Chapter IV
κ^+-Trees in L and the Fine Structure Theory

In this chapter we shall investigate natural generalisations of the Souslin and Kurepa hypotheses to cardinals above ω_1. In the case of the Souslin hypothesis this will require some combinatorial properties of L which we shall only be able to prove by developing the theory of the constructible hierarchy more thoroughly than hitherto. (This is the so-called "fine-structure theory".)

1. κ^+-Trees

Let κ be an infinite cardinal. The concept of a *κ-tree* was defined in Chapter III. By a *κ-Aronszajn tree* we mean a κ-tree with no κ-branch. A *κ-Souslin tree* is a κ-tree with no antichain of cardinality κ. Just as in III.1.2, every κ-Souslin tree is κ-Aronszajn. And by arguments as in III.1.3, if κ is regular, then any (κ, κ)-tree with unique limits which has no κ-branch has a subtree which is κ-Aronszajn; if in addition the original tree has no antichain of cardinality κ, it has a subtree which is κ-Souslin. The regularity of κ is essential here. Indeed, for singular κ, the notion of a κ-tree is somewhat pathalogical. For example, if κ is singular there is a (κ, κ)-tree with no κ-branch and no antichain of cardinality κ (namely the disjoint union of the well-ordered sets $(\kappa_\nu, \varepsilon)$, $\nu < \mathrm{cf}(\kappa)$, where $(\kappa_\nu \mid \nu < \mathrm{cf}(\kappa))$ is cofinal in κ), but every κ-tree has an antichain of cardinality κ (an easy exercise). We therefore restrict our attention to κ-trees for regular κ only. Since we shall be assuming $V = L$ for our main results, GCH will hold, and hence the only regular limit cardinals are the (strongly) inaccessible cardinals. In this context we may therefore expect the notion of a κ-tree for κ a regular limit cardinal to be bound up with the notion of large cardinals. As we shall see in Chapter VII, this is in fact the case. In this chapter we concentrate only upon the successor cardinals.

By a *κ^+-Kurepa tree* we mean a κ^+-tree with κ^{++} many κ^+-branches. (Adopting a similar definition of a "κ-Kurepa tree" for inaccessible κ does not lead to any interesting notions, as we see in Exercise 3. More care is required in order to define a reasonable notion of a κ-Kurepa tree in this case.) As in III.2.1, the existence of a κ^+-Kurepa tree can be shown to be equivalent to the existence of a certain kind of family of subsets of κ^+. Moreover, the proof that such a family exists in L is a straightforward generalisation of the proof for the ω_1 case, given in III.2.2.

However, when we try to construct a κ^+-Souslin tree in L we run into some difficulties. It turns out to be slightly easier to try to generalise the proof using $\diamondsuit$ (III.3.2 and III.3.3) rather than the original construction (III.1.5). Now, the proof of $\diamondsuit$ generalises from ω_1 to any uncountable regular cardinal in a straightforward manner. Hence the generalised construction of the tree hinges upon a generalisation of the proof of III.3.3. This is not so easy. For suppose we try to construct a κ^+-Souslin tree by recursion on the levels. Consideration of the proof of III.3.3 tells us that on a stationary set of levels we must be very restrictive in the choice of branches to extend, in order that all antichains be eventually "killed-off". But consider now some limit stage α "late" in the construction. We have defined the tree $\mathbf{T} \upharpoonright \alpha$ and wish to define T_α. Each point of T_α must extend some α-branch of $\mathbf{T} \upharpoonright \alpha$. But unless $\mathrm{cf}(\alpha) = \omega$, how can we be sure that $\mathbf{T} \upharpoonright \alpha$ has *any* α-branches? Our attempts to kill off antichains at earlier limit stages may have resulted in $\mathbf{T} \upharpoonright \alpha$ having a sort of "Aronszajn property". To overcome this problem we introduce a combinatorial principle, $\square_\kappa$ ("square κ"), which enables us to split the construction of the limit levels of the tree into two cases. At some limit stages we kill off antichains, using the generalised $\diamondsuit$ principle. At the remaining limit stages we ensure that enough branches are extended in order that the construction will never break down. The penalty we must pay in order to be able to do this lies in the proof of $\square_\kappa$. This requires a detailed analysis of the levels of the constructible hierachy (the "fine-structure theory"). This will occupy the later parts of this chapter.

2. κ^+-*Souslin Trees*

We prove that if $V = L$, then for all infinite cardinals κ there is a κ^+-Souslin tree. Our first step is to formulate and prove a generalisation of the combinatorial principle $\diamondsuit$.

Let κ be any uncountable regular cardinal, E a stationary subset of κ. By $\diamondsuit_\kappa(E)$ we mean the following assertion:

> *There is a sequence $(S_\alpha \mid \alpha \in E)$ such that $S_\alpha \subseteq \alpha$ for all α and whenever $X \subseteq \kappa$, the set $\{\alpha \in E \mid X \cap \alpha = S_\alpha\}$ is stationary in κ.*

We denote $\diamondsuit_\kappa(\kappa)$ by $\diamondsuit_\kappa$. Thus $\diamondsuit_{\omega_1}$ is the same as our original principle $\diamondsuit$.

In order to prove that $\diamondsuit_\kappa(E)$ is valid in L we require the following simple lemma.

2.1 Lemma. *Let κ be an uncountable regular cardinal, λ a limit ordinal greater than κ. Let $X \subseteq L_\lambda$, $|X| < \kappa$. Then there is an $N \prec L_\lambda$ such that $X \subseteq N$, $|N| < \kappa$, and $N \cap \kappa \in \kappa$.*

Proof. Let N_0 be the smallest $N \prec L_\lambda$ such that $X \subseteq N$, and set

$$\alpha_0 = \sup(N_0 \cap \kappa).$$

Since $|N_0| = \max(|X|, \omega) < \kappa$, and κ is regular, we have $\alpha_0 < \kappa$. Proceeding recursively now, let N_{n+1} be the smallest $N \prec L_\lambda$ such that $N_n \cup \alpha_n \subseteq N$, and set

$$\alpha_{n+1} = \sup(N_{n+1} \cap \kappa).$$

If $|N_n| < \kappa$ and $\alpha_n < \kappa$, then

$$|N_{n+1}| = \max(|N_n|, |\alpha_n|) < \kappa,$$

so as κ is regular, $\alpha_{n+1} < \kappa$.

Let

$$N = \bigcup_{n<\omega} N_n.$$

Then

$$X \subseteq N \prec L_\lambda,$$

and

$$N \cap \kappa = (\bigcup_{n<\omega} N_n) \cap \kappa = \bigcup_{n<\omega} (N_n \cap \kappa).$$

But for $n > 0$,

$$\alpha_{n-1} \subseteq N_n \cap \kappa \subseteq \alpha_n.$$

Hence

$$N \cap \kappa = \bigcup_{n<\omega} \alpha_n.$$

So, if we set

$$\alpha = \sup\nolimits_{n<\omega} \alpha_n,$$

we have $N \cap \kappa = \alpha$. But κ is regular. Thus $|N| < \kappa$ and $\alpha < \kappa$, so we are done. □

2.2 Theorem. *Assume $V = L$. Let κ be any uncountable regular cardinal, E a stationary subset of κ. Then $\diamondsuit_\kappa(E)$ is valid.*

Proof. By recursion on $\alpha \in E$, define (S_α, C_α) to be the $<_L$-least pair of subsets of α such that C_α is club in α and

$$\gamma \in C_\alpha \cap E \rightarrow S_\alpha \cap \gamma \neq S_\gamma,$$

provided $\lim(\alpha)$ and such a pair exists, and define $S_\alpha = C_\alpha = \emptyset$ in all other cases. We show that $(S_\alpha | \alpha \in E)$ satisfies $\diamondsuit_\kappa(E)$.

Suppose that $(S_\alpha | \alpha \in E)$ is not a $\diamondsuit_\kappa(E)$-sequence. Let (S, C) be the $<_L$-least pair of subsets of κ such that C is club in κ and

$$\gamma \in C \cap E \rightarrow S \cap \gamma \neq S_\gamma.$$

Now, the sequence $((S_\alpha, C_\alpha)|\alpha \in E)$ is clearly definable from E in L_{κ^+}. (The definition given above is absolute for L_{κ^+}.) Hence (S, C) is also definable from E in L_{κ^+}. Using 2.1, we now define a sequence of submodels $N_\nu \prec L_{\kappa^+}$ $(\nu < \kappa)$, by the following recursion:

$$N_0 = \text{the smallest } N \prec L_{\kappa^+} \text{ such that } |N| < \kappa,\ N \cap \kappa \in \kappa, \text{ and } E \in N;$$
$$N_{\nu+1} = \text{the smallest } N \prec L_{\kappa^+} \text{ such that } |N| < \kappa,\ N \cap \kappa \in \kappa, \text{ and } N_\nu \cup \{N_\nu\} \subseteq N;$$
$$N_\lambda = \bigcup_{\nu<\lambda} N_\nu, \text{ if } \lim(\lambda). \text{ (Clearly, } |N_\lambda| < \kappa \text{ and } N_\lambda \cap \kappa \in \kappa \text{ here also.)}$$

Set

$$\alpha_\nu = N_\nu \cap \kappa \quad (\nu < \kappa).$$

Then $(\alpha_\nu | \nu < \kappa)$ is a normal sequence in κ, so the set

$$Z = \{\alpha_\nu | \alpha_\nu = \nu\}$$

is club in κ. Hence

$$E \cap Z \cap C \neq \emptyset.$$

Let $\alpha_\nu \in E \cap Z \cap C$. Let

$$\pi\colon N_\nu \cong L_\beta.$$

Then,

$$\pi \upharpoonright L_\nu = \mathrm{id} \upharpoonright L_\nu, \quad \pi(\kappa) = \nu, \quad \pi(E) = E \cap \nu,$$
$$\pi((S_\alpha, C_\alpha) | \alpha \in E)) = ((S_\alpha, C_\alpha) | \alpha \in E \cap \nu), \quad \pi((S, C)) = (S \cap \nu, C \cap \nu).$$

Since $\pi^{-1}\colon L_\beta \prec L_{\kappa^+}$, $(S \cap \nu, C \cap \nu)$ is the $<_L$-least pair of subsets of ν such that $C \cap \nu$ is club in ν and

$$\gamma \in (C \cap \nu) \cap (E \cap \nu) \to (S \cap \nu) \cap \gamma \neq S_\gamma.$$

Hence $(S \cap \nu, C \cap \nu) = (S_\nu, C_\nu)$, and in particular $S \cap \nu = S_\nu$. But $\nu \in C \cap E$, so this contradicts the choice of (S, C), and we are done. □

Using $\diamondsuit_{\kappa^+}(E)$ for a suitable set E, in the case where κ is regular it is quite easy to construct a κ^+-Souslin tree in L. We take

$$E = \{\alpha \in \kappa^+ \,|\, \mathrm{cf}(\alpha) = \kappa\},$$

and construct the tree by recursion on the levels, following the pattern of III.3.3. At limit stages $\alpha \in E$ we extend branches to "kill off" S_α, if S_α happens to be a maximal antichain of $\mathbf{T} \upharpoonright \alpha$. At all other limit stages α we extend *all* α-branches of $\mathbf{T} \upharpoonright \alpha$, noting that as $\mathrm{cf}(\alpha) < \kappa$ in such cases, there are at most $\kappa^{\mathrm{cf}(\alpha)} = \kappa$ (by GCH) such branches, so that T_α will not be too big. We leave the details to the reader (see Exercise 2).

If κ is singular, however, the above idea will not work. It is in order to handle this case that we need to introduce the combinatorial principle $\square_\kappa(E)$. Using $\square_\kappa(E)$, we shall give a construction of a κ^+-Souslin tree which works in all cases.

Let κ be an infinite cardinal, E a subset of κ^+. By $\square_\kappa(E)$ we mean the following assertion:

There is a sequence $(C_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ *such that*:

(i) C_α *is club in* α;
(ii) $\mathrm{cf}(\alpha) < \kappa \to \mathrm{otp}(C_a) < \kappa$;
(iii) *if* $\bar{\alpha} < \alpha$ *is a limit point of* C_α, *then* $\bar{\alpha} \notin E$ *and* $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

Notice that by virtue of condition (iii), condition (ii) can be extended to give the implication

$$\text{(ii)}' \qquad \mathrm{cf}(\alpha) = \kappa \to \mathrm{otp}(C_a) = \kappa .$$

Notice also that if κ is singular, we shall have $\mathrm{cf}(\alpha) < \kappa$ for all relevant α, so $\mathrm{otp}(C_\alpha) < \kappa$ for all α.

For any set $E \subseteq \omega_1$, $\square_\omega(E)$ is a theorem of ZFC, since for each limit ordinal $\alpha < \omega_1$ we can take C_α to be any ω-sequence cofinal in α. But already $\square_{\omega_1}(E)$ is a significant proposition, not provable in ZFC alone.

We shall write $\square_\kappa$ in place of $\square_\kappa(\emptyset)$.

In 2.10 we shall prove that if $\square_\kappa$, then there is a stationary set $E \subseteq \kappa^+$ such that $\square_\kappa(E)$. And then in section 5 we shall prove the following theorem.

2.3 Theorem. *Assume $V = L$. Let κ be an infinite cardinal. Then $\square_\kappa$ is valid.* □

We are now ready to construct a κ^+-Souslin tree in L.

2.4 Theorem. *Assume $V = L$. Let κ be an infinite cardinal. Then there is a κ^+-Souslin tree.*

Proof. By 2.3 and 2.10, let $E \subseteq \kappa^+$ be stationary and let $(C_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ satisfy $\square_\kappa(E)$. By 2.2, let $(S_\alpha \mid \alpha \in E)$ satisfy $\diamondsuit_{\kappa^+}(E)$. We shall construct a κ^+-Souslin tree, $\mathbf{T}$, by recursion on the levels, ensuring as we proceed that for each infinite $\alpha < \kappa^+$, $\mathbf{T} \upharpoonright \alpha$ is a normal (α, α^+)-tree. The elements of $\mathbf{T}$ will be the ordinals in κ^+, and we shall ensure that

$$\alpha <_T \beta \to \alpha < \beta .$$

To commence, set

$$T_0 = \{0\} .$$

If $\mathbf{T} \upharpoonright \alpha + 1$ is defined, $T_{\alpha+1}$ is obtained by using new ordinals from κ^+ to provide each element of T_α with two successors in $T_{\alpha+1}$. There remains the case where $\lim(\alpha)$ and $\mathbf{T} \upharpoonright \alpha$ is defined. This is where we must proceed carefully.

For each $x \in T \upharpoonright \alpha$ we attempt to define an α-branch b_α^x of $\mathbf{T} \upharpoonright \alpha$ such that $x \in b_\alpha^x$. Let $(\gamma_\alpha(\nu) \mid \nu < \lambda_\alpha)$ be the monotone enumeration of C_α. Given $x \in T \upharpoonright \alpha$, let

$\nu_\alpha(x)$ be the least ν such that $x \in T \restriction \gamma_\alpha(\nu)$. Define a sequence $(p_\alpha^x(\nu) \mid \nu_\alpha(x) \leqslant \nu < \lambda_\alpha)$ of elements of $T \restriction \alpha$ as follows:

$$\begin{aligned} p_\alpha^x(\nu_\alpha(x)) &= \text{the least (as an ordinal) } y \in T_{\gamma_\alpha(\nu_\alpha(x))} \text{ such that } x <_T y; \\ p_\alpha^x(\nu+1) &= \text{the least } y \in T_{\gamma_\alpha(\nu+1)} \text{ such that } p_\alpha^x(\nu) <_T y; \\ p_\alpha^x(\eta) &= \text{the unique } y \in T_{\gamma_\alpha(\eta)} \text{ such that} \\ &\quad (\forall \nu < \eta)(\nu \geqslant \nu_\alpha(x) \rightarrow p_\alpha^x(\nu) <_T y), \\ &\quad \text{provided such a } y \text{ exists (otherwise undefined), if } \lim(\eta). \end{aligned}$$

Should the above construction prove impossible (because for some limit ordinal $\eta < \lambda_\alpha$, $p_\alpha^x(\eta)$ is not defined), the entire construction of $\mathbf{T}$ breaks down. But for the time being, let us assume that this is not the case and see how b_α^x is defined. Later on we shall prove (by induction on α) that the construction never breaks down. Set

$$b_\alpha^x = \{y \in T \restriction \alpha \mid (\exists \nu < \lambda_\alpha)(y \leqslant_T p_\alpha^x(\nu))\}.$$

Clearly, b_α^x is an α-branch of $\mathbf{T} \restriction \alpha$ which contains x and each point $p_\alpha^x(\nu)$ for $\nu_\alpha(x) \leqslant \nu < \lambda_\alpha$. We now define T_α as follows.

Suppose first that $\alpha \notin E$. In this case we use new ordinals from κ^+ to provide each branch b_α^x, $x \in T \restriction \alpha$, with an extension in T_α.

Now suppose that $\alpha \in E$, but that S_α is not a maximal antichain of $\mathbf{T} \restriction \alpha$. In this case construct T_α just as in the last case.

Finally, suppose that $\alpha \in E$ and that S_α is a maximal antichain of $\mathbf{T} \restriction \alpha$. Then use new ordinals from κ^+ to provide an extension in T_α of each branch b_α^x such that $x \in T \restriction \alpha$ lies above an element of S_α. (Since S_α is assumed to be a maximal antichain here, T_α will still contain a point above each member of $T \restriction \alpha$, so normality will be preserved.)

The definition is complete. We show that $\mathbf{T}$ is a κ^+-Souslin tree. It is clearly a κ^+-tree. So, given a maximal antichain, A, of $\mathbf{T}$, we must show that $|A| \leqslant \kappa$. Set

$$C = \{\alpha \in \kappa^+ \mid T \restriction \alpha \subseteq \alpha \wedge A \cap \alpha \text{ is a maximal antichain of } \mathbf{T} \restriction \alpha\}.$$

It is easily seen that C is club in κ^+. So by $\diamondsuit_{\kappa^+}(E)$ there is a limit ordinal $\alpha \in C \cap E$ such that $A \cap \alpha = S_\alpha$. Thus, in particular, S_α is a maximal antichain of $\mathbf{T} \restriction \alpha$. But $\alpha \in E$, so by construction every element of T_α lies above a member of $A \cap \alpha$. Thus $A \cap \alpha$ is a maximal antichain of $\mathbf{T}$. Hence $A = A \cap \alpha$, and we are done.

It remains to check that the construction of $\mathbf{T}$ never broke down. Suppose, on the contrary, that it did. Let α be the least limit ordinal for which we cannot define all the α-branches b_α^x, $x \in T \restriction \alpha$. Pick $x \in T \restriction \alpha$ so that b_α^x cannot be defined. Thus for some limit ordinal η, $\nu_\alpha(x) < \eta < \lambda_\alpha$, there is no point in $T_{\gamma_\alpha(\eta)}$ which extends all the points $p_\alpha^x(\nu)$ for $\nu_\alpha(x) \leqslant \nu < \eta$. Since $\lim(\eta)$, $\gamma_\alpha(\eta)$ is a limit point of C_α. Hence by the $\square_\kappa(E)$ properties, $\gamma_\alpha(\eta) \notin E$ and

$$C_{\gamma_\alpha(\eta)} = \gamma_\alpha(\eta) \cap C_\alpha = \{\gamma_\alpha(\nu) \mid \nu < \eta\}.$$

By this last equality, $b^x_{\gamma_\alpha(\eta)}$ contains all the points $p^x_\alpha(v)$ for $v_\alpha(x) \leqslant v < \eta$. But since $\gamma_\alpha(\eta) \notin E$, $b^x_{\gamma_\alpha(\eta)}$ has an extension in $T_{\gamma_\alpha(\eta)}$. But this extension is precisely what we assumed did not exist: an extension of each point $p^x_\alpha(v)$, $v_\alpha(x) \leqslant v < \eta$. This contradiction shows that the construction of **T** does not, in fact, break down, and thereby completes the proof. $\square$

Notice that what we have in fact just proved is the following result.

2.5 Theorem. *Let κ be an infinite cardinal. If there is a stationary set $E \subseteq \kappa^+$ such that both $\square_\kappa(E)$ and $\diamondsuit_{\kappa^+}(E)$ hold, then there is a κ^+-Souslin tree.* $\square$

Using 2.5, we shall show that κ^+-Souslin trees exist under much weaker assumptions than $V = L$. We need some preliminary combinatorial results.

By an argument as in III.3.4 we have:

2.6 Lemma. *Let κ be any infinite cardinal, and let $E \subseteq \kappa^+$ be stationary. Then $\diamondsuit_{\kappa^+}(E)$ is equivalent to the principle $\diamondsuit'_{\kappa^+}(E)$, which asserts the existence of a sequence $(S_\alpha \mid \alpha \in E)$ such that $S_\alpha \subseteq \mathscr{P}(\alpha)$, $|S_\alpha| \leqslant \kappa$, and whenever $X \subseteq \kappa^+$, the set $\{\alpha \in E \mid X \cap \alpha \in S_\alpha\}$ is stationary in κ^+.* $\square$

Using 2.6, we now prove (see also Exercise 7):

2.7 Lemma. *Assume* GCH. *Let κ be an infinite cardinal such that* $\mathrm{cf}(\kappa) > \omega$. *Let $W \subseteq \kappa^+$ be the stationary set*

$$W = \{\alpha \in \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}.$$

Then $\diamondsuit_{\kappa^+}(W)$ is valid.

Proof. By GCH there are exactly κ^+ many subsets of κ^+ of cardinality at most κ. Let $(X_v \mid v < \kappa^+)$ enumerate them in such a way that $X_v \subseteq v$ for each $v < \kappa^+$. For each $\alpha < \kappa^+$, set

$$\Gamma_\alpha = \{X_v \mid v < \alpha\}.$$

For each $\alpha \in W$, let

$$S_\alpha = \{\bigcup \mathrm{ran}(f) \mid f\colon \omega \to \Gamma_\alpha\}.$$

Since $|\Gamma_\alpha| \leqslant \kappa$ and $\mathrm{cf}(\kappa) > \omega$,

$$|S_\alpha| \leqslant |\Gamma_\alpha|^\omega \leqslant \kappa^\omega = \kappa.$$

And of course

$$S_\alpha \subseteq \mathscr{P}(\alpha).$$

We show that $(S_\alpha \mid \alpha \in W)$ is a $\diamondsuit'_{\kappa^+}(W)$-sequence (see 2.6).

Let $X \subseteq \kappa^+$ be given. Let $C \subseteq \kappa^+$ be club. We must find an $\alpha \in C \cap W$ such that $X \cap \alpha \in S_\alpha$. To this end, define a strictly increasing sequence $(\alpha_n \mid n < \omega)$ of elements of C as follows, by recursion. Let α_0 be the smallest infinite ordinal in C. If $\alpha_n \in C$ is defined, let α_{n+1} be the least element of C such that $\alpha_{n+1} > \alpha_n$ and $X \cap \alpha_n \in \Gamma_{\alpha_{n+1}}$. Let

$$\alpha = \sup_{n<\omega} \alpha_n.$$

Since C is closed in κ^+, $\alpha \in C$. Moreover, $\mathrm{cf}(\alpha) = \omega$, so $\alpha \in W$. Define $f\colon \omega \to \Gamma_\alpha$ by

$$f(n) = X \cap \alpha_n \qquad (n < \omega).$$

Clearly,

$$X \cap \alpha = \bigcup \mathrm{ran}(f) \in S_\alpha,$$

so we are done. $\square$

In the above proof, we used the assumption $\mathrm{cf}(\kappa) > \omega$ in order to ensure that the sets S_α had cardinality at most κ. But what about the status of $\diamondsuit_{\kappa^+}(W)$ when $\mathrm{cf}(\kappa) = \omega$? Well, if we assume $\square_\kappa$ in addition to GCH, we can modify the proof of 2.7 to cover this case also, as we show next. (See also Exercise 8.)

2.8 Lemma. *Assume* GCH. *Let κ be an uncountable cardinal such that* $\mathrm{cf}(\kappa) = \omega$, *and let $W \subseteq \kappa^+$ be the stationary set*

$$W = \{\alpha \in \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}.$$

If $\square_\kappa$ holds, then $\diamondsuit_{\kappa^+}(W)$ is valid.

Proof. Define Γ_α, $\alpha < \kappa^+$ as in 2.7. Let $(C_\lambda \mid \lambda < \kappa^+ \wedge \lim(\lambda))$ be a $\square_\kappa$-sequence, and for each λ let $(c_\nu^\lambda \mid \nu < \theta_\lambda)$ be the canonical enumeration of C_λ. (Thus $\theta_\lambda = \mathrm{otp}(C_\lambda)$.)

Let A_ν, $\nu < \kappa$, be disjoint subsets of κ of cardinality κ such that $\kappa = \bigcup_{\nu<\kappa} A_\nu$. For each $\delta < \kappa^+$ and each $\nu < \kappa$, let

$$f_\nu^\delta\colon \Gamma_\delta \xrightarrow{1-1} A_\nu.$$

Then for each limit $\lambda < \kappa^+$ we can define

$$f_\lambda\colon \Gamma_\lambda \xrightarrow{1-1} \kappa$$

by setting $f_\lambda(x) = f_\nu^{c_\nu^\lambda}(x)$ where $\nu < \theta_\lambda$ is least such that $x \in \Gamma_{c_\nu^\lambda}$. The important point to notice here is the following:

(*) If $\alpha < \lambda$ is a limit point of C_λ, then $f_\lambda \restriction \Gamma_\alpha = f_\alpha$.

(This is immediate from the fact that $C_\alpha = \alpha \cap C_\lambda$ in this case.)

For $\alpha \in W$ now, set

$$S_\alpha = \{\bigcup f_\alpha^{-1}[x] \mid x \text{ is a countable, bounded subset of } \kappa\}.$$

Then $S_\alpha \subseteq \mathscr{P}(\alpha)$, and, since the number of countable, *bounded* subsets of κ is κ, $|S_\alpha| \leqslant \kappa$. We show that $(S_\alpha \mid \alpha \in W)$ is a $\diamondsuit'_{\kappa^+}(W)$-sequence (as in 2.6).

Let $X \subseteq \kappa^+$ be given. Let $C \subseteq \kappa^+$ be club. We seek an $\alpha \in C \cap W$ such that $X \cap \alpha \in S_\alpha$. Define

$$A = \{\lambda \in \kappa^+ \mid (\forall \nu < \lambda)(X \cap \nu \in \Gamma_\lambda)\}.$$

Clearly, A is club in κ^+. Let λ be a limit point of $A \cap C$ such that $\mathrm{cf}(\lambda) = \omega_1$. Since C_λ is club in λ we can pick a strictly increasing, continuous sequence $(b_\nu \mid \nu < \omega_1)$ of elements of $A \cap C \cap C_\lambda$, cofinal in λ. Notice that

$$X \cap b_\nu \in \Gamma_{b_{\nu+1}}$$

for all $\nu < \omega_1$.

Let $(\kappa_n \mid n < \omega)$ be a strictly increasing sequence of cardinals, cofinal in κ. Define $h: \omega_1 \to \omega$ by:

$$h(\nu) = \text{the least } n \text{ such that } f_\lambda(X \cap b_\nu) < \kappa_n.$$

By Fodor's Theorem there is a stationary set $E \subseteq \omega_1$ such that for some fixed $n < \omega$, $h(\nu) = n$ for all $\nu \in E$. Let $(\gamma(i) \mid i < \omega)$ enumerate (in order) the first ω elements of E, and set $\gamma = \sup_{i<\omega} \gamma(i)$. Let $\alpha = b_\gamma$. Notice that $\mathrm{cf}(\gamma) = \omega$, so $\alpha \in W$. Moreover, by choice of the elements b_ν, α is a limit point of $C \cap C_\lambda$, and in particular $\alpha \in C$.

Now,

$$\alpha = b_\gamma = \sup_{i<\omega} b_{\gamma(i)},$$

so

$$X \cap \alpha = \bigcup_{i<\omega} (X \cap b_{\gamma(i)}).$$

Thus

$$X \cap \alpha = \bigcup f_\lambda^{-1}[x],$$

where $x \subseteq \kappa$ is defined by

$$x = \{f_\lambda(X \cap b_{\gamma(i)}) \mid i < \omega\}.$$

But by choice of E,

$$x \subseteq \kappa_n < \kappa,$$

so x is a countable, bounded subset of κ. Moreover, by (*),

$$f_\lambda \restriction \Gamma_\alpha = f_\alpha.$$

Hence

$$X \cap \alpha \in S_\alpha,$$

and we are done. □

2.9 Lemma. *Let κ be any infinite cardinal, and let $E \subseteq \kappa^+$ be stationary. Suppose that $\diamondsuit_{\kappa^+}(E)$ is valid. Let*

$$E = \bigcup_{\nu<\kappa} E_\nu$$

be a disjoint partition of E. Then for some $\nu < \kappa$, E_ν is stationary and $\diamondsuit_{\kappa^+}(E_\nu)$ is valid.

Proof. Much as in III.3.4, by $\diamondsuit_{\kappa^+}(E)$ we can find a sequence $(T_\alpha \mid \alpha \in E)$ such that $T_\alpha \subseteq \alpha \times \kappa$ and for each $X \subseteq \kappa^+ \times \kappa$, the set

$$\{\alpha \in E \mid X \cap (\alpha \times \kappa) = T_\alpha\}$$

is stationary in κ^+. For each $\nu < \kappa$, define $(S_\alpha^\nu \mid \alpha \in E_\nu)$ by

$$S_\alpha^\nu = T_\alpha'' \{\nu\}.$$

We show that for some $\nu < \kappa$, $(S_\alpha^\nu \mid \alpha \in E_\nu)$ is a $\diamondsuit_{\kappa^+}(E_\nu)$-sequence. (This will automatically entail that E_ν is stationary, of course.) Suppose that, on the contrary, no sequence $(S_\alpha^\nu \mid \alpha \in E_\nu)$ is a $\diamondsuit_{\kappa^+}(E_\nu)$-sequence. Then for each $\nu < \kappa$ we can find a set $X_\nu \subseteq \kappa^+$ and a club set $C_\nu \subseteq \kappa^+$ such that

$$\alpha \in C_\nu \cap E_\nu \rightarrow X_\nu \cap \alpha \neq S_\alpha^\nu.$$

Set

$$X = \bigcup_{\nu<\kappa} (X_\nu \times \{\nu\}),$$

$$C = \bigcap_{\nu<\kappa} C_\nu.$$

Then C is club in κ^+, and, since $X''\{\nu\} = X_\nu$ for each $\nu < \kappa$,

$$\alpha \in C \cap E \rightarrow X \cap (\alpha \times \kappa) \neq T_\alpha,$$

which is a contradiction. The lemma is proved. $\square$

2.10 Lemma. *Let κ be any uncountable cardinal for which $\square_\kappa$ is valid. Let $W \subseteq \kappa^+$ be the stationary set*

$$W = \{\alpha \in \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}.$$

Then there is a stationary set $E \subseteq W$ such that:

(i) *$\square_\kappa(E)$ is valid;*
(ii) *if $\diamondsuit_{\kappa^+}(W)$, then $\diamondsuit_{\kappa^+}(E)$.*

(*Thus, by* 2.7 *and* 2.8, *if* GCH *holds, then $\diamondsuit_{\kappa^+}(E)$ follows from* (ii).)

Proof. Let $(A_\lambda \mid \lambda < \kappa^+ \wedge \lim(\lambda))$ be a $\square_\kappa$-sequence. For each λ, let B_λ be the set of limit points of A_λ below λ. The sequence $(B_\lambda \mid \lambda < \kappa^+ \wedge \lim(\lambda))$ has the following properties:

(i) B_λ is a closed subset of λ;
(ii) if $\mathrm{cf}(\lambda) > \omega$, then B_λ is unbounded in λ;
(iii) $\gamma \in B_\lambda \to B_\gamma = \gamma \cap B_\lambda$;
(iv) $\mathrm{cf}(\lambda) < \kappa \to |B_\lambda| < \kappa$.

By (iii) and (iv), $\mathrm{otp}(B_\lambda) \leqslant \kappa$ for all λ, so we can define a partition

$$W = \bigcup_{\nu \leqslant \kappa} W_\nu$$

by setting

$$W_\nu = \{\lambda \in W \mid \mathrm{otp}(B_\lambda) = \nu\}.$$

Now, W is stationary, so for at least one $\nu \leqslant \kappa$, W_ν must be stationary. Indeed, by 2.9 we can pick a $\nu \leqslant \kappa$ such that W_ν is stationary and

$$\diamondsuit_{\kappa^+}(W) \to \diamondsuit_{\kappa^+}(W_\nu).$$

Let $E = W_\nu$ for such a ν. We prove that $\square_\kappa(E)$ holds. For each limit ordinal $\lambda < \kappa^+$, define D_λ as follows. If $\mathrm{otp}(B_\lambda) \leqslant \nu$, let $D_\lambda = B_\lambda$.Otherwise, let D_λ consist of all members of B_λ beyond the $(1 + \nu)$-th element, i.e.

$$D_\lambda = B_\lambda - \{\alpha \in B_\lambda \mid \mathrm{otp}(B_\alpha) \leqslant \nu\}.$$

It is easily checked that the sequence $(D_\lambda \mid \lambda < \kappa^+ \wedge \lim(\lambda))$ has properties (i)–(iv) above. And clearly, $D_\lambda \cap E = \emptyset$ for all λ. Define C_λ for limit $\lambda < \kappa^+$ by recursion on λ as follows:

$$C_\lambda = \begin{cases} \bigcup \{C_\gamma \mid \gamma \in D_\lambda\}, & \text{if } \sup(D_\lambda) = \lambda, \\ \bigcup \{C_\gamma \mid \gamma \in D_\lambda\} \cup \{\theta_n^\lambda \mid n < \omega\}, & \text{otherwise, where } (\theta_n^\lambda \mid n < \omega) \end{cases}$$
is any strictly increasing ω-sequence cofinal in λ such that $\theta_0^\lambda = \bigcup D_\lambda$. (By (ii) for D_λ, we have $\mathrm{cf}(\lambda) = \omega$ in case $\sup(D_\lambda) < \lambda$.)

We shall prove that $(C_\lambda \mid \lambda < \kappa^+ \wedge \lim(\lambda))$ is a $\square_\kappa$-sequence and that D_λ is the set of all limit points of C_λ below λ for each λ (which implies at once that $(C_\lambda \mid \lambda < \kappa^+ \wedge \lim(\lambda))$ is in fact a $\square_\kappa(E)$-sequence, since $D_\lambda \cap E = \emptyset$ for all λ).

A trivial induction on λ shows that C_λ is unbounded in λ for each λ. Now, by induction on λ, we prove:

(a) if $\gamma \in D_\lambda$, then $C_\gamma = \gamma \cap C_\lambda$.

Assume (a) holds below λ. Let $\gamma \in D_\lambda$. Then by definition of C_λ, $C_\gamma \subseteq C_\lambda$. So $C_\gamma \subseteq \gamma \cap C_\lambda$. To prove the reverse inclusion, let $\xi \in \gamma \cap C_\lambda$. We show that $\xi \in C_\gamma$. By the definition of C_λ, for some $\delta \in D_\lambda$ we have $\xi \in \gamma \cap C_\delta$. If $\delta = \gamma$ then $\xi \in C_\gamma$

is immediate. Suppose that $\delta < \gamma$. Since $\gamma \in D_\lambda$, we have $D_\gamma = \gamma \cap D_\lambda$. Thus $\delta \in D_\gamma$. So by definition of C_γ, $C_\delta \subseteq C_\gamma$. Thus $\xi \in C_\gamma$. Finally, suppose that $\delta > \gamma$. Then $\gamma \in \delta \cap D_\lambda$. But $\delta \in D_\lambda$, so $D_\delta = \delta \cap D_\lambda$. Thus $\gamma \in D_\delta$. So by induction hypothesis at δ, $C_\gamma = \gamma \cap C_\delta$. Thus $\xi \in C_\gamma$, and we are done.

The next step is to prove:

(b) D_λ is the set of all limit points of C_λ below λ.

Again we proceed by induction on λ. Assume that (b) holds below λ. Let $\xi \in D_\lambda$. Then by definition of C_λ, $C_\xi \subseteq C_\lambda$. But C_ξ is unbounded in ξ. Thus ξ is a limit point of C_λ. Conversely, let $\xi < \lambda$ be a limit point of C_λ. We consider first the case where $\sup(D_\lambda) < \lambda$. Then

$$C_\lambda = \bigcup \{C_\gamma \mid \gamma \in D_\lambda\} \cup \{\theta_n^\lambda \mid n < \omega\},$$

and so ξ must be a limit point of $\bigcup \{C_\gamma \mid \gamma \in D_\lambda\}$. Now, D_λ is closed in λ, so $\delta = \bigcup D_\lambda \in D_\lambda$. Thus $D_\delta = \delta \cap D_\lambda$ and

$$\bigcup \{C_\gamma \mid \gamma \in D_\lambda\} = (\bigcup \{C_\gamma \mid \gamma \in D_\delta\}) \cup C_\delta = C_\delta \cup C_\delta = C_\delta.$$

Thus ξ is a limit point of C_δ. Then by induction hypothesis at δ, $\xi \in D_\delta$. But $D_\delta = \delta \cap D_\lambda$. Thus $\xi \in D_\lambda$, as required. We turn to the other case, where $\sup(D_\lambda) = \lambda$. Let $\gamma \in D_\lambda$, $\gamma > \xi$. Thus ξ is a limit point of $\gamma \cap C_\lambda$. But by (a), $\gamma \cap C_\lambda = C_\gamma$. Thus by induction hypothesis at γ, $\xi \in D_\gamma$. But $\gamma \in D_\lambda$, so $D_\gamma = \gamma \cap D_\lambda$. Thus $\xi \in D_\lambda$, and again we are done.

By virtue of (a) and (b) we shall be done if we prove that each C_λ is closed in λ and that if $\mathrm{cf}(\lambda) < \kappa$ then $\mathrm{otp}(C_\lambda) < \kappa$. Well, we prove that C_λ is closed in λ by induction on λ. Assume it is true below λ. Let $\gamma < \lambda$ be a limit point of C_λ. We prove that $\gamma \in C_\lambda$. By (b), $\gamma \in D_\lambda$. If $\gamma = \bigcup D_\lambda$, then $\gamma = \theta_0^\lambda \in C_\lambda$ and we are done. Otherwise, there is an $\alpha \in D_\lambda$ such that $\alpha > \gamma$. By (a), $C_\alpha = \alpha \cap C_\lambda$. Thus γ is a limit point of C_α. So by induction hypothesis, $\gamma \in C_\alpha$. Thus $\gamma \in \quad \cap C_\lambda \subseteq C_\lambda$, and again we are done. Finally now, if $\mathrm{otp}(C_\lambda) \geqslant \kappa$, then C_λ must have at least κ limit points, so by (b), $|D_\lambda| \geqslant \kappa$. But if $\mathrm{cf}(\lambda) < \kappa$, this is not the case. The proof is complete. $\square$

Notice that in proving the above result, we have demonstrated that $\square_\kappa$ is equivalent to the existence of a sequence $(B_\lambda \mid \lambda < \kappa^+ \wedge \lim(\lambda))$ which satisfies (i)–(iv) as stated in that proof. A stronger result of this nature will be proved in section 5.

We are now ready to say a little more concerning the existence of κ^+-Souslin trees.

2.11 Theorem. *Assume* GCH. *Let κ be an uncountable cardinal for which $\square_\kappa$ holds. Then there exists a κ^+-Souslin tree.*

Proof. If $\mathrm{cf}(\kappa) > \omega$, then by 2.7, $\diamondsuit_{\kappa^+}(W)$ is valid, where

$$W = \{\alpha \in \kappa^+ \mid \mathrm{cf}(\alpha) = \omega\}.$$

If $\mathrm{cf}(\kappa) = \omega$, then by 2.8, $\diamondsuit_{\kappa^+}(W)$ is valid. Thus in all cases, $\diamondsuit_{\kappa^+}(W)$ holds. Hence by 2.10 there is a stationary set $E \subseteq \kappa^+$ such that both $\diamondsuit_{\kappa^+}(E)$ and $\square_\kappa(E)$ are valid. So by 2.5 there is a κ^+-Souslin tree. $\square$

3. κ^+-Kurepa Trees

A *κ^+-Kurepa tree*, it may be recalled, is a κ^+-tree with κ^{++} many κ^+-branches. A *κ^+-Kurepa family* is a family $\mathscr{F} \subseteq \mathscr{P}(\kappa^+)$ such that $|\mathscr{F}| = \kappa^{++}$ and for all $\alpha < \kappa^+$, $|\mathscr{F} \upharpoonright \alpha| \leqslant \kappa$, where

$$\mathscr{F} \upharpoonright \alpha = \{x \cap \alpha \mid x \in \mathscr{F}\}.$$

Exactly as in III.2.1, we can show that the existence of a κ^+-Kurepa tree is equivalent to the existence of a κ^+-Kurepa family. By generalising the proof of III.2.2 we shall show that if $V = L$, there is a κ^+-Kurepa family for every infinite cardinal κ. We require two lemmas, generalisations of II.5.10 and II.5.11, respectively.

3.1 Lemma. *Assume $V = L$. Let κ be an infinite cardinal. If*

$$\kappa \subseteq X \prec L_{\kappa^+},$$

then $X = L_\alpha$ for some $\alpha \leqslant \kappa^+$, $\alpha > \kappa$.

Proof. It suffices to prove that X is transitive, since the lemma then follows at once from the condensation lemma. But

$$\models_{L_{\kappa^+}} \forall x (|x| \leqslant \kappa),$$

so this is proved just as in II.5.10. $\square$

3.2 Lemma. *Assume $V = L$. Let κ be an infinite cardinal. If*

$$\kappa \subseteq X \prec L_{\kappa^{++}},$$

then $X \cap L_{\kappa^+} = L_\alpha$ for some $\alpha \leqslant \kappa^+$, $\alpha > \kappa$.

Proof. This follows from 3.1 in the same way that II.5.11 follows from II.5.10. $\square$

We can now prove:

3.3 Theorem. *Assume $V = L$. Let κ be any infinite cardinal. Then there is a κ^+-Kurepa tree.*

Proof. It suffices to construct a κ^+-Kurepa family. We proceed much as in III.2.2.

By 3.1 we can define a function $f\colon \kappa^+ \to \kappa^+$ by letting $f(\alpha)$ be the least ordinal such that

$$\kappa \cup \{\alpha\} \subseteq L_{f(\alpha)} \prec L_{\kappa^+}.$$

Set

$$\mathscr{F} = \{x \subseteq \kappa^+ \mid (\forall \alpha < \kappa^+)(x \cap \alpha \in L_{f(\alpha)})\}.$$

For each $\alpha < \kappa^+$, $|\mathscr{F} \upharpoonright \alpha| \leqslant \kappa$, so in order to show that $\mathscr{F}$ is a κ^+-Kurepa family we need only prove that $|\mathscr{F}| = \kappa^{++}$.

Suppose, on the contrary, that $|\mathscr{F}| \leqslant \kappa^+$, and let

$$X = (x_\alpha \mid \alpha < \kappa^+)$$

be the $<_L$-least κ^+-enumeration of $\mathscr{F}$. Since the function f is clearly definable in $L_{\kappa^{++}}$, so too are $\mathscr{F}$ and X.

By recursion, we define a sequence $(N_\nu \mid \nu < \kappa^+)$ of elementary submodels of $L_{\kappa^{++}}$ as follows:

$$N_0 = \text{the smallest } N \prec L_{\kappa^{++}} \text{ such that } \kappa \subseteq N;$$

$$N_{\nu+1} = \text{the smallest } N \prec L_{\kappa^{++}} \text{ such that } N_\nu \cup \{N_\nu\} \subseteq N;$$

$$N_\delta = \bigcup_{\nu<\delta} N_\nu, \qquad \text{if } \lim(\delta).$$

By 3.2,

$$\alpha_\nu = N_\nu \cap \kappa^+ \in \kappa^+,$$

for each $\nu < \kappa^+$. Clearly, $(\alpha_\nu \mid \nu < \kappa^+)$ is a normal sequence in κ^+.

Set

$$x = \{\alpha_\nu \mid \nu < \kappa^+ \wedge \alpha_\nu \notin x_\nu\}.$$

Since $x \neq x_\nu$ for each $\nu < \kappa^+$, $x \notin \mathscr{F}$, and we obtain our contradiction by proving that $x \cap \alpha \in L_{f(\alpha)}$ for all $\alpha < \kappa^+$.

Let $\alpha < \kappa^+$ be given. Let η be the largest limit ordinal such that $\alpha_\eta \leqslant \alpha$. (If no such η exists, then $x \cap \alpha$ is finite and we are done.) Since $x \cap \alpha$ differs from $x \cap \alpha_\eta$ by at most finitely many points, in order to show that $x \cap \alpha \in L_{f(\alpha)}$ it suffices to show that $x \cap \alpha_\eta \in L_{f(\alpha)}$. In fact we show that $x \cap \alpha_\eta \in L_{f(\alpha_\eta)}$, which is if anything a stronger result. Since we shall have no further recourse to the original α, let us write α for α_η from now on.

Now,

$$x \cap \alpha = \{\alpha_\nu \mid \nu < \eta \wedge \alpha_\nu \notin x_\nu\},$$

so if $(\alpha_\nu \mid \nu < \eta)$ and $(x_\nu \cap \alpha \mid \nu < \eta)$ are elements of $L_{f(\alpha)}$ we shall be done. (Recall that $L_{f(\alpha)}$ is a model of ZF^-, though nothing like the full power of ZF^- is required in order to define $x \cap \alpha$ from the above two sequences, of course.)

Let

$$\pi\colon N_\eta \cong L_\beta.$$

Clearly,

$$\pi \restriction L_\alpha = \mathrm{id} \restriction L_\alpha, \qquad \pi(\kappa^+) = \alpha, \qquad \pi(X) = (x_\nu \cap \alpha \mid \nu < \alpha).$$

Now,

$$\alpha \in L_{f(\alpha)} \prec L_{\kappa^+},$$

so

$$\models_{L_{f(\alpha)}} [|\alpha| \leqslant \kappa].$$

But

$$\alpha = (\kappa^+)^{L_\beta}.$$

Hence

$$\beta < f(\alpha).$$

So, as $\pi(X) = (x_\nu \cap \alpha \mid \nu < \alpha)$, we have

$$(x_\nu \cap \alpha \mid \nu < \alpha) \in L_{f(\alpha)}.$$

In particular,

$$(x_\nu \cap \alpha \mid \nu < \eta) \in L_{f(\alpha)}.$$

It remains to show that $(\alpha_\nu \mid \nu < \eta) \in L_{f(\alpha)}$. To this end, for $\nu < \eta$, let

$$\pi_\nu \colon N_\nu \cong L_{\beta(\nu)}.$$

For each ν,

$$\pi_\nu(\kappa^+) = \alpha_\nu,$$

so

$$\alpha_\nu = [\text{the largest cardinal}]^{L_{\beta(\nu)}}.$$

So, as $L_{f(\alpha)}$ is a model of ZF^-, it is sufficient to prove that

$$(\beta(\nu) \mid \nu < \eta) \in L_{f(\alpha)}.$$

We define, by recursion on ν, a sequence of elementary submodels $N'_\nu \prec L_\beta$, for $\nu < \eta' \leqslant \eta$, as follows (see below concerning η'):

$$N'_0 = \text{the smallest } N \prec L_\beta \text{ such that } \kappa \subseteq N;$$
$$N'_{\nu+1} = \text{the smallest } N \prec L_\beta \text{ such that } N'_\nu \cup \{N'_\nu\} \subseteq N;$$
$$N'_\delta = \bigcup_{\nu<\delta} N'_\nu, \quad \text{if } \lim(\delta).$$

The ordinal η' is the largest $\eta' \leqslant \eta$ for which the above recursion is possible. (We shall prove that $\eta' = \eta$.)

Clearly,

$$(N'_\nu \mid \nu < \eta') \in L_{f(\alpha)}.$$

Hence

$$(\beta'(\nu) \mid \nu < \eta') \in L_{f(\alpha)},$$

where we define

$$\pi'_\nu : N'_\nu \cong L_{\beta'(\nu)}$$

for each $\nu < \eta'$.

But

$$\nu < \eta \to N_\nu \prec N_\eta \prec L_{\kappa^{++}},$$

so in the definition of N_ν for $\nu < \eta$ we can replace $L_{\kappa^{++}}$ by N_η. That is:

$$\begin{aligned} N_0 &= \text{the smallest } N \prec N_\eta \text{ such that } \kappa \subseteq N; \\ N_{\nu+1} &= \text{the smallest } N \prec N_\eta \text{ such that } N_\nu \cup \{N_\nu\} \subseteq N; \\ N_\delta &= \bigcup_{\nu<\delta} N_\nu, \quad \text{if } \lim(\delta). \end{aligned}$$

But

$$\pi : N_\eta \cong L_\beta,$$

so an easy induction on ν now yields the result

$$\nu < \eta \to (\pi \restriction N_\nu) : N_\nu \cong N'_\nu .$$

Hence $\eta' = \eta$ and $\beta(\nu) = \beta'(\nu)$ for all $\nu < \eta$. In particular, we have $(\beta(\nu) \mid \nu < \eta) \in L_{f(\alpha)}$, so we are done. $\square$

By modifying the above proof along the lines of III.3.5 we may prove that $V = L$ implies $\diamondsuit^+_{\kappa^+}$ for all infinite cardinals κ, where $\diamondsuit^+_{\kappa^+}$ is obtained from $\diamondsuit^+$ by replacing ω_1 by κ^+ throughout (so $\diamondsuit^+$ is $\diamondsuit^+_{\omega_1}$). And an argument as in III.3.6 shows that $\diamondsuit^+_{\kappa^+}$ implies the existence of a κ^+-Kurepa family. (See Exercise 4.)

The notion of a κ-Kurepa tree and the principle $\diamondsuit^+_\kappa$ in the case of κ an innaccessible cardinal will be dealt with in Chapter VII.

4. The Fine Structure Theory

The deeper results concerning the constructible universe, including the proof that $\square_\kappa$ is valid in L, require a detailed study of the individual levels of the constructible hierarchy. (Actually, there is an alternative approach as far as $\square_\kappa$ is concerned: the so-called "Silver machine" method. This is described in Chapter IX.)

The detailed study of the individual levels of the constructible hierarchy needed to prove $\square_\kappa$ and related results was begun by Jensen in the late 1960's, and is known as the "fine structure theory". Initially this really was a study of the properties of the individual sets L_α as defined in Chapter II. However, it soon became clear that the sets L_α do not lend themselves easily to such a study. If one tries to carry out simple set theoretic arguments within an arbitrary L_α, then unless α is a limit ordinal one meets a host of minor, but troublesome difficulties. For instance, unless α is a limit ordinal, L_α is not closed under the formation of ordered pairs. Since the ordered pair function is one which is used all the time in even the most elementary set-theoretical arguments, this is an annoying problem. Certainly, it is possible to overcome this, and similar difficulties, but in so doing a great deal of cumbersome apparatus needs to be introduced, and much of the naturalness of set theory is lost. The difficulty is the more annoying because it arises for an essentially irrelevant reason. The very simple functions which we would like our levels to be closed under (ordered pairs, etc.) are all highly "constructible", and we only fail to achieve closure because they increase rank. And there lies the root of the problem. The trouble is, when we defined the constructible hierarchy, we mimicked the definition of the cumulative hierarchy, insisting that at each stage only *subsets* of the stage could appear at the next stage. But for constructibility the crucial point lies in our other requirement, that at each stage we allow only those new sets which are *constructible* from the sets already available. And there are many set-theoretic operations which are, under any definition, "constructible", but which increase rank by more than one level, and hence violate the "subsets only" requirement. The way out of this dilemma is easy. We modify the definition of the constructible hierarchy so that each level of the hierarchy is an amenable set. This was first done by Jensen, and we thus refer to the modified hierarchy as the *Jensen hierarchy*. It is this hierarchy whose "fine structure" is usually investigated. The α-th level of the Jensen hierarchy is denoted by J_α. Roughly speaking, J_α possesses all of the properties of the limit levels of the usual L_α-hierarchy of constructible sets. And we can think of J_α as being a "constructibly inessential" extension of the structure L_α. (By virtue of the closure properties we obtain for the sets J_α, this picture is not totally accurate, but by and large is the way in which the beginner should view matters: when you read "J_α", think "L_α, $\lim(\alpha)$"!)

In this section we outline the fine structure theory, developed to the stage where we can prove $\Box_\kappa$ (assuming $V = L$). However, by its very nature, the fine structure theory is very intricate, and some of the proofs tend to be long (though except for the early development they are rarely boring). Consequently, we omit practically all proofs in our outline. For applications of the fine structure theory of the type we shall consider, however, it is not at all necessary to know anything about these proofs, a knowledge of a few, readily appreciated key results being sufficient. So we do not lose a great deal by our approach. Then, in section 5, we use the fine structure theory outlined in order to give a rigorous proof of $\Box_\kappa$ in L. The interested reader may then investigate the fine structure theory itself in Chapter VI, where we develop the entire theory rigorously.

Now to our outline of the fine structure theory. Our first step is to define a new "constructible hierarchy". Since we are interested in functional closure of the levels of the hierarchy, rather than pure definability, our approach will be functional. We shall define the hierarchy by iteratively closing up under various set theoretical functions. All of these functions will be "constructible" in some sense. Moreover, they will be sufficient to ensure that at the very least we obtain all of the usual constructible sets at each stage, i.e. $L_\alpha \subseteq J_\alpha$. The collection we use is described below.

A function $f\colon V^n \to V$ is said to be *rudimentary* (*rud* for short) iff it is generated by the following schemas:

(i) $f(x_1, \ldots, x_n) = x_i \qquad (1 \leqslant i \leqslant n)$;

(ii) $f(x_1, \ldots, x_n) = \{x_i, x_j\} \qquad (1 \leqslant i, j \leqslant n)$;

(iii) $f(x_1, \ldots, x_n) = x_i - x_j \qquad (1 \leqslant i, j \leqslant n)$;

(iv) $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n))$, where $h, g_1, \ldots, g_k$ are rudimentary;

(v) $f(y, x_1, \ldots, x_{n-1}) = \bigcup_{z \in y} g(z, x_1, \ldots, x_{n-1})$, where g is rudimentary.

It is clear that rudimentary functions are "constructible", so that any hierarchy we define using them can reasonably be called a "constructible hierarchy". Indeed, it can be shown that all rudimentary functions are Σ_0^{ZF}. The converse to this is false, but if we define a relation $A \subseteq V^n$ to be *rudimentary* iff its characteristic function is rudimentary, then for relations the notions of being rudimentary and of being Σ_0^{ZF} do coincide. Another point which should perhaps be mentioned here is that although rudimentary functions increase rank they do so by a finite amount only.

If X is a set, the *rudimentary closure* of X is the smallest set $Y \supseteq X$ such that Y is closed under all rudimentary functions. If X is transitive, so is its rudimentary closure. For transitive sets X we set

$$\mathrm{rud}(X) = \text{the rudimentary closure of the transitive set } X \cup \{X\}.$$

That the rudimentary functions will constitute an ideal class for defining a constructible hierarchy which is only an "inessential" extension of the usual one follows from the fact that for any transitive set X,

$$\mathrm{rud}(X) \cap \mathscr{P}(X) = \mathrm{Def}(X).$$

(In fact,

$$\Sigma_0(\mathrm{rud}(X)) \cap \mathscr{P}(X) = \mathrm{Def}(X).)$$

The *Jensen hierarchy* is defined as follows:

$$J_0 = \emptyset;$$
$$J_{\alpha+1} = \mathrm{rud}(J_\alpha);$$
$$J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha, \qquad \text{if } \lim(\lambda).$$

Thus each J_α is transitive, the hierarchy is cumulative, and for each α, the rank of J_α is $\omega\alpha$, and

$$J_\alpha \cap \mathrm{On} = \omega\alpha.$$

(This last fact has the effect that in arguments involving the Jensen hierarchy, ordinals of the type $\omega\alpha$ appear all the time.) For $\alpha > 1$, J_α is amenable; the canonical LST formula which says

$$x = J_\alpha$$

is Σ_1^{ZF}; and $(J_\nu \mid \nu < \alpha)$ is uniformly $\Sigma_1^{J_\alpha}$ for $\alpha > 1$. For each α,

$$J_{\alpha+1} \cap \mathscr{P}(J_\alpha) = \mathrm{Def}(J_\alpha).$$

The relationship between the Jensen hierarchy and the usual constructible hierarchy is:

(i) $(\forall \alpha)(L_\alpha \subseteq J_\alpha \subseteq L_{\omega\alpha})$;
(ii) $L_\alpha = J_\alpha \quad$ iff $\omega\alpha = \alpha$.

(It is possible to say a little more, but the exact relationship between L_α and J_α is rather complicated, and in any case is of no use to us.) In particular,

$$L = \bigcup_{\alpha \in \mathrm{On}} J_\alpha.$$

There is a well-ordering $<_J$ of L which is definable by means of a Σ_1^{ZF} formula of LST which is absolute for L and for any set J_α, $\alpha > 1$, and which is such that $<_J \cap (J_\alpha \times J_\alpha)$ is uniformly $\Sigma_1^{J_\alpha}$ for all $\alpha > 1$.

There is a $\Sigma_1(J_\alpha)$ map of $\omega\alpha$ onto J_α for each $\alpha > 1$.

The concept of a Σ_n skolem function has been met in II.6, and in II.6.5 we proved that each limit L_α has a (uniformly Σ_1) Σ_1 skolem function. Essentially the same proof shows that each J_α, $\alpha > 1$, has a (uniformly Σ_1) Σ_1 skolem function. A rather more complicated proof shows that for $\alpha > 1$, J_α has a Σ_n skolem function for any $n \geqslant 1$. But there is no uniform Σ_n skolem function for J_α except for the case $n = 1$. (The proof developed in Exercise II.5 can be used to show that the Jensen hierarchy has no uniform Σ_2 skolem function.) This is a serious drawback. Even the rather simple result proved in II.6.8 shows how useful uniformity properties are in skolem function applications. And in order to prove results such as $\square_\kappa$, we need to be able to carry out Σ_n condensation arguments of a type generalising II.6.8 for any $n \geqslant 1$. In order to facilitate this, we proceed as follows.

Recall that a structure of the form $\langle M, A\rangle$ (i.e. $\langle M, \in, A\rangle$), where $A \subseteq M$, is said to be *amenable* iff M is an amenable set and

$$u \in M \rightarrow A \cap u \in M.$$

It is easily seen that most of the results about limit levels of the constructible hierarchy given in Chapter II are in fact valid (by almost the same proof in each case) for amenable structures of the form $\langle L_\alpha, A\rangle$. Moreover, each of these results has a valid analogue for amenable structures $\langle J_\alpha, A\rangle$. (In this connection, remember that J_α is an amenable set for all $\alpha > 1$.) In particular, there is a uniformly Σ_1 Σ_1-skolem function for the amenable structures $\langle J_\alpha, A\rangle$. The main idea behind the fine structure theory is to capitalise on this fact, by reducing Σ_n predicates over a J_α to Σ_1 predicates over some amenable structure $\langle J_\varrho, A\rangle$, and then working with $\langle J_\varrho, A\rangle$ instead of J_α.

Let $h_{\alpha, A}$ denote the canonical, uniform Σ_1 skolem function for any amenable structure $\langle J_\alpha, A\rangle$, and let $H_{\alpha, A}$ be the uniform $\Sigma_0^{\langle J_\alpha, A\rangle}$ preidcate on J_α such that

$$y \simeq h_{\alpha, A}(i, x) \leftrightarrow (\exists z \in J_\alpha) H_{\alpha, A}(z, y, i, x).$$

(The function $h_{\alpha, A}$ is defined in precisely the same manner as the canonical Σ_1 skolem function h_α for limit L_α in II.6. Thus,

$$h_{\alpha, A}(i, x) \simeq (r_{\alpha, A}(i, x))_0,$$

where

$$\begin{aligned} r_{\alpha, A}(i, x) \simeq {} & \text{the } <_J\text{-least } w \in J_\alpha \text{ such that} \\ & \models_{\langle J_\alpha, A\rangle} (\text{“}\mathring{w} \text{ is an ordered pair”}) \wedge \bar{\varphi}_i((\mathring{w})_0, \mathring{x}, (\mathring{w})_1), \end{aligned}$$

where $(\bar{\varphi}_i(v_0, v_1, v_2) \mid i < \omega)$ enumerates (in a uniformly $\Delta_1^{J_\alpha}$ fashion) all Σ_0 formulas of $\mathscr{L}(\mathring{A})$ having free variables amongst v_0, v_1, v_2.)

We now describe the means by which Σ_n predicates on a J_α can be coded as Σ_1 predicates on an amenable $\langle J_\varrho, A\rangle$.

Let $\alpha > 1$, $n > 0$. The *Σ_n-projectum* of α, denoted by ϱ_α^n, is the smallest $\varrho \leqslant \alpha$ such that there is a $\Sigma_n(J_\alpha)$ map f for which $f'' J_\varrho = J_\alpha$. It can be shown that ϱ_α^n is the largest $\varrho \leqslant \alpha$ such that $\langle J_\varrho, A\rangle$ is amenable for any $\Sigma_n(J_\alpha)$ subset A of J_ϱ. Moreover, ϱ_α^n equals the smallest ϱ such that $\Sigma_n(J_\alpha) \cap \mathscr{P}(\omega\varrho) \nsubseteq J_\alpha$.

It is easily seen that

$$m < n \rightarrow \varrho_\alpha^n \leqslant \varrho_\alpha^m.$$

For later convenience, we set

$$\varrho_\alpha^0 = \alpha.$$

For each $\alpha > 1$, $n \geqslant 0$, we can associate with α a *standard code*, A_α^n, and a *standard parameter* , p_α^n, with the following properties:

1. $A_\alpha^n \subseteq J_{\varrho_\alpha^n}$, $A_\alpha^n \in \Sigma_n(J_\alpha)$;
2. $\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle$ is amenable;
3. $A_\alpha^0 = p_\alpha^0 = \emptyset$;
4. For all $m > 0$,

$$\Sigma_m(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle) = \mathscr{P}(J_{\varrho_\alpha^n}) \cap \Sigma_{n+m}(J_\alpha);$$

5. p_α^{n+1} is the $<_J$-least $p \in J_{\varrho_\alpha^n}$ such that

$$J_{\varrho_\alpha^n} = h''_{\varrho_\alpha^n, A_\alpha^n}(\omega \times (J_{\varrho_\alpha^{n+1}} \times \{p\})).$$

By definition of the Σ_n-projectum, ϱ_α^n, there is a $\Sigma_n(J_\alpha)$ map f such that $f'' J_{\varrho_\alpha^n} = J_\alpha$. Suppose now that P is a $\Sigma_n(J_\alpha)$ predicate on J_α. Set

$$Q = \{x \in J_{\varrho_\alpha^n} \mid f(x) \in P\}.$$

Then Q is a $\Sigma_n(J_\alpha)$ subset of $J_{\varrho_\alpha^n}$. By Fact 4 in the above list, Q is $\Sigma_1(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle)$. In this way, instead of working with Σ_n predicates on J_α, we may work with "equivalent" (in the sense that Q and P are "equivalent" in the above discussion) Σ_1 predicates on $\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle$, thereby being able to utilise the uniform Σ_1 skolem function possessed by the structures $\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle$. (Actually, from the above account it would appear that the coding of a Σ_n predicate on J_α by a Σ_1 predicate on $\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle$ is via an arbitrary $\Sigma_n(J_\alpha)$ function f. In practice we use, in effect, a canonical such function constructed from the standard parameters and the canonical Σ_1 skolem functions. See the definition of the standard parameter p_α^{n+1} above.) What is now needed in order to make this procedure work is a suitable condensation lemma. For suppose that

$$\langle X, A_\alpha^n \cap X\rangle \prec_1 \langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle.$$

By the standard condensation lemma, there are unique $\bar{\varrho}, \bar{A}$ such that

$$\langle X, A_\alpha^n \cap X\rangle \cong \langle J_{\bar{\varrho}}, \bar{A}\rangle.$$

But if we are to be able to work with the structures $\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle$ instead of the original J_α, we shall require that the $\bar{\varrho}, \bar{A}$ obtained in this manner are of the form $\bar{\varrho} = \varrho_{\bar{\alpha}}^n$, $\bar{A} = A_{\bar{\alpha}}^n$ for some unique $\bar{\alpha}$. In other words, what we need is a condensation lemma for the hierarchy of structures

$$\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle \qquad (\alpha \in \mathrm{On}).$$

This is provided by the following property of the standard codes:

6. Let $\alpha > 1$, $m \geqslant 0$, $n \geqslant 1$. Let $\langle J_{\bar{\varrho}}, \bar{A}\rangle$ be amenable, and let

$$\pi\colon \langle J_{\bar{\varrho}}, \bar{A}\rangle \prec_m \langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle.$$

Then there is a unique $\bar{\alpha} \geqslant \bar{\varrho}$ such that $\bar{\varrho} = \varrho_{\bar{\alpha}}^n$, $\bar{A} = A_{\bar{\alpha}}^n$. Moreover, there is a unique $\tilde{\pi} \supseteq \pi$ such that

$$\tilde{\pi}\colon J_{\bar{\alpha}} \prec_{m+n} J_\alpha,$$

and such that for all $i = 1, \ldots, n$:

(a) $\quad \tilde{\pi}(p_{\bar{\alpha}}^i) = p_\alpha^i$;

(b) $\quad (\tilde{\pi} \restriction J_{\varrho_{\bar{\alpha}}^i})\colon \langle J_{\varrho_{\bar{\alpha}}^i}, A_{\bar{\alpha}}^i\rangle \prec_{m+n-i} \langle J_{\varrho_\alpha^i}, A_\alpha^i\rangle.$

The assertions concerning the extension $\tilde{\pi}$ here should not be too surprising, since A_α^n codes all the Σ_n information about J_α. The heart of assertion 6 is the fact that the standard codes are preserved under condensation arguments, indeed even under "Σ_0 condensation arguments". This is essentially the case because of the canonical manner in which the standard codes are defined. We fix some simple (hence uniformly Δ_1) enumeration $(\varphi_i \mid i < \omega)$ of the Σ_1 formulas of $\mathscr{L}(\mathring{A})$ with free variables v_0 and v_1, and define, by recursion on n:

$$A_\alpha^{n+1} = \{(i, x) \mid i \in \omega \wedge x \in J_{\varrho_\alpha^n} \wedge \models_{\langle J_{\varrho_\alpha^n}, A_\alpha^n \rangle} \varphi_i(\mathring{x}, \mathring{p}_\alpha^{n+1})\}.$$

5. The Combinatorial Principle $\Box_\kappa$

Using the fine structure theory outlined above, we shall prove that if $V = L$, then $\Box_\kappa$ is valid for all infinite cardinals κ. We begin by recalling the statement of $\Box_\kappa$.

$\Box_\kappa$: There is a sequence $(C_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ such that:

(i) C_α is a club subset of α;
(ii) $\mathrm{cf}(\alpha) < \kappa \rightarrow |C_\alpha| < \kappa$;
(iii) if $\bar{\alpha}$ is a limit point of C_α, then $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

Let $\Box'_\kappa$ assert the existence of a sequence $(B_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ such that:

(i) B_α is a closed subset of α such that $(\forall \gamma \in B_\alpha) \lim(\gamma)$;
(ii) $\mathrm{cf}(\alpha) > \omega \rightarrow B_\alpha$ is unbounded in α;
(iii) $\mathrm{otp}(B_\alpha) \leqslant \kappa$;
(iv) $\bar{\alpha} \in B_\alpha \rightarrow B_{\bar{\alpha}} = \bar{\alpha} \cap B_\alpha$.

5.1 Lemma. *Let κ be any uncountable cardinal. Then $\Box_\kappa$ and $\Box'_\kappa$ are equivalent.*

Proof. Before we commence, notice that a weaker version of this result was proved during the course of 2.10. The present proof is a refinement of the argument used there.

First of all, suppose $(C_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ is as in $\Box_\kappa$. For each α, let B_α be the set of limit points of C_α below α. It is clear that the sequence $(B_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ satisfies $\Box'_\kappa$.

Conversely, let $(B_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ be as in $\Box'_\kappa$. By recursion on α we define sets C_α as follows. If $\bigcup B_\alpha = \alpha$, set

$$C_\alpha = \bigcup \{C_\gamma \mid \gamma \in B_\alpha\}.$$

Otherwise, if B_α is not cofinal in α, then by (ii) of $\Box'_\kappa$, $\mathrm{cf}(\alpha) = \omega$, so we may fix some strictly increasing ω-sequence $(\theta_n^\alpha \mid n < \omega)$, cofinal in α, with $\theta_0^\alpha = \bigcup B_\alpha$, and set

$$C_\alpha = (\bigcup \{C_\gamma \mid \gamma \in B_\alpha\}) \cup \{\theta_n^\alpha \mid n < \omega\}.$$

The following are proved exactly as in 2.10:

(a) If $\gamma \in B_\alpha$, then $C_\gamma = \gamma \cap C_\alpha$.

(b) B_α is the set of all limit points of C_α below α.

(c) C_α is a club subset of α.

Moreover, we have

(d) $\operatorname{otp}(C_\alpha) \leqslant \kappa$.

For suppose, on the contrary, that $\operatorname{otp}(C_\alpha) > \kappa$. Then, since κ is an uncountable cardinal, it follows from (b) that $\operatorname{otp}(B_\alpha) > \kappa$. This is not the case, by choice of B_α.

Now, if κ is regular, then since C_α is cofinal in α, (d) implies that

$$\operatorname{cf}(\alpha) < \kappa \to |C_\alpha| < \kappa,$$

and hence $(C_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ satisfies $\square_\kappa$. On the other hand, if κ is singular, we must modify the sets C_α in order to obtain a $\square_\kappa$-sequence, as follows. Let $\bar{\kappa} = \operatorname{cf}(\kappa)$, and let $(\theta_\nu \mid \nu < \bar{\kappa})$ be a strictly increasing, continuous sequence of limit ordinals, cofinal in κ, with $\theta_0 = 0$. Set $\theta_{\bar{\kappa}} = \kappa$. Define sets C_α as follows. If there is a $\nu < \bar{\kappa}$ such that

$$\theta_\nu < \operatorname{otp}(C_\alpha) \leqslant \theta_{\nu+1},$$

set

$$C'_\alpha = \{\gamma \in C_\alpha \mid \operatorname{otp}(\gamma \cap C_\alpha) \geqslant \theta_\nu\}.$$

If no such ν exists, then we must have $\operatorname{otp}(C_\alpha) = \theta_\nu$ for some *limit* ordinal $\nu \leqslant \bar{\kappa}$, in which case we set

$$C'_\alpha = \{\gamma \in C_\alpha \mid (\exists \tau < \nu)(\operatorname{otp}(\gamma \cap C_\alpha) = \theta_\tau)\}.$$

It is routine to verify that $(C'_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ is a $\square_\kappa$-sequence. That completes the proof. $\square$

Assume $V = L$ from now on. We shall prove that $\square_\kappa$ holds for all infinite cardinals κ. Since $\square_\omega$ is trivially valid (in ZFC), we may ignore the case $\kappa = \omega$. By 5.1, given some uncountable cardinal κ, it suffices to prove $\square'_\kappa$. The basic idea is to construct sets B_α to satisfy (i)–(iii) of $\square'_\kappa$ by means of a construction which is sufficiently uniform to enable (iv) to be proved by a condensation argument. In order to do this we must set up some machinery.

Let α be a limit ordinal, and let $\omega\beta \geqslant \alpha$. We say that α is *singular over* J_β iff there is a J_β-definable map of a bounded subset of α cofinally into α; otherwise we say that α is *regular over* J_β. Let $n \geqslant 1$. We say that α is Σ_n*-singular over* J_β iff there is a $\Sigma_n(J_\beta)$ map of a bounded subset of α cofinally into α; otherwise we say that α is Σ_n*-regular over* J_β.

Clearly, α is regular over J_β iff it is Σ_n-regular over J_β for all n. If α is singular over J_β, then α is singular over J_γ for all $\gamma \geqslant \beta$. And if α is Σ_n-singular over J_β, then α is Σ_m-singular over J_β for all $m \geqslant n$. Moreover, by $V = L$, if α is singular, then there are β, n such that α is Σ_n-singular over J_β.

Let

$$S = \{\alpha \in \kappa^+ \mid (\alpha > \kappa) \wedge (\omega\alpha = \alpha) \wedge (\forall \gamma < \alpha)(|\gamma|^{J_\alpha} \leqslant \kappa)\}.$$

It is easily seen that S is a club subset of κ^+. We shall construct a sequence $(C_\alpha \mid \alpha \in S)$ such that:

(i) C_α is a closed subset of $S \cap \alpha$;
(ii) $\mathrm{cf}(\alpha) > \omega \to C_\alpha$ is unbounded in α;
(iii) $\mathrm{otp}(C_\alpha) \leqslant \kappa$;
(iv) $\bar{\alpha} \in C_\alpha \to C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

If we then identify S with $\{\alpha \in \kappa^+ \mid \lim(\alpha)\}$ in the obvious manner, we obtain a $\square'_\kappa$-sequence, as required.

Let $\alpha \in S$. Then α is a limit ordinal between κ and κ^+. So in particular, α is a
$\beta(\alpha)$ singular limit ordinal. Let $\beta(\alpha)$ be the least ordinal β such that α is singular over
$n(\alpha)$ J_β. Let $n(\alpha)$ be the least integer $n \geqslant 1$ such that α is Σ_n-singular over $J_{\beta(\alpha)}$. The definition of C_α splits into two cases, depending upon the nature of $\beta(\alpha)$ and $n(\alpha)$.

Define

Q $$Q = \{\alpha \in S \mid \beta(\alpha) \text{ is a successor ordinal and } n(\alpha) = 1\};$$

R $$R = S - Q.$$

5.2 Lemma. $\alpha \in Q \to \mathrm{cf}(\alpha) = \omega$.

Proof. Let $\beta = \beta(\alpha) = \gamma + 1$. Notice that as $\alpha \in S$, we have $\lim(\alpha)$, so we must have $\gamma \geqslant \alpha$ here. Let f be a $\Sigma(J_\beta)$ map of a subset, u, of an ordinal $\delta < \alpha$ cofinally into α. Let P be a $\Sigma_0(J_\beta)$ predicate such that

$$f(v) = \tau \leftrightarrow (\exists z \in J_\beta)\, P(z, \tau, v).$$

Now, $J_\beta = \mathrm{rud}(J_\gamma)$, so every element of J_β can be obtained by the successive application of finitely many rud functions to finitely many elements of the set $J_\gamma \cup \{J_\gamma\}$. But amongst the rud functions are the identity function, the pairing function, and the inverses to the pairing function. Moreover, the rud functions are closed under composition. Thus, given any $x \in J_\beta$ we can in fact find a single rud function g and a single element y of J_γ such that $x = g(y, J_\gamma)$. Hence, if $(g_i \mid i < \omega)$ is an enumeration of all the binary rud functions, we have:

$$(*) \qquad J_\beta = \{g_i(x, J_\gamma) \mid x \in J_\gamma \wedge i \in \omega\}.$$

For each $i < \omega$, define a partial function f_i on u by:

$$f_i(v) = \tau \leftrightarrow (\exists x \in J_\gamma)\, P(g_i(x, J_\gamma), \tau, v).$$

By $(*)$,

$$f = \bigcup_{i < \omega} f_i,$$

so

$$\sup_{i<\omega} \bigcup (f_i'' \delta) = \alpha.$$

Thus the lemma will be proved if we can show that $\bigcup(f_i''\delta) < \alpha$ for each $i < \omega$. Since α is regular over J_γ, it suffices to prove that for each $i < \omega, f_i$ is J_γ-definable.

Now, the predicate (of x, τ, v)

$$P(g_i(x, J_\gamma), \tau, v)$$

is $\Sigma_0(J_\beta)$ on J_γ. But one of the properties of the rudimentary functions that we mentioned in section 4 was that for any transitive set X,

$$\Sigma_0(\mathrm{rud}(X)) \cap \mathscr{P}(X) = \mathrm{Def}(X),$$

so in particular we have

$$\Sigma_0(J_\beta) \cap \mathscr{P}(J_\gamma) = \mathrm{Def}(J_\gamma).$$

Thus the predicate (of x, τ, v) $P(g_i(x, J_\gamma), \tau, v)$ is J_γ-definable. It follows at once that f_i is J_γ-definable, and we are done. $\square$

By virtue of 5.2, we may define

$$C_\alpha = \emptyset$$

for the case $\alpha \in Q$, and there is nothing further to check.

We consider now the case $\alpha \in R$.

5.3 Lemma. *If $\alpha \in R$, then $\varrho_{\beta(\alpha)}^{n(\alpha)} = \kappa$.*

Proof. Let $\beta = \beta(\alpha)$, $n = n(\alpha)$. Since κ is a cardinal, II.5.5 (for the Jensen hierar- β, n
chy) implies that $\mathscr{P}(\omega\gamma) \subseteq J_\kappa \subseteq J_\beta$ for all $\gamma < \kappa$. Thus we certainly have $\mathscr{P}(\omega\gamma) \cap \Sigma_n(J_\beta) \subseteq J_\beta$ for all $\gamma < \kappa$. Thus $\varrho_\beta^n \geqslant \kappa$.

Now, by choice of β, n there is a $\Sigma_n(J_\beta)$ map, f, of a bounded subset of α f
cofinally into α. We may code f as a subset of α in a simple fashion (e.g. using a $\Sigma_1(J_\alpha)$ map of α onto J_α). But as f is cofinal in α, $f \notin J_\alpha$, and if $\alpha < \beta$ then by definition of β, α is regular within J_β, so again as f is cofinal in α, we have $f \notin J_\beta$. Thus, in all cases, $\mathscr{P}(\omega\alpha) \cap \Sigma_n(J_\beta) \nsubseteq J_\beta$, and so $\varrho_\beta^n \leqslant \alpha$.

By definition of S, if $\mathrm{dom}(f) \subseteq \gamma < \alpha$, then $|\gamma|^{J_\alpha} \leqslant \kappa$, so J_α contains a map from κ onto γ. Consequently, by composing f with such a map if necessary, we may
assume that $\mathrm{dom}(f) \subseteq \kappa$. We may also assume (again by making trivial alter- f
ations to f if necessary) that $f(v) > \kappa$ for all $v \in \mathrm{dom}(f)$. f

Again, since $\alpha \in S$, for each γ such that $\kappa < \gamma < \alpha$ there is a function $g_\gamma \in J_\alpha$ such
that $g_\gamma\colon \kappa \leftrightarrow \gamma$. In fact we may take g_γ to be the $<_J$-least such map, and then the g_γ
sequence $(g_\gamma \mid \kappa < \gamma < \alpha)$ is $\Sigma_1(J_\alpha)$.

Let $(U_v \mid v < \kappa)$ be a J_κ-definable partition of κ into κ many disjoint sets of size κ, and let $(j_v \mid v < \kappa)$ be a J_κ-definable sequence of maps $j_v\colon U_v \leftrightarrow \kappa$. (Practically any $(U_v \mid v < \kappa)$ and $(j_v \mid v < \kappa)$ which are explicitly defined will be J_κ-definable.)

Set

$$k = \bigcup\{g_{f(v)} \circ j_v \mid v \in \mathrm{dom}(f)\}.$$

Clearly, k is a $\Sigma_n(J_\beta)$ map of κ onto α. Since $\varrho_\beta^n \leqslant \alpha$, it follows at once that $\varrho_\beta^n \leqslant \kappa$, and we are done. $\square$

For $\alpha \in R$, now, we set

$\varrho(\alpha)$, $A(\alpha)$
$$\varrho(\alpha) = \varrho_{\beta(\alpha)}^{n(\alpha)-1}, \qquad A(\alpha) = A_{\beta(\alpha)}^{n(\alpha)-1}.$$

If $n(\alpha) = 1$, then $\varrho(\alpha) = \beta(\alpha)$, so as $\alpha \in R$ we shall have $\lim(\varrho(\alpha))$. And if $n(\alpha) > 1$, then $\varrho(\alpha)$ will be admissible, so again $\lim(\varrho(\alpha))$. Thus $\varrho(\alpha)$ is a limit ordinal. Moreover $\varrho(\alpha) \geqslant \alpha$. For if $n(\alpha) = 1$, then $\varrho(\alpha) = \beta(\alpha) \geqslant \alpha$. On the other hand, suppose $n(\alpha) > 1$. Now, α is $\Sigma_{n(\alpha)-1}$-regular over $J_{\beta(\alpha)}$, so there is certainly no $\Sigma_{n(a)-1}(J_{\beta(\alpha)})$ map of a bounded subset of α cofinally into α. But by definition of $\varrho(\alpha)$, there is a $\Sigma_{n(\alpha)-1}(J_{\beta(\alpha)})$ map from a subset of $J_{\varrho(\alpha)}$ onto J_α, and hence there is a $\Sigma_{n(\alpha)-1}(J_{\beta(\alpha)})$ map from a subset of $\omega \cdot \varrho(\alpha)$ onto α. Thus $\omega \cdot \varrho(\alpha) \geqslant \alpha$. But $\omega\alpha = \alpha$. Hence $\varrho(\alpha) \geqslant \alpha$, as stated.

Fix $\alpha \in R$ now, and set

β, n, ϱ, A, h, H
$$\beta = \beta(\alpha), \quad n = n(\alpha), \quad \varrho = \varrho(\alpha), \quad A = A(\alpha), \quad h = h_{\varrho, A}, \quad H = H_{\varrho, A}.$$

So, in particular, h is the canonical Σ_1 skolem function for $\langle J_\varrho, A\rangle$ and H is a $\Sigma_0^{\langle J_\rho, A\rangle}$ predicate with the property that

$$y = h(i, x) \leftrightarrow (\exists z \in J_\varrho) H(z, y, i, x).$$

h_τ For $\tau < \varrho$, define a partial function h_τ from $\omega \times J_\tau$ into J_τ by

$$y = h_\tau(i, x) \leftrightarrow (\exists z \in J_\tau) H(z, y, i, x).$$

Now, the canonical Σ_1 skolem function $h_{\xi, U}$ is uniform for all amenable $\langle J_\xi, U\rangle$. In particular, whenever $\tau < \varrho$ is such that $\langle J_\tau, A \cap J_\tau\rangle$ is amenable, then the function h_τ defined above is its canonical Σ_1 skolem function, i.e. $h_\tau = h_{\tau, A \cap J_\tau}$.

Now, by definition of ϱ_β^n, together with 5.3 (and the J_γ-analogue of II.6.8) there

$f, \bar{f}$ is a $\Sigma_n(J_\beta)$ map f such that $f''\kappa = J_\beta$. Let $\bar{f} = f \cap (J_\varrho \times \kappa)$. Then $\bar{f}$ is a $\Sigma_n(J_\beta)$ subset of J_ϱ. So by the properties of the standard codes given in section 4, $\bar{f}$ is $\Sigma_1(\langle J_\varrho, A\rangle)$. Moreover, $\bar{f}''\kappa = J_\varrho$. By the properties of the Σ_1 skolem function, if $\bar{f}$ is $\Sigma_1^{\langle J_\rho, A\rangle}(\{p\})$, we will have

$$h''(\omega \times (\kappa \times \{p\})) = J_\varrho.$$

So we may define

p
$$p = p(\alpha) = \text{the } <_J\text{-least } p \in J_\varrho \text{ such that } J_\varrho = h''(\omega \times (\kappa \times \{p\})).$$

Define a map g from a subset of κ into J_ϱ by setting

g
$$g(\omega\nu + i) \simeq h(i, (\nu, p)).$$

G By choice of p, $g''\kappa = J_\varrho$. Moreover, g is $\Sigma_1^{\langle J_\rho, A\rangle}(\{p\})$. Let G be the canonical $\Sigma_0^{\langle J_\rho, A\rangle}(\{p\})$ predicate (obtained from H) such that

$$g(\nu) = x \leftrightarrow (\exists z \in J_\varrho) G(z, x, \nu).$$

By recursion we shall define functions $k\colon \theta \to \kappa$, $m\colon \theta \to \varrho$, $(X_\nu \mid \nu < \theta)$, $(\alpha_\nu \mid \nu < \theta)$, for some θ which is to be determined during the course of the definition. The exact order of this definition will be examined as soon as the definition has been given in full.

$k(\nu) =$ the least $\tau \in \mathrm{dom}(g)$ such that $\alpha_\nu < g(\tau) < \alpha$ and $|\alpha_\nu|^{J_{g(\tau)}} \leqslant \kappa$. $k(\nu)$

$m(0) =$ the least $\gamma \geqslant \kappa$ such that $p \in J_\gamma$; $m(\nu)$

$m(\nu + 1) =$ the least $\gamma > m(\nu), \alpha_\nu, g \circ k(\nu)$ such that:

(i) $A \cap J_{m(\nu)} \in J_\gamma$;

(ii) $m(\nu), \alpha_\nu, g \circ k(\nu) \in h''_\gamma(\omega \times (\kappa \times \{p\}))$;

(iii) $(\exists z \in J_\gamma)(G(z, g \circ k(\nu), k(\nu))$;

$m(\lambda) = \sup_{\nu < \lambda} m(\nu)$, if $\lim(\lambda)$ and this supremum is less than ϱ (undefined if the supremum equals ϱ).

$X_\nu = h''_{m(\nu)}(\omega \times (\kappa \times \{p\}))$. X_ν

$\alpha_\nu = \sup(X_\nu \cap \alpha)$. α_ν

Our $\square'_\kappa$ set C_α will be the set

$$C_\alpha = \{\alpha_\nu \mid \nu < \theta \wedge \lim(\nu)\},$$

where θ is the first ordinal for which the above definition breaks down. We shall show that $\lim(\theta)$ and that θ is the least ordinal such that $\sup_{\nu<\theta} m(\nu) = \varrho$. We shall also show that the function k is order-preserving, so $\theta \leqslant \kappa$ and $\mathrm{otp}(C_\alpha) \leqslant \kappa$. (The function g is used precisely in order to obtain this result.) A condensation argument will be used to show that $C_{\bar\alpha} = \bar\alpha \cap C_\alpha$ whenever $\bar\alpha \in C_\alpha$. The rather complicated definition of the function m is designed to facilitate this part of the proof. And now down to business. θ

Let us examine the way in which the above definition proceeds, and how it may break down. The definition of $m(0)$ comes first, and is unproblematical. Suppose now that $m(\nu)$ is defined for some ν. Then we may define X_ν and α_ν. We show that $\alpha_\nu < \alpha$. Suppose not. In other words, suppose that $h''_{m(\nu)}(\omega \times (\kappa \times \{p\})) \cap \alpha$ is cofinal in α. Now, $\langle J_\varrho, A\rangle$ is amenable, so $A \cap J_{m(\nu)} \in J_\varrho$. Thus $\langle J_{m(\nu)}, A \cap J_{m(\nu)}\rangle \in J_\varrho$. Thus $h_{m(\nu)} \in J_\varrho$. Now, there is a J_κ-definable map of κ onto $\omega \times (\kappa \times \{p\})$. Thus J_ϱ contains a map from a subset of κ cofinally into α. If $\varrho = \alpha$ this is already a contradiction. What if $\varrho > \alpha$? Well, in this case, since $\varrho \leqslant \beta(\alpha)$, α is a regular cardinal inside J_ϱ, and again we have a contradiction. Thus $\alpha_\nu < \alpha$. Since $\alpha \in S$, it follows that $k(\nu)$ is defined. And now we may define $m(\nu + 1)$ without any difficulty. Thus the only way in which the construction can break down is when a reach a limit ordinal θ such that $\sup_{\nu<\theta} m(\nu) = \varrho$.

For each $\nu < \theta$, by definition of $m(\nu + 1)$ we have $\alpha_\nu \in h''_{m(\nu+1)}(\omega \times (\kappa \times \{p\}))$, so $\alpha_\nu \in X_{\nu+1} \cap \alpha$. Thus $\alpha_\nu < \alpha_{\nu+1}$. Moreover, since the function m is continuous (by definition), for any limit ordinal $\lambda < \theta$ we have, by virtue of the manner in which the functions h_τ were defined, $X_\lambda = \bigcup_{\nu<\lambda} X_\nu$, and hence $\alpha_\lambda = \sup_{\nu<\lambda} \alpha_\nu$. Thus the sequence $(\alpha_\nu \mid \nu < \theta)$ is strictly increasing and continuous at limits. Again, since

$\sup_{\nu<\theta} m(\nu) = \varrho$, we have

$$\bigcup_{\nu<\theta} X_\nu = h_\varrho''(\omega \times (\kappa \times \{p\})) = J_\varrho,$$

so $\sup_{\nu<\theta} \alpha_\nu = \alpha$.

We show next that the function k is strictly increasing. Let $\nu < \tau < \theta$. By the definition of $k(\tau)$, $g \circ k(\tau) > \alpha_\tau$ and $|\alpha_\tau|^{J_{g \circ k(\tau)}} \leqslant \kappa$. So as $\alpha_\nu < \alpha_\tau$, we have $g \circ k(\tau) > \alpha_\nu$ and $|\alpha_\nu|^{J_{g \circ k(\tau)}} \leqslant \kappa$. So by the minimality of $k(\nu)$ in its definition, $k(\nu) \leqslant k(\tau)$. But by definition of $m(\nu + 1)$, $g \circ k(\nu) \in X_{\nu+1} \cap \alpha$, so $g \circ k(\nu) < \alpha_{\nu+1} \leqslant \alpha_\tau \leqslant g \circ k(\tau)$, and in particular $k(\nu) \neq k(\tau)$. Thus $k(\nu) < k(\tau)$.

Since k is strictly increasing from θ into κ, we must have $\theta \leqslant \kappa$.

We set

C_α

$$C_\alpha = \{\alpha_\nu \mid \nu < \theta \wedge \lim(\nu)\}.$$

By the above results, C_α is closed in α, has order-type at most κ, and if $\mathrm{cf}(\alpha) > \omega$ then C_α is unbounded in α. Moreover, by the definition of k and the inequality $\alpha_\nu < g \circ k(\nu) < \alpha_{\nu+1}$ (noted during the proof that k is increasing), we have $C_\alpha \subseteq S$. Thus, all that remains to be proved now is that if $\bar\alpha \in C_\alpha$, then $C_{\bar\alpha} = \bar\alpha \cap C_\alpha$.

$\bar\alpha$, λ Let $\bar\alpha \in C_\alpha$ be given. For some limit ordinal $\lambda < \theta$, $\bar\alpha = \alpha_\lambda$. Note that by the definition of m, $\lim(\lambda)$ implies that $\langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle$ is amenable.

5.4 Lemma. $\bar\alpha \subseteq X_\lambda$.

Proof. Since $\bar\alpha = \sup_{\nu<\lambda} \alpha_\nu$, it suffices to show that $\alpha_\nu \subseteq X_\lambda$ for all $\nu < \lambda$. So let $\nu < \lambda$. Then by definition of $m(\nu + 1)$, $\alpha_\nu \in X_{\nu+1} \subseteq X_\lambda$. But $\bar\alpha \in S$, so $|\alpha_\nu|^{J_{\bar\alpha}} \leqslant \kappa$. Hence $|\alpha_\nu|^{J_{m(\lambda)}} \leqslant \kappa$. Since

$$\{\alpha_\nu\} \cup \kappa \subseteq X_\lambda \prec_1 J_{m(\lambda)},$$

we have $\alpha_\nu \subseteq X_\lambda$, as required. □

By the condensation lemma, let

π, $\bar\varrho$, $\bar A$

$$\pi\colon \langle J_{\bar\varrho}, \bar A\rangle \cong \langle X_\lambda, A \cap X_\lambda\rangle.$$

Thus

$$\pi\colon \langle J_{\bar\varrho}, \bar A\rangle \prec_1 \langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle.$$

But by transitivity,

$$\langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle \prec_0 \langle J_\varrho, A\rangle.$$

Hence

$$\pi\colon \langle J_{\bar\varrho}, \bar A\rangle \prec_0 \langle J_\varrho, A\rangle.$$

It follows from the fine-structure theory (section 4) that there is a unique $\bar\beta$ such

$\bar\beta$, $\tilde\pi$ that $\bar\varrho = \varrho_{\bar\beta}^{n-1}$ and $\bar A = A_{\bar\beta}^{n-1}$, and a unique $\tilde\pi \supseteq \pi$ such that, in particular,

$$\tilde\pi\colon J_{\bar\beta} \prec_{n-1} J_\beta.$$

$\bar h$, $\bar p$ Let $\bar h = h_{\bar\varrho, \bar A}$, and set $\bar p = \pi^{-1}(p)$.

By 5.4, $\pi \upharpoonright \bar{\alpha} = \mathrm{id} \upharpoonright \bar{\alpha}$. If $\bar{\alpha} < \bar{\beta}$, then since $\bar{\alpha} = \alpha_\lambda = X_\lambda \cap \alpha = \mathrm{ran}(\pi) \cap \alpha$ and $\pi \subseteq \tilde{\pi}$, we have $\tilde{\pi}(\bar{\alpha}) \geqslant \alpha$. (If $\alpha \in \mathrm{ran}(\tilde{\pi})$, then in fact $\tilde{\pi}(\bar{\alpha}) = \alpha$, but we have no reason to suppose that this is the case.)

Define a map $\bar{g}_0$ from a subset of κ into $J_{m(\lambda)}$ by $\bar{g}_0$

$$\bar{g}_0(v) = x \leftrightarrow (\exists z \in J_{m(\lambda)})\, G(z, x, v).$$

Now, $\bar{g}_0$ is $\Sigma_1^{\langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle}(\{p\})$. Hence $\bar{g}_1 = \bar{g}_0 \cap (\bar{\alpha} \times \kappa)$ is $\Sigma_1^{\langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle}(\{p\})$. $\bar{g}_1$
But $\pi \upharpoonright \bar{\alpha} = \mathrm{id} \upharpoonright \bar{\alpha}$. Thus $\pi^{-1}{}''\bar{g}_1 = \bar{g}_1$, and so $\bar{g}_1$ is $\Sigma_1^{\langle J_{\bar{\varrho}}, \bar{A}\rangle}(\{\bar{p}\})$. So, as $\bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$, $\bar{A} = A_{\bar{\beta}}^{n-1}$, $\bar{g}_1$ is $\Sigma_n(J_{\bar{\beta}})$.

By definition of m, $k''\lambda \subseteq \mathrm{dom}(\bar{g}_1)$ and $\bar{g}_1 \upharpoonright (k''\lambda) = g \upharpoonright (k''\lambda)$. But $\alpha_v < g \circ k(v) < \alpha_{v+1}$ for all v. Thus $\bar{g}_1$ is cofinal in $\bar{\alpha}$.

5.5 Lemma. $\varrho_{\bar{\beta}}^n = \kappa$.

Proof. Since $\bar{\alpha} \in S$, we can use the function $\bar{g}_1$ to prove this by the same argument we used in 5.3. □

5.6 Lemma. $\bar{\beta} = \beta(\bar{\alpha})$.

Proof. The existence of $\bar{g}_1$ shows that $\beta(\bar{\alpha}) \leqslant \bar{\beta}$. If $\bar{\beta} = \bar{\alpha}$ we are done. So assume $\bar{\beta} > \bar{\alpha}$. Suppose that $\beta(\bar{\alpha}) < \bar{\beta}$. Then $J_{\bar{\beta}}$ will contain a map f from a subset of some $\gamma < \bar{\alpha}$ cofinally into $\bar{\alpha}$. Since $\tilde{\pi}$: $J_{\bar{\beta}} \prec_{n-1} J_\beta$, $\tilde{\pi}(f)$ will be a map from a subset of $\tilde{\pi}(\gamma) < \tilde{\pi}(\bar{\alpha})$ cofinally into $\tilde{\pi}(\bar{\alpha})$. Now, $\gamma \in \bar{\alpha}$, so $\tilde{\pi}(\gamma) = \gamma$. Thus $\tilde{\pi}(f)$ maps a subset of γ cofinally into $\tilde{\pi}(\bar{\alpha})$. But since $f \subseteq \bar{\alpha} \times \bar{\alpha}$, $f \subseteq \tilde{\pi}(f)$. So as $\mathrm{dom}(f) \subseteq \gamma$ and $\mathrm{dom}(\tilde{\pi}(f)) \subseteq \gamma$, we must have $\tilde{\pi}(f) = f$. Thus f maps a subset of γ cofinally into $\tilde{\pi}(\bar{\alpha})$. But $\mathrm{ran}(f) \subseteq \bar{\alpha} < \alpha \leqslant \tilde{\pi}(\bar{\alpha})$, so this is impossible. This proves the lemma. □

5.7 Lemma. $n = n(\bar{\alpha})$.

Proof. By 5.6 and the properties of $\bar{g}_1$, $n(\bar{\alpha}) \leqslant n$. If $n = 1$ we are done. So assume $n > 1$. We must prove that $\bar{\alpha}$ is Σ_{n-1} regular over $J_{\bar{\beta}}$. Suppose not. Then there is a $\Sigma_{n-1}(J_{\bar{\beta}})$ map of a bounded subset of $\bar{\alpha}$ cofinally into $\bar{\alpha}$. Since $\bar{\alpha} \in S$, an argument as in 5.3 now shows that $\varrho_{\bar{\beta}}^{n-1} = \kappa$. But $\varrho_{\bar{\beta}}^{n-1} = \bar{\varrho} \geqslant \bar{\alpha} > \kappa$, a contradiction. □

5.8 Lemma. $\bar{p} = p(\bar{\alpha})$.

Proof. By 5.5, 5.6, and 5.7, $p(\bar{\alpha})$ is (by definition) the $<_J$-least element of $J_{\bar{\varrho}}$ such that $J_\varrho = \bar{h}''(\omega \times (\kappa \times \{p(\bar{\alpha})\}))$. Now,

$$\pi\colon \langle J_{\bar{\varrho}}, \bar{A}\rangle \prec_1 \langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle,$$

so

$$\pi''[\bar{h}''(\omega \times (\kappa \times \{\bar{p}\}))] = h''_{m(\lambda)}(\omega \times (\kappa \times \{p\})) = X_\lambda.$$

Thus,

$$\bar{h}''(\omega \times (\kappa \times \{\bar{p}\})) = \pi^{-1}{}'' X_\lambda = J_{\bar{\varrho}}.$$

This proves that $p(\bar{\alpha}) \leqslant_J \bar{p}$. Let $p' = \pi(p(\bar{\alpha}))$. Pick $i \in \omega$, $v \in \kappa$ so that $\bar{p} = \bar{h}(i, (v, p(\bar{\alpha})))$. Applying π, we get $p = h(i, (v, p'))$. Thus

$$h''(\omega \times (\kappa \times \{p\})) \subseteq h''(\omega \times (\kappa \times \{p'\})).$$

Thus $J_\varrho = h''(\omega \times (\kappa \times \{p'\}))$. So by choice of p, $p \leqslant_J p'$. Applying π^{-1}, $\bar{p} \leqslant_J p(\bar{\alpha})$. The lemma is proved. □

$\bar{g}$ Now define $\bar{g}$ from $\bar{\alpha}$ exactly as g was defined from α.

5.9 Lemma. $\bar{g} \cap (\bar{\alpha} \times k''\lambda) = g \cap (\bar{\alpha} \times k''\lambda)$.

Proof. By virtue of 5.5 through 5.8, for $\nu < \kappa$, $\tau < \bar{\alpha}$, we have

$$\bar{g}(\omega\nu + i) = \tau \leftrightarrow \bar{h}(i, (\nu, \bar{p})) = \tau .$$

But $\pi\colon \langle J_\varrho, \bar{A}\rangle \prec_1 \langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle$ and $\pi \restriction \bar{\alpha} = \mathrm{id} \restriction \bar{\alpha}$. Hence

$$\bar{h}(i, (\nu, \bar{p})) = \tau \leftrightarrow h_{m(\lambda)}(i, (\nu, p)) = \tau .$$

Thus:

$$(*) \qquad \bar{g}(\omega\nu + i) = \tau \leftrightarrow h_{m(\lambda)}(i, (\nu, p)) = \tau .$$

Now, by the uniformity of the Σ_1 skolem function,

$$h_{m(\lambda)}(i, (\nu, p)) = \tau \quad \text{implies} \quad h(i, (\nu, p)) = \tau .$$

Thus by $(*)$,

$$\bar{g}(\omega\nu + i) = \tau \quad \text{implies} \quad g(\omega\nu + i) = \tau .$$

Suppose that, in addition, $\omega\nu + i \in k''\lambda$. Assume that $g(\omega\nu + i) = \tau$. Then by the definition of the function m,

$$(\exists z \in J_{m(\lambda)})\, G(z, \tau, \omega\nu + i) .$$

So by the canonical, uniform nature of the Σ_0 predicate G,

$$h_{m(\lambda)}(i, (\nu, p)) = \tau .$$

Thus by $(*)$, $\bar{g}(\omega\nu + i) = \tau$, and we are done. □

Now define $\bar{k}, \bar{m}, (\bar{X}_\nu \mid \nu < \bar{\theta})$, $(\bar{\alpha}_\nu \mid \nu < \bar{\theta})$ from $\bar{\alpha}$ exactly as we defined $k, m, (X_\nu \mid \nu < \theta), (\alpha_\nu \mid \nu < \theta)$ from α. Thus, in particular, provided that $\bar{\alpha} \in R$, we will have

$$C_{\bar{\alpha}} = \{\bar{\alpha}_\nu \mid \nu < \bar{\theta} \wedge \lim(\nu)\} .$$

5.10 Lemma. *For all* $\nu < \lambda$, $\bar{k}(\nu) = \pi(\bar{k}(\nu)) = k(\nu)$, $\pi(\bar{m}(\nu)) = m(\nu)$, $\pi''\bar{X}_\nu = X_\nu$, $\bar{\alpha}_\nu = \pi(\bar{\alpha}_\nu) = \alpha_\nu$.

Proof. Since $\pi\colon \langle J_{\bar{\varrho}}, \bar{A}\rangle \prec_1 \langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle$ and $\pi \restriction \bar{\alpha} = \mathrm{id} \restriction \bar{\alpha}$, this follows from 5.5 through 5.9 by a straightforward induction on ν. □

Since $(\alpha_\nu \mid \nu < \lambda)$ is cofinal in $\alpha_\lambda = \bar{\alpha}$, it follows from the above lemma that $\bar{\theta} = \lambda$. Hence,

$$\{\bar{\alpha}_\nu \mid \nu < \bar{\theta} \wedge \lim(\nu)\} = \{\alpha_\nu \mid \nu < \lambda \wedge \lim(\nu)\} = \bar{\alpha} \cap C_\alpha .$$

We shall be done provided that $\bar{\alpha} \in R$. Suppose that $n = 1$. Then $\bar{\varrho} = \varrho^0_{\bar{\beta}} = \bar{\beta}$. But $\pi\colon J_{\bar{\varrho}} \prec J_{m(\lambda)}$ and $\lim(m(\lambda))$. Hence $\lim(\bar{\varrho})$. Thus $n = 1$ implies $\lim(\bar{\beta})$. Thus $\bar{\alpha} \in R$, and we are done. $\square'_\kappa$ is proved.

Exercises

1. κ^+-*Aronszajn trees* (Section 2)

Let κ be an infinite cardinal. By a *special* κ^+-Aronszajn tree we mean a κ^+-tree $\mathbf{T}$ such that for each $\alpha < \kappa^+$, $T_\alpha \subseteq \{f \mid f\colon \alpha \xrightarrow{1-1} \kappa\}$, with the ordering $f <_{\mathbf{T}} g$ iff $f \subseteq g$. It is immediate that any such tree must be κ^+-Aronszajn, of course.

1 A. Prove that there is a special ω_1-Aronszajn tree, first of all by making a simple modification to the tree constucted in III.1.1, and then by means of a direct recursion on the levels, much as in the proof of III.1.1. (As then, the problem is to ensure that the construction does not break down at some stage.)

1 B. Prove that if κ is a regular cardinal such that $2^{<\kappa} = \kappa$, then there is a special κ^+-Aronszajn tree. (Generalise the direct proof of 1 A above. The hypotheses on κ are used to ensure that the construction does not break down.)

1 C. Prove that if $V = L$ (or more generally if $2^{<\kappa} = \kappa$ and $\square_\kappa$ holds), then for any infinite cardinal κ there is a special κ^+-Aronszajn tree. (The $\square_\kappa$-sequence is used to ensure that the construction does not break down. See the proof of 2.4.)

2. κ^+-*Souslin trees* (Section 2)

Let κ be an uncountable regular cardinal. Assume GCH together with $\diamondsuit_{\kappa^+}(E)$, where $E = \{\alpha \in \kappa^+ \mid \mathrm{cf}(\alpha) = \kappa\}$. Prove that there is a κ^+-Souslin tree.

3. κ-*Kurepa trees* (Section 1)

Show that if κ is inaccessible, there is a κ-tree with 2^κ many κ-branches. Suggest a definition of a κ-Kurepa tree which avoids this example (and indeed any other example one can construct in ZFC alone).

4. *The combinatorial principle* $\diamondsuit^+_{\kappa^+}$ (Section 3)

Formulate the principle $\diamondsuit^+_{\kappa^+}$ by analogy with $\diamondsuit^+$ for ω_1. Prove that $V = L$ implies $\diamondsuit^+_{\kappa^+}$ and that $\diamondsuit^+_{\kappa^+}$ implies the existence of a κ^+-Kurepa tree.

5. $\square_\kappa$ *in* $L[A]$ (Section 5)

Prove that $\square_\kappa$ holds if $V = L[A]$, where $A \subseteq \kappa^+$ is such that

$$(\forall\, \alpha < \kappa^+)[|\alpha|^{L[A \cap \alpha]} \leqslant \kappa].$$

(This requires some reworking of the fine-structure theory, and is quite a demanding exercise.)

6. *On the failure of* $\Box_\kappa$ (Section 5)

From the result of exercise 5 above, deduce that if $\Box_\kappa$ fails, then κ^+ is Mahlo in L. (It can be proved that if it is consistent with ZFC that a Mahlo cardinal exists, then it is consistent with ZFC that $\Box_{\omega_1}$ is false.)

7. *GCH and the principles* $\diamondsuit_{\kappa^+}$ (Section 2)

Prove the following generalisation of lemma 2.7: Assume $2^\kappa = \kappa^+$ and that $\lambda < \kappa$ is a regular cardinal such that either $\kappa^\lambda = \kappa$ or else $[\lambda \neq \mathrm{cf}(\kappa)$ and $(\forall \theta < \kappa) \cdot (\theta^\lambda \leqslant \kappa)]$. Then $\diamondsuit_{\kappa^+}(\{\delta < \kappa^+ \mid \mathrm{cf}(\delta) = \lambda\})$ holds. (Even better, conclude that $\diamondsuit^+_{\kappa^+}(\{\delta < \kappa^+ \mid \mathrm{cf}(\delta) = \lambda\})$ holds.)

8. $\Box_\kappa$ *and the principles* $\diamondsuit_{\kappa^+}$ (Section 2)

Prove the following generalisation of lemma 2.8: Assume $\Box_\kappa$ and that $(\forall \theta < \kappa) \cdot (\theta^{\mathrm{cf}(\kappa)} \leqslant \kappa)$ & $2^\kappa = \kappa^+$. Then $\diamondsuit_{\kappa^+}(\{\delta < \kappa^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\})$ holds. (Can the above be strengthened to get $\diamondsuit^*_{\kappa^+}(\{\delta < \kappa^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\})$?)