements of $\underset{\sim}{Q}_i$ is $\subseteq V$" (for each $i < \alpha$) *then* $|P_\alpha| \leq \lambda$.

2) Without this extra assumption, P_α has a dense subset of cardinality $\leq \lambda$.

3) In (1) if $Y \subseteq V$, $|Y| \leq \lambda$ then the number of canonical P_i-names of members of Y is $\leq \lambda$.

Proof. 1) We prove it by induction on α. For each $i < \alpha$, by the induction hypothesis, $|P_i| \leq \lambda$, by 2.8 the forcing notion P_i satisfies the c.c.c. so by I 3.6 there is a set $Y_i \in V$ of cardinality $\leq \lambda$ (in V) such that $\Vdash_{P_i}$ " every member of $\underset{\sim}{Q}_i$ belongs to Y_i". As in I §4 the number of canonical P_i-names of members of Y_i (hence of members of $\underset{\sim}{Q}_i$) is at most λ. Let $\hat{Q}_i$ be the set of canonical P_i-names of members of $\underset{\sim}{Q}_i$. So P_α is the set of functions f, with domain a finite subset of α, and $i \in \mathrm{Dom}(f) \Rightarrow f(i) \in \hat{Q}_i$. Clearly

$$|P_\alpha| \leq \sum_n |\alpha|^n \cdot (\sup_i |\hat{Q}_i|)^n \leq \lambda.$$

2) Let $\underset{\sim}{f}_\beta$ be a P_β-name for a one to one function for some ordinal ($\leq \lambda$) onto $\underset{\sim}{Q}_\beta$. Now $P'_\alpha = \{p \in P_\alpha :$ for each $\beta \in \mathrm{Dom}(p)$ for some canonical P_β-name $\underset{\sim}{\tau}$ of an ordinal, $p(\beta) = \underset{\sim}{f}_\beta(\underset{\sim}{\tau})\}$.

We prove by induction on α that P'_α is a dense subset α of cardinality $\leq \lambda$.

3) Left to the reader. $\square_{2.9}$

§3. Martin's Axiom and Few Applications

What is the meaning of MA (Martin's Axiom, discovered by Martin and Rowbottom independently). It says that we can find quite generic sets inside our universe. As we have noted before (see I 1.4), if P has no trivial generic subset (i.e., above any $p \in P$ there are two incompatible members of P) then we cannot find a generic subset of P over V. But we may well find such $G \subseteq P$ generic over some $V^\dagger \subseteq V$. So it is plausible that for any family of $< \kappa$ dense subsets of P there is a directed $G \subseteq P$ not disjoint to any of them. How can we build a model V satisfying such a requirement? We extend and re-extend the universe, in stage α we extend the universe we have got V_α, to $V_{\alpha+1} = V_\alpha[G_\alpha]$ by forcing by some forcing notion Q_α. The hope is that in the end $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ is as required, as if $R \in V_\lambda$ is a suitable forcing notion (satisfying the c.c.c. with elements from V and cardinality $< \lambda$ in our case) and $\mathcal{I}_i \subseteq R$ a dense subset for $i < i_0$, then R and $\langle \mathcal{I}_i : i < i_0 \rangle$ belongs to some V_i, and for some j, $i \leq j < \lambda$, $Q_j = R$. So the generic object $G_{j+1} \subseteq P_{j+1}$ will give a generic subset of $\underset{\sim}{Q}_j[G_{j+1} \cap P_j]$ as required: essentially $G_{j+1}/(G_{j+1} \cap P_j)$; so this construction is similar in some sense, to the consturction of λ-saturated models in model theory. Now $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ is impossible when $V_\alpha = V[G_\alpha]$, $G_\alpha \subseteq P_\alpha$ is generic over V, but $H(\lambda)^{V_\lambda} = \bigcup_{\alpha<\lambda} H(\lambda)^{V_\alpha}$ is reasonable and is enough.

This is carried out by iterated forcing.

3.1 Definition. Martins's Axiom for κ. MA_κ. If P is a forcing notion satisfying the c.c.c. and $\mathcal{I}_i \subseteq P$ is dense in P for $i < \kappa$ *then* there is a directed subset G of P such that for every $i < \kappa$ we have $G \cap \mathcal{I}_i \neq \emptyset$.

3.2 Observation. In order to have MA_κ it suffices to require that the definition of MA_κ hold only for forcing notions P such that $|P| \leq \kappa$.

Proof. To prove this let P be a forcing notion satisfying the c.c.c. and for $i < \kappa$ let $\mathcal{I}_i \subseteq P$ be a dense subset of P. For $i < \kappa$ let f_i be a function with domain P such that for $p \in P$ we have $f_i(p) \geq p$ and $f_i(p) \in \mathcal{I}_i$. Let g be a function with domain $P \times P$ such that for $p, q \in P$ if p and q are compatible then $g(p,q) \geq p, q$. Let p_0 be any member of P and let Q be the closure of $\{p_0\}$ under the functions f_i for $i < \kappa$ and g (i.e. this is its set of elements, the order is inherited.) We have $|Q| \leq \kappa$. Let us see that $\mathcal{I}_i \cap Q$ is dense in Q. Let $p \in Q$ then $f_i(p) \geq p, f_i(p) \in \mathcal{I}_i$, and $f_i(p) \in Q$ since Q is closed under f_i. Now let us prove that Q satisfies the c.c.c. Let A be an antichain in Q; we claim that A is also an antichain in P and hence $|A| \leq \aleph_0$. Let $p, q \in A$ be distinct, then p, q are incompatible in P since if p, q were compatible in P we would have $g(p,q) \geq p, q$, but Q is closed under g hence also $g(p,q) \in Q$ and p, q are compatible also in Q, contradicting the assumption that A is an antichain in Q. By the version of MA_κ for $|P| \leq \kappa$ there is a directed subset G of Q such that $G \cap \mathcal{I}_i \cap Q \neq \emptyset$ for all $i < \kappa$. This G is as required. $\square_{3.2}$

3.3 Definition. Martin's Axiom MA. $(\forall \kappa < 2^{\aleph_0})\mathrm{MA}_\kappa$.

$\mathrm{MA}_{\aleph_0}$ is true since if for $i < \omega$ the set $\mathcal{I}_i$ is a dense subset of P then let us pick by induction a sequence $\langle p_n : n < \omega \rangle$ such that $p_n \in \mathcal{I}_n$ and $p_{n+1} \geq p_n$, then choose $G = \{p_n : n < \omega\}$.

Therefore we have that the CH implies MA. However, usually one means by MA, MA with the negation of the CH.

3.4 Theorem. If $\aleph_0 < \lambda = \lambda^{<\lambda}$ then there is a forcing notion P, $|P| = \lambda$ which satisfies the c.c.c. and such that $\Vdash_P$ "$2^{\aleph_0} = \lambda$ & MA".

Proof. First we shall show that if $\lambda = \lambda^{<\lambda}$, $|P| \leq \lambda$ and P satisfies the c.c.c. then $\Vdash_P$ "$\lambda = \lambda^{<\lambda}$". Since P satisfies the c.c.c. and $\lambda > \aleph_0$, by I 3.6(i) the ordinal λ is an uncountable cardinal also in $V[G]$.

To prove the theorem we shall give a canonical name to every function from μ to λ, where μ is a cardinal $< \lambda$. The canonical names (for this context) will be as follows: such a name has the form

$$\{\langle p_{i,n}, \langle i, j_{i,n}\rangle\rangle : i < \mu \text{ and } n < \omega\}$$

where $j_{i,n} < \lambda$, $p_{i,n} \in P$. For each P-name $\underset{\sim}{\tau}$, $\Vdash_P$ "$\underset{\sim}{\tau}$ a function from μ to λ" choose for each $i < \mu$, a maximal antichain $\{p_{i,n} : n < \omega\} \subseteq P$ (possibly with repetitions) such that $p_{i,n} \Vdash_P$ "$\underset{\sim}{\tau}(i) = j_{i,n}$" for some $j_{i,n} < \lambda$ (this is possible as $\mathcal{I}_i = \{p : p \Vdash_P$ "$\underset{\sim}{\tau}(i) = j$" for some $j < \lambda\}$ is a dense open subset of P, as P satisfies the c.c.c.)(if the maximal antichain is finite we can change notation or use "possibly with repetitions"). Let

$$\underset{\sim}{\tau}^\dagger = \{\langle p_{i,n}, \langle i, j_{i,n}\rangle\rangle : i < \mu, n < \omega\}$$

Then clearly $\Vdash_P$ "$\underset{\sim}{\tau}^\dagger = \underset{\sim}{\tau}$". So we can consider only canonical $\underset{\sim}{\tau}$. What is their number? For each $i < \mu$ we choose two ω-sequences, one from P and one from λ, so we have $\leq \lambda^{\aleph_0}|P|^{\aleph_0}$; and so the numbers of such names is $\leq (\lambda^{\aleph_0}|P|^{\aleph_0})^\mu \leq \lambda^\mu \leq \lambda^{<\lambda} = \lambda$. Hence clearly in V^P, $\lambda^\mu = \lambda$ for $\mu < \lambda$. Now we return to the proof.

Let $\bar{S} = \langle S_{\gamma,\mu} : \gamma < \lambda,\ \mu < \lambda, \mu$ is a cardinal $\rangle$ be a partition of λ to disjoint sets each of cardinality λ such that $i \in S_{\gamma,\mu} \Rightarrow i \geq \gamma$. We shall define $\underset{\sim}{Q}_i$ by induction such that $\Vdash_{P_i}$ "the members of $\underset{\sim}{Q}_i$ are from V" and $|P_i| \leq \lambda$. Assuming we have arrived to α, we know P_α satisfies the c.c.c. by 2.8 (we carry the definition by 2.5). At stage α all the $\underset{\sim}{Q}_\beta$'s, for $\beta < \alpha$, are defined, and hence P_α is defined. Let $\langle \underset{\sim}{\leq}_\xi : \xi \in S_{\alpha,\mu} \rangle$ be a list of the canonical names for the forcing P_α of quasi-orders of μ. We shall use below $\leq_\xi$ in the ξ-th stage of the construction but since for $\xi \in S_{\alpha,\mu}$ we have $\xi \geq \alpha$ this does not spoil the induction. But can we find such a list, i.e. is $S_{\alpha,\mu}$ large enough? So how many such canonical names are there? By the induction hypothesis and 2.9, $|P_\alpha| \leq \lambda$. A quasi-order on μ is a function on $\mu \times \mu$ into $\{0,1\}$, and as we saw in I 4.2 the number of canonical names in the forcing P_α is $\leq \lambda^{<\lambda} = \lambda$, since $|P_\alpha| \leq \lambda$.

Now by the choice of $\bar{S}$ there are unique $\gamma_\alpha < \lambda$ and $\mu_\alpha < \lambda$ (a cardinal) such that $\alpha \in S_{\gamma_\alpha,\mu_\alpha}$. By the demand above this implies $\alpha \geq \gamma_\alpha$, hence $\langle \underset{\sim}{\leq}_\xi : \xi \in S_{\gamma_\alpha,\mu_\alpha} \rangle$ is already well defined. So in particular $\underset{\sim}{\leq}_\alpha$ is a P_{γ_α}-name of a partial order on μ_α. As $\gamma_\alpha \leq \alpha$ we know $P_{\gamma_\alpha} \lessdot P_\alpha$ hence $\underset{\sim}{\leq}_\alpha$ is also a P_α-name of partial order on μ_α.

We define now

$$\underset{\sim}{Q}_\alpha = \begin{cases} \langle \mu_\alpha, \underset{\sim}{\leq}_\alpha \rangle & \text{if } \Vdash_{P_\alpha} \text{ "}\langle \mu_\alpha, \underset{\sim}{\leq}_\alpha \rangle \text{ satisfies the c.c.c."} \\ \langle 1, \{\langle 1,1 \rangle\} \rangle & \text{otherwise} \end{cases}$$

It is now obvious that $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ is a forcing notion, it satisfies the c.c.c. and its elements are ordinals " since if this does not hold for $(\mu_\alpha, \underset{\sim}{\leq}_\alpha)$ then we have chosen Q_α as $\langle 1, \{\langle 1,1 \rangle\} \rangle$ which obviously satisfies the c.c.c. So we have carried the induction. Therefore $P \overset{\text{def}}{=} P_\lambda$ also satisfies the c.c.c. by 2.8. Our argument above that $|P_\alpha| \leq \lambda$ works also for $\alpha = \lambda$ hence $|P| \leq \lambda$. It is true also that $2^{\aleph_0} \geq \lambda$ since, as we shall see in the next lemma $\mathrm{MA}_\mu \Rightarrow 2^{\aleph_0} > \mu$ and of course $2^{\aleph_0} \leq \lambda$ so equality holds.

Let $G \subseteq P$ $(= P_\lambda)$ be generic over V, we should prove $V[G] \models$ "MA_μ" for a given $\mu < \lambda$. Let $\mu < \lambda$, and let R be a c.c.c. forcing notion in $V[G]$ and let $\mathcal{I}_i$, for $i < \mu$, be dense subsets of R in $V[G]$. As we saw by 3.2 we can assume without loss of generality that the set of members of R is μ (if $|R| < \mu$ we can introduce many "copies" of a single member and setting each of them $\leq$ than all others on μ). Let $\mathcal{I} = \{\langle i,j \rangle : i \in \mathcal{I}_j\} \subseteq \mu \times \mu$. So for some P-names $\underset{\sim}{R}$, $\underset{\sim}{\mathcal{I}}$, $\underset{\sim}{\mathcal{I}}_i$ we have $R = \underset{\sim}{R}[G]$ and $\mathcal{I} = \underset{\sim}{\mathcal{I}}[G]$, $\mathcal{I}_i = \underset{\sim}{\mathcal{I}}_i[G]$, w.l.o.g. $\Vdash_P$ "$\underset{\sim}{R}$ is a quasi order on μ and $\underset{\sim}{\mathcal{I}}$ is a subset of $\mu \times \mu$ and $\mathcal{I}_i = \{j : \langle i,j\rangle \in \underset{\sim}{\mathcal{I}}\}$ is a dense subset of $\underset{\sim}{R}$" (we could have add "and $\underset{\sim}{R}$ satisfies the c.c.c." and slightly save later). For each pair $\langle i,j\rangle \in \mu \times \mu$ there is a maximal antichain $\mathcal{I}_{i,j}$ in P which determines the truth value of $\underset{\sim}{R} \models i \leq j$ and a maximal antichain $\mathcal{J}_{i,j}$ in P which determines the membership of $\langle i,j\rangle$ in $\mathcal{I}$. Now $\bigcup\{\mathrm{Dom}(p) : p \in \bigcup_{i,j<\mu} \mathcal{I}_{i,j} \cup \mathcal{J}_{i,j}\}$ is a subset of λ in V of cardinality $\leq \mu < \lambda$. Now $\lambda = \lambda^{<\lambda}$ hence λ is regular (since $\lambda^{\mathrm{cf}\lambda} > \lambda$) and therefore $\gamma \stackrel{\mathrm{def}}{=} \sup \bigcup\{\mathrm{Dom}(p) : p \in \bigcup_{i,j<\mu} \mathcal{I}_{i,j} \cup \mathcal{J}_{i,j}\} + 1 < \lambda$, and $\mathcal{I}_{i,j}, \mathcal{J}_{i,j} \subseteq P_\gamma$ for $i,\ j < \mu$, and so the P-names $\underset{\sim}{R}$ and $\underset{\sim}{\mathcal{I}}$ of R and $\mathcal{I}$ are names for the forcing notion P_γ. For the generic subset G of P let $G_\gamma = \{p{\restriction}\gamma : p \in G\}$. So G_γ is a generic subset of P_γ. This can be shown in any one of the following two ways. One way is to use the fact that $P_\lambda \approx P_\gamma * \underset{\sim}{P}_{\gamma,\lambda}$ (see 2.4) and then G_γ is the first component of $G \subseteq P_\gamma * \underset{\sim}{P}_{\gamma,\lambda}$ and we have already proved that it is generic in P_γ. Another way is to prove directly that G_γ is a generic subset of P_γ using 2.2A. Since in computing $\underset{\sim}{R}[G] = R$ and $\underset{\sim}{\mathcal{I}}[G] = \mathcal{I}$ only G_γ is used we have $\underset{\sim}{R}[G_\gamma] = R$, $\underset{\sim}{\mathcal{I}}[G_\gamma] = \mathcal{I}$ and

$$V[G_\gamma] \models \text{"}\underset{\sim}{R} \text{ is a quasi-order with set of elements } \mu \,\&\, \underset{\sim}{\mathcal{I}}_i \text{ for } i < \mu \text{ is dense in } \underset{\sim}{R} \text{ and } \underset{\sim}{R} \text{ satisfies the c.c.c. ".}$$

Hence there is a $p \in G_\gamma$ such that $p \Vdash_{P_\gamma}$ "$\underset{\sim}{R}$ is a quasi-order $\mu \,\&\, R$ satisfies the c.c.c. ".

$$\text{Let } \underset{\sim}{\leq}^* \stackrel{\mathrm{def}}{=} \begin{cases} \underset{\sim}{R} & \text{if } p \in G_\gamma \\ \in {\restriction}\mu & \text{otherwise} \end{cases}$$

then $\Vdash_{P_\gamma}$ "$\underset{\sim}{\leq}^*$ is a quasi-order with set of elements μ satisfying the c.c.c. ". Therefore there is a $\xi \in S_{\gamma,\mu}$ such that $\underset{\sim}{\leq}_\xi = \underset{\sim}{\leq}^*$. Since $\xi \in S_{\gamma,\mu}$ we know $\xi \geq \gamma$ hence $G_\xi \supseteq G_\gamma$ and since $p \in G_\gamma$ we have $\underset{\sim}{Q}_\xi[G_\xi] = \underset{\sim}{\leq}_\xi[G_\xi] = R$. $P_{\xi+1} = P_\xi * \underset{\sim}{Q}_\xi$ hence $G^* = G_{\xi+1}/G_\xi$ is a generic subset of $\underset{\sim}{Q}_\xi[G_\xi] = R$ over $V[G_\xi]$ (provided that $(\mu, \underset{\sim}{\leq}_\xi)$ satisfies the c.c.c. in $V[G_\xi]$, but this follows from $V[G] \models$ "R satisfies the c.c.c."). For $i < \mu$ we know $\mathcal{I}_i \in V[G_\gamma] \subseteq V[G_\xi]$ and $\mathcal{I}_i$ is dense in $\underset{\sim}{Q}_\xi[G_\xi] = R$ hence $G^* \cap \mathcal{I}_i \neq \emptyset$, and G^* is a directed subset of R.

3.4A. Lemma. $\mathrm{MA}_\mu \to \mu < 2^{\aleph_0}$.

Proof. Assume $\mu \geq 2^{\aleph_0}$. Let P be the forcing notion of all finite functions from ω into $\{0,1\}$, with proper inclusion as the partial order (i.e. the Cohen forcing). For each $\eta \in {}^{\omega}\{0,1\}$ let $\mathcal{I}_\eta = \{p \in P : p \not\subseteq \eta\}$, and for each $n < \omega$ let $\mathcal{I}_n = \{p \in P : n \in \mathrm{Dom}(p)\}$. Obviously each $\mathcal{I}_\eta$ and $\mathcal{I}_n$ is dense (and open) and there are $2^{\aleph_0} \le \mu$ such sets. By MA_μ there is a directed subset G of P such that $G \cap \mathcal{I}_\eta \ne \emptyset$ for each $\eta \in {}^{\omega}\{0,1\}$ and $G \cap \mathcal{I}_n \ne 0$ for each $n < \omega$. Since G is directed $g = \bigcup G$ is a function, since $G \cap \mathcal{I}_n \ne 0$ we know $n \in \mathrm{Dom}(g)$ for each $n < \omega$, hence $g \in {}^{\omega}\{0,1\}$. Now $G \cap \mathcal{I}_g = \emptyset$ since for every $p \in G$ we know $p \subseteq \bigcup G = g$ hence $p \notin \mathcal{I}_g$, but this contradicts $G \cap \mathcal{I}_g \ne \emptyset$ which we get by MA_μ. $\square_{3.4A,3.4}$

Some Applications of $\mathrm{MA}+2^{\aleph_0} > \aleph_1$.

3.5 Theorem. Assume MA and let λ be a cardinal $\aleph_0 \le \lambda < 2^{\aleph_0}$ and let $\langle A_i : i < \lambda \rangle$ be a family of infinite pairwise almost disjoint subsets of ω (i.e., if $i \ne j$ and $i, j < \lambda$ then $A_i \cap A_j$ is finite), and let S be a subset of λ. There is a function f on ω into $\{0,1\}$ such that for all $i < \lambda$ we have: $f \restriction A_i =_{ae} 1_{A_i}$ iff $i \in S$, where 1_{A_i} is the function on A_i with the fixed value 1 and $=_{ae}$ denotes that two functions have the same values for all elements of their domain except (possibly) finitely many.

Proof. Let

$P = \{f : f$ is a function such that $\mathrm{Dom}(f) = A_{i_1} \cup \cdots \cup A_{i_n} \cup w$ for some $i_1, \ldots, i_n \in S$ and a finite $w \subseteq \omega$ and for $1 \le \ell \le n$ we have $f \restriction A_{i_\ell} =_{ae} 1_{A_{i_\ell}}\}$.

The partial order on P is inclusion.

Since, for each $f \in P, f^{-1}[\{0\}]$ is finite we can take for each $f \in P$ a finite $w_f \supseteq f^{-1}[\{0\}]$ to play the role of w in the definition of P.

To see that P satisfies the c.c.c. let $\langle f_i : i < \aleph_1 \rangle$ be a sequence of members of P. Since all the $f \restriction w_f$ belong to the countable set of all finite functions from ω into $\{0,1\}$ we have $i \ne j$, $i, j < \aleph_1$ such that $f_i \restriction w_{f_i} = f_j \restriction w_{f_j}$. Obviously $f_i \bigcup f_j$ is a function and a member of P and above f_i and f_j, hence f_i and f_j are compatible.

We shall specify below λ dense subsets of P (called $\mathcal{I}^*_n, \mathcal{I}_i, \mathcal{I}_{n,i}$). By MA there is a directed $G \subseteq P$ such that $\mathcal{I} \cap G \ne \emptyset$ for each one of the specified dense sets $\mathcal{I}$. Let $g = \bigcup G$. Since G is directed every two members of G are compatible and g is a function from a subset of ω into $\{0,1\}$. We establish now the following properties of G.

1. $\mathrm{Dom}(g) = \omega$. For $n < \omega$ let $\mathcal{I}_n^* = \{f \in P : n \in \mathrm{Dom}(f)\}$. Now $\mathcal{I}_n^*$ is dense since for $f \in P$ if $f \notin \mathcal{I}_n^*$ then $f \bigcup \{\langle n, 1\rangle\} \in \mathcal{I}_n^*$. Since $G \cap \mathcal{I}_n^* \neq \emptyset$ there is an $f \in G$ such that $n \in \mathrm{Dom}(f)$, hence $n \in \mathrm{Dom}(f) \subseteq \mathrm{Dom}(g)$.

2. If $i \in S$ then $g \restriction A_i =_{ae} 1_{A_i}$. For $i \in S$ let $\mathcal{I}_i = \{f \in P : \mathrm{Dom}(f) \supseteq A_i\}$. To see that $\mathcal{I}_i$ is dense in P let $f \in P$ then $\mathrm{Dom}(f) = A_{i_1} \cup \cdots \cup A_{i_n} \cup w$ where $i_1, \ldots, i_n \in S$ and w is finite. If $i \in \{i_1, \ldots, i_n\}$ then $f \in \mathcal{I}_i$. If $i \notin \{i_1, \ldots, i_n\}$ then $A_i \cap \mathrm{Dom}(f) \subseteq \bigcup_{1 \leq \ell \leq n} (A_i \cap A_{i_\ell}) \bigcup w$ and each set participating in this union is finite. It follows immediately now that $f \cup [(A_i \setminus \mathrm{Dom}(f)) \times \{1\}] \in P$ and this member of P obviously is above f and belongs to $\mathcal{I}_i$. Hence we have now $G \cap \mathcal{I}_i \neq \emptyset$. Let $f \in G \cap \mathcal{I}_i$, $f \restriction A_i =_{ae} 1_{A_i}$ and since $g \supseteq f$ we have $g \restriction A_i =_{ae} 1_{A_i}$.

3. If $i \notin S$ then $g \restriction A_i$ obtains the value 0 for infinitely many members of A_i. Let $\mathcal{I}_{i,n} = \{f \in P : (\exists m \in A_i)\ (m \geq n \,\&\, f(m) = 0)\}$. To see that $\mathcal{I}_{i,n}$ is dense let $f \in P$, and let f be as in the definition of P. Since $i \notin S$, neccessarily $i \notin \{i_1, \ldots, i_n\}$, therefore, as we saw above, $\mathrm{Dom}(f) \cap A_i$ is finite. Since A_i is infinite there is an $m \in A_i \setminus \mathrm{Dom}(f)$ such that $m \geq n$. Now $f \cup \{\langle m, 0\rangle\} \in P$ hence $f \cup \{\langle m, 0\rangle\} \in \mathcal{I}_{n,i}$ also $f \leq f \cup \{\langle m, 0\rangle\}$; hence we have shown $\mathcal{I}_{n,i}$ is dense. Since $\mathcal{I}_{n,i} \cap G \neq \emptyset$ there is an $f \in G$ such that there is an $m \in A_i$ satisfying $m \geq n$ and $f(m) = 0$, hence $g(m) = 0$. Thus $g(m) = 0$ for arbitrarily large $m \in A_i$. $\square_{3.5}$

3.5A. Conclusion. MA_λ implies $2^\lambda = 2^{\aleph_0}$. $\square_{3.5A}$

It is natural to ask

3.6 Question. Under the assumptions of Theorem 3.5, is there an $f : \omega \to \{0,1\}$ such that $f \restriction A_i =_{ae} 1_{A_i}$ for $i \in S$ and $f \restriction A_i =_{ae} 0_{A_i}$ for $i \notin S$?

We shall return to this later. Note however

3.7 Theorem. In Theorem 3.5 we can omit the requirement "$A_i \subseteq \omega$", requiring only $|A_i| = \aleph_0$.

Proof. We let $P = \{f : f$ is a function whose domain is $A_{i_1} \cup \cdots \cup A_{i_n}$ and whose range is $\subseteq \{0,1\}$ where for some $n < \omega$ we have $i_1 \in S, \dots, i_n \in S$, $f^{-1}[\{0\}]$ finite $\}$, ordered by inclusion.

Let $p_i \in P$ (for $i < \aleph_1$) be $\aleph_1$ conditions, $\mathrm{Dom}(p_i) = \bigcup_{\alpha \in u_i} A_\alpha$ with $u_i < \lambda$ is finite, so w.l.o.g. $i \neq j \Rightarrow u_i \cap u_j = u^*$. By the definition of P, there are only countably many possible $p_i \restriction \bigcup_{\alpha \in u^*} A_\alpha$, so w.l.o.g. $p_i \restriction \bigcup_{\alpha \in u^*} A_\alpha = f$ for every $i < \aleph_1$. Let $w_i \overset{\mathrm{def}}{=} p_i^{-1}[\{0\}]$, it is a finite set so w.l.o.g. $i \neq j \rightarrow w_i \cap w_i = w^*$ & $|w_i| = \ell(*)$. So for each i the cardinality of $\{j : \mathrm{Dom}(p_i) \cap w_j \setminus w^* \neq \emptyset\}$ is at most the cardinality of $\mathrm{Dom}(p_i)$ hence this set is countable, so w.l.o.g. $i < j \Rightarrow \mathrm{Dom}(p_i) \cap w_j \subseteq w^*$.

Now if $i < j$ and p_i, p_j are incompatible, then there is $x \in \mathrm{Dom}(f_i) \cap \mathrm{Dom}(f_j)$ such that $f_i(x) \neq f_j(x)$, so $0 \in \{f_i(x), f_j(x)\}$ hence $x \in w_i \cup w_j$, in fact $x \in (w_i \setminus w^*) \cup (w_j \setminus w^*)$; but by the previous sentence $x \notin w_j \setminus w^*$ so $x \in w_i \setminus w^*$; also $x \notin \mathrm{Dom}(f)$ (as $p_i \restriction \mathrm{Dom}(f) = p_j \restriction \mathrm{Dom}(f)$) so $x \in \bigcup_{\alpha \in u_j \setminus u^*} A_\alpha$, hence $(w_i \setminus w^*) \cap (\bigcup_{\alpha \in u_j \setminus u^*} A_\alpha) \neq \emptyset$. Let $w_i \setminus w^* = \{x_{i,\ell} : \ell < \ell(*)\}$; so rephrasing if $i < j < \aleph_1$ and p_i, p_j are incompatible then for some $\alpha(i,j) \in u_j \setminus u^*$ and $\ell(i,j) < \ell(*)$ we have $x_{i,\ell(i,j)} \in A_{\alpha(i,j)}$. Let D be a nonprincipal ultrafilter on ω, so for each $j \in [\omega, \omega_1)$ for some $\ell_j < \ell(*)$ we have $\{i < \omega : \ell(i,j) = \ell\} \in D$ and as u_j is finite, for some $\alpha_j \in u_j \setminus u^*$ we have $\{i < \omega : \alpha(i,j) = \alpha_j\} \in D$. The number of possible ℓ_j is $\ell(*)$ so for some $j(1) \neq j(2) \in [\omega, \omega_1)$ we have $\ell_{j(1)} = \ell_{j(2)}$. Hence

$$A = \{i < \omega : \ell(i, j(1)) = \ell_{j(1)} = \ell_{j(2)} = \ell(i, j(2)) \text{ and}$$

$$\alpha(i, j(1)) = \alpha_{j(1)} \text{ and } \alpha(i, j(2)) = \alpha_{j(2)}\}$$

belongs to D, so $A_{\alpha_{j(1)}} \cap A_{\alpha_{j(2)}}$ includes $\{x_{i,\ell_{j(1)}} : i \in A\}$ which is infinite, hence we have $\alpha_{j(1)} = \alpha_{j(2)}$ and $(u_{j(1)} \setminus u^*) \cap (u_{j(2)} \setminus u^*) \neq \emptyset$ contradicting the choice of u^*. So P satisfies the c.c.c.

Now if $i < \lambda$, $i \notin S$ and $A_i \cap (\bigcup_{j \in S} A_j)$ is infinite, then $\mathcal{I}_{i,n} \stackrel{\text{def}}{=} \{p \in P : A_i \cap p^{-1}(\{0\})$ has cardinality $\geq n\}$ is a dense subset of P. Also for each $x \in \bigcup\limits_{i<\lambda} A_i$ we have $\mathcal{J}_x \stackrel{\text{def}}{=} \{p \in P : x \in \mathrm{Dom}(p)\}$ is a dense subset of P. Lastly for $i \in S$ we have $\mathcal{I}_i = \{p \in P : A_i \subseteq \mathrm{Dom}(p)\}$ is a dense subset of P.

So there is a directed $G \subseteq P$ such that $G \cap \mathcal{I}_{i,n} \neq \emptyset$ for $i < \lambda$, $i \notin S$, $n < \omega$ and $G \cap \mathcal{I}_i \neq \emptyset$ for $i \in S$, and $G \cap \mathcal{J}_x \neq \emptyset$ for $x \in \bigcup\limits_{i<\lambda} A_i$. Let f^* be $\bigcup_{f \in G} f$ and f^{**} be the function extending f^* with domain $\bigcup\limits_{i<\lambda} A_i$ and being constantly zero on $\bigcup\limits_{i<\lambda} A_i \setminus \bigcup\limits_{i \in S} A_i$. It is easy to check f^{**} is as required. $\square_{3.7}$

A partial answer to the Question 3.6 is

3.8 Definition. A sequence $\langle A_i : i < \lambda \rangle$ of infinite pairwise almost disjoint subset of ω is called a *tree* if for any $i, j < \lambda$ if $n \in A_i \cap A_j$, then $A_i \cap n = A_j \cap n$.

An example of a tree of $2^{\aleph_0}$ subsets of the set of all finite sequences of 0's and 1's is the set $\{T_f : f \in {}^\omega 2\}$ where $T_f = \{f \restriction n : n < \omega\}$.

3.9 Theorem. Assume MA, let λ be a cardinal such that $\aleph_0 \leq \lambda < 2^{\aleph_0}$ and let $\langle A_i : i < \lambda \rangle$ be a family of infinite pairwise almost disjoint subsets of ω which is a tree. Let $S \subseteq \lambda$ then there is an $f : \omega \to \{0, 1\}$ such that for all $i \in S$ we have $f \restriction A_i =_{ae} 1_{A_i}$ and for all $i \in \lambda \setminus S$ we have $f \restriction A_i =_{ae} 0_{A_i}$.

Proof. Let

$P = \{f : f$ is a function and for some $i_1, \ldots, i_n < \lambda$ and a finite subset w of ω we have $\mathrm{Dom}(f) = A_{i_1} \cup \cdots \cup A_{i_n} \cup w$, and for $\ell = 1, \ldots, n$: $f \restriction A_{i_\ell} =_{ae} 1_{A_{i_\ell}}$ if $i_\ell \in S$ and $f \restriction A_{i_\ell} =_{ae} 0_{A_{i_\ell}}$ if $i_\ell \notin S\}$.

The partial order on P is inclusion.

Let us prove now that P satisfies the c.c.c. Let $\langle f_i : i < \aleph_1 \rangle$ be a sequence of members of P. For f_i let $\mathrm{Dom}(f_i) = A_{\alpha(i,1)} \cup \cdots \cup A_{\alpha(i,n(i))} \cup w_i$, where $\alpha(i, 1)$, $\alpha(i, 2), \ldots < \lambda$. Let $k_i < \omega$ be such that k_i is a strict upper bound of w_i and for each ℓ, $1 \leq \ell \leq n(i)$, $A_{\alpha(i,\ell)} \cap k_i$ contains a number m such that $f_i(m') = f_i(m)$ for every number $m' \in A_{\alpha(i,\ell)}$ which is $> m$. There is such a k_i since each $f_i \restriction A_{\alpha(i,\ell)}$ has the value 0 almost everywhere or the value 1 almost everywhere. We demand in addition that for $1 \leq \ell$, $m \leq n(i)$, $\ell \neq m$,

$A_{\alpha(i,\ell)} \cap k_i \not\subseteq A_{\alpha(i,m)} \cap k_i$ and $A_{\alpha(i,\ell)} \cap A_{\alpha(i,m)} \subseteq k_i$; without loss of generality we can assume that for all $i < \aleph_1$ $n(i)$ is fixed, i.e., $n(i) = n$ for $i < \aleph_1$, k_i is fixed, i.e., $k_i = k^*$ for $i < \aleph_1$ and for $1 \leq \ell \leq n$, the set $A_{\alpha(i,\ell)} \cap k^*$ is fixed (for each ℓ) and $f_i \restriction k^*$ is fixed and w_i is fixed. We shall now prove that for $i \neq j$ $i, j < \aleph_1$ the conditions f_i and f_j are compatible. Consider the union $f_i \cup f_j$. If this union is a function it obviously belongs to P and is above f_i and f_j (thus completing the proof of c.c.c.). Suppose it is not a function. Since $f_i \restriction k^* = f_j \restriction k^*$ there is a $k \geq k^*(k < \omega)$ such that $f_i(k) \neq f_j(k)$. Since $w_i = w_j \subseteq k^*$ clearly $k \notin w_i = w_j$, hence for some $1 \leq \ell_0, \ell_1 \leq n$, $k \in A_{\alpha(i,\ell_0)}$ and $k \in A_{\alpha(j,\ell_1)}$. Let $m_0 = \text{Max } A_{\alpha(i,\ell_0)} \cap k^*$, since $k \cap A_{\alpha(i,\ell_0)}$ contains a number m such that we have $(\forall m')[m \leq m' \in A_{\alpha(i,\ell_0)}, f_i(m') = f_i(m)]$, clearly $f_i(m_0) = f_i(k)$. Note $\ell_0 \neq \ell_1 \Rightarrow A_{\alpha(i,\ell_0)} \cap k^* \neq A_{\alpha(i,\ell_1)} \cap k^* = A_{\alpha(j,\ell_1)} \cap k^*$. Since $\langle A_i : i < \aleph_1 \rangle$ is a tree and $k \in A_{\alpha(i,\ell_0)} \cap A_{\alpha(j,\ell_1)}$, $k \geq k^*$, we have $\ell_0 = \ell_1$ and therefore $m_0 = \text{Max}(A_{\alpha(j,\ell_1)} \cap k^*)$ and, as for i, $f_j(k) = f_j(m_0)$. Since $m_0 < k^*$ and $f_i \restriction k^* = f_j \restriction k^*$ we have $f_i(m_0) = f_j(m_0)$ hence $f_i(k) = f_j(k)$, contradicting our choice of k.

Looking at the proof of Theorem 3.5 it is clear how to choose dense subsets $\mathcal{I}$ of P such that if G is a directed subset of P which intersects each one of them then $g = \bigcup G$ has the following properties: $\text{Dom}(g) = \omega$, for $i \in S$ we have $g \restriction A_i =_{ae} 1_{A_i}$ and for $i \in \lambda \setminus S$ we have $g \restriction A_i =_{ae} 0_{A_i}$. $\square_{3.9}$

3.10 Theorem. In 3.9 "$A_i \subseteq \omega$" is not required, provided $|A_i| \leq \aleph_0$.

Proof. The proof is analogous to the proof of Theorem 3.7. $\square_{3.10}$

§4. The Uniformization Property

Let us present now the setting with which we shall deal with the problem 3.6.

4.1 Definition. Let $\bar{A} = \langle A_i : i < \alpha \rangle$ be a sequence of sets (or we use a family of sets), each of which is (usually countable and always) infinite and h a function from $\bigcup_{i<\alpha} A_i$ to the class of nonzero ordinals. We say that $(\bar{A}, h)$ has the *uniformization property* if for every family of functions $\langle f_i : i < \alpha \rangle$ such that $f_i : A_i \to \text{Ord}$ and $f_i(a) < h(a)$, there is a function $f : \bigcup_{i<\alpha} A_i \to \text{Ord}$ such that for each $i < \alpha$ the function f_i is almost included in f (i.e., included except for $< |\text{Dom}(f_i)|$ members a of $\text{Dom}(f_i)$; which usually means except for finitely many $a \in \text{Dom}(f)$). We shall denote $\bigcup_{i<\alpha} A_i$ with $D(\bar{A})$, so $\text{Dom}(h) = D(\bar{A})$. If h is constantly λ we may write $(\bar{A}, \lambda)$ and if $\lambda = 2$ we may write $\bar{A}$. We shall say that $\bar{A}$ is a tree if there is a partial order $<$ on $D(\bar{A})$ such that:

a) $(D(\bar{A}), <)$ is a tree such that for all $x \in D(\bar{A})$ the set $\{y : y < x\}$ is a finite set linearly ordered by $<$.

b) For each i the set A_i is a branch in $D(\bar{A})$, i.e., if $x \in A_i$ and $y < x$ then also $y \in A_i$ and any two members of A_i are $<$-comparable. Since each A_i is infinite and since by clause (a) the tree $(D(A), <)$ is of height $\leq \omega$, each A_i is a maximal branch, i.e., it goes all the way up.

We shall denote $(D(\bar{A}), <)$ with T and T_n will denote the n-th level of T, that is $T_n = \{x : |\{y : y < x| = n\}$. For a branch A we denote with $(A)_{[n]}$ the only member of A in the level n, i.e., the only member of $A \cap T_n$.

4.2 Claim. If $\bar{A} = \langle A_i : i < \alpha \rangle$, $\alpha \geq \omega_1$, is a tree in the above sense and $D(\bar{A}) \subseteq {}^{\omega >}2$ and the tree order is $\triangleleft$ *then* $\bar{A}$ does not have the uniformization property.

Proof. So let $\eta_i \in {}^{\omega}2$ be such that $A_i = \{\eta_i \restriction n : n < \omega\}$ so $(A_i)_{[n]} = \eta_i \restriction n$. Let us define the function f_i on A_i as follows. $f_i(\eta_i \restriction n) = \eta_i(n)$. The value of f_i for this member of $D(\bar{A})$ tells us which way the branch goes at $\eta_i \restriction n$, right or left, since

$$\eta_i \restriction (n+1) = \eta_i \,\hat{}\, \langle \eta_i(n) \rangle$$

Assume now that there is a function f uniformizing all f_j's, or even all f_i's for some uncountable set $I \subseteq \alpha$. For each $i \in I$ let $n(i)$ be such that $f_i(x) = f(x)$ for every $x \in A_i$ of level $\geq n(i)$. Since I is uncountable there is an $n^* < \omega$ such that $\{i \in I : n(i) = n^*\}$ is uncountable, we denote this set with I_1. Since $D(\bar{A})$ has at most 2^{n^*} members of level n^* there is a $\rho \in {}^{n^*}2$ such that $I_2 = \{i \in I_1 : \eta_i{\restriction}n^* = \rho\}$ is uncountable; we denote this set with I_2. For all $i, j \in I_2$ we have $\eta_i{\restriction}n^* = \rho = \eta_j{\restriction}n^*$ and since A_i and A_j are branches we have $\eta_i(k) = \eta_j(k)$ for all $k < n^*$. For $i \neq j$ such that $i, j \in I_2$ we know that A_i and A_j are different branches hence there is a least ℓ such that $\eta_i(\ell) \neq \eta_j(\ell)$, since $\ell < n^* \Rightarrow n(i) = n(j)$ clearly $\ell \geq n^*$ hence $f_i(\eta_i{\restriction}\ell) = f(\eta_i{\restriction}\ell)$ and $f_j(\eta_j{\restriction}\ell) = f(\eta_j{\restriction}\ell)$. By the definition of f_i and f_j we have $\eta_i{\restriction}(\ell+1) = (\eta_i{\restriction}\ell)\hat{\ }\langle\eta_i(\ell)\rangle = (\eta_i{\restriction}\ell)\hat{\ }\langle f(\eta_i{\restriction}\ell)\rangle = (\eta_j{\restriction}\ell)\hat{\ }\langle f(\eta_j{\restriction}\ell)\rangle$ (since $\eta_i{\restriction}\ell = \eta_j{\restriction}\ell$ by the minimality of ℓ) $= (\eta_j{\restriction}\ell)\hat{\ }\langle\eta_j(\ell)\rangle = \eta_j{\restriction}(\ell+1)$, contradicting the choice of ℓ. $\square_{4.2}$

4.3 Theorem. Assume MA. Suppose $\bar{A} = \langle A_i : i < \lambda\rangle$ is a family of pairwise almost disjoint countable sets and $\lambda < 2^{\aleph_0}$. *Then* $(\bar{A}, \aleph_0)$ has the uniformization property provided that:

(∗) for every countable A, $\{i < \lambda : A \cap A_i \text{ is infinite }\}$ is countable.

Proof. Let $f_i : A_i \to \omega$ for $i < \lambda$, and we shall find f, almost extending each f_i. Let $P = \{f :$ there are $i_1, \ldots, i_n < \lambda$ such that $\mathrm{Dom}(f) = A_{i_1} \cup \cdots \cup A_{i_n}$, $\mathrm{Range}(f) \subseteq \omega$ and f almost extend f_i for $\ell = 1, \ldots, n\}$ ordered by inclusion.

It suffices to prove that P satisfies the c.c.c., and proceed as in e.g. 3.5.; but this is not hard (proof similar to 3.9). $\square_{4.3}$

4.4 Discussion. Theorem 4.2 asserts that if $D(\bar{A})$ is a tree which at each node branches into at most two branches then $\bar{A}$ does not have the uniformization property (the fact that in the theorem $D(\bar{A})$ was actually a subset of ${}^{\omega}2$ is obviously irrelevant). The assumption in the theorem that $D(\bar{A})$ is a tree can be replaced by the following weaker assumption. For each $i < \alpha$ let A_i be given as a sequence of length ω enumerating its elements with no repetitions, which we write as $\langle a_{i,0}, a_{i,1}, \ldots\rangle$ so $\{a_{i,n} : n < \omega\} = A_i$. The weaker hypothesis is that $\bar{A}$ is uncountable and for each sequence $\langle b_0, \ldots, b_{n-1}\rangle$ of members of $D(\bar{A})$ the set $\{a_{i,n} : \langle a_{i,0}, \ldots, a_{i,n-1}\rangle = \langle b_0, \ldots, b_{n-1}\rangle\}$ has at most two members, i.e., for all A_i given $a_{i,0}, \ldots, a_{i,n-1}$ there are at most two possibilities for $a_{i,n}$. (This assumption obviously holds if $D(\bar{A})$ is a tree with branching as above and each sequence $\langle a_{i,0}, a_{i,1}, \ldots\rangle$ is such that $a_{i,n} = (A_i)_{[n]}$ for all $n < \omega$). We shall see that also under this weaker assumption $\bar{A}$ does not have the uniformization property.

Let $\bar{A}$ be as above. For each $i < \alpha$ let $B_i = \{\langle a_{i,0}, \ldots, a_{i,n-1}\rangle : n < \omega\}$, $\bar{B} = \langle B_i : i < \alpha\rangle$. Now $\bar{B}$ is obviously a tree which at each node branches to at most two branches, and for $i \neq j$ we have $B_i \neq B_j$ since $A_i \neq A_j$. We can prove by induction on n that $\{a_{i,n} : i < \alpha\}$ has $\leq 2^n$ elements, hence $\bigcup_{i<\alpha} B_i$ is countable. Let $\langle f_i : i < \alpha\rangle$ where $f_i : B_i \to \{0,1\}$ be a family of functions which cannot be uniformized, by claim 4.2. We define a family $\langle g_i : i < \omega_1\rangle$ where $g_i : A_i \to \{0,1\}$ as follows. Let $g_i(a_{i,n}) = f_i(\langle a_{i,0}, \ldots, a_{i,n}\rangle)$. We shall see that this family too cannot be uniformized. Suppose $g : D(\bar{A}) \to \{0,1\}$ uniformizes this family. Define f on $D(\bar{B})$ as follows: $f(\langle a_{i,0}, \ldots, a_{i,n-1}\rangle) = g(a_{i,n-1})$ if $n > 0$ and $f(\langle\rangle)$ can be given any value. We shall see that f is an uniformization of $\langle f_i : i < \alpha\rangle$. Given $i < \alpha$ we know that for some $k < \omega$, for all $n > k$ we have $g_i(a_{i,n}) = g(a_{i,n})$ hence, by the definitions of g_i and f we have $f_i(\langle a_{i,0}, \ldots, a_{i,n}\rangle) = f(\langle a_{i,0}, \ldots, a_{i,n}\rangle)$.

Let us remark now that the theorem is true also for a tree $D(\bar{A})$ which has a bounded branching at each node (and hence also in the more general case mentioned above). We shall show it here for the case where $D(\bar{A}) \subseteq {}^{\omega>}4$, which is sufficiently general to exhibit the proof of the general case. For a natural number k let $d_\ell(k)$ denote the coefficient of 2^ℓ in the binary expansion of k, $d_\ell(k) \in \{0,1\}$. We define now on the branch A_i of ${}^{\omega>}4$ two functions f_i^0 and f_i^1 by setting $f_i^\ell((A_i)_{[n]}) = d_\ell((A_i)_{[n+1]})$, for $\ell = 0, 1$. Let f^ℓ be the function uniformizing the functions $\langle f_i^\ell : i < \alpha\rangle$ for $\ell = 0, 1$. Let n^*, z and I be such that $I \subseteq \alpha$, $|I| > \aleph_0$, $(A_i)_{[n^*]} = z$ for $i \in I$ and $f_i^\ell(n) = f^\ell(n)$ for $n \geq n^*$ (as in the proof of the theorem). We get now a contradiction from the fact that for $i, j \in I$ $i \neq j$ there is a least level where A_i and A_j go to their separate ways, while at each node (from level n^* and above) f^0 and f^1 determine completely the way the branch goes in the next level so A_i and A_j must go the same way.

We deal here exclusively with trees $\bar{A}$ and other systems where $|D(\bar{A})| = \aleph_0$. In 3.8 we dealt with a different definition of a tree, namely we called a family w of subsets of ω a tree if for all $x, y \in w$ if $n \in x \cap y$ then $x \cap n = y \cap n$.

In this case let us define $<^*$ on ω by $k <^* \ell$ if $k < \ell$ and for every $A \in w$ if $\ell \in A$ then also $k \in A$. It is easily seen that $<^*$ is a partial order; the set $\{k : k <^* n\}$ is linearly ordered by $<^*$ which coincides with $<$ on this set and that each $A \in w$ is a branch in the tree $\langle \omega, <^* \rangle$. Thus a tree in the sense of 3.8 is also a tree in the present sense.

Now let $\bar{A}$ be a tree as defined here with $|D(\bar{A})| = \aleph_0$ with $<^*$ as the partial order relation of the tree. Identify the members of $D(\bar{A})$ with the natural numbers in such a way that if $k <^* \ell$ then $k < \ell$. Now suppose $n \in A_i \cap A_j$ and $k < n$, $k \in A_i$. Since A_i is a branch of the tree we have $k <^* n$ or $n <^* k$; $n <^* k$ would imply $n < k$, contradiction; hence $k <^* n$. Since also A_j is a branch, $n \in A_j$ and $k <^* n$ we get also $k \in A_j$. Since this works in both directions we have $A_i \cap n = A_j \cap n$. Thus a tree in the present sense is isomorphic to a tree in the sense of Definition 3.8.

4.5 Theorem. Assuming MA+$2^{\aleph_0} > \aleph_1$. Let $\bar{A} = \langle A_i : i < \aleph_1 \rangle$, $A_i = \{a_{i,n} : n < \omega\}$ be a set of $\aleph_1$ countably infinite pairwise distinct sets such that $|D(\bar{A})| = \aleph_0$. *Then* there is an uncountable subset $W \subseteq \aleph_1$ which has the property that for every sequence $\langle b_0, \ldots, b_{n-1} \rangle$ the set $\{a_{i,n} : \langle a_{i,0}, \ldots, a_{i,n-1} \rangle = \langle b_0, \ldots, b_{n-1} \rangle$, and $i \in W\}$ has at most two members. As a consequence $\bar{A}$ does not have the uniformization property, since $\bar{A} \restriction W$ does not have it.

Proof. Remember A_i is $\{a_{i,0}, a_{i,1}, \ldots\}$ (with no repetitions). Let $P \stackrel{\text{def}}{=} \{w \subseteq \aleph_1 : w$ is finite and for all $\langle b_0, \ldots, b_{n-1} \rangle$ we have

$$|\{a_{n,i} : i \in w \text{ and } \langle a_{i,0}, \ldots, a_{i,n-1} \rangle = \langle b_0, \ldots, b_{n-1} \rangle\}| \leq 2\}$$

and P is partially ordered by inclusion. By deleting at most $\aleph_0$ members of $\bar{A}$ we can make sure that for all $i < \aleph_1$ and $n < \omega$:

$$|\{j < \aleph_1 : \langle a_{j,0}, \ldots, a_{j,n-1} \rangle = \langle a_{i,0}, \ldots, a_{i,n-1} \rangle\}| = \aleph_1$$

[For every finite sequence $\langle b_0, \ldots, b_{n-1} \rangle$ of members of $D(\bar{A})$, and there are $\aleph_0$ such sequences, if $0 < |\{j < \aleph_1 : \langle a_{j,0}, \ldots, a_{j,n-1} \rangle = \langle b_0, \ldots, b_{n-1} \rangle\}| \leq \aleph_0$ we delete all the A_j's with j in this set, so altogether we deleted countably many sets.]

Let us see now that P satisfies the c.c.c. Let $\langle w_t : t < \aleph_1\rangle$ be a sequence of members of P. Let us write w_t as $\{i_1^t, i_2^t, \ldots, i_{n(t)}^t\}$. Let $k(t)$ be the least number k such that the sequences $\langle a_{i_r^t,0}, a_{i_r^t,1}, \ldots, a_{i_r^t,k-1}\rangle$ for $r = 1, \ldots, n(t)$ are pairwise distinct. Without loss of generality we can assume that for all $t < \aleph_1$ we have $n(t) = n$, $k(t) = k$ and for each $r = 1, \ldots, n$ the sequence $\langle a_{i_r^t,0}, a_{i_r^t,1}, \ldots, a_{i_r^t,k-1}\rangle$ is fixed for all $t < \aleph_1$, we shall now see that for $s, t < \aleph_1$ we have $w_s \cup w_t \in P$ and hence we have w_s and w_t are compatible. Proving $w_s \cup w_t \in P$ is equivalent to showing that for all sequences $\langle b_0, \ldots, b_{m-1}\rangle$ we have $|\{a_{i,m} : i \in w_s \cup w_t \text{ and } \langle a_{i,0}, \ldots, a_{i,m-1}\rangle = \langle b_0, \ldots, b_{m-1}\rangle\}| \leq 2$. If $m < k$ then the initial $(m+1)$- tuples $\langle a_{i,0}, \ldots, a_{i,m-1}, a_{i,m}\rangle$ of the A_i's for $i \in w_s \bigcup w_t$ are exactly these $(m+1)$ - tuples for $i \in w_s$ since for $m = 1, \ldots, n$ we have $\langle a_{i_m^s,0}, \ldots, a_{i_m^s,k-1}\rangle = \langle a_{i_m^t,0}, \ldots, a_{i_m^t,k-1}\rangle$, hence $\{a_{i,m} : i \in w_s \bigcup w_t$ and $\langle a_{i,0}, \ldots, a_{i,m-1}\rangle = \langle b_0, \ldots, b_{m-1}\rangle\} = \{a_{i,m} : i \in w_s$ and $\langle a_{i,0}, \ldots, a_{i,m-1}\rangle = \langle b_0, \ldots, b_{m-1}\rangle\}$, and the latter set has at most two members since $w_s \in P$. If $m \geq k$ then since the initial sequences of length k of all A_i for $i \in w_s$ are all distinct, and the same holds for all $i \in w_t$ so $\langle b_0, \ldots, b_{m-1}\rangle$ can be an initial sequence of A_i for at most one $i \in w_s$ and at most one $i \in w_t$ so that $|\{i \in w_s \bigcup w_t : \langle a_{i,0}, \ldots, a_{i,m-1}\rangle = \langle b_0, \ldots, b_{m-1}\rangle\}| \leq 2$ hence $|\{a_{i,m} : i \in w_s \bigcup w_t$, and $\langle a_{i,0}, \ldots, a_{i,m-1}\rangle = \langle b_0, \ldots, b_{m-1}\rangle\}| \leq 2$. So P satisfies the c.c.c.

For $\varepsilon < \aleph_1$ let $\mathcal{I}_\varepsilon$ be the subset of P, $\mathcal{I}_\varepsilon = \{w \in P : (\exists j > \varepsilon)[j \in w]\}$. Let us prove that $\mathcal{I}_\varepsilon$ is a dense subset of P. Let $w \in P$ and let k be such that the sequences $\langle a_{i,0}, a_{i,1}, \ldots, a_{i,k-1}\rangle$ are all distinct for different $i \in w$. Take a fixed $i \in w$ (the case where $w = \emptyset$ is trivial). By our assumption there are $\aleph_1$ ordinals j such that $\langle a_{j,0}, \ldots, a_{j,k-1}\rangle = \langle a_{i,0}, \ldots, a_{i,k-1}\rangle$, hence there is such a $j >$ max(w), ε. We shall see that $w \cup \{j\} \in P$, to prove that we have to show that for all $\langle b_0, \ldots, b_{m-1}\rangle$

$$(*) \qquad |\{a_{\gamma,m} : \gamma \in w \textstyle\bigcup \{j\} \text{ and } \langle a_{\gamma,0}, \ldots, a_{\gamma,m-1}\rangle = \langle b_0, \ldots, b_{m-1}\rangle\}| \leq 2.$$

If $m < k$ then since the first k members of A_j are the first k members of A_i, where $i \in w$, the left side of $(*)$ remains unchanged if $w \cup \{j\}$ is replaced by w and since the inequality holds for w, as $w \in P$, it holds also for $w \cup \{j\}$. If $m \geq k$

then, since the sequences $\langle a_{\gamma,0}, a_{\gamma,1}, \ldots, a_{\gamma,k-1}\rangle$ are all distinct for different $\gamma \in w$ there is at most one $\gamma \in w$ such that $\langle a_{\gamma,0}, \ldots, a_{\gamma,n-1}\rangle = \langle b_0, \ldots, b_{m-1}\rangle$, hence there are at most two $\gamma \in w \cup \{j\}$ which satisfy this equality and $(*)$ follows immediately.

By MA there is a directed subset G of P which intersects each $\mathcal{I}_\varepsilon$. Therefore $W = \bigcup G$ is a cofinal subset of $\aleph_1$. We take now $\bar{A}^\dagger = \langle A_i : i \in W\rangle$. We still have to prove that for all $\langle b_0, \ldots, b_{m-1}\rangle$ we have $|\{a_{i,m} : i \in W$ and $\langle a_{i,0}, \ldots, a_{i,m-1}\rangle = \langle b_0, \ldots, b_{m-1}\rangle\}| \leq 2$. Assume that for some $\langle b_0, \ldots, b_{m-1}\rangle$ this set has three members $a_{i_1,m}, a_{i_2,m}, a_{i_3,m}$ where $i_1, i_2, i_3 \in W$. Since $W = \bigcup G$ and G is directed we have $i_1, i_2, i_3 \in w$ for some $w \in G \subseteq P$. Then we have $|\{a_{i,m} : i \in w$ and $\langle a_{i,0}, \ldots, a_{i,m-1}\rangle = \langle b_0, \ldots, b_{m-1}\rangle\}| \geq 3$, contradicting $w \in P$. $\square_{4.5}$

4.6 Theorem. It is consistent that there is a tree $\bar{A} = \langle A_\alpha : \alpha < \omega_1\rangle$ such that $|D(\bar{A})| = \aleph_0$ and $\bar{A}$ has the uniformization property.

4.6A Remark. We can phrase a condition on forcing notions preserved by FS iteration (it depends on an $\bar{A}$ with the relevant property) and by this prove the consistency of an axiom; see [Sh:98].

Proof. Let T be a tree with ω levels, $|T| = \aleph_0$, T_n (the n'th level) is finite for $n < \omega$ and for all $x \in T_n$ we have $|\{y \in T_{n+1} : y > x\}| \geq 2^{n^2}$. We shall obtain a generic extension of V in which there is a tree $\bar{A}$ of length $\aleph_1$ which has the uniformization property. We saw above (in 4.4) that if the branching of a tree at each node is bounded the tree does not have the uniformization property. Here we shall see that if at level n each node branches to 2^{n^2} branches the tree may have the uniformization property. There is a gap here which one should narrow or even eliminate.

First one introduces $\aleph_1$ generic branches of T by the following set Q_0 of conditions

$Q_0 = \{f : f$ is a finite function on $\omega_1 \times \omega$ such that:

$$\langle \alpha, n\rangle \in \mathrm{Dom}(f) \Rightarrow f(\alpha, n) \in T_n,$$

$$\langle \alpha, n+1\rangle \in \mathrm{Dom}(f) \Rightarrow \langle \alpha, n\rangle \in \mathrm{Dom}(f),$$
$$f(\alpha, n) < f(\alpha, n+1) \text{ (when defined)}\}$$

The partial order relation on Q_0 is inclusion. If G is a generic subset of Q_0 let $F = \bigcup G$, $A_\alpha = \{F(\alpha, n) : n < \omega\}$ and $\bar{A} = \langle A_\alpha : \alpha < \omega_1\rangle$. Now $\bar{A}$ will be the tree which has the uniformization property (it is obvious that for $\alpha \neq \beta$, $\alpha, \beta < \omega_1$ we have $A_\alpha \neq A_\beta$, since the subset $\{f \in Q_0 : \exists n[\langle \alpha, n\rangle, \langle \beta, n\rangle \in \mathrm{Dom}(f) \,\&\, f(\alpha, n) \neq f(\beta, n)]\}$ of Q_0 is dense in Q_0). '

The next step is to carry out an iterated forcing so that at each step a different system $\bar{g} = \langle g_i : i < \omega_1\rangle$ where $g_i : A_i \to \{0,1\}$ gets uniformized. Eventually we obtain that $\bar{A}$ has the uniformization property if all appropriate $\bar{g}$'s appear. We shall describe first only a single step of this iterated forcing. The conditions we use now are the following. Let $\bar{g}$ be a name of a system of functions as above in the forcing Q_0. We define $Q(\bar{g}) = \{h : h$ is a finite function from $\aleph_1$ to ω and for every $\alpha, \beta \in \mathrm{Dom}(h)$ the functions $g_\alpha \restriction \{x \in A_\alpha : \mathrm{height}(x) \geq h(\alpha)\}$ and $g_\beta \restriction \{x \in A_\beta : \mathrm{height}(x) \geq h(\beta)\}$ are compatible $\}$.

The partial order relation on $Q(\bar{g})$ is inclusion. This forcing is done over $V[G_0]$, where G_0 is a generic subset of Q_0. Let G be a generic subset of $Q(\bar{g})$ and $H = \bigcup G$. We can now define the uniformizing function $g : T \to \{0,1\}$ as follows. For $x \in T$ if $x \in A_\alpha$ and $\mathrm{heigth}(x) \geq H(\alpha)$ then $g(x) = g_\alpha(x)$; if x belongs to no A_α such that $H(\alpha) \leq \mathrm{height}(x)$ we define $g(x)$ arbitrarily. If $x \in A_\alpha$, $\mathrm{height}(x) \geq H(\alpha)$ and also, $\beta \neq \alpha$, $x \in A_\beta$ and $\mathrm{height}(x) \geq H(\beta)$ then since G is directed and $H = \bigcup G$ there is an $h \in G$ such that $\alpha, \beta \in \mathrm{Dom}(h)$, $h(\alpha) = H(\alpha)$ and $h(\beta) = H(\beta)$. Since $h \in Q(\bar{g})$, $h(\alpha), h(\beta) \leq \mathrm{height}(x)$ we, by the definition of $Q(\bar{g})$, have $g_\alpha(x) = g_\beta(x)$, so that $g(x)$ is well defined. It is obvious from the definition of g that it uniformizes all g_α's for $\alpha < \aleph_1$.

Let us consider now the set $Q_0 * Q(\underset{\sim}{\bar{g}})$ and prove that it satisfies the c.c.c. Let W be an uncountable subset of $Q_0 * Q(\underset{\sim}{\bar{g}})$ and we shall prove that there are two distinct members which are compatible. Each $\langle p, \underset{\sim}{q}\rangle \in W \subseteq Q_0 * Q(\underset{\sim}{\bar{g}})$ is first extended as we shall tell. If two of the extended conditions will turn out to be compatible then the corresponding original conditions are compatible too. First we extend p to p_1 which decides exactly the value of $\underset{\sim}{q}$ ($\underset{\sim}{q}$ is the name of a finite subset of $\omega_1 \times \omega$), so we shall now regard q as a finite function from ω_1 to ω and not as a name. Then we extend p_1 to p_2 so that the $\mathrm{Dom}^\dagger(p_2) \overset{\mathrm{def}}{=} \{\alpha : (\exists n)[\langle \alpha, n\rangle \in \mathrm{Dom}(p_2)]\}$ (which is the set of branches on which p "speaks") will include the domain of q. Let u_p denote $\mathrm{Dom}^\dagger(p)$ i.e., the set of indices of the branches about which p contains information. For a sufficiently large $c < \omega$ we can extend p_2 to p_3 so that $c > |u_{p_2}|$, $\mathrm{Dom}(p_3) = u_{p_2} \times \{0, \ldots, c\}$ and $p_3(i, c) \neq p_3(j, c)$ for $i, j \in u_{p_2}$ such that $i \neq j$ (i.e., different $i \in u_{p_2}$ are indices of branches of $\bar{A}$ which branch off from each other at a level up to level c), and $c > q(i)$ for $i \in u_p$. ¿From now on we shall assume that all the members of W have the properties of $\langle p_{3,} q\rangle$.

Using the standard technique we can delete members from W so as to obtain a set W with the following properties. There are a $c < \omega$, a finite subset $v \subseteq \aleph_1$; finite sequences $v^\alpha = \langle j_1^\alpha, \ldots, j_n^\alpha\rangle$ for $\alpha < \aleph_1$ of n different members such that the sets $\{j_1^\alpha, \ldots, j_n^\alpha\}$ (which we shall also call v^α) are disjoint from each other for different α's and are disjoint from v; a function p with domain $v \times \{0, \ldots, c\}$, a function ρ with domain $\{1, \ldots, n\} \times \{0, \ldots, c\}$, a partial function q with domain v and a partial function h on $\{1, \ldots, n\}$ such that $W = \{\langle p_\alpha, q_\alpha\rangle : \alpha < \aleph_1\}$ and $\mathrm{Dom}(p_\alpha) = (v \cup v_\alpha) \times \{0, \ldots, c\}$, $p = p_\alpha \restriction (v \times \{0, \ldots, c\})$, $p_\alpha \restriction (v_\alpha \times \{0, \ldots, c\}) = \{\langle\langle j_k^\alpha, m\rangle, \rho(k, m)\rangle : 1 \leq k \leq n\}$ and $q \leq q_\alpha$ and $q_\alpha \setminus q = \{\langle j_k^\alpha, h(k)\rangle : 1 \leq k \leq n\}$. The members $\rho(1, c), \rho(2, c), \ldots, \rho(n, c)$ and $p(\alpha, c)$ (for $\alpha \in V$) of T_c are pairwise distinct (by the properties of $\langle p, q\rangle$ obtained above). Above each one of them there are at least 2^{c^2} different members of T_{c+1}. Let $\langle t_{k,\ell} : 1 \leq \ell \leq 2^{c^2}\rangle$ be a sequence of 2^{c^2} different members of T_{c+1} which are above $\rho(k, c)$. For $\alpha, \ell = 1, \ldots, 2^{c^2}$ let $p^*_{\alpha,\ell} = p_\alpha \bigcup \{\langle\langle j_k^\alpha, c+1\rangle, t_{k,\ell}\rangle : 1 \leq k \leq n\}$. Notice that for different α's the initial parts of the branches with indices $j_1^\alpha, \ldots, j_n^\alpha$ were the same, but this is not the case for the $p^*_{\alpha,\ell}$'s since we introduced different branchings at level $c+1$. Let $p^*_\ell = p^*_{\ell,\ell}$ for $\ell = 1, \ldots, 2^{c^2}$. Since all the p^*_ℓ's behave the same way on v and have otherwise disjoint domains also $\bigcup_{1 \leq \ell \leq 2^{c^2}} p^*_\ell \in Q_0$. Extend this condition to a condition p^* which determines the values of $\underset{\sim}{g}_{j_m^\alpha}(\rho(m, k))$ for all $1 \leq \alpha \leq 2^{c^2}$, $1 \leq m \leq n$ and $1 \leq k \leq c$. If we keep α fixed and let k, m vary we have here 2^{c^2} functions on a set with at most $n \times c < c^2$ members into $\{0,1\}$, hence there must be two different $\beta \neq \gamma$, $1 \leq \beta, \gamma \leq 2^{c^2}$ such that $\underset{\sim}{g}_{j_k^\beta}(\rho(k,m)) = \underset{\sim}{g}_{j_k^\gamma}(\rho(k,m))$ for all $1 \leq k \leq n$, $1 \leq m \leq c$ (they are Q_0-names but we mean the values p^* force for them). We claim that $\langle p^*, q_\beta \cup q_\gamma \rangle \in Q_0 * Q(\underset{\sim}{\bar{g}})$, hence $\langle p_\beta, q_\beta \rangle$ and $\langle p_\gamma, q_\gamma \rangle$ are compatible. What may prevent $\langle p^*, q_\beta \cup q_\gamma \rangle$ from being a condition when each of $\langle p^*, q_\beta \rangle$ and $\langle p^*, q_\gamma \rangle$ is (since $\langle p_\beta, q_\beta \rangle$ and $\langle p_\gamma, q_\gamma \rangle$ are conditions and $p_\beta \leq p^*$, $p_\gamma \leq p^*$ by the choice of p^*)? There may be k_1, m, k_2, such that $m \geq h(k_1)$, $m \geq h(k_2)$, $\underset{\sim}{F}(j_{k_1}^\beta, m) = \underset{\sim}{F}(j_{k_2}^\gamma, m)$ such that $\underset{\sim}{g}_{j_{k_1}^\beta}(\rho(k_1,m)) \neq \underset{\sim}{g}_{j_{k_2}^\gamma}(\rho(k_2,m))$ (where $\underset{\sim}{F} = \bigcup\{f : f \in \underset{\sim}{G}_{Q_0}\}$, see the choice of the A_j's). But by the choice of p^* this can occur only for $m \leq c$, so we get $\rho(k_1, m) = \rho(k_2, m)$. This is the case where the corresponding functions g_j give different values to the same member $\rho(k_1, m) = \rho(k_2, m)$ of T and this member is above the place in the two branches with indices $j_{k_1}^\beta$ and $j_{k_2}^\gamma$ where the uniformization is supposed to occur. By our choice of β and γ we have $g_{j_{k_2}^\beta}(\rho(k_2,m)) = g_{j_{k_2}^\gamma}(\rho(k_2,m))$, hence $g_{j_{k_1}^\beta}(\rho(k_1,m)) \neq g_{j_{k_2}^\beta}(\rho(k_2,m))$ while $\rho(k_1,m) = \rho(k_2,m)$, but this shows that $\langle p^*, q_\beta \rangle$ is not a condition in $Q_0 * Q(\gamma)$, which is a contradiction.

This of course does not yet prove Theorem 4.6. If we want to carry on the iteration, somehow imitating the scheme of Martin's axiom, we should do two things: first isolate some property of $\langle A_i : i < \omega_1 \rangle$ in V^{Q_0}, a property which is the "reason" why $Q(\bar{g})$ is c.c.c. Then we should formulate a property of forcing notion which would ensure the property of $\langle A_i : i < \omega_1 \rangle$ is preserved, at last we should show by induction on $\alpha \leq \omega_2$ that the iteration of $Q(\bar{g})$ over all names for candidates has the desired property. Where by candidate we mean a $\bar{g} = \langle g_i : i < \omega_1 \rangle, g_i : A_i \to \{0,1\}$.

Assume T was chosen such that:

$$(\forall m)(\exists n > m)(\forall x \in T_n)[|\{y \in T_{n+1} : y > x\}| \geq 2^{|\{y \in T : (y \leq x)\}|^m}]$$

What we get from the genericity of the $\aleph_1$-branches is:

⊗ for every $k < \omega$ and distinct A_i^ℓ for $i < \omega_1, \ell < k$ from $\bar{A}$ there are $n < \omega$, and pairwise distinct $a_1, \ldots, a_k \in T_n$ and $w \subseteq \omega_1$ such that:

(i) $a_i = (A^\ell_i)_{[n]}$ for every $i \in w$

(ii) $i \neq j \in w \Rightarrow (A^\ell_i)_{[n+1]} \neq (A^\ell_j)_{[n+1]}$

(iii) $|w| > \prod_{\ell<k} 2^{|\{y|y<a_\ell\}|} = i(a_0, \ldots, a_{k-1})$ (the i is for notational convenience)

For $\bar g = \langle g_\alpha : \alpha < \omega_1\rangle$, $g_\alpha : A_\alpha \to \{0,1\}$ the forcing notion $Q(\bar g)$ is defined above, let $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \alpha^*\rangle$ denote a Finite Support iteration of length α^* such that Q_0 is as above and for $\alpha > 0$, $\underset{\sim}{Q}_\alpha = Q(\underset{\sim}{\bar g}_\alpha)$ uniformizes a candidate $\underset{\sim}{\bar g}_\alpha$ where $\bar g_\alpha = \langle g^\alpha_\xi : \xi < \omega_1\rangle$ such that $\Vdash_{P_\alpha}$ "$\bar g^\alpha$ is a candidate ". We prove by induction on $\alpha \le \alpha^*$ the following condition.

$(*)_\alpha$: if $k < \omega$ and $p_i \in P_\alpha$, $A^\ell_i \in \bar A$ for $i < \omega_1$, $\ell < k$, then $\forall n_1 < \omega \exists n < \omega$, $n > n_1$ $\exists a_0, \ldots, a_{k-1} \in T$ $\exists w \subseteq \omega_1$ such that

(i) $a_\ell = (A^\ell_i)_{[n]}$ for $\ell < k, i \in w$ and $a_{\ell_1} = a_{\ell_2} \Leftrightarrow (\forall i \in w)(A^{\ell_1}_i = A^{\ell_2}_i)$.

(ii) for $i \neq j \in w$ we have $(A^\ell_i)_{[n+1]} \neq (A^\ell_j)_{[n+1]}$ or $A^\ell_i = A^\ell_j$.

(iii) $|w| > i(a_0, \ldots, a_{k-1})$

(iv) $\exists q \in P_\alpha$ such that $p_i \le q$ for each $i \in w$

Note that $(*)_\alpha \Rightarrow P_\alpha$ satisfies the c.c.c. So if we succeed to prove that, the rest of the proof is like the proof of 3.4. Also note that proving $(*)_\alpha$ w.l.o.g. for each i the sequence $\langle A^\ell_i : \ell < k\rangle$ is without repetitions. So let us carry the induction.

Case (1) $\alpha = 0$ nothing to prove.

Case (2) $\mathrm{cf}(\alpha) \ge \aleph_2$. Then for some $\beta < \alpha$ we have: $i < \omega_1 \Rightarrow p_i \in P_\beta$ so $(*)_\beta$ gives the conclusion.

Case (3) α limit, $\mathrm{cf}(\alpha) = \omega$.

Let $\alpha = \bigcup_{n<\omega} \alpha_n$ then for each $i < \omega_1$ there is $n(i) < \omega$ such that: $p_i \in P_{\alpha_{n(i)}}$ so for some n we have $|\{i : n(i) = n\}| = \aleph_1$ then $(*)_{\alpha_{n(i)}}$ gives the conclusion.

Case (4) α limit, $\mathrm{cf}(\alpha) = \aleph_1$ so $\alpha = \bigcup_{i<\omega_1} \alpha_i$ and $\langle \alpha_i : i < \omega_1\rangle$ is increasing and continuous; for each i we let $h(i) = \mathrm{Min}\{j : \mathrm{Dom}(p_i \restriction \alpha_i) \subseteq \alpha_j\}$ this is a pressing down function on the set of limit ordinals $< \omega_1$, so by Fodor's Lemma for some i_0 we have: $S \overset{\text{def}}{=} \{i : h(i) \le \alpha_{i_0}\}$ is stationary. W.l.o.g. $p_i \in P_{\alpha_{i+1}}$, let $p'_i = p_i \restriction \alpha_i$, so $p'_i \in P_{\alpha_{i_0}}$. Now for any finite $w \subseteq \omega_1$, if $\{p'_i : i \in w\}$ has an upper bound p^* in P_{i_0}, then $\{p_i : i \in w\}$ has a common upper bound p in P_α: $p^* \cup \bigcup_{i\in w} (p_i\restriction[\alpha_i, \alpha_{i+1}))$. Hence by $(*)_{\alpha_{i_0}}$ (which holds by the induction hypothesis) we can complete the proof of $(*)_\alpha$.

Case (5) $\alpha = \beta + 1$.

If $\alpha + 1$ this is clear, so assume $\beta > 0$. Remember that $Q(\underset{\sim}{\bar{g}}_\beta)$ is the set of functions f such that : $\mathrm{Dom}(f)$ is a finite subset of ω_1 and for $\xi \in \mathrm{Dom}(f)$, $f(\xi) < \omega$ and

$$(\forall \zeta, \xi \in \mathrm{Dom}(f))(\forall n)\Big[f(\zeta) \leq n \wedge f(\xi) \leq n$$

$$\& \, (A_\zeta)_{[n]} = (A_\xi)_{[n]} \Rightarrow g_\beta((A_\zeta)_{[n]}) = g_\beta((A_\xi)_{[n]})\Big]$$

W.l.o.g. for each $i < \omega_1$ we have $\beta \in \mathrm{Dom}(p_i)$, so $p_i(\beta)$ is (the name of) a finite function from ω_1. W.l.o.g. $p_i(\beta)$ is an actual function, and $|\mathrm{Dom}(p_i(\beta))|$ is constant so let $\mathrm{Dom}(p_i(\beta)) = \{A_i^\ell : k \leq \ell < k(0)\}$ (where k comes from the case of $(*)_\alpha$ we are trying to prove) w.l.o.g. $p_i(\beta)(A_i^\ell)$ depends on ℓ only. We apply $(*)_\beta$ to $k(0)$, $p_i\restriction\beta \in P_\beta$ and $A_i^\ell(i < \omega_1, \ell < k(0))$; we thus get n, $a_\ell(\ell < k(0))$ w_0 and q_0. Clearly we can find $q_1 \in P_\beta$, $q_0 \leq q_1$ such that for each $i \in w_0$, $k \leq \ell < k(0)$ and $m \leq n$ we have $q_1 \Vdash_{P_\beta}$ "$\underset{\sim}{g}_\gamma(m) = c_i(\ell, m)$" where $A_i^\ell = A_\gamma$; of course $c_i(\ell, m) \in \{0, 1\}$. The number of possible functions is $i(a_k, \ldots, a_{k(0)-1})$ as $|w_0| > i(a_0, \ldots, a_{k(0)-1})$ so for some c, $w = \{i \in w_0 : c_i = c\}$ has cardinality $> i(a_0, \ldots, a_{k-1})$. Now $q = q_1 \cup \{\langle \beta, \bigcup_{i\in w_0} p_i(\beta)\rangle\} \in P_\alpha$ and q, $a_\ell(\ell < k)$, w exemplify $(*)_\alpha$. $\square_{4.6}$

§5. Maximal Almost Disjoint Families of Subsets of ω

5.0 Definition. By a mad (maximally almost disjoint) subset of $\mathcal{P}(\omega)$ we mean an infinite subset F of $\mathcal{P}(\omega)$ such that for all $x, y \in F$ we have $|x| = \aleph_0$ but $|x \cap y| < \aleph_0$ and for every $z \in \mathcal{P}(\omega)$ there exists $x \in F$ such that $|z \cap x| = \aleph_0$.

5.1 Claim. There is a mad subset of $\mathcal{P}(\omega)$ of cardinality $2^{\aleph_0}$.

Proof. Replace ω by the set of all nodes of the full binary tree, i.e., with the set of all finite sequences of 0's and 1's. This tree has $2^{\aleph_0}$ branches and every two of them are almost disjoint. Extend the set of all branches to a mad set by Zorn's lemma. $\square_{5.1}$

5.2 Observation. No countable subset of $\mathcal{P}(\omega)$ is mad.

Proof. Let $\langle a_i : i < \omega \rangle$ be a sequence of infinite pairwise almost disjoint subsets of ω. For each $i < \omega$ we have $a_i \setminus \bigcup_{j<i} a_j = a_i \setminus \bigcup_{j<i} a_j \cap a_i$, and since each set $a_j \cap a_i$ is finite and a_i is infinite there is an $x_i \in a_i \setminus \bigcup_{j<i} a_j$. For $j < i$ we have $x_j \in a_j$ while $x_i \notin a_j$ hence $x_j \neq x_i$, therefore the set $b = \{x_i : i < \omega\}$ is an infinite set. Since for $j < i$, $x_i \notin a_j$ we have $b \cap a_j \subseteq \{x_0, \ldots, x_j\}$. Thus the intersection of b with each a_j is finite and therefore $\{a_i : i < \omega\}$ is not mad.

$\square_{5.2}$

So if the continuum hypothesis holds then there are mad sets of cardinality $2^{\aleph_0} = \aleph_1$ and of no other cardinality. If the continuum hypothesis does not hold we are faced with the question whether there are mad sets of cardinalities $\geq \aleph_1$ but $< 2^{\aleph_0}$.

5.3 Theorem. Martin's axiom implies that every mad set is of cardinality $2^{\aleph_0}$.

Proof. Let $\mathcal{A}$ be an infinite set of infinite pairwise almost disjoint subsets of ω, $|\mathcal{A}| < 2^{\aleph_0}$. Let $P_{\mathcal{A}} = \{\langle a, t\rangle : a$ is a finite subset of ω and t is a finite subset of $\mathcal{A}\}$. For $p = \langle a, t\rangle$ and $q = \langle b, s\rangle$ in $P_{\mathcal{A}}$ we define: $p \leq q$ if: $a \subseteq b$, $s \subseteq t$ and for each $u \in t$ we have $b \cap u = a \cap u$. Now $\leq$ is easily seen to be a partial order. The meaning of the condition $\langle a, t\rangle$ is that the set w which we are constructing will include a, but for every $u \in t$ we will have $w \cap u = a \cap u$, so no additional members of u will be in w (other than those already in a).

If $\langle a, t\rangle$, $\langle a, s\rangle$ are in $P_{\mathcal{A}}$ then, obviously, also $\langle a, t \bigcup s\rangle \in P_{\mathcal{A}}$ hence $\langle a, t\rangle$ and $\langle a, s\rangle$ are compatible. Therefore incompatible members of $P_{\mathcal{A}}$ must have different first components. Since there are only $\aleph_0$ finite subsets of ω every subset of $P_{\mathcal{A}}$ of pairwise incompatible members must be countable, and the c.c.c. holds.

For every $u \in \mathcal{A}$ let $\mathcal{I}_u = \{\langle a, t\rangle \in P_{\mathcal{A}} : u \in t\}$. Now $\mathcal{I}_u$ is a dense subset of $P_{\mathcal{A}}$ since for every $\langle b, s\rangle \in P_{\mathcal{A}}$ we have $\langle b, s \cup \{u\}\rangle \in \mathcal{I}_u$. For $n < \omega$ let $\mathcal{I}_n = \{\langle a, t\rangle \in P_{\mathcal{A}} : (\exists k \geq n) k \in a\}$. Let us prove that $\mathcal{I}_n$ is a dense subset of $P_{\mathcal{A}}$. Let $\langle b, s\rangle \in P_{\mathcal{A}}$. Since s is finite and $\mathcal{A}$ infinite there is a $v \in \mathcal{A} \setminus s$. Now $v \cap (\bigcup s) = v \cap \bigcup\{v \cap u : u \in s\}$ is finite hence $v \setminus \bigcup s$ is infinite. Let $k \in v \setminus \bigcup s$, $k \geq n$, we shall see that $\langle b \cup \{k\}, s\rangle \geq \langle b, s\rangle$. To prove that let $u \in s$; now $(b \cup \{k\}) \cap u = b \cap u$ since $k \notin \bigcup s$ hence $k \notin u$.

By Martin's axiom there is a directed subset G of $P_{\mathcal{A}}$ which intersects each $\mathcal{I}_u$ and each $\mathcal{I}_n$. Let A be the union of the first components of G. Let $u \in \mathcal{A}$, let $\langle a, s\rangle \in \mathcal{I}_u \cap G$, then $u \in s$. We shall see that $A \cap u = a \cap u$ and hence $A \cap u$ is finite. We know $\langle a, s\rangle \in G$ so $a \subseteq A$ hence $a \cap u \subseteq A \cap u$; assume that $m \in A$, then $m \in b$ for some $\langle b, t\rangle \in G$. Since G is directed there is an $\langle c, r\rangle \in G$ such that $\langle c, r\rangle \geq \langle a, s\rangle$, $\langle b, t\rangle$ so $m \in b \subseteq c$ but $m \in u$ by its choice so $m \in c \cap u$. Since $u \in s$ and $\langle c, r\rangle \geq \langle a, s\rangle$ we have $u \cap c = u \cap a$, hence $m \in u \cap a$. So we have proved $m \in A \cap u \Rightarrow m \in a \cap u$; hence $A \cap u \subseteq a \cap u$; so by a previous sentence $A \cap u = a \cap u$, hence $A \cap u$ is finite. A is infinite since for every n we know G contains a member $\langle a, t\rangle$ of $\mathcal{I}_n$ hence A contains a number $\geq n$.

Thus we have constructed an infinite set A which is almost disjoint from every member of $\mathcal{A}$, and hence $\mathcal{A}$ is not mad. $\square_{5.3}$

5.4 Theorem. If V does not satisfy the continuum hypothesis and λ is a cardinal such that $\aleph_1 \leq \lambda \leq 2^{\aleph_0}$ in V *then* V can be generically extended by a c.c.c. extension which preserves also $2^{\aleph_0}$ (automatically it still is a cardinal) to an extension $V[G]$ where there is a mad subset of $\mathcal{P}(\omega)$ of cardinality λ.

Proof. We start in V with a sequence $\langle A_\alpha : \alpha < \lambda\rangle$ of almost disjoint subsets of ω (there is such by claim 5.1). Now we proceed with a system of FS iterated forcing of length ω_1 as follows. At step α we assume that we have constructed a sequence $\langle A_i : i < \lambda + \alpha\rangle$ of pairwise almost disjoint subsets of ω. We take $F_\alpha = \{A_i : i < \lambda + \alpha\}$ and we use at this step the forcing notion P_{F_α}, where

$P_{\mathcal{A}}$ is as in the proof of Theorem 5.3, and we introduce by means of it a new infinite subset $A_{\lambda+\alpha}$ of ω almost disjoint with each member of F_α. (This follows immediately from the proof of Theorem 5.3.) Since P_{F_α} is a c.c.c. forcing, by 2.8 our iterated forcing is c.c.c. For P_{F_α} we can use, instead of pairs $\langle a, t\rangle$ where a is a finite subset of ω and t is a finite subset of F_α, such pairs where t is a finite subset of $\lambda + \alpha$ and each $i \in t$ stands for the corresponding A_i. Thus each P_{F_α} will consist of elements from V (while its set of elements is not necessarily in V), and the cardinality of P_{F_α} is therefore $\mathrm{Max}(\alpha, \lambda) = \lambda$. For the iterated forcing we can use only standard names in the set P of conditions, hence $|P| = \lambda$. Since P is a c.c.c. forcing, standard arguments (as in 3.4) show that $|\mathcal{P}(\omega)|^{(V[G])} = [(\lambda)^{\aleph_0}]^V \leq (\lambda^{\aleph_0})^{V[G]} \leq [(2^{\aleph_0})^{\aleph_0}]^V = [2^{\aleph_0}]^V$, hence in $V[G]$ we know $2^{\aleph_0}$ is the same as in V.

Finally let us prove that $\{A_\alpha : \alpha < \lambda + \omega_1\}$ is a mad subset of $\mathcal{P}(\omega)$. The "almost disjoint" is trivial. Let $A \subseteq \omega$, $A \in V[G]$. For each $n \in \omega$ let $\mathcal{I}_n$ be a maximal antichain in P of conditions which decide $\underset{\sim}{n} \in \underset{\sim}{A}$, where $\underset{\sim}{A}$ is a name of A. Since P satisfies the c.c.c. we know $|\mathcal{I}_n| = \aleph_0$. Let $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$, so $|\mathcal{I}| = \aleph_0$. For each $q \in \mathcal{I}$ the set $\mathrm{Dom}(q)$ is a finite subset of ω_1, hence there is an $\alpha < \omega_1$ such that for all $q \in \mathcal{I}$ we have $\mathrm{Dom}(q) \subseteq \alpha$. If G_α is the component of G in the iterated forcing up to and not including α, we have $A \in V[G_\alpha]$. We shall see that $A \cap A_\beta$ is infinite for some $\beta \leq \lambda + \alpha$ which establishes that $\{A_i : i < \lambda + \omega_1\}$ is mad. Moreover, we shall show that if $A \cap A_\beta$ is finite for every $\beta < \lambda + \alpha$ then $A \cap A_\alpha$ is infinite. This follows from the following lemma, which ends the proof. $\square_{5.4}$

5.5 Lemma. Let F be a set of pairwise almost disjoint subsets of ω, and let $B \subseteq \omega$ be such that for every finite subset $F^\dagger$ of F the set $B \setminus \bigcup F^\dagger$ is infinite. Let G be a generic subset of P_F over V and let A be the union of the first components of the members of G, *then* $A \cap B$ is infinite.

Proof. For every n let

$$\mathcal{I}_n = \{\langle a, t\rangle \in P_F : (\exists m \geq n)[m \in B \cap a]\}.$$

We shall see that $\mathcal{I}_n$ is dense. Let $\langle b, t\rangle \in P_F$. By our assumption $B \setminus \bigcup t$ is infinite hence it contains an $m \geq n$. Now $\langle a \cup \{m\}, t\rangle \in \mathcal{I}_n$ and since $m \notin \bigcup t$ $\langle a \cup \{m\}, t\rangle \leq \langle a, t\rangle$. Since $\mathcal{I}_n$ is dense, G contains some $\langle a, t\rangle \in \mathcal{I}_n$, hence $A \supseteq a$ and so A contains an $m \in B$ such that $m \geq n$. Since this holds for every $n < \omega$ we have $A \cap B$ is infinite. $\square_{5.5}$

Discussion. We shall prove that essentially all countable forcing notions are equivalent. If one carries out the proof of the last theorem for the case where $\alpha = \aleph_1$, but one starts with a sequence $\langle A_i : i < \omega\rangle$ of pairwise almost disjoint subsets of ω then each P_F is a countable forcing and therefore equivalent to the addition of a Cohen real. Thus the FS iterated addition of $\aleph_1$ Cohen reals yields an extension of V with a mad set of cardinality $\aleph_1$. But the iterated extension of $\aleph_1$ Cohen reals is the same as the simultaneous addition of $\aleph_1$ Cohen reals (as in the proof of I 3.2).

5.6 Theorem. Every two countable forcing notions where above every condition there are two incompatible conditions are equivalent.

Proof. Let P be the set of all finite sequences of natural numbers ordered by proper inclusion and let Q be a countable forcing as in the theorem. We can assume that Q has a minimal element $\emptyset_Q$ (or change the proof slightly). We shall show that P and Q are equivalent by constructing an isomorphism F from P onto a dense subset of Q. We shall define $F(\eta)$, for $\eta \in P$, by induction on the length of η. Let $F(\langle\rangle) = \emptyset_Q$, where $\emptyset_Q$ is the minimal member of Q. Let $Q = \{q_n : n < \omega\}$. We shall take care to satisfy the following in the definition.

(i) For every $\eta \in P$ the set $\{F(\eta\hat{\ }\langle i\rangle) : i < \omega\}$ is a maximal set of incompatible members of Q greater than $F(\eta)$.

(ii) For every n there is an $\eta \in P$ of length $n+1$ such that $F(\eta) \geq q_n$.

It is easy to check that this suffice. We assume, as an induction hypothesis, that $\{F(\eta) : \ell g(\eta) = n\}$ is pre-dense in P. Since, by clause (i), for every $\eta \in \bigcup\limits_{\ell<n} {}^{\ell}\omega$ we have $\{F(\eta\hat{\ }\langle i\rangle) : i < \omega\}$ is a maximal set of incompatible members of

Q above $F(\eta)$, the set $\{F(\nu) : \ell g(\nu) = n\}$ is a maximal antichain of Q. To define $F(\eta\hat{}\langle i\rangle)$, for all $i < \omega$ and $\eta \in {}^n\omega$, define a set $\mathcal{I}^*_\eta$ of pairwise incompatible members of Q above $F(\eta)$ as follows. If $\ell g(\eta) = n$ and $F(\eta)$ is compatible with q_n let $s_0 \geq F(\eta), q_n$ (in Q) otherwise let $s_0 \geq F(\eta)$. Let t_1, s_1 be two incompatible members of Q greater than s_0 and, by induction, let t_{n+1}, s_{n+1} be two incompatible members of Q greater than s_n. Take $\mathcal{I}_\eta = \{t_n : 1 \leq n < \omega\}$, $\mathcal{I}_\eta$ is a set of pairwise incompatible members, and if $F(\eta)$ is compatible with q_n then each member of $\mathcal{I}_\eta$ is $\geq q_n$. Let $\mathcal{I}^*_\eta$ be a maximal set of pairwise incompatible members above $F(\eta)$ which includes $\mathcal{I}_\eta$. Now $\mathcal{I}^*_\eta$ is countable, since Q is, and we define the $F(\eta\hat{}\langle i\rangle)$'s so that $\{F(\eta\hat{}\langle i\rangle) : i < \omega\} = \mathcal{I}^*_\eta$. Since (ii) holds $\{F(\eta) : \eta \in P\}$ is a dense subset of Q. $\square_{5.6}$

5.7 Discussion. Using the method of Theorem 5.4 we can extend V so that in the extension for every λ such that $\aleph_0 < \lambda \leq 2^{\aleph_0}$ there is a mad subset of cardinality λ but $2^{\aleph_0}$ is preserved. To do this let $\langle \lambda_\alpha : \alpha < \mu\rangle$ be the cardinals $\aleph_0 < \lambda < 2^{\aleph_0}$ in increasing order. For $\alpha < \mu$ we construct a mad subset $\{A^\alpha_i : i < \lambda_\alpha + \omega_1\}$ of $\mathcal{P}(\omega)$ as follows. $\{A^\alpha_i : i < \lambda_\alpha\} \in V$ is a family of pairwise almost disjoint subsets of ω. We extend V by iterating $\mu \times \omega_1$ times and at the step of ordinal $\mu \times i + \alpha$ we add the set $A^\alpha_{\lambda+i} \subseteq \omega$ as an almost disjoint set of $\{A^\alpha_j : j < \lambda_\alpha + i\}$ as in the proof 5.5: forcing by $P_{\{A^\alpha_j : j < \lambda_\alpha + i\}}$.