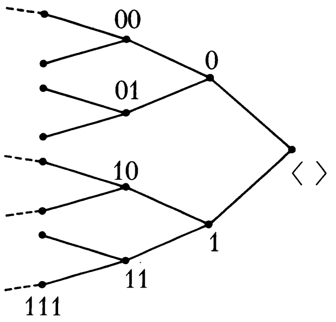
Chapter VIII

Strict Π_1^1 Predicates and König Principles

1. The König Infinity Lemma

In this section we discuss some of the uses of the Infinity Lemma in ordinary recursion theory. The applications chosen for discussion are those which become important new "axioms" or *König Principles*, when stated in the abstract.

Let $\mathcal{T} = \langle T, \prec \rangle$ be the *full binary tree*, as pictured below.



The set T is the set of nodes (finite sequences of 0's and 1's) ordered by

$$d' \prec d$$

if the sequence d' properly extends the sequence d. If $S \subseteq T$ is such that $d_0 \in S$ and $d_0 \prec d_1$ implies $d_1 \in S$, then $\mathcal{S} = \langle S, \prec \upharpoonright S \rangle$ is called a *subtree* of $\mathcal{T}$. If $\mathcal{S}$ is a subtree then any maximal $\prec$-linearly ordered subset b of S is called a *branch through* $\mathcal{S}$.

1.1 König Infinity Lemma. *Let* $\mathcal{S} = \langle S, \prec \upharpoonright S \rangle$ *be any subtree of the full binary tree. The following are equivalent:*

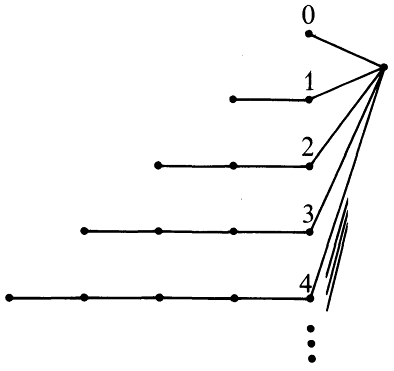
(i) *$\mathcal{S}$ has no infinite branch,*
(ii) *$\mathcal{S}$ is well founded,*
(iii) *$\mathcal{S}$ is well founded and has finite rank,*
(iv) *S is finite.*

Proof. Each of the implications (iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i) is completely trivial so we need only prove (i)$\Rightarrow$(iv), or equivalently, $\neg$(iv)$\Rightarrow\neg$(i). Suppose S is infinite. Let $d_0 = \langle\ \rangle \in S$. Either 0 or 1 has infinitely many predecessors in S so let d_1 be the least of these which has infinitely many predecessors in S. Let $d_n \in S$ have infinitely many predecessors in S and let d_{n+1} be the first of $d_n\frown 0$, $d_n\frown 1$ which has infinitely many predecessors in S. One of them must. Then

$$b = \{d_0, d_1, d_2, \ldots\}$$

is an infinite branch through S. □

One can generalize 1.1 trivially by allowing each node to have any finite number of immediate predecessors, instead of exactly two, but once you allow infinitely many, the theorem becomes false, as the following tree shows.



Indeed, the Infinity Lemma is so tied to the notion of finiteness and the integers that it is difficult to generalize in a really useful way. So, rather than generalize the Infinity Lemma itself, we go back and look for useful consequences of the Infinity Lemma. Three of these consequences have turned out to play important roles when generalized to other admissible sets. In this section we prove these three results.

A predicate $P(x, f)$ of integers x and number theoretic functions f is r.e. iff there is a recursive predicate $R(x, y)$ of integers such that

$$P(x, f) \leftrightarrow \exists n\, R(x, \bar{f}(n))$$

where $\bar{f}(n)$ is (a code for) the finite sequence $\langle f(0), \ldots, f(n-1)\rangle$ and $R(x, \bar{f}(n))$ implies $R(x, \bar{f}(m))$ for all $m \geqslant n$. (This may be taken as the definition or verified easily from any other reasonable definition. This is the natural extension of r.e. to predicates $P(x, f)$ of numbers and functions.)

1.2 Definition. A predicate $S(\vec{x})$ on the integers is *strict*-Π^1_1 (or s-Π^1_1 for short) if it can be written in the form

$$S(\vec{x}) \leftrightarrow \forall f \in 2^\omega\, P(\vec{x}, f)$$

where P is r.e.

Here we use 2^ω to denote the set of characteristic functions, i. e., those functions mapping ω into $2=\{0,1\}$. The word "strict" refers to the fact that f ranges only over 2^ω, not over all number theoretic functions.

Our first application of the Infinity Lemma is to prove the following result.

1.3 Theorem (s-Π^1_1 = r.e., on ω). *A predicate P on ω is strict-Π^1_1 iff P is r.e.*

Proof. To prove ($\Leftarrow$) just add a superfluous function quantifier. To prove ($\Rightarrow$) write

$$(1) \qquad P(x) \leftrightarrow \forall f \in 2^\omega\, \exists n\, R(x, \bar{f}(n))$$

where R is recursive and satisfies

$$R(x, \bar{f}(n)) \wedge m \geqslant n \Rightarrow R(x, \bar{f}(m)).$$

For $f \in 2^\omega$, each $\bar{f}(n)$ is a sequence of 0's and 1's and so is really just a node on the full binary tree T. The condition on R above asserts that

$$R(x,d) \wedge d' \prec d \Rightarrow R(x,d')$$

or, turning it around,

$$\neg R(x,d') \wedge d' \prec d \Rightarrow \neg R(x,d).$$

Thus, $S_x = \{d \mid \neg R(x,d)\}$ is a subtree of T. If we restate (1) in terms of trees, it becomes

(2) $\qquad P(x)$ iff S_x has no infinite path,

which becomes, by the Infinity Lemma,

$$\begin{aligned} P(x) &\leftrightarrow S_x \text{ is finite} \\ &\leftrightarrow \exists N\, \forall d \text{ of length } N,\ d \notin S_x \\ &\leftrightarrow \exists N\, \forall d \text{ of length } N,\ R(x,d) \\ &\leftrightarrow \exists N\, R'(x,N), \end{aligned}$$

where R' is recursive. More informally, $P(x)$ holds iff you can find a finite subtree such that $R(x,d)$ holds for every end-node d on S. ☐

In proving 1.3 we also proved the next theorem. This (or a relativized form of it) is what Shoenfield [1967] refers to as the Brouwer-König Infinity Lemma.

1.4 Theorem (s-Π_1^1 Reflection for ω). *Let*

$$P(x) \leftrightarrow \forall f \in 2^\omega\, \exists n\, R(x, \bar{f}(n))$$

define a strict-Π_1^1 *predicate. Then for any* x

$$P(x) \leftrightarrow \exists N\, \forall f \in 2^\omega\, \exists n \leqslant N\, R(x, \bar{f}(n)).$$

Proof. This is contained in the proof of 1.3. $\square$

We will see in § 4 that the equation "s-$\Pi_1^1 = $r.e." can be viewed as abstract formulation of the completeness theorem for $\mathsf{L}_{\omega\omega}$ and that "s-Π_1^1 Reflection for ω" corresponds to the compactness theorem for $\mathsf{L}_{\omega\omega}$. The fact that our proof of 1.3 also gives 1.4 corresponds to the fact that most proofs of the completeness theorem yield compactness, but not vice versa.

Our final application of the Infinity Lemma is to the notion of implicit ordinal. We state the definition in general to save repeating the definition in § 5.

1.5 Definition. Let $\mathfrak{M}$ be a structure for some language L, let R, S be two new n-ary relation symbols and let $\varphi(\mathsf{R},\mathsf{S})$ be a sentence of $\mathsf{L}(\mathsf{R},\mathsf{S})$, possibly containing parameters from $\mathfrak{M}$. Let α be an ordinal. The sentence $\varphi(\mathsf{R},\mathsf{S})$ *implicitly defines* α *over* $\mathfrak{M}$ if the relation $\prec_\varphi$ defined by

$$R \prec_\varphi S \quad \text{iff} \quad (\mathfrak{M}, R, S) \models \varphi(\mathsf{R},\mathsf{S})$$

is well founded and α is its rank, i. e., $\alpha = \rho(\prec_\varphi)$.

Our final application of the Infinity Lemma shows that if a Π_1^0 relation $\varphi(\mathsf{R},\mathsf{S})$ on ω implicitly defines an ordinal, then that ordinal is finite. In § 6 we will learn that any α implicitly defined by even a Σ_1^1 sentence on ω is just the order type of a recursive (explicit) well-ordering of ω. These two facts explain why the notion of implicitly defined ordinal does not arise explicitly in ordinary recursion theory.

A predicate $\varphi(R,S)$ on ω is Π_1^0 iff $\neg\varphi(R,S)$ is r.e. (To fit this into our definition of r.e. replace R, S by their characteristic functions.)

1.6 Theorem. *Let* $\varphi(R,S)$ *be a* Π_1^0 *predicate of n-ary relations on* ω. *If the relation* $\prec_\varphi$ *defined by*

$$R \prec_\varphi S \quad \textit{iff} \quad \varphi(R,S)$$

is well founded, then its rank is finite.

Proof. By use of pairing functions we can assume $n=1$, i.e., that R, S range over subsets of ω. Assume $\prec_\varphi$ is well founded so that

$$(2) \qquad \forall R_1, R_2 \ldots \exists n \neg\varphi(R_{n+1}, R_n).$$

For any $f \in 2^\omega$ let $(f)_n = \{x \mid f(2^x 3^n) = 1\}$. We can restate (2) as

$$(3) \qquad \forall f \in 2^\omega \, \exists n \neg\varphi((f)_{n+1}, (f)_n).$$

Since φ is Π_1^0, the predicate $\neg\varphi((f)_{n+1}, (f)_n)$ is an r.e. predicate of f, n. By s-Π_1^1 reflection there is an $N < \omega$ such that

$$\forall f \, \exists n \leqslant N \neg\varphi((f)_{n+1}, (f)_n)$$

which says that there is no sequence

$$R_{N+1} \prec_\varphi R_N \prec_\varphi \cdots \prec_\varphi R_1 \prec_\varphi R_0.$$

Thus $\rho(\prec_\varphi) \leqslant N+1$. □

1.7—1.10 Exercises

1.7. Prove the relativized version of the theorems of this section. (The fact that s-Π_1^1 Reflection holds relativized to any relation R on ω is expressed by saying that ω is *strict*-Π_1^1 *indescribable.*)

1.8. Let $R \prec S$ iff $R, S \subseteq \omega$ and the least member of R is less than the least member of S. Show that this is an r.e. predicate of R, S and that it implicitly defines ω.

1.9. Let $R \prec S$ iff $R, S \subseteq \omega \times \omega$, R, S are well-orderings and R is a proper initial segment of S. Show that this is a Π_1^1 relation which implicitly defines the first uncountable ordinal.

1.10. Let $\prec$ be a well-founded relation on subsets of ω defined by a Σ_1^1 sentence φ. Show that the rank of $\prec$ is $< \omega_1^c$. [Hint: Show that $\rho(\prec)$ can be pinned down by a sentence of $\mathsf{L}_{\omega_1^c}$.]

1.11 Notes. The equation "s-$\Pi_1^1 =$ r.e., on ω" was first observed by Kreisel in the proof of the Kreisel Basis Theorem (cf. p. 187 of Shoenfield [1967]).

2. *Strict Π_1^1 Predicates: Preliminaries*

Over ω, or $\mathbb{HF}$, the strict-Π_1^1 predicates coincide with the r.e. predicates (by 1.3) so it is difficult to see the exact role that the notion of strict-Π_1^1 plays in traditional model theory and recursion theory. In general, however, strict-Π_1^1 does not coincide with Σ_1. By studying the s-Π^1_1 predicates in the general case, then, we see more clearly the role they play over ω.

Let $\mathsf{L}^* = \mathsf{L}(\in, \ldots)$ be the language for KPU. We assume that there are only relation and constant symbols in L^*, no function symbols. (This is not an essential restriction—see Exercise V.1.8.) Let $\mathsf{R}_1, \mathsf{R}_2, \ldots$ be an infinite list of new relation symbols, an infinite number of arity n for each $n < \omega$. Let $\mathsf{L}^*(\vec{\mathsf{R}})$ be the expanded language.

2.1 Definition. i) The *strict-Π^1_1 formulas* (s-Π^1_1 for short) of $\mathsf{L}^*(\vec{\mathsf{R}})$ form the smallest class containing the Δ_0 formulas of $\mathsf{L}^*(\vec{\mathsf{R}})$ closed under $\wedge$, $\vee$, $\forall u \in v$, $\exists u \in v$, $\exists u$ and the clause

$$\text{if } \Phi(\mathsf{R}_i) \text{ is strict-}\Pi^1_1 \text{ so is } \forall \mathsf{R}_i\, \Phi(\mathsf{R}_i).$$

The strict-Π^1_1 formulas of L^* consist of those s-Π^1_1 formulas of $\mathsf{L}^*(\vec{\mathsf{R}})$ which have only quantified occurrences of the new relation symbols $\mathsf{R}_1, \mathsf{R}_2, \ldots$.

ii) The *strict-Σ^1_1 formulas* form the dual class; that is, they form the smallest class containing the Δ_0 formulas closed under $\wedge$, $\vee$, $\forall u \in v$, $\exists u \in v$, $\forall u$, $\exists \mathsf{R}_i$.

There are two essential restrictions in the definition of strict-Π^1_1 formula. First, only existential quantifiers over individuals are permitted. Second, only universal second order quantifiers are allowed, and then only over relations, not over functions. If we were to allow universal second order quantification over functions, then we could build in first order universal quantification (by the manipulations discussed in § IV.2). These observations are summarized by the diagram:

$$\begin{tikzcd}
 & \text{strict-}\Pi^1_1 \text{ formulas} \arrow[dr, phantom, "\subset" sloped] & \\
\Sigma \text{ formulas} \arrow[ur, phantom, "\subset" sloped] & & \Pi^1_1 \text{ formulas} \\
 & \text{first order formulas} \arrow[ul, phantom, "\subset" sloped] \arrow[ur, phantom, "\subset" sloped] &
\end{tikzcd}$$

All inclusions are proper.

Don't forget that L^* may have extra relation symbols (like a symbol for the power set relation) which are allowed to occur in Δ_0, hence in s-Π^1_1, formulas.

Satisfaction of s-Π^1_1 and s-Σ^1_1 formulas is defined in the classical second order manner. Thus

$$\mathfrak{M} \models \forall \mathsf{R}\, \Phi(\mathsf{R})$$

means that for every relation R on $\mathfrak{M}$ (of the correct number of places)

$$(\mathfrak{M}, R) \models \Phi(\mathsf{R}).$$

The study of s-Π^1_1 predicates is one of the few places in logic where the difference between relation symbols and function symbols really matters. In § 1 we defined s-Π^1_1 over ω in terms of quantification over *characteristic* functions, rather than the relations they describe, just to fit with standard practice in ordinary recursion theory. Here the approach with relation symbols is more natural.

The following simple lemma expresses one of the most crucial properties of strict-Π_1^1 formulas.

2.2 Lemma. *Strict-Π_1^1 formulas persist upwards under end extensions. That is, if $\mathfrak{A}_{\mathfrak{M}}, \mathfrak{B}_{\mathfrak{N}}$ are L^*-structures with $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{B}_{\mathfrak{N}}$, and if $\Phi(v_1,\ldots,v_n)$ is a s-Π_1^1 formula of L^* then*

$$\mathfrak{A}_{\mathfrak{M}} \models \Phi[x_1,\ldots,x_n] \quad \textit{implies} \quad \mathfrak{B}_{\mathfrak{N}} \models \Phi[x_1,\ldots,x_n]$$

for all $x_1,\ldots,x_n \in \mathfrak{A}_{\mathfrak{M}}$.

Proof. We need to prove a bit more to keep the induction on s-Π_1^1 formulas going. Let $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{B}_{\mathfrak{N}}$ be given. We prove by induction on s-Π_1^1 formulas $\Phi(\mathsf{R}_1,\ldots,\mathsf{R}_m,v_1,\ldots,v_n)$ of $\mathsf{L}^*(\vec{\mathsf{R}})$ that for all relations $R_1,\ldots,R_m$ on $\mathfrak{B}_{\mathfrak{N}}$ and all $x_1,\ldots,x_n \in \mathfrak{A}_{\mathfrak{M}}$

$$(\mathfrak{A}_{\mathfrak{M}}, R_1 \restriction \mathfrak{A}_{\mathfrak{M}},\ldots,R_m \restriction \mathfrak{A}_{\mathfrak{M}}) \models \Phi[x_1,\ldots,x_n]$$

implies

$$(\mathfrak{B}_{\mathfrak{N}}, R_1,\ldots,R_m) \models \Phi[x_1,\ldots,x_n].$$

The proof is just the proof of persistence of Σ formulas with a new case for $\forall\mathsf{S}$ thrown in. Suppose

$$(1) \qquad (\mathfrak{A}_{\mathfrak{M}}, R_1 \restriction \mathfrak{A}_{\mathfrak{M}},\ldots,R_m \restriction \mathfrak{A}_{\mathfrak{M}}) \models \forall\mathsf{S}\, \Phi(\vec{x},\vec{\mathsf{R}},\mathsf{S}).$$

Let S be any relation on $\mathfrak{B}_{\mathfrak{M}}$ of the correct number of places. By (1)

$$(2) \qquad (\mathfrak{A}_{\mathfrak{M}}, R_1 \restriction \mathfrak{A}_{\mathfrak{M}},\ldots,R_m \restriction \mathfrak{A}_{\mathfrak{M}}, S \restriction \mathfrak{A}_{\mathfrak{M}}) \models \Phi(\vec{x},\vec{\mathsf{R}},\mathsf{S})$$

so, by the induction hypothesis,

$$(\mathfrak{B}_{\mathfrak{N}}, R_1,\ldots,R_m,S) \models \Phi(\vec{x},\vec{\mathsf{R}},\mathsf{S}),$$

as desired. Notice that if we had allowed quantification over function symbols step (2) would fail; just because S is a total function on $\mathfrak{B}_{\mathfrak{N}}$ is no reason to suppose that $S \restriction \mathfrak{A}_{\mathfrak{M}}$ is a total function. $\square$

Let $\mathfrak{A}_{\mathfrak{M}}$ be a structure for L^*. A relation P on $\mathfrak{A}_{\mathfrak{M}}$ is s-Π_1^1 if it can be defined by a s-Π_1^1 formula of L^* with parameters from $\mathfrak{A}_{\mathfrak{M}}$. P is s-Σ_1^1 if P can be defined by a s-Σ_1^1 formula with parameters. P is strict-Δ_1^1 if P is both s-Π_1^1 and s-Σ_1^1.

A function is s-Π_1^1, s-Σ_1^1 on s-Δ_1^1 iff its graph is s-Π_1^1, s-Σ_1^1 or s-Δ_1^1 respectively.

2.3 Lemma. *If a total function f on $\mathfrak{A}_{\mathfrak{M}}$ is s-Π_1^1 then it is s-Δ_1^1.*

Proof. Since f is total,

$$f(x_1,\ldots,x_n)\neq y \quad \text{iff} \quad \exists z\,[f(x_1,\ldots,x_n)=z \wedge z\neq y].$$

Replacing $f(\vec{x})=z$ by its s-Π_1^1 definition gives us a s-Π_1^1 definition of $f(\vec{x})\neq y$; or equivalently, a s-Σ_1^1 definition of the graph of f. □

2.4 Examples. Let $\mathbb{A}$ be an admissible set. We give three examples of strict-Π_1^1 predicates which are not, in general, Σ_1. Note, however, that if A is countable then these relations are Σ_1 (for rather trivial reasons).

(i) Define $P(a,b)$ iff $\operatorname{card}(a)<\operatorname{card}(b)$. Then P is s-Π_1^1 on $\mathbb{A}$.

(ii) Define $P(a)$ iff $\operatorname{card}(a)<\operatorname{card}(A)$. Then P is s-Π_1^1 on $\mathbb{A}$.

(iii) Define $P(a,b)$ iff $b=\operatorname{Power}(a)$, the real power set of a. P is s-Π_1^1 on $\mathbb{A}$. If $\mathbb{A}$ is closed under the power set operation then P is s-Δ_1^1 on $\mathbb{A}$.

Proof. (i) We can write $\operatorname{card}(a)<\operatorname{card}(b)$ as

$$\begin{aligned}\forall \mathsf{R}\,[\mathsf{R}\subseteq b\times a \wedge \forall x\in b\ \exists y\in a\ \mathsf{R}(x,y)&\\ \to \exists x,x'\in b\ \exists y\in a\,(x\neq x' \wedge \mathsf{R}(x,y)\wedge \mathsf{R}(x',y))]&\end{aligned}$$

which asserts that no relation on $b\times a$ can be a one-one map of b into a. Schematically, we can rewrite this as

$$\forall \mathsf{R}\,[\Pi_1(\mathsf{R})\wedge\Delta_0(\mathsf{R})\to\Delta_0(\mathsf{R})].$$

Replacing → by $\vee$ we get

$$\forall \mathsf{R}\,[\Sigma_1(\mathsf{R})\vee\Delta_0(\mathsf{R})\vee\Delta_0(\mathsf{R})]$$

which is s-Π_1^1. The proof of (ii) is much the same. To prove that $b=\operatorname{Power}(a)$ is s-Π_1^1, note that $b=\operatorname{Power}(a)$ iff

$$\forall x\in b\,(x\subseteq a)\wedge\forall \mathsf{R}\ \exists y\in b\ \forall x\in a\,(x\in y\leftrightarrow \mathsf{R}(x)).$$

The second sentence of (iii) follows from 2.3. □

A formula is in s-Π_1^1 *normal form* if it is of the form

$$\forall \mathsf{R}\ \exists y_1,\ldots,y_m\ \varphi(v_1,\ldots,v_n,y_1,\ldots,y_m,\mathsf{R})$$

where φ is Δ_0. The next lemma states that every s-Π_1^1 formula is logically equivalent to one in normal form.

2.5 s-Π_1^1 Normal Form Lemma. *Assume that* L^* *has a constant symbol* 0. *For every* s-Π_1^1 *formula* $\Phi(\vec{x},\vec{\mathsf{R}})$ *of* $\mathsf{L}^*(\vec{\mathsf{R}})$ *there is a* s-Π_1^1 *formula* Φ' *in normal form, with exactly the same free variables and free relation symbols, such that for all* L^* *structures* $\mathfrak{A}_{\mathfrak{M}}$:

$$\mathfrak{A}_{\mathfrak{M}}\models\forall\vec{\mathsf{R}}\ \forall\vec{x}\ [\Phi(\vec{x},\vec{\mathsf{R}})\leftrightarrow\Phi'(\vec{x},\vec{\mathsf{R}})].$$

Proof. We describe five quantifier-pushing manipulations which allow us to put any s-Π_1^1 formula in normal form.

(i) $\forall \mathsf{R}_1 \, \forall \mathsf{R}_2 \, \Phi(\mathsf{R}_1, \mathsf{R}_2) \leftrightarrow \forall \mathsf{S} \, \Phi'(\mathsf{S})$

where S is $n+m$-ary, n being the arity of R_1, m the arity of R_2, and where $\Phi'(\mathsf{S})$ results from Φ by replacing

$$\mathsf{R}_1(t_1,\dots,t_n) \quad \text{by} \quad \mathsf{S}(t_1,\dots,t_n,\mathbf{0},\dots,\mathbf{0}),$$
$$\mathsf{R}_2(t_1,\dots,t_m) \quad \text{by} \quad \mathsf{S}(\mathbf{0},\dots,\mathbf{0},t_1,\dots,t_m).$$

(ii) $\forall \mathsf{R}_1 \, \exists \vec{x} \, \varphi \wedge \forall \mathsf{R}_2 \, \exists \vec{y} \, \psi \leftrightarrow \forall \mathsf{R}_1 \, \forall \mathsf{R}_2 \, \exists \vec{x} \, \exists \vec{y} \, (\varphi \wedge \psi)$

as long as the various symbols are distinct. Similarly for $\vee$.

(iii) $\forall x \, \exists \mathsf{R} \, \Phi(x, \mathsf{R}) \leftrightarrow \exists \mathsf{R}' \, \forall x \, \Phi'(x, \mathsf{R}')$

where Φ' results from Φ by replacing $\mathsf{R}(t_1,\dots,t_n)$ by $\mathsf{R}'(x,t_1,\dots,t_n)$. Taking negations on both sides of (iii) we get

(iv) $\exists x \, \forall \mathsf{R} \, \Phi(x, \mathsf{R}) \leftrightarrow \forall \mathsf{R}' \, \exists x \, \Phi'(x, \mathsf{R}')$

which lets us pull $\forall \mathsf{R}$ out in front of $\exists x$. The bounded existential quantifier step follows from (ii) $\wedge$ (iv). The only remaining step is the bounded universal quantifier.

(v) $\forall x \in a \, \forall \mathsf{R} \, \exists y \, \Phi(x, \mathsf{R}, y)$
$$\leftrightarrow \forall \mathsf{U} \, \forall \mathsf{R} \, \exists y \, [\operatorname{card}(a \cap \mathsf{U}) \leqslant 1 \rightarrow \forall x \in a \, (\mathsf{U}(x) \rightarrow \Phi(x, \mathsf{R}, y))].$$

The part in brackets is Δ_0 since it can be written

$$\forall x \in a \, \forall z \in a \, (\mathsf{U}(x) \wedge \mathsf{U}(z) \rightarrow x = z) \rightarrow \forall x \in a \, (\mathsf{U}(x) \rightarrow \Phi(x, \mathsf{R}, y)).$$

It is now clear, by induction on s-Π_1^1 formulas, that we can put every s-Π_1^1 formula in normal form. $\square$

The Normal Form Lemma is quite useful. We use it in proving the next theorem, and repeatedly in this sections to come.

Recall, from § IV.3, that for countable structures $\mathfrak{M}$

$$\Pi_1^1 \text{ on } \mathfrak{M} = \Sigma_1 \text{ on } \mathbb{HYP}_{\mathfrak{M}}.$$

We proved an absolute version of this theorem in § VI.5, by showing that

$$\text{inductive* on } \mathfrak{M} = \Sigma_1 \text{ on } \mathbb{HYP}_{\mathfrak{M}}.$$

We close this section by proving a different generalization.

2.6 Theorem. *Let $\mathfrak{M}=\langle M,R_1,\dots,R_l\rangle$ be an infinite structure. A relation S on $\mathfrak{M}$ is Π_1^1 on $\mathfrak{M}$ iff S is strict-Π_1^1 on $\mathbb{HYP}_{\mathfrak{M}}$.*

To see that this is a generalization of the countable result we need to know that, for $\mathfrak{M}$ countable,

$$s\text{-}\Pi_1^1 \text{ on } \mathbb{HYP}_{\mathfrak{M}} = \Sigma_1 \text{ on } \mathbb{HYP}_{\mathfrak{M}}.$$

This follows from Theorem 3.1 of the next section.

Proof of Theorem 2.6. We first prove the easy half ($\Rightarrow$). Suppose

$$S(x)\leftrightarrow\mathfrak{M}\models\forall \mathsf{R}\, \varphi(x,\mathsf{R}).$$

Then

$$S(x)\leftrightarrow\mathbb{HYP}_{\mathfrak{M}}\models\forall \mathsf{R}\, [\mathsf{R}\subseteq M^n\rightarrow\varphi^{(M)}(x,\mathsf{R})].$$

The part within brackets is Σ_1 since all quantifiers in $\varphi^{(M)}$ are bounded by M, an element of $\mathbb{HYP}_{\mathfrak{M}}$. To prove ($\Leftarrow$) we must reexamine the proof of Theorem IV.3.3, the result that

$$\Sigma_1 \text{ on } \mathbb{HYP}_{\mathfrak{M}} \Rightarrow \Pi_1^1 \text{ on } \mathfrak{M}$$

regardless of $\mathfrak{M}$'s cardinality. Suppose S is s-Π_1^1 on $\mathbb{HYP}_{\mathfrak{M}}$, $S\subseteq\mathfrak{M}$. By the Normal Form Lemma we can write

$$S(p) \quad\text{iff}\quad \mathbb{HYP}_{\mathfrak{M}}\models\forall \mathsf{P}\, \exists\vec{y}\, \varphi(p,\vec{y},\mathsf{P},\vec{z})$$

for some $\vec{z}=z_1,\dots,z_k\in\mathbb{HYP}_{\mathfrak{M}}$. As in the proof mentioned above, we can replace all parameters z_i by good Σ_1 definitions and so assume all parameters are from $M\cup\{M\}$. Let's say

$$S(p) \quad\text{iff}\quad \mathbb{HYP}_{\mathfrak{M}}\models\forall P\, \exists\vec{y}\, \varphi(p,\vec{y},\mathsf{P},q,M).$$

By the persistence of s-Π_1^1 formulas under end extensions, and by the truncation lemma, $S(p)$ is equivalent to

(1′) $(\mathfrak{A}_{\mathfrak{M}},P)\models\exists\vec{y}\,\varphi(p,\vec{y},\mathsf{P},q,M)$ for all P and all models $\mathfrak{A}_{\mathfrak{M}}$ of KPU$^+$ (of cardinality card($\mathfrak{M}$)).

From here the proof proceeds exactly like the proof of IV.3.3, by coding up (1′) on M, with the extra $\forall P$ riding along for free. ☐

2.7 Exercise. Let $\mathfrak{A}_{\mathfrak{M}}$ be any structure for L^* and let Γ be a s-Π_1^1 inductive definition on $\mathfrak{A}_{\mathfrak{M}}$; i. e.

$$\vec{x} \in \Gamma(R) \quad \text{iff} \quad (\mathfrak{A}_{\mathfrak{M}}, R) \models \Phi(x, \mathsf{R}_+)$$

where Φ is s-Π_1^1. Show that the fixed point I_Γ is s-Π_1^1 on $\mathfrak{A}_{\mathfrak{M}}$.

2.8 Notes. The only theorem of this section comes from Barwise-Gandy-Moschovakis [1971].

3. König Principles on Countable Admissible Sets

Strict-Π_1^1 formulas give us a language for expressing important new principles, or axioms, for admissible sets; principles that isolate important aspects of the Infinity Lemma.

In this section we discuss three König principles and show that they hold on all countable admissible sets. Their role in the general case is discussed in the remaining sections of this chapter.

K_1: *An admissible set* $\mathbb{A}$ *satisfies*

$$s\text{-}\Pi_1^1 = \Sigma_1$$

if every strict-Π_1^1 *relation on* $\mathbb{A}$ *is already a* Σ_1 *relation on* $\mathbb{A}$.

It is important to remember that this equation (s-$\Pi_1^1 = \Sigma_1$) depends very much on just what extra relations may be part of our admissible set $\mathbb{A} = (\mathfrak{M}; A, \in, \ldots)$ in those three little dots. Add a new relation to $\mathbb{A}$ and you increase both the number of s-Π_1^1 formulas and the number of Σ_1 formulas. It should also be kept in mind that the Σ_1 formula defining a s-Π_1^1 predicate P may have parameters not appearing in a given s-Π_1^1 definition of P.

3.1 Theorem. *Every countable admissible set satisfies* s-$\Pi_1^1 = \Sigma_1$.

Proof. We will prove more; namely, that every Σ_1 complete admissible set $\mathbb{A}$ satisfies s-$\Pi_1^1 = \Sigma_1$. Let P be s-Π_1^1 on $\mathbb{A}$. By the Normal Form Lemma we can write P in the form

$$P(x) \quad \text{iff} \quad \mathbb{A} \models \forall \mathsf{R}\, \varphi(x, \mathsf{R})$$

for some Σ_1 formula φ. Let T be the usual infinitary diagram of $\mathbb{A}$:

$$\text{diagram}(\mathbb{A}),$$
$$\forall v \left[v \in \mathsf{a} \rightarrow \bigvee_{x \in a} v = \mathsf{x} \right].$$

By the persistence of s-Π_1^1 formulas under end extensions (Lemma 2.2) the following are equivalent:

$$\begin{aligned}
&P(x),\\
&\mathbb{A}\models\forall \mathsf{R}\,\varphi(\mathsf{R},x),\\
&\mathfrak{B}\models\forall \mathsf{R}\,\varphi(\mathsf{R},x) \quad \text{for all} \quad \mathfrak{B}\supseteq_{\text{end}}\mathbb{A},\\
&(\mathfrak{B},R)\models\varphi(\mathsf{R},\mathsf{x}) \quad \text{for all} \quad \mathfrak{B}\supseteq_{\text{end}}\mathbb{A} \quad \text{and all} \quad R\subseteq\mathfrak{B}^n,\\
&T\models\varphi(\mathsf{R},\mathsf{x}).
\end{aligned}$$

If $\mathbb{A}$ is Σ_1 complete then the set of x such that $T\models\varphi(\mathsf{R},\mathsf{x})$ is a Σ_1 set. □

Before stating the second König Principle, K_2, we need to define the notation $\Phi^{(a)}$ for second order formulas Φ. To obtain $\Phi^{(a)}$ one relativizes all unbounded first order quantifiers to a (replace $\exists u$ by $\exists u\in a$, $\forall u$ by $\forall u\in a$) and replaces

$$\begin{aligned}
&\exists\mathsf{R}(\ldots) \quad \text{by} \quad \exists\mathsf{R}\,[\mathsf{R}\subseteq a^n\wedge(\ldots)],\\
&\forall\mathsf{R}(\ldots) \quad \text{by} \quad \forall\mathsf{R}\,[\mathsf{R}\subseteq a^n\rightarrow(\ldots)].
\end{aligned}$$

Note that if Φ is strict-Π_1^1, or even Π_1^1, then $\Phi^{(a)}$ is strict-Π_1^1 with free variables those of Φ and the new variable a.

3.2 Lemma. *For every structure* $\mathfrak{A}_{\mathfrak{M}}$ *for* L^* *and every* s-Π_1^1 *formula* $\Phi(v_1,\ldots,v_n)$ *of* L^*, *the following are true in* $\mathfrak{A}_{\mathfrak{M}}$:

(i) $\forall a\,\forall v_1,\ldots,v_n\in a\,[\mathrm{Tran}(a)\wedge\Phi^{(a)}(\vec{v})\rightarrow\Phi(\vec{v})]$;

(ii) $\forall a,b,\forall v_1,\ldots,v_n\in a\,[\mathrm{Tran}(a)\wedge a\subseteq b\wedge\Phi^{(a)}(\vec{v})\rightarrow\Phi^{(b)}(\vec{v})]$.

Proof. This is just another version of the persistence of s-Π_1^1 formulas under end extensions. It can be proved directly or deduced from Lemma 2.2. □

K_2: *An admissible set* $\mathbb{A}$ *satisfies strict-*Π_1^1 *reflection if for every* s-Π_1^1 *formula* $\Phi(v_1,\ldots,v_n)$ *and every* $x_1,\ldots,x_n\in\mathbb{A}$, $\mathbb{A}$ *satisfies*

$$\Phi(\vec{x})\rightarrow\exists a\,[\mathrm{Tran}(a)\wedge x_1,\ldots,x_n\in a\wedge\Phi^{(a)}(\vec{x})].$$

We will see in §§ 4, 6 and 7 that s-Π_1^1 reflection is a strong assumption. For now we prove that it holds in all countable admissible sets.

3.3 Theorem. *Every countable admissible set satisfies* s-Π_1^1 *reflection.*

Proof. Again we prove more with an eye toward the next section. This time we prove that if $\mathbb{A}$ is Σ_1 compact then $\mathbb{A}$ satisfies s-Π_1^1 reflection. Let $\Phi(v_1,\ldots,v_n)$ be s-Π_1^1. By the Normal Form Lemma there is a formula $\Psi(v_1,\ldots,v_n)$ in s-Π_1^1 normal form logically equivalent to Φ. It follows that $\Psi^{(a)}(v_1,\ldots v_n)$ is logically equivalent to $\Phi^{(a)}(v_1,\dots,v_n)$ so it suffices to prove reflection for formulas in s-Π_1^1 normal form. So suppose $\Phi(\vec{v})$ is $\forall \mathsf{R}\ \varphi(\vec{v},\mathsf{R})$ and that $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}$ and

$$\mathbb{A}\models\forall \mathsf{R}\ \varphi(x_1,\dots,x_n,\mathsf{R})$$

where φ is a Σ_1 formula. Let T be the infinitary diagram of $\mathbb{A}$, as in 3.2. As we saw in that proof,

$$T\models\varphi(\mathsf{x}_1,\dots,\mathsf{x}_n,\mathsf{R}).$$

By Σ_1 compactness there is a $T_0\subseteq T$, $T_0\in\mathbb{A}$ such that

$$T_0\models\varphi(\mathsf{x}_1,\dots,\mathsf{x}_n,\mathsf{R}).$$

Let $a_0=\{y\,|\,\mathsf{y}\text{ occurs in }T_0\}\cup\{x_1,\dots,x_n\}$ and let $a=\mathrm{TC}(a_0)$ so that $a\in\mathbb{A}$. Then

$$(\mathfrak{M}\cap a;\,a,\in,\dots)\models T_0$$

so

$$(\mathfrak{M}\cap a;\,a,\in,\dots)\models\forall \mathsf{R}\ \varphi(x_1,\dots,x_n,\mathsf{R})$$

which is another way of saying that $\Phi^{(a)}[x_1,\dots,x_n]$ holds. ☐

The third König principle concerns the notion of implicit ordinal introduced in 1.5 and is suggested by Theorem 1.6.

An ordinal α is a Π *implicit ordinal* over $\mathbb{A}$ if there is a Π sentence $\varphi(\mathsf{R},\mathsf{S})$, possibly containing parameters from $\mathbb{A}$, which implicitly defines α over $\mathbb{A}$ (in the sense of 1.5). The notion of a s-Σ_1^1 implicit ordinal is defined in a parallel fashion. (It will turn out that every s-Σ_1^1 implicit ordinal is less than some Π implicit ordinal; see 3.11 or 6.3). It is easy to see that every $\beta<o(\mathbb{A})$ is a Π implicit ordinal over $\mathbb{A}$.

K_3: *The third König principle asserts that every* Π *implicit ordinal over* $\mathbb{A}$ *is an element of* $\mathbb{A}$.

One might paraphrase K_3 by saying that the Π implicit ordinals over $\mathbb{A}$ are explicitly in $\mathbb{A}$.

3.4. Theorem. *Every countable admissible set satisfies the third König principle.*

Proof. Since $h_\Sigma(\mathbb{A})=o(\mathbb{A})$ for countable $\mathbb{A}$, 3.4 follows from 3.5. ☐

Admissible sets do not, in general, satisfy K_3. In general, the Π implicit ordinals know new bounds.

3.5 Theorem. *Let* $\mathbb{A}$ *be admissible and let* α *be a* s-Σ_1^1 *implicit ordinal over* $\mathbb{A}$. *If* $\beta=h_\Sigma(\mathbb{A})$ *then* $\alpha<\beta$.

Proof. Let $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}$ be admissible and let α be the rank of the well-founded relation $\prec_\Phi$ where Φ is s-Σ_1^1; say

$$R\prec_\Phi S \quad \text{iff} \quad (\mathbb{A},R,S)\models\exists\mathsf{Q}\,\varphi(\mathsf{Q},\mathsf{R},\mathsf{S})$$

and φ is a Π sentence. We can assume that Q, R, S are all unary (i. e. range over subsets of $\mathbb{A}$) since the pairing function $\langle x,y\rangle$ is $\mathbb{A}$-recursise. We need to find a Σ_1 theory T of $\mathsf{L}_{\mathbb{A}}$ which pins down α. The crucial observation is contained in (1).

$$(1)\begin{cases}\text{If } \mathbb{A}_{\mathfrak{M}}\subseteq_{\text{end}}\mathfrak{B}_{\mathfrak{N}} \text{ and } (\mathfrak{B}_{\mathfrak{N}},R,S)\models\exists\mathsf{Q}\,\varphi(\mathsf{Q},\mathsf{R},\mathsf{S}), \text{ and if } R_0=R\cap\mathbb{A}_{\mathfrak{M}},\\ S_0=S\cap\mathbb{A}_{\mathfrak{M}} \text{ then } R_0\prec_\Phi S_0.\end{cases}$$

This follows from $(\mathbb{A}_{\mathfrak{M}},R_0,S_0)\subseteq_{\text{end}}(\mathfrak{B}_{\mathfrak{N}},R,S)$ by the s-Σ_1^1 version of Lemma 2.2. From (1) we get (2).

$$(2)\begin{cases}\text{Let } \mathbb{A}_{\mathfrak{M}}\subseteq_{\text{end}}\mathfrak{B}_{\mathfrak{N}}. \text{ The relation } \prec' \text{ on subsets of } \mathfrak{B}_{\mathfrak{N}} \text{ defined by}\\ \qquad R\prec' S \quad \text{iff} \quad (\mathfrak{B}_{\mathfrak{N}},R,S)\models\exists\mathsf{Q}\,\varphi(\mathsf{Q},\mathsf{R},\mathsf{S})\\ \text{is well founded. Hence any subrelation } \prec'' \text{ of } \prec' \text{ is well founded.}\end{cases}$$

For, by (1), any infinite descending sequence in $\prec'$ would give rise to an infinite descending sequence in $\prec_\Phi$.

It is pretty obvious how to use (2) to pin down the ordinal α by building the hypothesis of (2) into a Σ_1 theory $T=T(<,\ldots)$. The language for T will contain the symbols of $\mathsf{L}^*=\mathsf{L}(\in,\ldots)$; a constant symbol x for each $x\in A$; unary symbols A (for A), P (for Power(A)), U (for α); binary symbols E (for $\in\cap(A\times P)$), $\prec$ (for $\prec_\Phi$), $<$ (for $\in\restriction\alpha$); and a function symbol F. The intended model for T, the one with $<^{\mathfrak{M}}$ of order type α, is:

$$\mathfrak{M}=\langle A\cup\text{Power}(A)\cup\alpha;\,A,\ldots;\,\text{Power}(A),\in\cap(A\times\text{Power}(A)),\prec_\Phi;\,\alpha,<,F\rangle$$

where $F(x)=0$ for $x\notin\text{field}(\prec_\Phi)$, $F(R)=\prec_\Phi$-rank of R for R in field of $\prec_\Phi$. Hence $\text{rng}(F)=\alpha$ and $R\prec_\Phi S$ implies $F(R)<F(S)$. The theory T contains:

$\forall x\,[\mathsf{A}(x)\vee\mathsf{P}(x)\vee\mathsf{U}(x)]$,

Infinitary diagram of $\mathbb{A}$,

Extensionality for E,

"$\mathsf{E}\subseteq\mathsf{A}\times\mathsf{P}$".

(3) $\forall r,s\,[r\prec s\leftrightarrow\mathsf{P}(r)\wedge\mathsf{P}(s)\wedge\exists y\,(\mathsf{P}(q)\wedge\varphi(q,r,s))]$.

(4) "$<$ linearly orders U, $\text{rng}(\mathsf{F})=\mathsf{U}$, $\mathsf{F}(x)=0$ for $x\notin\text{field}(\prec)$, and $\mathsf{F}(s)=<\text{-sup}\{\mathsf{F}(r)+1 : r\prec s\}$ for $s\in\text{field}(\prec)$".

In line (3), $\varphi(q,r,s)$ denotes the result of replacing $\mathsf{R}(x)$ by $x\mathsf{E}r$, $\neg\mathsf{R}(x)$ by $\neg(x\mathsf{E}r)$ and similar for Q, S. (We are abusing notation since q does not range over urelements here.) To see that T pins down α it remains only to prove that for any other model $\mathfrak{M}$ of T, $<^{\mathfrak{M}}$ is well ordered. Let $\mathfrak{M}$ be any model of T. We can obviously assume $\mathfrak{M}$ has the form

$$\mathfrak{M}=\langle\mathfrak{B}_{\mathfrak{N}}\cup P_1\cup U;\mathfrak{B}_{\mathfrak{N}},\ldots;P,\in\cap(\mathfrak{B}_{\mathfrak{N}}\times P),\prec'',U,<,F\rangle$$

where $\mathbb{A}_{\mathfrak{M}}\subseteq_{\text{end}}\mathfrak{B}_{\mathfrak{N}}$ and $P\subseteq\text{Power}(\mathfrak{B}_{\mathfrak{N}})$. To see that $<$ is well ordered it suffices, by (4), to prove that $\prec''$ is well founded. But this is immediate from (2) and (3). ▯

3.6 Corollary. *If* $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ *is countable and* α *is a first order, or even* Σ_1^1, *implicit ordinal over* $\mathfrak{M}$ *then* $\alpha<O(\mathfrak{M})$. *In particular,* α *is countable.*

Proof. If α is Σ_1^1 over $\mathfrak{M}$ then it is s-Σ_1^1 over $\mathbb{HYP}_{\mathfrak{M}}$ so the result follows from 3.5. ▯

3.7 Corollary. *Every* Σ_1^1 *implicit ordinal over* ω *is less than* ω_1^c.

Proof. Immediate from 3.6 since $\omega_1^c=O(\mathcal{N})$. ▯

As we mentioned in § 1, Theorem 1.6 and Corollary 3.7 together account for the fact that implicit ordinals seldom appear in ordinary recursion theory on ω. They *do* appear in parts of mathematics far removed from the theory of admissible sets. We present one example suggestive of others.

3.8 Example. Let $\mathfrak{M}$ be a Noetherian module (over a ring with identity), that is, a module with no infinite chain

$$M_0\subset M_1\subset M_2\subset\cdots$$

of submodules. Then

$$M'\prec M'' \quad \text{iff} \quad M',M'' \text{ are submodules,} \quad M''\subset M'$$

defines a first order implicit well-founded relation. Its rank $\alpha=\rho(\prec)$ is called the length of $\mathfrak{M}$, $\alpha=l(\mathfrak{M})$. Thus $l(\mathfrak{M})$ is a first order implicit ordinal over M. This ordinal plays an important role in the structure theory of Noetherian modules.

3.9—3.12 Exercises

3.9. Prove a uniform version of 3.1. That is, show that for every s-Π_1^1 formula $\Phi(v_1,\ldots,v_n)$ there is a Σ_1 formula $\varphi(v_1,\ldots,v_n)$ such that for every countable admissible set $\mathbb{A}$,

$$\mathbb{A}\models\forall\vec{v}\,[\Phi(\vec{v})\leftrightarrow\varphi(\vec{v})].$$

3.10. Prove directly that every s-Σ_1^1 implicit ordinal is $\leqslant$ some Π implicit ordinal over $\mathbb{A}$.

3.11. Let $\mathfrak{M}=\langle M,R_1,\dots,R_l\rangle$ and let α be a Σ_1^1 implicit ordinal over $\mathfrak{M}$. Improve 3.5 to show that $\alpha\leqslant h(\mathbb{HYP}_{\mathfrak{M}})$.

3.12. Prove that on the class of admissible sets $\mathbb{A}$ and relations P on $\mathbb{A}$, .

"P is Σ_1 on $\mathbb{A}$"

is the absolute version of

"P is s-Π_1^1 on $\mathbb{A}$".

4. *König Principles* K_1 *and* K_2 *on Arbitrary Admissible Sets*

To summarize, the three König Principles introduced in § 3 are:

K_1: strict-$\Pi_1^1=\Sigma_1$;
K_2: strict-Π_1^1 reflection;
K_3: Every Π implicit ordinal over $\mathbb{A}$ is an element of $\mathbb{A}$.

These three principles are generalized recursion theoretic statements which attempt to capture different aspects of the Infinity Lemma. Each of them has a model-theoretic counterpart for the infinitary logic $\mathsf{L}_{\mathbb{A}}$. In this section we discuss the counterparts of K_1 and K_2.

The basic tool for the study of all three of these principles is the Weak Completeness Theorem of § VII.2. Our first theorem explains the reason for referring to that result as a completeness theorem.

4.1 Theorem. *Let $\mathbb{A}$ be admissible and let T be a set of sentence of $\mathsf{L}_{\mathbb{A}}$ which is strict-Π_1^1 definable on $\mathbb{A}$. The set*

$$\mathrm{Cn}(T)=\{\varphi\in\mathsf{L}_{\mathbb{A}}: T\models\varphi\}$$

is also strict-Π_1^1 on $\mathbb{A}$.

Theorem 4.1 will follow from the Weak Completeness Theorem and the next Lemma.

4.2 Lemma. *Let $\mathbb{A}$ be admissible and let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment which is Δ_1 on $\mathbb{A}$ (in the sense of Lemma VII.2.4). There is a Π sentence $\varphi(\mathsf{D})$ such that for any $\mathscr{D}\subseteq\mathbb{A}$:*

$$\mathscr{D} \text{ is an s.v.p. for } \mathsf{L}_{\mathbb{A}} \quad \text{iff} \quad (\mathbb{A},\mathscr{D})\models\varphi(\mathsf{D}).$$

Proof. Since $\mathsf{L}_{\mathbb{A}}$ is Δ_1 on $\mathbb{A}$, T_{Skolem} is a Δ_1 subset of $\mathbb{A}$. $\mathscr{D}$ is an s.v.p. for $\mathsf{L}_{\mathbb{A}}$ iff $(\mathbb{A}, \mathscr{D})$ satisfies all the following conditions:

$$\mathscr{D} \subseteq \mathsf{L}_{\mathbb{A}},$$

$$\forall \varphi\, [(\varphi \text{ an axiom (A 1)—(A 7) of } \mathsf{L}_{\mathbb{A}}) \to (\varphi \in \mathscr{D})],$$

$$\mathscr{D} \text{ is closed under (R 1)—(R 3)},$$

$$\forall \varphi\, [\varphi \in \mathscr{D} \to (\neg \varphi) \notin \mathscr{D}],$$

$$T_{\text{Skolem}} \subseteq \mathscr{D},$$

$$\forall \Phi\, [\bigvee \Phi \in \mathscr{D} \wedge (\bigvee \Phi \text{ a sentence}) \to \exists \varphi \in \Phi (\varphi \in \mathscr{D})].$$

Each of these conditions is naturally expressed as a Π condition on $\mathscr{D}$, so the lemma is proved. Note the important role played here by Skolem fragments. If we had to do without "$T_{\text{Skolem}} \subseteq \mathscr{D}$", we would have to add the clause

$$\forall x [x = \varphi(v) \wedge (\exists v\, \varphi(v)) \in \mathscr{D} \to \exists t\, [\varphi(t/v) \in \mathscr{D}]]$$

which is not Π due to the unbounded $\exists t$. □

Proof of Theorem 4.1. We may assume, by VII.2.4, that $\mathsf{L}_{\mathbb{A}}$ is a Skolem fragment Δ_1 on $\mathbb{A}$ and that every model of T can be expanded to a Skolem model. By the Weak Completeness Theorem we have $T \models \varphi$ iff

$$\forall \mathscr{D}\, [\mathscr{D} \text{ an s.v.p. for } \mathsf{L}_{\mathbb{A}} \wedge T \subseteq \mathscr{D} \to \varphi \in \mathscr{D}].$$

By 4.2 this takes the form

$$\forall \mathscr{D}\, [\Pi(\mathscr{D}) \wedge \forall x\, (\Phi(x) \to x \in \mathscr{D}) \to \varphi \in \mathscr{D}]$$

where $\Phi(v)$ defines the s-Π_1^1 theory T. The hypothesis of the outer implication is s-Σ_1^1 so the whole becomes a s-Π_1^1 predicate of φ. □

4.3 Corollary. *An admissible set* $\mathbb{A}$ *satisfies* s-$\Pi_1^1 = \Sigma_1$ *iff* $\mathbb{A}$ *is* Σ_1 *complete.*

Proof. The implication ($\Rightarrow$) follows from 4.1. The other direction was proved explicitly in the proof of Theorem 3.1. □

4.4 Corollary. *The set of valid sentences of the admissible fragment* $\mathsf{L}_{\mathbb{A}}$ *is always* s-Π_1^1 *on* $\mathbb{A}$.

Proof. Let $T = 0$ in 4.1. □

At various places in the book we have referred to Σ_1 as a syntactic generalization of r.e. on ω and to strict-Π_1^1 as a semantic version of r.e. on ω. The next corollary of 4.1 makes this precise.

A subset $X \subseteq \mathbb{A}$ is a *complete* Σ_1 *set (or complete strict-*Π_1^1 *set)* for the admissible set $\mathbb{A}$ if X is Σ_1 (resp. s-Π_1^1), and for any other Σ_1 set (resp. s-Π_1^1 set) Y on $\mathbb{A}$ there is a one-one total $\mathbb{A}$-recursive function F such that

$$y \in Y \quad \text{iff} \quad F(y) \in X$$

for all $y \in \mathbb{A}$.

Recall that $T \vdash \varphi$ means that φ is provable from T in the sense of $\mathsf{L}_{\mathbb{A}}$. (This notation occurs in § III.5.)

4.5 Corollary. *Let* $\mathbb{A}$ *be admissible. Let* L' *contain* $\mathsf{L}^*(\vec{\mathsf{R}})$ *and a symbol* x *for each* $x \in \mathbb{A}$ *and let* $\mathsf{L}'_{\mathbb{A}}$ *be the admissible fragment given by* $\mathbb{A}$. *Let* T *be the infinitary diagram of* $\mathbb{A}$.

(i) *The set* $X_0 = \{\varphi \in \mathsf{L}'_{\mathbb{A}} \mid T \vdash \varphi\}$ *is complete* Σ_1 *for* $\mathbb{A}$.

(ii) *The set* $X_1 = \{\varphi \in \mathsf{L}'_{\mathbb{A}} \mid T \models \varphi\}$ *is complete* s-Π_1^1 *for* $\mathbb{A}$.

Proof. (i) is implicit in much of Chapters V and VI. It can also be obtained simply as the absolute version of (ii). To prove (ii) note that X_1 is s-Π_1^1 by 4.1 and that every s-Π_1^1 set is one-one reducible to X_1 by the proof of 3.1. $\square$

An analogous proof shows that on "bad" admissible sets, s-Π_1^1 can be as far from Σ_1 as is conceivable.

4.6 Corollary. *Let* $\mathbb{A}$ *be a self-definable admissible set. Then* $\Pi_1^1 =$ *strict-*Π_1^1 *on* $\mathbb{A}$. *That is, every* Π_1^1 *relation on* $\mathbb{A}$ *can be defined by a strict-*Π_1^1 *formula.*

Proof. Let T be a Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$ which self-defines $\mathbb{A}$. By 4.1, $\mathrm{Cn}(T)$ is s-Π_1^1. But by Lemma VII.1.9, every Π_1^1 relation on $\mathbb{A}$ is one-one reducible to $\mathrm{Cn}(T)$ so every Π_1^1 relation is s-Π_1^1. $\square$

For example, $\Pi_1^1 = \text{strict-}\Pi_1^1$ on $H(\aleph_{\alpha+1})$ for all $\alpha \geqslant 0$, by VII.1.4.

We now turn to consider the logical role of strict-Π_1^1 reflection.

4.7 Theorem. *An admissible set is* Σ_1 *compact iff it satisfies strict-*Π_1^1 *reflection.*

Proof. The implication ($\Rightarrow$) was proved explicitly in the proof of Theorem 3.3. To prove the converse, suppose that $\mathbb{A}$ is admissible and satisfies s-Π_1^1 reflection and that T is a Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$. Assume further that every $T_0 \subseteq T$, $T_0 \in \mathbb{A}$ has a model. By Lemma VII.2.4 we may assume that $\mathsf{L}_{\mathbb{A}}$ is a Skolem fragment and that every $T_0 \subseteq T$, $T_0 \in \mathbb{A}$ has a Skolem model. We will prove that T has a Skolem model. Suppose, aiming at a contradiction, that T has no Skolem model. By the Weak Completeness Theorem, no s.v.p. $\mathscr{D}$ for $\mathsf{L}_{\mathbb{A}}$ can contain T as a subset. Hence $(\mathbb{A}, T)$ satisfies the s-Π_1^1 sentence $\Psi(\mathsf{T})$ expressing:

$$\forall \mathscr{D}\, [\mathscr{D} \text{ an s.v.p. for } \mathsf{L}_{\mathbb{A}} \rightarrow \exists x\, (x \in \mathsf{T} \wedge x \notin \mathscr{D})].$$

Let $\theta(v)$ be the Σ_1 formula defining T. The line above becomes

$$\mathbb{A} \models \forall \mathscr{D}\, [\mathscr{D} \text{ an s.v.p. for } \mathsf{L}_{\mathbb{A}} \rightarrow \exists x\, (\theta(x) \wedge x \notin \mathscr{D})]$$

which is a s-Π_1^1 sentence $\Phi(\vec{y})$ with parameters $\vec{y}$ those of $\theta(v)=\theta(v,y_1,\ldots,y_k)$. By s-Π_1^1 reflection there is a transitive set $a\in\mathbb{A}$ with $\vec{y}\in a$ such that $\mathbb{A}\models\Phi^{(a)}[\vec{y}]$. Let $\mathbb{A}_0=(\mathfrak{M}\cap a;\,a,\in,\ldots)$ and let

$$T_0=\{x\in a\,|\,\theta^{(a)}(x)\}$$

so that $T_0\in\mathbb{A}$ by Δ_0 Separation and $T_0\subseteq T$ since θ is Σ_1. Since $\Phi^{(a)}[\vec{y}]$ holds we have

$$(\mathbb{A}_0,T_0)\models\Psi(\mathsf{T}).$$

We don't really know what $\Psi(\mathsf{T})$ says on $\mathbb{A}_0$, but $(\mathbb{A}_0,T_0)\subseteq_{\text{end}}(\mathbb{A},T_0)$ so, by the persistence of s-Π_1^1 formulas

$$(\mathbb{A},T_0)\models\Psi(\mathsf{T}).$$

But this says that T_0 is not a subset of any s.v.p. $\mathscr{D}$ for $\mathsf{L}_{\mathbb{A}}$. Hence T_0 has no Skolem model, a contradiction. □

Thus we see that two different aspects of the König Infinity Lemma, those expressed by K_1 and K_2, reflect themselves in related but apparently distinct aspects of first order logic. K_1 is responsible for the Completeness Theorem, K_2 for the Compactness Theorem.

One usually thinks of the Completeness Theorem as implying the Compactness Theorem. The corollary to the next result shows this to be the case for *resolvable* admissible sets. The general case is still open.

4.8 Proposition. *The resolvable admissible sets are divided into two disjoint classes: those that are Σ_1 compact and those that are self-definable.*

Proof. Proposition VII.1.3 shows that no self-definable $\mathbb{A}$ can be Σ_1 compact. Now let $\mathbb{A}$ be a resolvable, admissible set which is not Σ_1 compact. We must show that it is self-definable. Since $\mathbb{A}$ is resolvable there is a total $\mathbb{A}$-recursive function $J\colon o(\mathbb{A})\to\mathbb{A}$ such that

$$\alpha<\beta\Rightarrow J(\alpha)\in J(\beta),$$

$$J(\alpha)\text{ is transitive, for all }\alpha,$$

$$\mathbb{A}=\bigcup_{\alpha<o(\mathbb{A})}J(\alpha).$$

Since $\mathbb{A}$ is not Σ_1 compact, $\mathbb{A}$ does not satisfy s-Π_1^1 reflection, by 4.7. Thus there is a s-Π_1^1 formula $\Phi(v)$, and an $x\in\mathbb{A}$ such that $\mathbb{A}$ satisfies:

$$\Phi(x),$$

$$\neg\exists b\,[\mathrm{Tran}(b)\wedge x\in b\wedge\Phi^{(b)}(x)].$$

The second formula is s-Σ_1^1 and so is logically equivalent to a s-Σ_1^1 formula

$$\exists \mathsf{R}\ \varphi(x, \mathsf{R})$$

where φ is Π_1. Let $\sigma(u,v)$ be a Σ_1 formula defining J:

$$J(\alpha)=y \quad \text{iff} \quad \mathbb{A}\models\sigma(\alpha,y).$$

The Σ_1 theory used to self-define $\mathbb{A}$ consists of:

$$\begin{array}{l} \text{The infinitary diagram of } \mathbb{A}, \\ \forall u,v\,[\mathsf{J}(u)=v\leftrightarrow\sigma(u,v)], \\ \forall u,u'\,[\mathrm{Ord}(u)\wedge\mathrm{Ord}(u')\wedge u<u'\rightarrow\mathsf{J}(u)\in\mathsf{J}(u')], \\ \forall x\,\exists u\,[\mathrm{Ord}(u)\wedge x\in\mathsf{J}(u)], \\ \forall u\,[\mathrm{Tran}(\mathsf{J}(u))], \\ \varphi(\mathsf{x},\mathsf{R}). \end{array}$$

Let $(\mathfrak{B}_{\mathfrak{N}},J)$ be any model of T. We can assume $\mathbb{A}_{\mathfrak{M}}\subseteq_{\text{end}}\mathfrak{B}_{\mathfrak{N}}$. We need to show that $\mathbb{A}_{\mathfrak{M}}=\mathfrak{B}_{\mathfrak{N}}$. If not, let $x\in\mathfrak{B}_{\mathfrak{N}}-\mathbb{A}_{\mathfrak{M}}$. Then by the axioms on σ,

$$\mathfrak{B}_{\mathfrak{N}}\models\exists a\,[\mathrm{Ord}(a)\wedge x\in\mathsf{J}(a)].$$

Pick such an a. Then a is an ordinal of $\mathfrak{B}_{\mathfrak{N}}$ but $a\notin\mathbb{A}_{\mathfrak{M}}$, for if $a\in\mathbb{A}_{\mathfrak{M}}$ then $J(a)\in\mathbb{A}$ which implies $x\in\mathbb{A}$. Thus $\alpha>\beta$ for all $\beta\in\mathbb{A}$. But then $J(\beta)\subseteq J(a)$ holds in $\mathfrak{B}_{\mathfrak{N}}$, for each $\beta\in\mathbb{A}$. Thus $\mathbb{A}\subseteq_{\text{end}}\langle J(a),E\rangle$ and so $\Phi^{(J(a))}(x)$ holds. This contradicts

$$\mathfrak{B}_{\mathfrak{N}}\models\exists\mathsf{R}\ \varphi(\mathsf{x},\mathsf{R})$$

since this asserts that $\Phi^{(a)}$ fails for all transitive b, in particular for $b=J(a)$. ☐

4.9 Corollary. *Every Σ_1 complete resolvable admissible set is Σ_1 compact. In other words, K_1 implies K_2 on resolvable admissible sets.*

Proof. If s-$\Pi_1^1=\Sigma_1$ then s-$\Pi_1^1\neq\Pi_1^1$ and hence $\mathbb{A}$ cannot be self-definable, by 4.6. Then by 4.8, $\mathbb{A}$ must be Σ_1 compact. ☐

What is wrong with the following argument? If s-$\Pi_1^1=\Sigma_1$ then (since Σ reflection holds in all admissible sets) we must have s-Π_1^1 reflection. If you try to fill in the steps in this argument you see that one is missing a certain uniformity in the equation s-$\Pi_1^1=\Sigma_1$. This uniformity is captured by the next definition.

Let $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}$ be transitive, let $\Phi(v_1,\ldots,v_n)$ be strict-Π_1^1 and let $\varphi(v_1,\ldots,v_n)$ be Σ_1, where extra parameters from $\mathbb{A}$ are permitted in φ. We say that φ is *uniformly equivalent* to Φ on $\mathbb{A}$ if

$$\mathbb{A}\models\forall v_1,\ldots,v_n\,[\Phi(\vec{v})\rightarrow\varphi(v)], \tag{1}$$

$$\mathbb{A}\models\forall a\,\forall v_1,\ldots,v_n\in a\,[\mathrm{Tran}(a)\wedge\varphi^{(a)}(\vec{v})\rightarrow\Phi^{(a)}(\vec{v})]. \tag{2}$$

4.10 Lemma. *Let* $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}$ *be transitive, closed under pairs and* TC. *Let* $\varphi(v_1,\dots,v_n)$ *be a* Σ_1 *formula which is uniformly equivalent to the* s-Π_1^1 *formula* $\Phi(v_1,\dots,v_n)$ *on* $\mathbb{A}$. *For all* $x_1,\dots,x_n \in \mathbb{A}$, $\mathbb{A}$ *satisfies:*

$$\begin{aligned}&\Phi(\vec{x}) \leftrightarrow \varphi(\vec{x})\\ &\quad\leftrightarrow \exists a\,[\mathrm{Tran}(a) \wedge \vec{x} \in a \wedge \varphi^{(a)}(\vec{x})]\\ &\quad\leftrightarrow \exists a\,[\mathrm{Tran}(a) \wedge \vec{x} \in a \wedge \Phi^{(a)}(\vec{x})].\end{aligned}$$

Proof. Write $\varphi(v_1,\dots,v_n)$ as $\exists y\,\psi(v_1,\dots,v_n,y)$ where y is Δ_0. By (1),

$$\mathbb{A} \models (\Phi(\vec{x}) \rightarrow \exists y\,\psi(\vec{x},y)).$$

If $\mathbb{A} \models \psi(\vec{x},y)$ then let $a=\mathrm{TC}(\{y,x_1,\dots,x_n\})$. Then $a \in \mathbb{A}$ and $\varphi^{(a)}(\vec{x})$ holds so

$$\varphi(\vec{x}) \rightarrow \exists a\,[\mathrm{Tran}(a) \wedge \vec{x} \in \varphi^{(a)}(\vec{x})].$$

By (2), the right hand side of the above line implies

$$\exists a\,[\mathrm{Tran}(a) \wedge \vec{x} \in a \wedge \Phi^{(a)}(\vec{x})].$$

By Lemma 3.2 (i), the above implies $\Phi(\vec{x})$. ◻

4.11 Definition. An admissible set $\mathbb{A}$ satisfies

$$s\text{-}\Pi_1^1=\Sigma_1 \textit{ uniformly}$$

if for each s-Π_1^1 formula $\Phi(v_1,\dots,v_n)$ of L* there is a Σ_1 formula $\varphi(v_1,\dots,v_n)$, possibly with additional parameters from $\mathbb{A}$, such that φ is uniformly equivalent to Φ on $\mathbb{A}$.

4.12 Theorem. *An admissible set* $\mathbb{A}$ *satisfies* s-$\Pi_1^1=\Sigma_1$ *uniformly iff* $\mathbb{A}$ *satisfies* s-$\Pi_1^1=\Sigma_1$ *and* s-Π_1^1 *Reflection.*

Proof. The implication ($\Rightarrow$) is immediate from Lemma 4.10. To prove that converse, assume that $\mathbb{A}$ satisfies K_1 and K_2 and that $\Phi(v_1,\dots,v_n)$ is s-Π_1^1. We must find a Σ_1 formula $\varphi(v_1,\dots,v_n)$ uniformly equivalent to $\Phi(v_1,\dots,v_n)$ on $\mathbb{A}$. Let $\Psi(v_1,\dots,v_n,b)$ be the s-Π_1^1 formula

$$[\mathrm{Tran}(b) \wedge v_1,\dots,v_n \in b \wedge \Phi^{(b)}(v_1,\dots,v_n)].$$

Since s-$\Pi_1^1=\Sigma_1$ there is a Σ_1 formula $\psi(v_1,\dots,v_n,b)$ equivalent to $\Psi(v_1,\dots,v_n,b)$ on $\mathbb{A}$. Let $\varphi(v_1,\dots,v_n)$ be

$$\exists b\,\psi(v_1,\dots,v_n,b).$$

To prove $\varphi(\vec{v})$ uniformly equivalent to $\Phi(\vec{v})$ first suppose that $\Phi(\vec{x})$ holds in $\mathbb{A}$. By s-Π_1^1 Reflection there is a $b \in \mathbb{A}$ such that $\Psi(\vec{x},b)$ holds in $\mathbb{A}$. But then $\psi(\vec{x},b)$

holds so $\varphi(\vec{x})$ holds. To prove (2), suppose that $a \in \mathbb{A}$ is transitive, that $x_1, \ldots, x_n \in a$ and that $\varphi^{(a)}(\vec{x})$ holds. Then there is a $b \in a$ such that $\psi(\vec{x}, b)^{(a)}$ holds in $\mathbb{A}$. Hence $\psi(\vec{x}, b)$ holds in $\mathbb{A}$ and so

$$\operatorname{Tran}(b) \wedge x_1, \ldots, x_n \in b \wedge \Phi^{(b)}(\vec{x}).$$

Since $b \in a$ and a is transitive, $b \subseteq a$ so, by 3.2 (ii), $\Phi^{(a)}(\vec{x})$ holds, as desired. □

4.13 Corollary. *An admissible set* $\mathbb{A}$ *satisfies* $s\text{-}\Pi_1^1 = \Sigma_1$ *uniformly iff* $\mathbb{A}$ *is* Σ_1 *complete and* Σ_1 *compact.* □

4.14 Corollary. *On resolvable admissible sets the condition* $s\text{-}\Pi_1^1 = \Sigma_1$ *is equivalent to the condition* $s\text{-}\Pi_1^1 = \Sigma_1$ *uniformly.*

Proof. By 4.9 and 4.12. □

The condition $s\text{-}\Pi_1^1 = \Sigma_1$ uniformly clearly captures a great deal of the recursion theoretic and logical content of the Infinity Lemma. If you state it in the "$s\text{-}\Sigma_1^1 = \Pi_1$ *uniformly*" version, it even looks like the Infinity Lemma, at least from one point of view. We will use it in § 6 to help us find interesting uncountable Σ_1 complete and Σ_1 compact admissible sets.

4.15—4.21 Exercises

4.15. Let α be admissible but not recursively inaccessible, let $\mathbb{A} = L(\alpha)$. Prove that the valid sentences of $\mathsf{L}_{\mathbb{A}}$ form a complete $s\text{-}\Pi_1^1$ set. Show that for any admissible $\beta \leqslant \omega_1$, β is recursively inaccessible iff the set of valid sentences of L_β is β-recursive.

4.16. A subset X of an admissible set $\mathbb{A}$ in *bounded* if $X \subseteq a$ for some $a \in \mathbb{A}$. Let $\mathbb{A}$ be admissible and satisfy $s\text{-}\Pi_1^1$ Reflection. Let T be a set of sentences of $\mathsf{L}_{\mathbb{A}}$ which is $s\text{-}\Pi_1^1$ on $\mathbb{A}$. Prove that if every bounded subset $T_0 \subseteq T$ has a model then T has model. (It is open whether one can improve this by replacing "bounded" by "$\mathbb{A}$-finite".)

4.17. Let $\mathbb{A}$ be admissible. Suppose that for every Δ_0 theory T of $\mathsf{L}_{\mathbb{A}}$, if every $T_0 \subseteq T$, $T_0 \in \mathbb{A}$ has a model, then T has a model. Show that $\mathbb{A}$ is Σ_1 compact. [Show that $s\text{-}\Pi_1^1$ Reflection holds.]

4.18. Let $\mathbb{A}$ be admissible, $\alpha = o(\mathbb{A})$. $\mathbb{A}$ is $s\text{-}\Delta_1^1$ *resolvable* if there is a limit ordinal $\lambda \leqslant \alpha$ and a $s\text{-}\Delta_1^1$ function $J: \lambda \to \mathbb{A}$ such that

$$\beta < \xi \Rightarrow J_\beta \in J_\xi \quad \text{for} \quad \beta, \xi < \lambda,$$

$$J_\beta \quad \text{is transitive for all} \quad \beta < \lambda,$$

$$\mathbb{A} = \textstyle\bigcup_{\beta < \lambda} J_\beta .$$

The ordinal λ is said to be the *length* of the hierarchy J on $\mathbb{A}$.

(i) Prove that if $\alpha=\beth_\alpha$ then $H(\beth_\alpha)$ is s-Δ_1^1 resolvable. [Hint: If $\alpha=\beth_\alpha$ then $H(\beth_\alpha)=\mathrm{V}(\alpha)$. Let $J_\beta=\mathrm{V}(\beta)$.]

(ii) Prove that if $\mathbb{A}$ is s-Δ_1^1 resolvable and if J is as above with $\lambda<o(\mathbb{A})$ then $\mathbb{A}$ is self-definable.

(iii) Strengthen Proposition 4.8 to: The class of s-Δ_1^1 resolvable admissible sets are divided into disjoint two classes, the Σ_1 compact and the self-definable.

(iv) Let $\kappa=\beth_\kappa$. Show that $\langle H(\kappa),\in,\mathscr{P}\rangle$ satisfies K_1 iff it satisfies K_2.

4.19. Kunen [1968] introduced an invariant definability approach to generalized recursion theory on admissible sets by introducing the notions of a.i.d., i.i.d., and s.i.i.d. (see below) as generalizations of the concepts of finite, recursive and r.e. In Barwise [1968], [1969b] we showed that s-Π_1^1=s.i.i.d. (see (ii)). (This leads to the formulation of s-Π_1^1 Reflection and the results of this section in Barwise [1968], [1969b].) Let $\mathbb{A}$ be admissible and let P be an n-ary relation on $\mathbb{A}$. P is

(a) absolutely implicitly definable (a.i.d.) on $\mathbb{A}$,

(b) invariantly implicitly definable (i.i.d.) on $\mathbb{A}$,

(c) semi-invariantly implicitly definable (s.i.i.d.) on $\mathbb{A}$

if there is a finitary first order sentence $\theta(\mathsf{P},\mathsf{S}_1,\ldots,\mathsf{S}_k)$ of $\mathsf{L}^*(\mathsf{P},\ldots)$ such that

$$(\mathbb{A},P)\models\exists\mathsf{S}_1,\ldots,\mathsf{S}_k,\theta(\mathsf{P},\mathsf{S}_1,\ldots,\mathsf{S}_k)$$

and such that if $\mathbb{A}\subseteq_{\mathrm{end}}\mathfrak{B}$ and $P'\subseteq\mathfrak{B}^n$ satisfies

$$(\mathfrak{B},P')\models\exists\mathsf{S}_1,\ldots,\mathsf{S}_k,\theta(\mathsf{P},\mathsf{S}_1,\ldots,\mathsf{S}_k)$$

then

(a) $P=P'$,

(b) $P=P'\cap\mathbb{A}^n$,

(c) $P\subseteq P'\cap\mathbb{A}^n$.

The sentence θ may contain parameters from $\mathbb{A}$.

(i) Prove that P is i.i.d. iff P, $\neg P$ are s.i.i.d.

(ii) Prove that P is s.i.i.d. iff P is s-Π_1^1. [One half of this uses 4.1.]

(iii) Prove that if $\mathbb{A}$ satisfies s-$\Pi_1^1=\Sigma_1$ uniformly then

$$\begin{aligned}\text{s.i.i.d.}&=\Sigma_1 \quad \text{on} \quad \mathbb{A},\\ \text{i.i.d.}&=\Delta_1 \quad \text{on} \quad \mathbb{A},\\ \text{a.i.d.}&=\text{element} \quad \text{of} \quad \mathbb{A}.\end{aligned}$$

(iv) Prove that $\mathbb{A}$ is self-definable iff $\mathbb{A}$ is a.i.d. on $\mathbb{A}$.

(v) Prove that if $\mathbb{A}$ satisfies K_2 then every a.i.d. subset of $\mathbb{A}$ is bounded.

4.20. The notion of uniformity given by Definition 4.11 is really suggested by the notion of proof. Prove directly, using the Extended Completeness Theorem that if $\mathbb{A}$ is countable and admissible then $\mathbb{A}$ satisfies $s\text{-}\Pi_1^1 = \Sigma_1$ uniformly.

4.21. A more recursion theoretic version of the uniformity discussed in 4.11 and 4.20 was discovered by Nyberg. Prove that the admissible set $\mathbb{A}$ satisfies $s\text{-}\Pi_1^1 = \Sigma_1$ uniformly iff for every $s\text{-}\Pi_1^1$ formula $\Phi(x, \mathsf{T}_+)$ in an extra relation symbol T there is a Σ_1 formula $\varphi(x, \mathsf{T}_+)$ such that for all Σ_1 relations T on $\mathbb{A}$, $(\mathbb{A}, T)$ satisfies

$$\forall x \, [\Phi(x, \mathsf{T}_+) \leftrightarrow \varphi(x, \mathsf{T}_+)].$$

[Show that this condition is equivalent to $\mathrm{K}_1 \wedge \mathrm{K}_2$. Note that in the proof of 4.7, T occurs positively in $\Psi(\mathsf{T})$.]

4.22 Notes. See Exercise 4.19 for the way $s\text{-}\Pi_1^1$ predicates found their way into the subject. Corollary 4.9 was observed by Nyberg. For the record, it is still open whether every Σ_1 complete admissible set is Σ_1 compact. (Surely not!) It follows from Theorem 8.3 (applied to $L(\alpha)$) that there are lots of resolvable Σ_1 compact sets which are not Σ_1 complete.

5. *König's Lemma and Nerode's Theorem: a Digression*

In this section we interupt our study to apply the condition

$$s\text{-}\Pi_1^1 = \Sigma_1 \quad \text{uniformly}$$

to notions of relative definability.

One of the starkest applications of the Infinity Lemma in ordinary recursion theory is the proof of Nerode's Theorem:

> *B is truth table reducible to C iff there is a total general recursive operator $\mathfrak{F}$ with $\mathfrak{F}(K_C) = K_B$.*

Here $B, C \subseteq \omega$ and K_B is the characteristic function of B. This says, in effect, that the total general recursive operators rather trivial.

Since Nerode's Theorem uses so little about ω, other than the Infinity Lemma, it becomes a good test case for abstract versions of the Infinity Lemma, the matter which concerns us in this chapter.

Turing reducibility breaks up into many non-equivalent notions over an arbitrary set. We discuss generalizations of Nerode's Theorem for three of these:

$\leqslant_{\mathrm{d}}$ is "Δ definable from",

$\leqslant_{\mathrm{w}}$ is "weakly meta-recursive in",

$\leqslant_{\mathrm{mr}}$ is "meta-recursive in".

5.1 Definition. Let $\mathbb{A}$ be admissible and let $\varphi(x,\mathsf{C})$, $\psi(x,\mathsf{C})$ be Σ formulas, possibly containing parameters from $\mathbb{A}$. Let B, C be subsets of $\mathbb{A}$. We say that $B \leqslant_{\mathrm{d}} C$ (via $\langle\varphi,\psi\rangle$) if for all $x \in \mathbb{A}$:

$$x \in B \quad \text{iff} \quad (\mathbb{A}, C) \models \varphi(x,\mathsf{C}),$$

$$x \notin B \quad \text{iff} \quad (\mathbb{A}, C) \models \psi(x,\mathsf{C}).$$

If, for every C there is a B such that $B \leqslant_{\mathrm{d}} C$ via $\langle\varphi,\psi\rangle$ then the pair $\langle\varphi,\psi\rangle$ is called a *general* Δ *definability operator* $\mathfrak{F}$ over $\mathbb{A}$, and we write $\mathfrak{F}(C)=B$.

By the relativized version of Theorem II.2.3, if $\mathbb{A}=\mathbb{HF}$ then $B \leqslant_{\mathrm{d}} C$ iff B is recursive in C, so that $\leqslant_{\mathrm{d}}$ coincides with Turing reducibility.

What is the most obvious way to define a general Δ definability operator? It seems to be captured by the following definition. If $\mathbb{A}$ is admissible then $\Delta_0(\mathbb{A})$ denotes the Δ_0 formulas when all total $\mathbb{A}$-recursive functions are denoted by terms of the language.

5.2 Definition. Let $\mathbb{A}$ be admissible and let $\varphi(x,\mathsf{C})$ be a $\Delta_0(\mathbb{A})$ formula. Then $B \leqslant^{\mathrm{tt}}_{\mathrm{d}} C$ via φ if, for all $x \in \mathbb{A}$,

$$x \in B \leftrightarrow (\mathbb{A}, C) \models \varphi(x,\mathsf{C}).$$

5.3 Lemma. *Let* $\mathbb{A}$ *be admissible.*

(i) *Every* $\Delta_0(\mathbb{A})$ *formula* $\varphi(x,\mathsf{C})$ *defines a general* Δ *definability operator.*

(ii) *If* F *is* $\mathbb{A}$*-recursive and*

$$x \in B \quad \textit{iff} \quad F(x) \in C$$

then $B \leqslant^{\mathrm{tt}}_{\mathrm{d}} C$.

(iii) $\mathbb{A} - B \leqslant^{\mathrm{tt}}_{\mathrm{d}} B$.

(iv) $\leqslant^{\mathrm{tt}}_{\mathrm{d}}$ *is transitive.*

Proof. They are all trivial. For example, to prove (i) you simply replace any function symbol in φ by its definition as in § I.4. Note that

$$x \notin B \quad \text{iff} \quad (\mathbb{A}, C) \models \neg\varphi(x,\mathsf{C})$$

and $\neg\varphi$ is also a $\Delta_0(\mathbb{A})$ formula. $\square$

5.4 Theorem. *Let* $\mathbb{A}$ *be a resolvable, admissible set satisfying* $s\text{-}\Pi^1_1=\Sigma_1$ *uniformly. Let* $\mathfrak{F}$ *be any general* Δ *definability operator over* $\mathbb{A}$*. There is a* $\Delta_0(\mathbb{A})$ *formula* $\varphi(\mathsf{C})$ *such that for all* $C \subseteq \mathbb{A}$

$$\mathfrak{F}(C) \leqslant^{\mathrm{tt}}_{\mathrm{d}} C \quad \textit{via} \quad \varphi.$$

Proof. Let $\theta(x,\mathsf{C})$, $\psi(x,\mathsf{C})$ be Σ formulas such that $\mathfrak{F}$ is defined by

$$x \in \mathfrak{F}(C) \quad \text{iff} \quad (\mathbb{A}, C) \models \theta(x,\mathsf{C}),$$

$$x \notin \mathfrak{F}(C) \quad \text{iff} \quad (\mathbb{A}, C) \models \psi(x,\mathsf{C}).$$

Then, for every x, the following s-Π_1^1 formula $\Phi(x)$ holds on $\mathbb{A}$:

$$\forall \mathsf{C}\,[\theta(x,\mathsf{C}) \vee \psi(x,\mathsf{C})].$$

Let $\varphi(x)$ be a Σ_1 formula uniformly equivalent to $\Phi(x)$ on $\mathbb{A}$. Since $\mathbb{A}$ is resolvable there is an $\mathbb{A}$-recursive function $J: o(\mathbb{A}) \rightarrow \mathbb{A}$ such that $\mathbb{A} = \bigcup_{\alpha < o(\mathbb{A})} J(\alpha)$ and $J(\alpha)$ is transitive for all α. Now for each x, $\Phi(x)$ holds so there is an $\alpha \in \mathbb{A}$ such that $\varphi(x)^{(J(\alpha))}$. Define

$$G(x) = J(\text{least}\,\alpha\,[\varphi(x)^{(J(\alpha))}]).$$

Then G is $\mathbb{A}$-recursive and total, $G(x)$ is always transitive and

$$\varphi(x)^{(G(x))}.$$

But then for every x, $\Phi^{(G(x))}(x)$, by the uniform equivalence of φ and Φ. Thus, for every $C \subseteq \mathbb{A}$, either $\theta(x,C)^{(G(x))}$ or $\psi(x,C)^{(G(x))}$. We claim that

$$(1) \qquad x \in \mathfrak{F}(C) \quad \text{iff} \quad \theta(x,C)^{(G(x))}.$$

For if $x \in \mathfrak{F}(C)$ then $(\mathbb{A},C) \models \theta(x,\mathsf{C})$ so $\psi(x,C)^{(G(x))}$ cannot hold so $\varphi(x,C)^{(G(x))}$ must hold. Similarly, if $x \notin \mathfrak{F}(C)$ then $\theta(x,C)^{(G(x))}$ cannot hold. Let $\sigma(v,\mathsf{C})$ be $\theta(v,\mathsf{C})^{(F(v))}$. Then $\sigma(v,\mathsf{C})$ is $\Delta_0(\mathbb{A})$ and $\mathfrak{F}(C) \leqslant_d^{u} C$ via σ. □

Since "$s-\Pi_1^1 = \Sigma_1$" implies "$s-\Pi_1^1 = \Sigma_1$ uniformly" on resolvable admissible sets, we could have used the weaker condition in the statement of the theorem. This seems to conceal the main point of the theorem, though, since it is the uniformity which really matters in the above proof. Since the above proof is virtually identical (in outline) to the proof of Nerode's Theorem (in, say Rogers [1967]) this gives further support to the feeling that "$s-\Pi_1^1 = \Sigma_1$ uniformly" captures a great deal of the recursion theoretic content of the Infinity Lemma.

The relation $\leqslant_{\mathrm{d}}$ is quite sensible from a definability point of view. It has been studied very little, however, because one does not have the tools from ordinary recursion theory available. Put another way, the relation $B \leqslant_{\mathrm{d}} C$ is not sensible in terms of computation if the expanded structure $(\mathbb{A},C)$ fails to be admissible, for then in checking whether or not $x \in B$ one may have to use all of C, not just an $\mathbb{A}$-finite amount of information about C. This never comes up for $\mathbb{HF}$, or for any other $H(\kappa)$, κ-regular, since every expansion of $H(\kappa)$ is still admissible.

These observations prompt one to define a new notion of reducibility, one where an answer to "$x \in B$?" is determined by an $\mathbb{A}$-finite amount of information about C. Let K_C be the characteristic function of C:

$$K_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ 1 & \text{if } x \notin C \end{cases}$$

and let $\mathrm{Ch}_{\mathbb{A}}(C) = \{f \in A \mid f \subseteq K_C\}$. Thus $\mathrm{Ch}_{\mathbb{A}}(C)$ is the set of all $\mathbb{A}$-finite bits of information about membership in C.

5.5 Definition. Let $\mathbb{A}$ be admissible and let $\varphi(x,f)$, $\psi(x,f)$ be Σ_1 formulas with parameters from $\mathbb{A}$. We say that B is *weakly metarecursive in C via* $\langle\varphi,\psi\rangle$, written $B\leqslant_{\mathrm{w}}C$ via $\langle\varphi,\psi\rangle$, if for all $x\in\mathbb{A}$

$$\begin{aligned} x\in B \quad &\text{iff} \quad \exists f\in \mathrm{Ch}_{\mathbb{A}}(C)[\mathbb{A}\models\varphi(x,f)],\\ x\notin B \quad &\text{iff} \quad \exists f\in \mathrm{Ch}_{\mathbb{A}}(C)[\mathbb{A}\models\psi(x,f)]. \end{aligned}$$

If for every C there is a B such that $B\leqslant_{\mathrm{w}}C$ via $\langle\varphi,\psi\rangle$ then the pair $\langle\varphi,\psi\rangle$ is called a *general weak metarecursive operator* $\mathfrak{F}$ and we write $\mathfrak{F}(C)=B$.

The notion of tt-reducibility corresponding to $\leqslant_{\mathrm{w}}$ is complicated by the following observations. On $\mathbb{HF}$ one can define a recursive function H by

$$H(x)=\{f\mid f \text{ is a characteristic function with } \mathrm{dom}(f)=x\}.$$

Then given any recursive predicate P of finite functions one can "split" it by

$$\begin{aligned} F(x)&=\{f\in H(x)\mid P(f)\},\\ G(x)&=\{f\in H(x)\mid \neg P(f)\}. \end{aligned}$$

Then F,G are recursive and, for each $x\in\mathbb{HF}$ and each $C\subseteq\mathbb{HF}$, $\mathrm{Ch}_{\mathbb{HF}}(C)$ meets (has nonempty intersection with) exactly one of the sets $F(x)$, $G(x)$ (depending on whether or not $K_C\restriction x$ satisfies P or not). This triviality simplifies a lot of the recursion theory on $\mathbb{HF}$, especially when contrasted with a general admissible set $\mathbb{A}$ where $H(x)$ need not be a subset of $\mathbb{A}$, let alone an element of $\mathbb{A}$.

5.6 Definition. Let $\mathbb{A}$ be admissible. An *$\mathbb{A}$-recursive splitting* is a pair F,G of total $\mathbb{A}$-recursive functions such that

(i) for each $x\in\mathbb{A}$, $F(x)$, $G(x)$ are sets of $\mathbb{A}$-finite characteristic functions,
(ii) for each $x\in\mathbb{A}$ and each $C\subseteq\mathbb{A}$, $\mathrm{Ch}_{\mathbb{A}}(C)$ meets exactly one of $F(x)$, $G(x)$.

5.7 Lemma. *Let $\mathbb{A}$ be admissible and let F,G be an $\mathbb{A}$-recursive splitting. Define*

$$\mathfrak{F}(C)=\{x\mid \mathrm{Ch}_{\mathbb{A}}(C)\cap F(x)\neq 0\}.$$

Then $\mathfrak{F}$ is a general weak metarecursive operator on $\mathbb{A}$.

Proof. Let $\varphi(x,f)$ be $f\in F(x)$, $\psi(x,f)$ be $f\in G(x)$. Then

$$\begin{aligned} x\in\mathfrak{F}(C) \quad &\text{iff} \quad \exists f\in \mathrm{Ch}_{\mathbb{A}}(C)\,\varphi(x,f),\\ x\notin\mathfrak{F}(C) \quad &\text{iff} \quad \exists f\in \mathrm{Ch}_{\mathbb{A}}(C)\,\psi(x,f) \end{aligned}$$

so $\mathfrak{F}(C)\leqslant_{\mathrm{w}}C$ via $\langle\varphi,\psi\rangle$, for all $C\subseteq\mathbb{A}$. □

The next theorem shows that for some admissible sets, every general weak metarecursive operator arises as in the above lemma.

5.8 Theorem. *Let $\mathbb{A}$ be a resolvable admissible set satisfying $s-\Pi_1^1=\Sigma_1$ uniformly. Let $\mathfrak{F}$ be any general weak metarecursive operator. There is an $\mathbb{A}$-recursive splitting F, G such that for all C,*

$$\mathfrak{F}(C)=\{x \mid \mathrm{Ch}_{\mathbb{A}}(C)\cap F(x)\neq 0\}.$$

Proof. The proof is very much like the proof of Theorem 5.4. Let $\theta(x,f)$, $\psi(x,f)$ be Σ_1 formulas such that $\mathfrak{F}$ is defined by

$$x\in\mathfrak{F}(C) \quad\text{iff}\quad \exists f\in\mathrm{Ch}_{\mathbb{A}}(C)[A\models\theta(x,f)],$$

$$x\notin\mathfrak{F}(C) \quad\text{iff}\quad \exists f\in\mathrm{Ch}_{\mathbb{A}}(C)[A\models\psi(x,f)].$$

Then for each $x\in\mathbb{A}$ the following $s-\Pi_1^1$ formula $\Phi(x)$ holds:

$$\forall C\,\exists f[f\in\mathrm{Ch}_{\mathbb{A}}(C)\wedge[\theta(x,f)\vee\psi(x,f)]].$$

Let $\varphi(x)$ be uniformly equivalent to $\Phi(x)$ on $\mathbb{A}$. Let $J\colon o(\mathbb{A})\to\mathbb{A}$ be as in the proof of 5.4 and define

$$H(x)=J(\text{least}\,\alpha\,\varphi(x)^{(J(\alpha))}).$$

As in the proof of 5.4 we see that H is a total $\mathbb{A}$-recursive function, that $H(x)$ is always transitive, and $\Phi(x)^{(H(x))}$; i.e.,

$$\text{(2)}\qquad \forall C\,\exists f\in H(x)[f\in\mathrm{Ch}_{\mathbb{A}}(C)\wedge\theta(x,f)\vee\psi(x,f)]^{(H(x))}.$$

Let

$$F(x)=\{f\in H(x)\mid f \text{ is a characteristic function }\wedge\theta(x,f)^{(H(x))}\},$$

$$G(x)=\{f\in H(x)\mid f \text{ is a characteristic function }\wedge\psi(x,f)^{(H(x))}\}.$$

We claim that for all x, C

$$x\in\mathfrak{F}(C) \quad\text{iff}\quad F(x)\cap\mathrm{Ch}_{\mathbb{A}}(C)\neq 0,$$

$$x\notin\mathfrak{F}(C) \quad\text{iff}\quad G(x)\cap\mathrm{Ch}_{\mathbb{A}}(C)\neq 0.$$

This will prove that F, G is an $\mathbb{A}$-recursive splitting and the conclusion of the theorem. First suppose $x\in\mathfrak{F}(C)$. From line (2) we see that $F(x)\cap\mathrm{Ch}_{\mathbb{A}}(C)\neq 0$. But line (2) also implies that $G(x)\cap\mathrm{Ch}_{\mathbb{A}}(C)=0$ for if $f\in\mathrm{Ch}_{\mathbb{A}}(C)\wedge\psi(x,f)^{(H(x))}$ then $\psi(x,f)$ holds in $\mathbb{A}$, since ψ is Σ_1, so $x\notin\mathfrak{F}(C)$. The other half is similar. □

The relation $\leqslant_{\mathrm{w}}$ has been studied a fair amount by the Sacks school (on admissible sets of the form $\mathrm{L}(\alpha)$). In particular, it has shown that $\leqslant_{\mathrm{w}}$ is not transitive. This is not too surprising given the disparity between the amount of information used about C (namely $f\in\mathrm{Ch}_{\mathbb{A}}(C)$) and the amount of information received ($x\in B$ or $x\notin B$). Thus Sacks defines

$$B\leqslant_{\mathrm{mr}}C \quad\text{iff}\quad \mathrm{Ch}_{\mathbb{A}}(B)\leqslant_{\mathrm{w}}C.$$

This is equivalent to the existence of a single Σ_1 formula $\varphi(x,f)$ such that

$$(3) \qquad g \in \mathrm{Ch}_{\mathbb{A}}(B) \quad \text{iff} \quad \exists f \in \mathrm{Ch}_{\mathbb{A}}(C)[\mathbb{A} \models \psi(x,f)].$$

There doesn't seem to be a very elegant notion of tt-reducibility to go along with $\leqslant_{\mathrm{mr}}$, but we do get one out of Theorem 5.8.

5.9 Corollary. *Let $\mathbb{A}$ be resolvable and satisfy $s-\Pi_1^1=\Sigma_1$ uniformly. Let ψ be such that for every C there is a B satisfying line* (3) *above. There is an $\mathbb{A}$-recursive splitting F,G such that for all C,B as in* (3)

$$g \in \mathrm{Ch}_{\mathbb{A}}(B) \quad \mathit{iff} \quad F(g) \cap \mathrm{Ch}_{\mathbb{A}}(C) \neq 0 . \quad \square$$

5.10 Exercise (R. Shore). Show that if $\mathrm{V}=\mathrm{L}$ then the conclusions of 5.4 and 5.8 fail for $\mathbb{A}=\mathrm{L}(\omega_1)$. [For 5.8 define $\mathfrak{F}(C)=\mathbb{A}$ if $C \cap \omega$ is infinite, $=0$ otherwise. For 5.4 let $R \subset \mathscr{P}(\omega)$ by Δ_1 on $\mathbb{A}$ but not Δ_0 and define $\mathfrak{F}(C)=\mathbb{A}$ if $R(C \cap \omega)$, $=0$ otherwise.]

5.11 Notes. The reader should consult Simpson's forthcoming book in this series for more about reducibilities on admissible sets.

6. Implicit Ordinals on Arbitrary Admissible Sets

For the model theory of an admissible fragment $\mathsf{L}_{\mathbb{A}}$, the ordinal $h_\Sigma(\mathbb{A})$ plays a more important role than $o(\mathbb{A})$. For countable $\mathbb{A}$ we have $h_\Sigma(\mathbb{A})=o(\mathbb{A})$. In general, we will see that this condition again goes back to the König Infinity Lemma.

6.1 Theorem. *An admissible set satisfies the third König principle iff $h_\Sigma(\mathbb{A})=o(\mathbb{A})$.*

Proof. This is an immediate consequence of the next theorem. $\square$

The ordinal $h_\Sigma(\mathbb{A})$ is not an absolute notion. That is, the size (cardinality) of $h_\Sigma(\mathbb{A})$ may vary drastically from one model of set theory to another (cf. Theorem 4.2 in Barwise-Kunen [1971]). The important point for application, though, is that $h_\Sigma(\mathbb{A})$ has a precise description in terms of the generalized recursion theory of $\mathbb{A}$.

6.2 Theorem. *Let $\mathbb{A}$ be admissible:*

$$h_\Sigma(\mathbb{A})=\sup\{\xi : \xi \text{ is a } \Pi \text{ implicit ordinal over } \mathbb{A}\} .$$

Proof. The inequality $\geqslant$ follows from Theorem 3.5. To prove the theorem it suffices to prove that every ordinal $\beta<h_\Sigma(\mathbb{A})$ is less than some Π implicit ordinal ξ.

We can read this off the proof of Theorem VII.3.1. Since $\beta < h_\Sigma(\mathbb{A})$, β can be pinned down by some Σ_1 theory T of $\mathsf{L}_{\mathbb{A}}$. Now consider the proof of VII.3.1 for this particular theory T. In particular, consider the well-founded relation $\langle \mathfrak{S}, \prec \rangle$ constructed there. Since every ordinal pinned down by T is less than the rank $\xi = \rho(\prec)$ of this well-founded relation, it suffices to prove that this ξ is Π implicit over $\mathbb{A}$ or, at least less than or equal to some Π implicit ordinal.

Case 1. If $o(\mathbb{A}) > \omega$ then ξ is a Π implicit ordinal.
For if $o(\mathbb{A}) > \omega$. then we can write

$$\mathscr{D} \in \mathfrak{S} \wedge \mathscr{D}' \in \mathfrak{S} \wedge \mathscr{D} \prec \mathscr{D} \tag{1}$$

out as a Π sentence $\varphi(\mathscr{D}', \mathscr{D})$ using 4.2:

$$\begin{aligned} \exists n < \omega\, \exists m < \omega [n > m \wedge T \subseteq \mathscr{D} \subseteq \mathscr{D}' \wedge \mathscr{D}' &\text{ is an s.v.p. for } \mathsf{L}_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n), \\ \mathscr{D} &\text{ is an s.v.p. for } \mathsf{L}_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_m) \\ &\wedge \forall i < n [0 < i \rightarrow (\mathsf{c}_{i+1} < \mathsf{c}_i) \in \mathscr{D}']\,. \end{aligned}$$

All these clauses are Π and the others follows from these. Thus $\varphi(\mathscr{D}', \mathscr{D})$ is a Π sentence which implicitly defines $\xi = \rho(\prec)$.

Case 2. If $o(\mathbb{A}) = \omega$ then $\xi \leqslant \xi'$ for some Π implicit ordinal ξ'.
Let $\psi(\mathscr{D}', \mathscr{D})$ be the Π sentence expressing the following:

$\mathscr{D}, \mathscr{D}'$ are sets of sentences of $\mathsf{L}_{\mathbb{A}}(C)$,
$T \subseteq \mathscr{D} \subseteq \mathscr{D}'$,
$\mathscr{D} \cap \mathsf{L}_{\mathbb{A}}$ is an s.v.p. for $\mathsf{L}_{\mathbb{A}}$,
$\forall m[(\mathsf{c}_m = \mathsf{c}_m) \in \mathscr{D} \rightarrow \mathscr{D} \cap \mathsf{L}_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_m)$ is an s.v.p.
for $L_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_m)$ and $\forall i < m[0 < i \rightarrow (\mathsf{c}_{i+1} < \mathsf{c}_i) \in \mathscr{D}$,
the same sentence for $\mathscr{D}'$,
$(\mathsf{c}_1 = \mathsf{c}_1) \in \mathscr{D}'$,
$\forall m[(\mathsf{c}_m = \mathsf{c}_m) \in \mathscr{D} \rightarrow (\mathsf{c}_{m+1} = \mathsf{c}_{m+1}) \in \mathscr{D}']$.

If $\mathscr{D}, \mathscr{D}' \in S$ and $\mathscr{D}' \prec \mathscr{D}$ then $\psi(\mathscr{D}', \mathscr{D})$. If

$$\mathscr{D}' \prec' \mathscr{D} \quad \text{iff} \quad (\mathbb{A}, \mathscr{D}, \mathscr{D}') \models \psi(\mathscr{D}', \mathscr{D})$$

defines a well-founded relation then its rank is $\geqslant \xi = \rho(\prec)$ since $\prec$ is a subrelation. So we need only prove that $\prec'$ is well founded. Suppose not. That is, suppose

$$\cdots \prec' \mathscr{D}_{n+1} \prec' \mathscr{D}_n \prec' \cdots \prec' \mathscr{D}_1 \prec' \mathscr{D}_0\,.$$

Let $\bar{\mathscr{D}}_n = \mathscr{D}_n \cap \mathsf{L}_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n)$. Since $\psi(\mathscr{D}_1, \mathscr{D}_0)$ holds it follows that $\bar{\mathscr{D}}_0$ is an s.v.p. for $\mathsf{L}_{\mathbb{A}}$, that $(\mathsf{c}_1 = \mathsf{c}_1) \in \bar{\mathscr{D}}_2$ and hence that $\bar{\mathscr{D}}_1$ is an s.v.p. for $\mathsf{L}_{\mathbb{A}}(\mathsf{c}_1)$. That is, $\bar{\mathscr{D}}_0 \in \mathfrak{S}_0$, $\mathscr{D}_1 \in \mathfrak{S}_1$ and $\bar{\mathscr{D}}_1 \prec \bar{\mathscr{D}}_0$. By induction on n we see that $\bar{\mathscr{D}}_n \in \mathfrak{S}_n$ and $\bar{\mathscr{D}}_{n+1} \prec \bar{\mathscr{D}}_n$.

This contradicts the well-foundedness of $\prec$. Thus $\prec'$ is well founded. Since ψ is a Π formula, the rank $\xi' = \rho(\prec')$ is a Π implicit ordinal and $\beta < \xi \leqslant \xi'$. □

Theorem 6.2 would have simplified the proofs of the theorems in § VII.4 since it is usually easier to show that a given well-founded relation $\prec$ is definable by a Π sentence than to prove $\rho(\prec) < h_\Sigma(\mathbb{A})$.

6.3 Corollary. *Every* $s-\Sigma_1^1$ *implicit ordinal over the admissible set* $\mathbb{A}$ *is less than some* Π *implicit ordinal over* $\mathbb{A}$.

Proof. Immediate from 3.5 and 6.2. □

6.4 Corollary. *If* $\mathbb{A}$ *is a resolvable admissible set and* $h_\Sigma(\mathbb{A}) = o(\mathbb{A})$ *then* $\mathbb{A}$ *is* Σ_1 *compact; i.e.,* K_3 *implies* K_2 *on resolvable admissible sets.*

Proof. By 4.8, if $\mathbb{A}$ fails to be Σ_1 compact then $\mathbb{A}$ is self-definable, and hence $h_\Sigma(\mathbb{A}) > o(\mathbb{A})$ by Proposition VII.1.5. □

6.5 Corollary. *Let* $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ *and let* $\mathbb{A} = \mathbb{HYP}_{\mathfrak{M}}$. *Then* $\mathbb{A}$ *is* Σ_1 *compact iff* $h_\Sigma(\mathbb{A}) = o(\mathbb{A})$.

Proof. One half follows from 6.4, since $\mathbb{HYP}_{\mathfrak{M}}$ is resolvable; the other half ($\Rightarrow$) from VII.3.8. □

We conclude this section with a theorem that explains why Π implicit ordinals are so important from a theoretical, not just a practical, point of view.

Let $\varphi(v, \mathsf{R}, \mathsf{S})$ be a formula with R, S n-ary, v a free variable, which may contain parameters from $\mathbb{A}$. For $x \in \mathbb{A}$ we write $\prec_\varphi^x$ for the relation defined by

$$R \prec_\varphi^x S \quad \text{iff} \quad (\mathbb{A}, R, S) \models \varphi(x, \mathsf{R}, \mathsf{S}).$$

6.6 Lemma. *If* $\varphi(v, \mathsf{R}, \mathsf{S})$ *is a* Π (*or even* $\mathrm{s}-\Sigma_1^1$) *formula then*

$$P(x) \quad \textit{iff} \quad \prec_\varphi^x \text{ is well founded}$$

defines a $s-\Pi_1^1$ *predicate* P *over the admissible set* A.

Proof. $P(x)$ holds iff

$$\forall \mathsf{Q}\, \exists m \neg \varphi(x, (\mathsf{Q})_{m+1}, (\mathsf{Q})_m)$$

where Q is $n+1$ ary and $\varphi(x, (\mathsf{Q})_{m+1}, (\mathsf{Q})_m)$ denotes the result of replacing $\mathsf{R}(x_1, \ldots, x_n)$ by $\mathsf{Q}(x_1, \ldots, x_n, m+1)$, $\mathsf{S}(x_1, \ldots, x_n)$ by $\mathsf{Q}(x_1, \ldots, x_m, m)$. □

6.7 Theorem. *Let* $\mathbb{A}$ *be admissible. There is a* Π *formula* $\varphi(v, \mathsf{R}, \mathsf{S})$ *such that*

$$\{x : \prec_\varphi^x \text{ is well founded}\}$$

is a complete strict$-\Pi_1^1$ *set for* $\mathbb{A}$.

Proof. The set in question is always $s-\Pi_1^1$ by 6.6. Let X_1 be the complete $s-\Pi_1^1$ set defined in Corollary 4.5: $X_1=\{\varphi\in\mathsf{L}'_{\mathbb{A}} \mid T\models\varphi\}$. We can assume $\mathsf{L}'_{\mathbb{A}}$ (of 4.5) is a Skolem fragment which is Δ_1 on $\mathbb{A}$ and that every model of T can be expanded to a Skolem model. We will show how to write "Is $x\in X_1$?" in terms of asking whether or not a certain tree of theories of $\mathsf{L}'_{\mathbb{A}}$ is well founded. Let $\varphi(x,\mathsf{T}'',\mathsf{T}')$ express:

$$\begin{array}{l}
T', T'' \text{ are sets of sentences of } \mathsf{L}'_{\mathbb{A}},\\
T\cup\{\neg x\}\subseteq T'\subseteq T'',\\
\forall y[y\in T''\rightarrow(\neg y)\notin T''],\\
\forall y[y=\bigwedge\Phi\in T'\rightarrow\forall\psi\in\Phi(\psi\in T'')],\\
\forall y[y=\bigvee\Phi\in T'\rightarrow\exists\psi\in\Phi(\psi\in T'')],\\
\forall y[y=\forall v\varphi(v)\in T'\rightarrow\forall t(t \text{ a closed term of } \mathsf{L}'\rightarrow\varphi(t/v)\in T'')],\\
\forall y[y=\exists v\varphi(v)\in T'\rightarrow\varphi(\mathsf{F}_{\exists v\varphi}/v)\in T''],\\
\forall y,z[y,z \text{ closed terms of } \mathsf{L}'\wedge(y=z)\in T'\rightarrow(z=y)\in T''],\\
\forall y,z,w[w=\varphi(v)\in\mathsf{L}'_{\mathbb{A}}\wedge y,z \text{ closed terms of } \mathsf{L}'\wedge\varphi(y/v)\in T'\\
\qquad\qquad\wedge(z=y)\in T'\rightarrow\varphi(z/v)\in T''].
\end{array}$$

If $x\notin X_1$ (i.e. $T\not\models x$) then $T\cup\{\neg x\}$ has a Skolem model $\mathfrak{M}$. Let T' be the set of sentences true in $\mathfrak{M}$. Then $\psi(x,T',T')$ holds so $\prec_\varphi^x$ is not well founded. Now suppose $\prec_\varphi^x$ is not well founded, so there is a sequence

$$\cdots\prec_\varphi^x T_{n+1}\prec_\varphi^x T_n\prec\cdots\prec_\varphi^x T_1\,.$$

Let $T_\omega=\bigcup_n T_n$. Then T satisfies all the conditions of Lemma VII.2.9, so T_ω has a model. But $T\cup\{\neg x\}\subseteq T_\omega$ so $T\not\models x$. Thus

$$x\in X_1 \quad\text{iff}\quad \prec_\varphi^x \text{ is well founded}\,. \quad\square$$

6.8—6.10 Exercises

6.8. Let $\mathfrak{M}$ be infinite. Show that if $P\subseteq\mathfrak{M}$ is Π_1^1 on $\mathfrak{M}$ then there is a first order formula $\varphi(v,\mathsf{R},\mathsf{S})$ such that

$$P(x) \text{ iff } \prec_\varphi^x \text{ is well founded}\,.$$

This is analogous to the normal form for Π_1^1 relations on $\mathcal{N}$.

6.9. Show that if α is the rank of some well-founded relation on Power($\mathbb{A}$) then $\alpha<(2^{\mathrm{card}(\mathbb{A})})^+$. Conclude that $h_\Sigma(\mathbb{A})<(2^{\mathrm{card}(\mathbb{A})})^+$.

6.10 (Open). Prove that $h_\Sigma(\mathbb{HYP}_{\mathfrak{M}})=\sup\{\xi:\xi$ is a first order implicit ordinal over $\mathfrak{M}\}$.

7. *Trees and Σ_1 Compact Sets of Cofinality ω*

The results of this chapter would be vacuous if there were no uncountable admissible sets sytisfying the König principles K_1—K_3. We exhibit such admissible sets in this and the next section.

A set $\mathbb{A}$ is *essentially uncountable* if every countable subset of $\mathbb{A}$ is an element of $\mathbb{A}$. All of the Σ_1 compact sets exhibited in this section have *cofinality* ω in the sense that

$$\mathbb{A} = \bigcup_{n<\omega} A_n$$

where each $A_n \in \mathbb{A}$. Hence none of them is essentially uncountable. We give a proof of the existence of essentially uncountable Σ_1 compact sets in the next section, though no explicit such sets are known. An explanation for this phenomenon will be found in § 9.

Let us return to our discussion of trees from § 1, and attempt to give a generalization of the Infinity Lemma soley in terms of trees.

In this section we turn the full binary tree around and think of it as pictured in Fig. 7A.

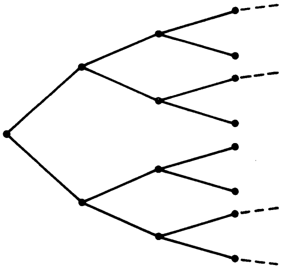


Fig. 7A. Another view of the full binary tree

Another tree, one with paths of length ω^2, is pictured in Fig. 7B.

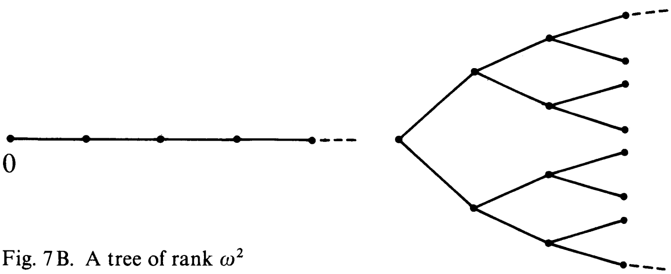


Fig. 7B. A tree of rank ω^2

In general, a *tree* is a well-founded partial ordering $\mathscr{T} = \langle T, \prec \rangle$, with a least element (usually denoted by 0), such that for each $x \in T$, the set $\{y \in T : y \prec x\}$

of predecessors of x is well ordered by $\prec$. A subset $C \subseteq T$ is a *chain* in $\mathcal{T}$ if for each $x, y \in C$,

$$x \prec y \quad \text{or} \quad x = y \quad \text{or} \quad y \prec x .$$

A *path thru* $\mathcal{T}$ is a maximal chain. Thus every path is well ordered by $\prec$.

Let $\mathcal{T} = \langle T, \prec \rangle$ be a tree. Since $\prec$ is well founded we have the usual rank function $\rho = \rho_{\prec}$ associated with $\mathcal{T}$:

$$\rho(x) = \sup\{\rho(y) + 1 : y \prec x\}$$

and $\mathcal{T}$ has a *rank* $\rho(\mathcal{T})$:

$$\begin{aligned} \rho(\mathcal{T}) &= \rho(\prec) \\ &= \sup\{\rho(x) + 1 : x \in T\} . \end{aligned}$$

A *branch* thru the tree $\mathcal{T}$ is a path of length $\rho(\mathcal{T})$. Not every tree has a branch, as Fig. 7C demonstrates.

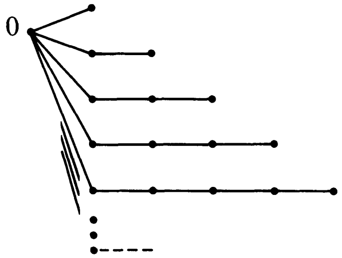


Fig. 7C. A tree with no branch

This tree has rank $\rho(\mathcal{T}) = \omega$ but every path is finite. Thus $\mathcal{T}$ has no branch. We call the elements x of a tree $\mathcal{T}$ with $\rho(x) = \beta$ the *nodes of level* β. Thus $\mathcal{T}$ has nodes of every level $\beta < \rho(\mathcal{T})$. Let $\text{lev} = \text{lev}_{\mathcal{T}}$ be the function with domain $\rho(\mathcal{T})$ defined by

$$\text{lev}(\beta) = \{x \in T : \rho(x) = \beta\} .$$

Let $\mathbb{A}$ be an admissible set. A tree $\mathcal{T} = \langle T, \prec \rangle$ is an $\mathbb{A}$-*tree* if $T \subseteq \mathbb{A}$, T, $\prec$, $\text{lev}_{\mathcal{T}}$ are $\mathbb{A}$-recursive and the rank $\rho(\mathcal{T})$ of $\mathcal{T}$ is $o(\mathbb{A})$. In particular, for each $\beta < o(\mathbb{A})$, T has nodes of level β (since $\rho(\mathcal{T}) = o(\mathbb{A})$) but the set of all nodes of level β is $\mathbb{A}$-finite (since $\text{lev}(\beta) \in \mathbb{A}$).

The König Infinity Lemma can be restated as follows. Let $\mathbb{A} = \langle \mathbb{HF}, \in, R \rangle$. Then every $\mathbb{A}$-tree has a branch.

7.1 Theorem. *If* $\mathbb{A}$ *is a* Σ_1 *compact admissible set then every* $\mathbb{A}$*-tree has a branch.*

Proof. It is easy to prove this by means of Σ_1 compactness, by constructing a nonstandard extension of the tree, picking a node d of nonstandard length and letting the branch be defined by

$$B=\{x\in T \mid x\prec d\}.$$

An even easier proof, though, is by means of $s-\Pi_1^1$ Reflection. Let $\mathcal{T}=\langle T,\prec\rangle$ be an $\mathbb{A}$-tree and suppose $\mathcal{T}$ has no branch, i.e. no path of length $\rho(\mathcal{T})=o(\mathbb{A})$. Then $\mathbb{A}$ satisfies the $s-\Pi_1^1$ sentence:

$$\forall C[C \text{ is a chain} \rightarrow \exists\beta\,\forall x\in\operatorname{lev}(\beta)\,(x\notin C)].$$

By $s-\Pi_1^1$ Reflection there is a $\gamma\in\mathbb{A}$ such that

$$(1)\qquad \forall C[C \text{ is a chain} \rightarrow \exists\beta<\gamma\,\forall x\in\operatorname{lev}(\beta)\,(x\notin C)].$$

But then lev(γ) must be empty, for if $y\in\operatorname{lev}(\gamma)$ then

$$C=\{x\in T \mid x\prec y\}$$

would violate (1). But then $\rho(\mathcal{T})\leqslant\gamma<o(\mathbb{A})$, contradicting the definition of $\mathbb{A}$-tree. □

The hypothesis "every $\mathbb{A}$-tree has a branch" looks like it ought to be called a König principle. The next theorem shows that it is in fact too weak to be of general interest.

7.2 Theorem. *Let $\mathbb{A}$ be an admissible set whose ordinal $\alpha=o(\mathbb{A})$ has cofinality ω. Then every $\mathbb{A}$-tree has a branch.*

Proof. Since this is a direct generalization of the Infinity Lemma it is not surprising that the proof is an amplification of the proof of that lemma. Let $\alpha=\sup\{\alpha_n\colon n<\omega\}$ where $\alpha_0<\alpha_1<\cdots<\alpha_n<\cdots<\alpha$. Let $\mathcal{T}=\langle T,\prec\rangle$ be an $\mathbb{A}$-tree. We claim that we can find $x_0,x_1,\ldots$ such that $x_n\in\operatorname{lev}(\alpha_n)$ and $x_0\prec x_1\prec\ldots$. If we can do this, then

$$B=\{y\in T\colon y\prec x_n \text{ for some } n\}$$

will be a branch thru $\mathcal{T}$. To find the x's, let $x_0\in\operatorname{lev}(\alpha_0)$ be such that

$$\forall\beta>\alpha_0\,\exists z\in\operatorname{lev}(\beta)\quad(x_0\prec z).$$

(We must see that there is such an x_0.) Given x_0, let $x_1\succ x_0$ be choosen so that $x_1\in\operatorname{lev}(\alpha_1)$ and

$$\forall\beta>\alpha_1\,\exists z\in\operatorname{lev}(\beta)\,[x_1\prec z].$$

Continuing in this way gives the desired sequence of x_n's. Let us now prove that x_0 exists. (The proof that given x_n we can find x_{n+1} as above is almost identical.) Suppose there were no such x_0. Then

$$\forall x \in \mathrm{lev}(\alpha_0)\, \exists \beta > \alpha_0\, \forall z \in \mathrm{lev}(\beta)\, [x \not\prec z].$$

By Σ_1 Reflection there is a $\gamma \in \mathbb{A}$ such that

$$\forall x \in \mathrm{lev}(\alpha_0)\, \exists \beta < \gamma\, (\alpha_0 < \beta \wedge \forall z \in \mathrm{lev}(\beta)\, [x \not\prec z]. \tag{2}$$

Let $w \in \mathrm{lev}(\gamma)$. Now w has a predecessor x of level α_0 and a predecessor z_β for each $\beta < \gamma$. But then $x \prec z_\beta$, contradicting (2). □

Thus, e. g., $\mathbb{A} = H(\beth_{\omega+\omega})$ satisfies "Every $\mathbb{A}$-tree has a branch" but it does not satisfy s-Π_1^1 Reflection. Still, we did use the Infinity Lemma to prove s-Π_1^1 Reflection in § 1, so there should be some context in which the tree proof generalizes. If you analyze that proof you see that we also used two other facts: every subset of an $a \in \mathbb{HF}$ is in $\mathbb{HF}$ and, moreover, we can effectively find the set of all subsets of a. It easy to see that a pure admissible set $\mathbb{A}$ such that

$$b \subseteq a \in \mathbb{A} \Rightarrow b \in \mathbb{A}$$

(sometimes called *supertransitive*) must be of the form $H(\kappa)$ for some κ, so we restrict attention to $H(\kappa)$'s for the time being. In order for $H(\kappa)$ to be closed under $\mathcal{P}$, the power set, it is necessary and sufficient that κ be a strong limit cardinal ($\lambda < \kappa \Rightarrow 2^\lambda < \kappa$). Note that $H(\beth_{\omega+\omega})$ is closed under the power set but that

$$\langle H(\beth_{\omega+\omega}), \in, \mathcal{P} \rangle$$

is *not* admissible (for the same reason that $L(\omega+\omega)$ is not admissible). We write $\langle H(\kappa), \in, \mathcal{P} \rangle$ rather than the correct $\langle H(\kappa), \in, \mathcal{P} \cap H(\kappa)^2 \rangle$.

7.3 Theorem. *Let κ be a strong limit cardinal and suppose that $\mathbb{A} = \langle H(\kappa), \in, \mathcal{P}, R \rangle$ is admissible. Then $\mathbb{A}$ is Σ_1 compact iff every $\mathbb{A}$-tree has a branch.*

Proof. We have ($\Rightarrow$) by 7.1. To prove the converse we assume that every $\mathbb{A}$-tree has a branch and prove s-Π_1^1 Reflection. Since $H(\kappa)$ is closed under the power set and since $\langle H(\kappa), \in, \mathcal{P} \rangle$ is admissible, the usual definition by recursion of $\mathrm{V}(\alpha)$ shows that $\alpha \mapsto \mathrm{V}(\alpha)$ is an $\mathbb{A}$-recursive function of α. The usual "ZF-proof" that every set is in some $\mathrm{V}(\alpha)$ shows that $\mathrm{V}(\kappa) = H(\kappa)$. Let

$$\forall \mathsf{S}\, \exists y\, \varphi(\mathsf{S}, x, y) \tag{3}$$

be a typical s-Π_1^1 formula *true in* $\mathbb{A}$, where φ is Δ_0 and we assume S is unary, for simplicity. Let $\xi = \mathrm{rk}(x)$ so that $x \in \mathrm{V}(\alpha)$ for all $\alpha > \xi$. We suppose that for each $\alpha, \xi < \alpha < \kappa$,

$$\neg \forall S \subseteq \mathrm{V}(\alpha)\, \exists y \in \mathrm{V}(\alpha)\, \varphi(S, x, y)$$

and get a contradiction, thus establishing s-Π_1^1 Reflection. Thus we are assuming that for each $\alpha, \xi < \alpha < \kappa$

$$(4) \qquad \exists S \subseteq \mathrm{V}(\alpha)\, \forall y \in \mathrm{V}(\alpha) \neg\varphi(S,x,y).$$

We define a tree $\mathscr{T} = \langle T, \prec \rangle$ by

$$T = \{\langle \alpha, S\rangle : S \subseteq \mathrm{V}(\alpha) \wedge (\xi < \alpha \rightarrow \forall y \in \mathrm{V}(\alpha) \neg\varphi(S,x,y))\}$$

$$\langle \alpha, S\rangle \prec \langle \beta, S'\rangle \quad \text{iff} \quad \alpha < \beta \quad \text{and} \quad S = S' \cap \mathrm{V}(\alpha).$$

Each such S is an element of $H(\kappa)$, by supertransitivity. The least member of T is $\langle 0,0 \rangle$. The predecessors of some $\langle \alpha, S \rangle \in T$ are just the pairs of the form $\langle \beta, S \cap \mathrm{V}(\beta)\rangle$ for $\beta < \alpha$ and hence have order type α under $\prec$. Thus the level of a pair $\langle \alpha, S\rangle$ is just α. Furthermore, since

$$\mathrm{lev}(\alpha) \subseteq \{\alpha\} \times \mathscr{P}(\mathrm{V}(\alpha)),$$

$\mathrm{lev}(\alpha) \in \mathbb{A}$ and, by the above comments, lev is $\mathbb{A}$-recursive. Line (4) says that $\mathrm{lev}(\alpha) \neq 0$ for all $\alpha < \kappa$. Thus $\mathscr{T}$ is an $\mathbb{A}$-tree. Let B be a branch thru $\mathscr{T}$, that is, a path of order type κ. B is a set of pairs

$$\langle \alpha, S_\alpha \rangle \in T,$$

exactly one pair for each $\alpha < \kappa$, linearly ordered by $\prec$. Furthermore, $\alpha < \beta$ implies $S_\alpha = S_\beta \cap \mathrm{V}(\alpha)$. Let $S = \bigcup_{\alpha < \kappa} S_\alpha$. Then

$$S_\alpha = S \cap \mathrm{V}(\alpha)$$

for each α. We claim that $(\mathbb{A}, S)$ satisfies

$$\neg \exists y\, \varphi(S,x,y)$$

contradicting (3). For let $y \in \mathbb{A}$ be arbitrary. Pick $\alpha < \kappa$ such that $\xi < \alpha$ and $y \in \mathrm{V}(\alpha)$. Since $\langle \alpha, S_\alpha \rangle \in T$

$$\langle \mathrm{V}(\alpha), \in, \mathscr{P} \restriction \mathrm{V}(\alpha), R \restriction \mathrm{V}(\alpha), S_\alpha \rangle \models \neg\varphi(\mathrm{S},x,y).$$

But $(\mathbb{A}, S)$ is an end extension of this structure so it also satsfies the Δ_0 formula $\neg\varphi(\mathrm{S},x,y)$, establishing our contradiction to (3). □

A cardinal is said to be *(strongly) inaccessible* if κ is a regular strong limit cardinal. It follows from Theorem II.3.2 that if κ is inaccessible then

$$\langle H(\kappa), \in, \mathscr{P}, R \rangle$$

is admissible for all $R \subseteq H(\kappa)$, so that Theorem 7.3 applies. A simple Löwenheim-Skolem argument shows that one can find $\lambda < \kappa$ such that

$$\langle H(\lambda), \in, \mathscr{P}, R \upharpoonright H(\lambda) \rangle$$

is admissible and $\mathrm{cf}(\lambda) = \omega$. Alternatively, one can drop all talk of inaccessibles and prove directly (using the reflection theorem of Lévy) that for any definable class R there are cardinals λ with $\mathrm{cf}(\lambda) = \omega$ such that $\langle H(\lambda), \in, \mathscr{P}, R \upharpoonright H(\lambda) \rangle$ is admissible. Thus the hypothesis of the next theorem is not vacuous. It is this theorem which has been the aim of the first part of this section.

7.4 Theorem. *Let κ be a strong limit cardinal of cofinality ω and assume that $\mathbb{A} = \langle H(\kappa), \in, \mathscr{P}, R \rangle$ is admissible. Then $\mathbb{A}$ is Σ_1 compact.*

Proof. This is immediate from 7.2 and 7.3. □

Exercise 7.10 shows that $\mathbb{A}$ is also Σ_1 complete. Exercise 7.11 shows that it satisfies K_3.

The urelement versions of 7.3 and 7.4 are not very interesting since 7.3 only goes through for $\langle \mathfrak{M}; H(\kappa)_{\mathfrak{M}}, \in, \mathscr{P} \rangle$ when $\mathrm{card}(\mathfrak{M}) < \kappa$, in which case $\mathfrak{M}$ is already contained in $H(\kappa)$, up to isomorphism.

We will return briefly to the notion of tree in § 9. Now we go on to discuss two rather different examples of Σ_1 compact admissible sets.

The following theorem of Nyberg gives quite concrete examples of Σ_1 compact and Σ_1 complete admissible sets.

7.5 Theorem. *Let α be a limit ordinal of cofinality ω, let $\mathbb{A}$ be of the form $\langle H(\beth_\alpha), \in, R \rangle$ and let $\mathbb{B}$ be admissible with $\mathbb{A} \in \mathbb{B}$, $\mathbb{B}$ projectible into $\mathbb{A}$. Then $\mathbb{B}$ satisfies $s\text{-}\Pi_1^1 = \Sigma_1$ uniformly and hence is Σ_1 complete and Σ_1 compact. ($\mathbb{A}$ is not necessarily admissible.)*

The proof of this theorem is sketched in Exercise 7.16. Note that it applies to $\mathbb{H}\mathrm{YP}(H(\beth_\alpha))$ whenever $\mathrm{cf}(\alpha) = \omega$. This is a resolvable admissible set satisfying $s\text{-}\Pi_1^1 = \Sigma_1$ uniformly (and hence the theorems of the previous sections). On the other hand, if $\mathrm{cf}(\alpha) > \omega$ then $\mathbb{H}\mathrm{YP}(H(\beth_\alpha))$ is strongly self-definable, hence not Σ_1 complete or Σ_1 compact.

We conclude this section with a different kind of example.

A structure $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ is *recursively Σ_1^1 saturated* iff for every finite expansion $\mathsf{L}' = \mathsf{L}(\mathsf{S}_1, \ldots, \mathsf{S}_k)$ of L and every recursive (equivalently, r.e.) set $\Phi(v_1, \ldots, v_n, \mathsf{S}_1, \ldots, \mathsf{S}_k)$ of formulas of $\mathsf{L}'_{\omega\omega}$, $\mathfrak{M}$ is a model of:

$$\forall \vec{v}\, [\textstyle\bigwedge_{\Phi_0 \in S_\omega(\Phi)} \exists \mathsf{S}_1, \ldots, \mathsf{S}_k \bigwedge \Phi_0(\vec{v}, \vec{\mathsf{S}}) \to \exists \mathsf{S}_1, \ldots, \mathsf{S}_k \bigwedge \Phi(\vec{v}, \vec{\mathsf{S}})].$$

It is easy to see that every recursively Σ_1^1 saturated structure is recursively saturated. Theorem IV.5.7 shows that if $\mathfrak{M}$ is countable and recursively saturated then it is recursively Σ_1^1 saturated. The following theorem characterizes the recursively Σ_1^1 saturated structures among the class of recursively saturated structures.

7.6 Theorem. *Let $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ be an infinite recursively saturated structure of any cardinality. Then $\mathfrak{M}$ is recursively Σ_1^1 saturated iff $\mathbb{HYP}_{\mathfrak{M}}$ is Σ_1 compact.*

Proof. The "if" part of the theorem is established by the very proof of Theorem IV.5.7. To prove the converse, we assume that $\mathfrak{M}$ is recursively Σ_1^1 saturated and prove that $\mathbb{HYP}_{\mathfrak{M}}$ satisfies s-Π_1^1 Reflection. Suppose

$$(5) \qquad \mathbb{HYP}_{\mathfrak{M}} \models \forall \mathsf{R}\, \varphi(\mathsf{R},\vec{x})$$

where φ is a Σ_1 formula. Since $\mathbb{HYP}_{\mathfrak{M}}=\mathrm{L}(\mathfrak{M},\omega)$, we need to exclude the possibility that for every $n<\omega$

$$(6)_n \qquad \mathrm{L}(\mathfrak{M},n)\models \exists \mathsf{R}\, \neg\varphi(\mathsf{R},\vec{x}).$$

Since each $x\in\mathbb{HYP}_{\mathfrak{M}}$ has a good Σ_1 definition in terms of parameters from $M\cup\{M\}$, we may assume that each x_i in the sequence x is either in M or is M itself. Let us rewrite (5) as

$$(7) \qquad \mathrm{L}(\mathfrak{M},\omega)\models \forall \mathsf{R}\, \exists y\, \psi(\mathsf{R},p,y,M)$$

where ψ is a Δ_0 formula with no other parameters. We can rewrite (6) as: for every $n<\omega$

$$(8)_n \qquad \mathrm{L}(\mathfrak{M},\omega)\models \exists \mathsf{R}\, \forall y\in \mathrm{L}(M,n)\, \neg\psi(\mathsf{R},p,y,M).$$

Let $\Phi(p)$ be the set of formulas in $\mathsf{L}(\mathsf{U},\mathsf{A},\mathsf{E},\mathsf{F},\mathsf{R},\mathsf{R}_1',\ldots,\mathsf{R}_l')$ which express the following about $\mathfrak{M},p$:

KPU$^+$ relativized to U (for urelements), S (for sets), E (for $\in$),
"$\mathsf{F}:\langle M,\mathsf{R}_1,\ldots,\mathsf{R}_l\rangle\cong\langle \mathsf{U},\mathsf{R}_1',\ldots,\mathsf{R}_l'\rangle$",
$\forall y\in \mathrm{L}(\mathsf{U},n)\neg\psi(\mathsf{R},\mathsf{F}(p),y,\mathsf{U})$ (for all $n<\omega$).

Every finite subset $\Phi_0(p)$ of $\Phi(p)$ is satisfiable on $\mathfrak{M}$ by choosing relations which code up $\mathbb{HYP}_{\mathfrak{M}}$ on $\mathfrak{M}$ itself and using $(8)_n$ to satisfy the last sentence in $\Phi_0(p)$. Since $\mathfrak{M}$ is recursively Σ_1^1 saturated there are relations on $\mathfrak{M}$ which make the whole set $\Phi(p)$ true:

$$(9) \qquad \langle M,R_1,\ldots,R_l,\,U,R_1',\ldots,R_l',\,F,A,E,R\rangle\models\Phi(p).$$

But then $(\langle U,R_1',\ldots,R'\rangle; A,E,R)$ is isomorphic to $(\mathfrak{A}_{\mathfrak{M}},R)$ for some $\mathfrak{A}_{\mathfrak{M}}\supseteq_{\mathrm{end}}\mathbb{HYP}_{\mathfrak{M}}$.

Let $R^* = R\restriction \mathbb{H}\mathrm{YP}_{\mathfrak{M}}$. By (7) there is a $y \in \mathbb{H}\mathrm{YP}_{\mathfrak{M}}$ such that

$$(\mathrm{L}(\mathfrak{M},\omega), R^*) \models \psi(R,p,y,M).$$

But $y \in \mathrm{L}(M,n)$ for some $n<\omega$. Since ψ is Δ_0 and

$$(\mathbb{H}\mathrm{YP}_{\mathfrak{M}}, R^*) \subseteq_{\mathrm{end}} (\mathfrak{A}_{\mathfrak{M}}, R),$$

we have

$$(\mathfrak{A}_{\mathfrak{M}}, R) \models y \in \mathrm{L}(M,n) \wedge \psi(\mathrm{R},p,y,M)$$

contradicting (9), since (9) asserts, among other things, that

$$(\mathfrak{A}_{\mathfrak{M}}, R) \models \forall y \in \mathrm{L}(M,n) \neg\psi(\mathrm{R},p,y,M). \quad \Box$$

To see that this result gives us lots of uncountable Σ_1 compact sets, we must know that there are lots of recursively Σ_1^1 saturated models. We assume the reader is familiar with saturated or special models, referring him to Chang-Keisler [1973] for the relevant definitions and properties.

7.7 Proposition. *Every saturated (or even every special) model* $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ *is recursively* Σ_1^1 *saturated.*

Proof. If we assume the GCH we can get rid of the requirement that the set of formulas is recursive; the proof not involving the GCH is sketched in Exercise 7.17. Let $\mathfrak{M}$ be a special model and let $\Phi(\bar{p},S)$ be a set of sentences such that for each finite $\Phi_0 \subseteq \Phi$,

$$(\mathfrak{M},p) \models \exists \mathrm{S} \bigwedge \Phi_0(\bar{\mathrm{p}},\mathrm{S}).$$

Then the first order theory $Th(\mathfrak{M},p) \cup \Phi(\bar{\mathrm{p}},\mathrm{S})$ is consistent and so has a special model $(\mathfrak{M}',p',S')$ of power $\mathrm{card}(\mathfrak{M})$, by the GCH.

But then $(\mathfrak{M},p) \equiv (\mathfrak{M}',p')$ $(\mathrm{L}_{\omega\omega})$, and both models are special so

$$(\mathfrak{M},p) \cong (\mathfrak{M}',p').$$

Hence

$$(\mathfrak{M},p) \models \exists \mathrm{S} \bigwedge \Phi(\bar{p},\mathrm{S}). \quad \Box$$

7.8—7.19 Exercises

7.8. Prove that the pure admissible set $\mathbb{A}$ is supertransitive iff $\mathbb{A} = H(\kappa)$ for some cardinal κ.

7.9. Prove the following: *Let* $\mathbb{A}$ *be pure, admissible, supertransitive and* Σ_1 *compact. There is a cardinal* $\kappa = \beth_\kappa$ *such that* $\mathbb{A} = H(\kappa)$. *Let* $\mathbb{A}' = (\mathbb{A}, \mathscr{P})$. *Then* $\mathbb{A}'$ *is admissible and satisfies* s-$\Pi_1^1 = \Sigma_1$ *uniformly.* More slowly, prove:

(i) $\mathbb{A}$ is closed under $\mathscr{P}$, using s-Π_1^1 Reflection.
(ii) $\mathbb{A}'$ satisfies s-Π_1^1 Reflection ($\mathscr{P}$ is s-Δ_1^1 on $\mathbb{A}$).
(iii) $\mathbb{A}'$ is admissible (using (ii)).
(iv) $\mathbb{A}'$ satisfies s-$\Pi_1^1=\Sigma_1$.

7.10. Prove that the following are equivalent, where κ is a strong limit cardinal and $\mathbb{A}=\langle H(\kappa),\in,\mathscr{P},R\rangle$ is admissible:
(i) $\mathbb{A}$ is Σ_1 compact (s-Π_1^1 Reflection),
(ii) $\mathbb{A}$ is Σ_1 complete (s-$\Pi_1^1=\Sigma_1$),
(iii) Every $\mathbb{A}$-tree has a branch.

7.11. Let $\mathbb{A}=\langle H(\kappa),\in,R\rangle$ be Σ_1 compact. Prove that $h_\Sigma(\mathbb{A})=\kappa$. [Use 7.9 and s-Π_1^1 Reflection plus trivial cardinality considerations.]

7.12. Let $\lambda=\text{card}(\mathfrak{M})$ and let κ be a limit ordinal. Prove that the following are equivalent:
(i) $(\mathfrak{M}; \mathrm{V}_{\mathfrak{M}}(\kappa),\in)$ is admissible,
(ii) $\kappa=\beth_\kappa(\lambda)$,
(iii) κ is a cardinal and $\mathrm{V}_{\mathfrak{M}}(\kappa)=H(\kappa)_{\mathfrak{M}}$.

7.13. Prove in ZFC that there are arbitrarily large cardinals $\kappa=\beth_\kappa$ of cofinality ω such that $\langle H(\kappa),\in,\mathscr{P}\rangle$ is admissible.

7.14. Let κ be the Hanf number of second order logic. Show that $\langle H(\kappa),\in,\mathscr{P}\rangle$ satisfies the hypothesis of 7.4.

7.15. Let α be a limit ordinal, let $\mathbb{A}$ be admissible and let $\mathrm{V}(\alpha)\in\mathbb{A}$. Prove that $H(\beth_\alpha)\in\mathbb{A}$. [Consider the set $X=\{E\in\mathrm{V}(\alpha)\}$: E is well-founded$\}$.]

7.16. Theorem 7.5 follows from the following result of Nyberg. Prove that if $\mathfrak{M}$ is a uniform Kleene structure and $\mathbb{A}_{\mathfrak{M}}$ is admissible above $\mathfrak{M}$ and projectible into $\mathfrak{M}$ then $\mathbb{A}_{\mathfrak{M}}$ satisfies s-$\Pi_1^1=\Sigma_1$ uniformly. [Use the alternate form of "s-$\Pi_1^1=\Sigma_1$ uniformly" given in Exercise 4.21.]

7.17. A structure $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ is resplendent if for every finitary Σ_1^1 sentence $\exists \mathrm{S}\,\varphi(\mathrm{S})$ with constants from $\mathfrak{M}$, if $\mathfrak{N}\models\exists\mathrm{S}\,\varphi(\mathrm{S})$ for some $\mathfrak{N}\succ\mathfrak{M}$, then $\mathfrak{M}\models\exists\mathrm{S}\,\varphi(\mathrm{S})$.

(i) Prove that every special model is ω-resplendent (Kueker [1971]).

(ii) Prove that every resplendent model is recursively Σ_1^1 saturated. [Use the techniques of IV.2.]

(iii) Associate with any finitary Σ_1^1 formula $\Phi(x)$ a recursive closed game formula $\mathscr{G}_\Phi(x)$ such that

$$\mathfrak{M}\models\forall x\,(\Phi(x)\rightarrow\mathscr{G}_\Phi(x))$$

for all $\mathfrak{M}$ and, for $\mathfrak{M}$ countable,

$$(10) \qquad \mathfrak{M} \models \forall x\,(\mathcal{G}_{\Phi}(x) \leftrightarrow \Phi(x)).$$

Such a $\mathcal{G}_{\Phi}$ is given by (the proof of) Svenonius's Theorem. Prove that if $\mathfrak{M}$ is recursively saturated then $\mathfrak{M}$ is resplendent iff

$$\mathfrak{M} \models \forall x\,(\mathcal{G}_{\Phi}(x) \leftrightarrow \Phi(x))$$

for all Σ_1^1 formulas Φ.

(iv) Prove that if $\mathfrak{M}$ is resplendent then Π_1^1 on $\mathfrak{M} = \Sigma_1$ on $\mathbb{HYP}_{\mathfrak{M}}$.

(v) (Schlipf). Improve (iv) by showing that if $\mathfrak{M}$ is resplendent then $\mathbb{HYP}_{\mathfrak{M}}$ satisfies K_1.

7.18. (Open). Characterize those $\mathfrak{M}$ such that $\mathbb{HYP}_{\mathfrak{M}}$ is Σ_1 compact.

7.19. (Open). Characterize those $\mathfrak{M}$ such that $\mathbb{HYP}_{\mathfrak{M}}$ is Σ_1 complete.

7.20 Notes. Theorem 7.4 is due to Barwise [1968] and, independently, and by a completely different proof, to Karp [1968]. Theorem 7.3 is a refinement of a classical result about weakly compact cardinals, contained in Theorem 9.10.

8. Σ_1 Compact Sets of Cofinality Greater than ω

In this section we prove an existence theorem which shows that there are many Σ_1 compact admissible sets besides those exhibited in the previous section. In particular, we prove the existence of essentially uncountable Σ_1 compact admissible sets.

Let κ be an uncountable regular cardinal. A subset C of κ is *closed in* κ if for each initial segment C_0 of C,

$$(\sup C_0) < \kappa \quad \text{implies} \quad (\sup C_0) \in C.$$

This says that C is closed in the order topology on κ. C is *unbounded in* κ if

$$\forall \beta < \kappa\, \exists \gamma \in C\, (\beta < \gamma).$$

A set C is *c.u.b. in* κ if $C \subseteq \kappa$ and C is closed and unbounded in κ.

8.1 Lemma. *Let $\kappa > \omega$ be regular. If C_0, C_1 are c.u.b. in κ then so is $C_0 \cap C_1$. In particular, $C_0 \cap C_1$ is nonempty.*

Proof. The intersection $C_0 \cap C_1$ is closed since the intersection of closed sets is closed. To see that $C_0 \cap C_1$ is unbounded, let $\beta < \kappa$ be given. Let $\gamma_1 > \beta$ be in C_1. Let $\gamma_2 > \gamma_1$ be in C_0. Let $\gamma_3 > \gamma_2$ be in C_1 and so on for each $n<\omega$. Then $\gamma = \sup_{n<\omega} \gamma_n$ is less than κ since κ is regular. Since $\gamma = \sup_{n<\omega} \gamma_{2n}$ and C_0 is closed, $\gamma \in C_0$. Since $\gamma = \sup_{n<\omega} \gamma_{2n+1}$ and C_1 is closed, $\gamma \in C_1$. Thus $\beta < \gamma$ and $\gamma \in (C_0 \cap C_1)$. ☐

Thus, by 8.1, the collection

$$\mathfrak{F} = \{C \subseteq \kappa : C_0 \subseteq C \text{ for some } C_0 \text{ c.u.b. in } \kappa\}$$

defines a filter on the subsets of κ, called the c.u.b. filter on κ. We say that $P(\alpha)$ *holds for almost all* $\alpha < \kappa$ if

$$\{\alpha < \kappa : P(\alpha)\}$$

is a member of the c.u.b. filter on κ.

8.2 Lemma. *Let λ, κ be regular cardinals, $\omega \leqslant \lambda < \kappa$. If $P(\alpha)$ holds for almost all $\alpha < \kappa$ then $P(\alpha)$ holds for some α with $\mathrm{cf}(\alpha) = \lambda$.*

Proof. Let C be c.u.b. in κ be a subset of

$$\{\alpha < \kappa : P(\alpha)\}.$$

Let γ be the λ^{th} member of C, enumerated in the natural order. There is such a λ^{th} member since κ is regular and C is unbounded in κ. It is clear that $\mathrm{cf}(\gamma) = \mathrm{cf}(\lambda) = \lambda$ since λ is regular. ☐

In reading the next theorem, the student should think of J_α as $H(\aleph_\alpha)$ or $\mathrm{L}(\omega\alpha)$ or $\mathrm{L}(a, \omega\alpha)$, since these are the usual applications.

8.3 Theorem. *Let κ be an uncountable regular cardinal, let $R \subseteq H(\kappa)$ and let $J: \kappa \to H(\kappa)$ have the following properties:*

- (i) *J_α is transitive and closed under pairs and union, for all $\alpha < \kappa$;*
- (ii) *$\alpha < \beta < \kappa$ implies $J_\alpha \in J_\beta$;*
- (iii) *if $\lambda < \kappa$ is a limit ordinal then $J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha$; and*
- (iv) *for each $\alpha < \kappa$, the structure*

$$\mathbb{J}_\alpha = \langle J_\alpha, \in, R \cap J_\alpha \rangle$$

satisfies Δ_0 Separation. Then, for almost all $\alpha < \kappa$, $\mathbb{J}_\alpha$ is a Σ_1 compact admissible set.

Proof. The idea for this proof goes back to the notion of stable ordinal. For the purposes of this proof we call an ordinal α *β-superstable* if $\alpha < \beta < \kappa$ and for every s-Π_1^1 formula $\Phi(v_1, \ldots, v_n)$ and every $a_1, \ldots, a_n \in J_\alpha$,

$$\text{if } \mathbb{J}_\beta \models \Phi(a_1, \ldots, a_n) \text{ then } \mathbb{J}_\alpha \models \Phi(a_1, \ldots, a_n).$$

We first prove:

(1) if α is β-superstable then $\mathbb{J}_\alpha$ is a Σ_1 compact admissible set.

So suppose α is β-superstable. Since Δ_0 Collection follows from s-Π_1^1 Reflection (in fact from Σ Reflection) it suffices to prove that $\mathbb{J}_\alpha$ satisfies s-Π_1^1 Reflection. Let $\Phi(a_1,\ldots,a_n)$ be a s-Π_1^1 formula true in J_α.

Then

$$\mathbb{J}_\beta \models \Phi^{(J_\alpha)}(a_1,\ldots,a_n)$$

and hence $\mathbb{J}_\beta$ is a model of the s-Π_1^1 formula $\Psi(a_1,\ldots,a_n)$

$$\exists b\,[\mathrm{Tran}(b) \wedge a_1,\ldots,a_n \in b \wedge \Phi^{(b)}(a_1,\ldots,a_n)]$$

since $J_\alpha \in J_\beta$. But then by superstability, $\mathbb{J}_\alpha \models \Psi(a_1,\ldots,a_n)$, so $\mathbb{J}_\alpha$ satisfies s-Π_1^1 Reflection, proving (1).

We will prove the theorem by proving that almost every $\alpha<\kappa$ is β-superstable for every $\beta, \alpha<\beta<\kappa$. To prove this we use normal functions. (A function $f\colon \kappa \to \kappa$ is *normal* if f is increasing $(\alpha<\beta<\kappa \Rightarrow f(\alpha)<f(\beta))$ and continuous (λ a limit $<\kappa \Rightarrow f(\lambda)=\sup\{f(\alpha)\colon \alpha<\lambda\}$). If $f\colon \kappa\to\kappa$ is normal then the set of fixed points of f,

$$\{\alpha<\kappa\colon f(\alpha)=\alpha\},$$

is always c.u.b. in κ, as is easily seen.) We define a normal function f such that $f(\alpha)=\alpha$ implies α is β-superstable for all β between α and κ. This will prove the theorem. Let $P(\alpha,\beta)$ be the following condition on $\alpha,\beta<\kappa$:

> for all $\beta', \beta \leqslant \beta' < \kappa$, and for all s-Π_1^1 sentences $\Phi(a_1,\ldots\ldots,a_n)$ with constants from J_α, if $\mathbb{J}_{\beta'} \models \Phi(a_1,\ldots,a_n)$ then $\mathbb{J}_\beta \models \Phi(a_1,\ldots,a_n)$.

Note that $P(\alpha,\beta_0)$ implies $P(\alpha,\beta_1)$ for all β_1 between β_0 and κ. Since $\mathrm{card}(J_\alpha)<\kappa$ there are $<\kappa$ s-Π_1^1 formulas $\Phi(\vec{a})$ so a trivial cardinality argument proves that $\forall\alpha<\kappa\,\exists\beta<\kappa\,P(\alpha,\beta)$. Now define f by

$$f(\alpha) = \text{least}\,\beta\,[\beta > f(\gamma) \text{ for all } \gamma<\alpha, \text{ and } P(\alpha,\beta)].$$

Since κ is regular, $f(\alpha)$ is defined for all $\alpha<\kappa$. Thus $f\colon\kappa\to\kappa$ and f is increasing by definition. Let us prove that f is continuous. Let $\lambda<\kappa$ be a limit ordinal. Let $\beta=\sup\{f(\alpha)\colon\alpha<\lambda\}$. We need to verify $P(\lambda,\beta)$. Thus let $\beta'\geqslant\beta$ and let Φ be a s-Π_1^1 sentence with parameters from J_λ which is true in $J_{\beta'}$. We must see that Φ is true in J_β. But Φ is defined in J_α for some $\alpha<\lambda$ so Φ is true in $J_{f(\alpha)}$ and hence in J_β by persistence of s-Π_1^1 formulas. Thus f is normal.

Now suppose $f(\alpha)=\alpha$. Then $P(\alpha,\alpha)$ holds so α is β'-stable for all $\beta'>\alpha$, $\beta'<\kappa$. By (1) this shows that almost every $\alpha<\kappa$ has $\mathbb{J}_\alpha$ Σ_1 compact. $\square$

8.4 Corollary. *Let $\kappa>\omega$ be regular. Then for almost all $\alpha<\kappa$,* $\mathrm{L}(\alpha)$ *is a Σ_1 compact admissible set.*

Proof. Apply 8.3 with $J_\alpha=\mathrm{L}(\omega\alpha)$. Then for almost all $\alpha<\kappa$, $\mathrm{L}(\omega\alpha)$ is Σ_1 compact. But $\omega\alpha=\alpha$ for almost all $\alpha<\kappa$ since $f(\alpha)=\omega\alpha$ is a normal function. ☐

8.5 Corollary. *Let $\kappa>\omega$ be regular and let $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ be a structure of power less than κ. Then for almost all $\alpha<\kappa$,* $\mathrm{L}(\mathfrak{M},\alpha)$ *is Σ_1 compact.*

Proof. Similar to 8.4. Since there is isomorphic copy of $\mathfrak{M}$ in $H(\kappa)$. ☐

The next result gives us essentially uncountable Σ_1 compact admissible sets, when one applies Lemma 8.2 and the observation that $H(\kappa)$ is essentially uncountable iff $\mathrm{cf}(\kappa)>\omega$. $\mathscr{P}$ denotes the power set operation (restricted to $H(\lambda)$ in 8.6).

8.6 Theorem. *Let κ be inaccessible, $\kappa>\omega$. Let $R\subseteq H(\kappa)$. Then for almost all $\lambda<\kappa$, $\langle H(\lambda),\in,\mathscr{P},R\cap H(\lambda)\rangle$ is Σ_1 compact.*

Proof. Let $J_\alpha=H(\beth_\alpha)$. Then $J:\kappa\to H(\kappa)$ since $\kappa=\beth_\kappa$, and $\mathrm{card}(H(\beth_\alpha)\leqslant\beth_{\alpha+1}<\kappa$. Thus, for almost all $\alpha<\kappa$, $\langle H(\beth_\alpha),\in,\mathscr{P},R\cap H(\beth_\alpha)\rangle$ is Σ_1 compact. But $f(\alpha)=\beth_\alpha$ is a normal function so almost all $\alpha<\kappa$ have $\beth_\alpha=\alpha$. Thus almost all $\lambda<\kappa$ have

$$\langle H(\lambda),\in,\mathscr{P},R\cap H(\lambda)\rangle$$

Σ_1 compact. ☐

We can reinterpret all of the above by thinking of the class of all ordinals as an inaccessible cardinal. We can restate Theorem 8.6 in this case as a result in ZFC.

8.7 Corollary. *Let R be any class. The class of λ such that $\langle H(\lambda),\in,\mathscr{P},R\cap H(\lambda))$ is Σ_1 compact contains a closed proper class of cardinals. Hence for any regular κ there are arbitrarily large such λ's of cofinality κ.*

Proof. The last sentence follows from 8.2. ☐

A cardinal κ is a *Mahlo cardinal* if every c.u.b. set $C\subseteq\kappa$ contains an inaccessible cardinal (and hence contains κ such inaccessible cardinals $\lambda<\kappa$).

8.8 Corollary. *Let κ be a Mahlo cardinal and let $R\subseteq H(\kappa)$. There are κ inaccessible cardinals $\lambda<\kappa$ such that $\langle H(\lambda),\in,\mathscr{P},R\cap H(\lambda)\rangle$ is Σ_1 compact.*

Proof. Immediate from 8.6. ☐

8.9 Exercise. Suppose $\langle H(\kappa),\in\rangle$ is Σ_1 compact. Prove that κ is not the first inaccessible. Prove, in fact, that if κ is inaccessible then κ is the κ^{th} inaccessible. [Use s-Π_1^1 Reflection.]

8.10 Notes. Theorem 8.3 is contained in Barwise [1969b].

9. Weakly Compact Cardinals

In this final section we consider weakly compact cardinals and their relationship to Σ_1 compact admissible sets.

Let L be a language with $\leqslant\kappa$ symbols coded as a Δ_1 subset of $H(\kappa)$. The language $\mathsf{L}_{\kappa\omega}$ consists of those $\varphi\in\mathsf{L}_{\infty\omega}$ with less than κ subformulas.

9.1 Definition. A cardinal $\kappa\geqslant\omega$ is *weakly compact* (*for* $\mathsf{L}_{\kappa\omega}$) if for every set $T\subseteq H(\kappa)$ of sentences of $\mathsf{L}_{\kappa\omega}$, if every subset $T_0\subseteq T$ of power $<\kappa$ has a model then T has a model.

This definition is usually expressed in terms of a stronger language $\mathsf{L}_{\kappa\kappa}$ (defined in Exercise 9.14) and it is usually assumed that κ is inaccessible in which case $H(\kappa)$ has power κ and hence T has power $\leqslant\kappa$. We will see that both of these apparent strengthenings follow from Definition 9.1. Note that ω is weakly compact.

9.2 Lemma. *Let* $\kappa\geqslant\omega$ *be a cardinal.*

(i) $\mathsf{L}_{\kappa\omega}=\mathsf{L}_{\infty\omega}\cap H(\kappa)$.

(ii) *If* κ *is regular then* $\mathsf{L}_{\kappa\omega}$ *is the least subset of* $\mathsf{L}_{\infty\omega}$ *containing* $\mathsf{L}_{\omega\omega}$ *closed under* $\neg$, $\forall$, $\exists$ *and*

$$\textit{if}\quad \Phi\subseteq\mathsf{L}_{\kappa\omega}\quad \textit{and}\quad \operatorname{card}(\Phi)<\kappa\quad \textit{then}\quad \bigwedge\Phi\quad \textit{and}\quad \bigvee\Phi\in\mathsf{L}_{\kappa\omega}.$$

(iii) *If* $\kappa>\omega$ *is a limit cardinal then*

$$\mathsf{L}_{\kappa\omega}=\bigcup_{\lambda<\kappa}\mathsf{L}_{\lambda\omega}$$

where the union is over all infinite cardinals $\lambda<\kappa$.

(iv) κ *is weakly compact iff* $\langle H(\kappa),\in,R\rangle$ *is* Σ_1 *compact for every relation* $R\subseteq H(\kappa)$.

Proof. (i), (iii) and (iv) are immediate from the definitions. To prove (ii) let $\mathsf{L}'_{\kappa\omega}$ be the least class described. It is clear that $\mathsf{L}_{\kappa\omega}\subseteq\mathsf{L}'_{\kappa\omega}$. To prove $\mathsf{L}_{\kappa\omega}=\mathsf{L}'_{\kappa\omega}$ it suffices to prove that $\mathsf{L}_{\kappa\omega}$ is closed under $\neg$, $\forall$, $\exists$ and the clause

$$\text{if}\quad \Phi\subseteq\mathsf{L}_{\kappa\omega}\quad \text{and}\quad \operatorname{card}(\Phi)<\kappa\quad \text{then}\quad \bigwedge\Phi,\ \bigvee\Phi\in\mathsf{L}_{\kappa\omega}.$$

The first part is trivial. So suppose $\Phi\subseteq\mathsf{L}_{\kappa\omega}$ and $\operatorname{card}(\Phi)<\kappa$. We must verify that

$$\operatorname{card}(\operatorname{sub}(\bigwedge\Phi))<\kappa.$$

But

$$\operatorname{sub}(\bigwedge\Phi)=\{\bigwedge\Phi\}\cup\bigcup\{\operatorname{sub}(\varphi)\colon\varphi\in\Phi\}.$$

Since $\Phi\subseteq\mathsf{L}_{\kappa\omega}$ each $\operatorname{sub}(\varphi)$ has power $<\kappa$ for $\varphi\in\Phi$. But $\operatorname{card}(\Phi)<\kappa$ and κ is regular so $\operatorname{card}(\bigwedge\Phi)<\kappa$. Similarly, $\operatorname{card}(\bigvee\Phi)<\kappa$. □

Part (iv) of this lemma shows that the notion of weakly compact cardinal is just the relativization of the concept of Σ_1 compact admissible set to an arbitrary $R \subseteq H(\kappa)$.

Before we see just how strong the assumption that κ is weakly compact and uncountable is, let us stop to examine the plausibility of the existence of such cardinals. We want to show that the same kind of intuition which prompts one to admit ω, inaccessible cardinals and Mahlo cardinals as legitimate abstract objects also prompts one to admit weakly compact cardinals as legitimate objects in the hierarchy of sets.

There was a time when the existence of ω was considered problematic. One must accept each natural number, but it took years for the limit, the set of natural numbers, to be accepted as a legitimate abstract object, suitable for use in mathematics.

Once one accepts the basic principles of set theory, one sees how to generate many cardinal numbers, which must be accepted. Only fairly recently have inaccessible cardinals begun to be considered as the natural limit of the accessible cardinals and hence suitable for use in mathematics.

We saw in Corollary 8.7 that for any class R, almost all cardinals κ have the property that $\langle H(\kappa), \in, R \cap H(\kappa)\rangle$ is Σ_1 compact. Given any collection $\mathscr{R}$ of classes that can be coded by a single class, we see that almost all κ are such that $\langle H(\kappa), \in, R \cap H(\kappa)\rangle$ is Σ_1 compact for all $R \in \mathscr{R}$. A natural limiting assumption is that $\langle H(\kappa), \in, R\rangle$ should by Σ_1 compact for *all* $R \subseteq H(\kappa)$. This is the assumption that κ is weakly compact.

(Another argument that is often given for the existence of weakly compact cardinals, as well as measurable cardinals and strongly compact cardinals, cardinals we can see no argument for at all, is that they should exist "by analogy with ω". This seems like a very weak argument. The results of § 7 suggest that the crucial property of $\kappa = \omega$ for compactness is that $\mathrm{cf}(\kappa) = \omega$, whereas weakly compact cardinals are always inaccessible and hence regular. Of course ω is the only regular cardinal κ with $\mathrm{cf}(\kappa) = \omega$.)

Call κ a Σ_1 *compact cardinal* if $\langle H(\kappa), \in\rangle$ is Σ_1 compact. Call κ a $\Sigma_1(R)$ *compact cardinal* if $\langle H(\kappa), \in, R\rangle$ is Σ_1 compact. Thus κ is weakly compact iff κ is $\Sigma_1(R)$ compact for every $R \subseteq H(\kappa)$. We remind the reader once again that $\omega = \beth_\omega$.

9.3 Proposition. *Let* $\kappa \geqslant \omega$.

(i) *If* κ *is* Σ_1 *compact then* $\kappa = \beth_\kappa$.

(ii) *If* κ *is weakly compact then* κ *is inaccessible.*

Proof. Part (i) is a small part of Exercise 7.9 but we include its proof for completeness sake. Suppose κ is Σ_1 compact. We will first prove that

(1) $H(\kappa)$ is closed under the power set.

Suppose $a \in H(\kappa)$. Then $H(\kappa)$ satisfies the s-Π^1_1 formula

$$\forall U\, \exists b\, [b \subseteq a \wedge \forall x \in a\, (x \in b \leftrightarrow U(x))].$$

By s-Π_1^1 Reflection, $\mathscr{P}(a) \subseteq c$ for some $c \in H(\kappa)$ so $\mathscr{P}(a) \in H(\kappa)$. For (1) we see that $\kappa = \beth_\alpha$ for some limit ordinal α. Suppose $\alpha < \kappa$. Then $H(\kappa)$ satisfies the s-Π_1^1 formula expressing:

$$\begin{aligned} \forall \beta < \alpha\, \exists f\, [\mathrm{fun}(f) \wedge \mathrm{dom}(f) = \beta + 1& \\ f(0) = 0& \\ f(\gamma+1) = \mathscr{P}(f(\gamma)) \quad &\text{for} \quad \gamma < \beta \\ f(\lambda) = \textstyle\bigcup_{\alpha<\lambda} f(\alpha) \quad &\text{for limit} \quad \lambda \leqslant \beta]. \end{aligned}$$

Then s-Π_1^1 Reflection gives a contradiction since one would have an $a \in H(\kappa)$ such that $\mathrm{V}(\alpha) \subseteq a$. This proves (i). To prove (ii) we need only see that κ is regular. Suppose $f: \alpha \to \kappa$ where $\alpha < \kappa$ and $\kappa = \sup\{f(\beta): \beta < \alpha\}$. We claim that $\langle H(\kappa), \in, f \rangle$ does not satisfy s-Π_1^1 Reflection. In fact it does not even satisfy Σ Reflection and hence is not admissible, since it satisfies the Σ formula

$$\forall \beta < \alpha\, \exists \gamma\, (f(\beta) = \gamma)$$

but there can be no bound $\xi < \kappa$ for the ordinals γ. □

There are many characterizations of the class of weakly compact cardinals which fall out of our study. An admissible set $\mathbb{A}$ is *strict-*Π_1^1 *indescribable* if $(\mathbb{A}, R)$ satisfies s-Π_1^1 Reflection for every $R \subseteq \mathbb{A}$. κ is s-Π_1^1 *indescribable* iff $\langle H(\kappa), \in \rangle$ is s-Π_1^1 indescribable.

9.4 Theorem. *An infinite cardinal* κ *is weakly compact iff it is strict-*Π_1^1 *indescribable.*

Proof. Immediate from Theorem 4.7. □

An admissible set $\mathbb{A}$ satisfies Π_1^1 *Reflection* if for every Π_1^1 formula $\Phi(x_1, \ldots, x_n)$, $\mathbb{A}$ satisfies

$$\Phi(\vec{x}) \to \exists a\, [\mathrm{Tran}(a) \wedge x_1, \ldots, x_n \in a \wedge \Phi^{(a)}(\vec{x})].$$

$\mathbb{A}$ is Π_1^1 *indescribable* if $(\mathbb{A}, R)$ satisfies Π_1^1 Reflection for every $R \subseteq \mathbb{A}$. κ is Π_1^1 indescribable iff $\langle H(\kappa), \in \rangle$ is Π_1^1 indescribable. $\mathbb{HF}$ does not satisfy Π_1^1 Reflection or, for that matter, Π_2^0 Reflection since

$$\mathbb{HF} \models \forall x\, \exists y \quad (x \in y)$$

but no finite set can satisfy this sentence. Thus ω is certainly not Π_1^1 indescribable. We will see, however, that for κ with $\mathrm{cf}(\kappa) > \omega$, s-Π_1^1 Reflection implies Π_1^1 Reflection and s-Π_1^1 indescribability implies Π_1^1 indescribability. The secret to understanding this and a number of other facts is contained in the following surprising result.

Let the language L (of $\mathsf{L}_{\infty\omega}$) contain a distinguished binary relation symbol E. A *well-founded* L*-structure* is an L-structure $\mathfrak{M}$ with $\mathsf{E}^{\mathfrak{M}}$ well-founded.

9.5 Theorem. *Let* $\mathbb{A}$ *be an essentially uncountable* Σ_1 *compact admissible set. Let* T *be a* Σ_1 *theory of* $\mathsf{L}_{\mathbb{A}}$. *If every* $\mathbb{A}$*-finite* $T_0 \subseteq T$ *has a well-founded model then* T *has a well-founded model.*

Proof. Recall that $\mathbb{A}$ is essentially uncountable iff every countable subset of $\mathbb{A}$ is an element of $\mathbb{A}$. We know that $\mathbb{A}$ satisfies s-Π_1^1 Reflection since $\mathbb{A}$ is Σ_1 compact. The proof of this theorem is exactly like the proof that s-Π_1^1 Reflection implies Σ_1 compactness, once we have the following definitions and lemma. ☐

We may assume that $\mathsf{L}_{\mathbb{A}}$ is a Skolem fragment which is Δ_1 on $\mathbb{A}$. Call an s.v.p. $\mathscr{D}$ for $\mathsf{L}_{\mathbb{A}}$ *well-founded* if there is no infinite sequence $\langle t_n : n<\omega\rangle$ of closed terms of L_A such that $(t_{n+1}\,\mathsf{E}\,t_n)\in\mathscr{D}$ for all $n<\omega$.

9.6 Lemma. *Let* $\mathbb{A}$ *be an essentially uncountable admissible set.*

(i) *There is a* Π *sentence* $\varphi(\mathsf{D})$ *such that for all* $\mathscr{D}\subseteq\mathbb{A}$,

$$(\mathbb{A},\mathscr{D})\models\varphi(\mathsf{D}) \text{ iff } \mathscr{D} \text{ is a well-founded s.v.p. for } \mathsf{L}_{\mathbb{A}}.$$

(ii) *If* $\mathfrak{M}$ *is a well-founded Skolem structure for* $\mathsf{L}_{\mathbb{A}}$ *then the* s.v.p. $\mathscr{D}_{\mathfrak{M}}$ *given by* $\mathfrak{M}$ *is well-founded.*

(iii) *If* $\mathscr{D}$ *is a well-founded* s.v.p. *for* $\mathsf{L}_{\mathbb{A}}$ *then* $\mathscr{D}$ *has a well-founded model.*

Proof. (i) Since $\mathbb{A}$ is essentially uncountable, every sequence $\langle t_n : n<\omega\rangle$ of terms of $\mathsf{L}_{\mathbb{A}}$ is actually an element of $\mathsf{L}_{\mathbb{A}}$. Thus the condition that $\mathscr{D}$ be well-founded is expressed by a universal quantifier over $\mathbb{A}$. The proof of (ii) is trivial. To prove (iii) let $\mathscr{D}$ be a well-founded s.v.p. By the Weak Completeness Theorem, $\mathscr{D}$ has a model $\mathfrak{M}_1$. Let $\mathfrak{M}$ be the smallest submodel of $\mathfrak{M}_1$. Then

$$\mathfrak{M}\prec\mathfrak{M}_1 \quad (\mathsf{L}_{\mathbb{A}}).$$

By Exercise VII.2.14 every element of $\mathfrak{M}$ is denoted by a closed term of $\mathsf{L}_{\mathbb{A}}$. Thus $\mathfrak{M}$ is well-founded and a model of the sentences in $\mathscr{D}$. ☐

This lemma can also be used to prove a completeness theorem. See Exercise 9.11.

Theorem 9.5 explains why none of the explicitly described Σ_1 compact sets given in § 7 were essentially uncountable. The conclusion of Theorem 9.5 is so strong that it makes such sets very hard to find.

Our first use of Theorem 9.5 is to prove the results referred to above.

9.7 Theorem. *Let* κ *be a cardinal with* $\mathrm{cf}(\kappa)>\omega$.

(i) *If* $\langle H(\kappa),\in,R\rangle$ *satisfies* s-Π_1^1 *Reflection then it satisfies* Π_1^1 *Reflection.*

(ii) *If* κ *is* s-Π_1^1 *indescribable then* κ *is* Π_1^1 *indescribable.*

Proof. Part (ii) follows immediately from (i). To prove (i) let $\langle H(\kappa),\in,R\rangle$ satisfy s-Π_1^1 Reflection. By 9.3, $\kappa=\beth_\kappa$. Since $H(\kappa)$ is closed under $\mathscr{P}$, the graph of $\mathscr{P}$ is s-Π_1^1 on $H(\kappa)$ so $\mathbb{A}=\langle H(\kappa),\in,\mathscr{P},R\rangle$ also satisfies s-Π_1^1 Reflection and in particular, is admissible. Thus the definition of $\mathrm{V}(\alpha)$ is $\mathbb{A}$-recursive and $H(\kappa)=\mathrm{V}(\kappa)$. Suppose

$$\langle \mathrm{V}(\kappa),\in,R\rangle \models \forall \mathsf{S}\ \psi(\mathsf{S})$$

where ψ is first order but that for all $\alpha, \alpha_0 \leqslant \alpha < \kappa$,

$$\langle \mathrm{V}(\alpha),\in,R\cap \mathrm{V}(\alpha)\rangle \models \exists \mathsf{S}\, \neg\psi(\mathsf{S})$$

where α_0 is large enough so that all parameters in ψ are in $\mathrm{V}(\alpha_0)$. Let T be the following Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$:

$\mathrm{KP}+\mathrm{Power}$,

Infinitary diagram of $\langle \mathbb{A},\mathscr{P}\rangle$,

"c is an ordinal",

$(\mathsf{c}>\overline{\beta})$ for all $\beta<\kappa=o(\mathbb{A})$,

$\forall\alpha\leqslant \mathsf{c}\ \exists \mathsf{S}\in \mathrm{V}(\alpha+1)\, \neg\psi(\mathsf{S})^{(\mathrm{V}(\alpha))}$.

Every $\mathbb{A}$-finite subset of T has a well-founded model; one simply interprets c as some large $\alpha<\kappa$. By Theorem 9.5, T has a well-founded model $\mathfrak{M}$. Since it is well founded we can assume it is transitive. But then $\mathsf{c}^{\mathfrak{M}}$ is a real ordinal $\beta\geqslant\kappa$ and the last axiom of T implies that there is an $S\subseteq \mathrm{V}(\kappa)$ such that

$$\langle \mathrm{V}(\kappa),\in,R,S\rangle \models \neg\psi(\mathsf{S}). \quad \square$$

Theorem 9.7 is really rather remarkable since if κ is Σ_1 compact then s-$\Pi_1^1=\Sigma_1(\mathscr{P})$ and hence s-$\Pi_1^1\neq\Pi_1^1$.

9.8 Corollary. *If κ is weakly compact and greater than ω then κ is Mahlo.*

Proof. Since κ is weakly compact it is inaccessible. Since $\kappa>\omega$, 9.7 applies so κ is Π_1^1 indescribable. Let $C\subseteq\kappa$ be c.u.b. in κ. We must prove that there is a $\lambda<\kappa$ such that λ is inaccessible and $\lambda\in C$. Let $\mathbb{A}=\langle H(\kappa),\in,\mathscr{P},C\rangle$ and consider the Π_1^1 sentence Φ true in $\mathbb{A}$:

$$\forall \mathsf{F}\ \forall\alpha\,[\mathsf{F}\text{ a function }\wedge \mathrm{dom}(\mathsf{F})=\alpha\wedge\forall\beta<\alpha\,(\mathsf{F}(\beta)\text{ is an ordinal}) \rightarrow \exists\gamma\ \forall\beta<\alpha\,(\mathsf{F}(\beta)<\gamma)]. \tag{2}$$

$$\forall a\ \exists b\ \exists\beta\ \exists f\,[b=P(a)\wedge f\colon b\xrightarrow{1-1}\beta]. \tag{3}$$

$$\forall\alpha\ \exists\beta\,(\alpha<\beta\wedge\beta\in C). \tag{4}$$

The $\forall \mathsf{F}$ in (2) is the only second order quantifier; so Φ is Π_1^1 (but not s-Π_1^1). By Π_1^1 Reflection, there is transitive $B \in H(\kappa)$ such that $\Phi^{(B)}$ holds. Let $\lambda = o(B) = B \cap \mathrm{Ord}$. By (2), λ is a regular cardinal. By (3), λ is a strong limit cardinal. By (4), λ is the sup of elements of C. Since C is closed, $\lambda \in C$. ☐

We can connect weakly compact cardinals with trees as follows. A tree $\mathscr{T} = \langle T, \prec \rangle$ is a *κ-tree* if the rank of $\mathscr{T}$ is κ and for each $\alpha < \kappa$, $\mathscr{T}$ has less than κ nodes of level α. A cardinal κ has the *tree property* iff every κ-tree has a branch, that is, a path of length κ.

9.9 Theorem. *Let $\kappa \geqslant \omega$ be inaccessible. Then κ is weakly compact iff κ has the tree property.*

Proof. By Theorem 7.3 we see that, for κ inaccessible, κ is weakly compact iff for every $\mathbb{A}$ of the form $\langle H(\kappa), \in, \mathscr{P}, R \rangle$, every $\mathbb{A}$-tree has a branch. Clearly every such $\mathbb{A}$-tree is a κ-tree. Conversely, if $\mathscr{T}$ is a κ-tree then $\mathscr{T}$ is isomorphic to a tree on $H(\kappa)$. Thus T is isomorphic to an $\mathbb{A}$-tree for some expansion $\langle H(\kappa), \in, R \rangle$ of $H(\kappa)$. ☐

We summarize the characterizations of weakly compact cardinals obtained in the above by means of the following statement. We say that κ *satisfies* s-$\Pi_1^1(R) = \Sigma_1(R)$ *uniformly in R* if for every s-Π_1^1 formula $\Phi(v_1, \ldots, v_n, \mathsf{P}, \mathsf{R})$ there is a Σ_1 formula $\varphi(v_1, \ldots, v_n, \mathsf{P}, \mathsf{R})$ such that

$$\langle H(\kappa), \in, \mathscr{P}, R \rangle \models \forall \vec{v}\, [\Phi(\vec{v}, \mathsf{R}) \leftrightarrow \varphi(\vec{v}, \mathsf{R})]$$

for all $R \subseteq H(\kappa)$. (This is a different use of the word "uniformly".) We say that κ is *weakly compact* for $\mathsf{L}_{\kappa\omega}(\mathscr{W\!f})$ if for every $T \subseteq H(\kappa)$, if every subset of T_0 of power $< \kappa$ has a well-founded model, then T has a well-founded model.

9.10 Theorem (Summary). *Let κ be an infinite cardinal. The following are equivalent:*

(i) *κ is weakly compact for $\mathsf{L}_{\kappa\omega}$.*

(ii) *$\kappa = \omega$ or κ is weakly compact for $\mathsf{L}_{\kappa\omega}(\mathscr{W\!f})$.*

(iii) *κ is s-Π_1^1 indescribable.*

(iv) *$\kappa = \omega$ or κ is Π_1^1 indescribable.*

(v) *κ is inaccessible and has the tree property.*

(vi) *κ is inaccessible and for every $R \subseteq H(\kappa)$, $\langle H(\kappa), \in, R \rangle$ has a proper elementary end extension.*

(vii) *κ is inaccessible and $\kappa = \omega$ or else for every $R \subseteq H(\kappa)$, $\langle H(\kappa), \in, R \rangle$ has a proper well-founded elementary end extension.*

(viii) *κ is inaccessible and satisfies s-$\Pi_1^1(R) = \Sigma_1(R)$, uniformly in R.*

Proof. We list below the equivalences which have been already stated or else are immediate consequences of earlier results.

(i) $\Longleftrightarrow$ (ii) ($\Rightarrow$ by 9.5; $\Leftarrow$ by just adding E to a theory not mentioning it),
(i) $\Longleftrightarrow$ (iii) (by 9.4),
(iii) $\Longleftrightarrow$ (iv) (by 9.7 ii),
(i) $\Longleftrightarrow$ (v) (by 9.3 and 9.9).

The following implications are trivial:

(ii) $\Longrightarrow$ (vii) (trivial compactness argument),
(vii) $\Longrightarrow$ (vi) (trivial for $\kappa>\omega$, the case $\kappa=\omega$ follows from compactness of $\mathsf{L}_{\omega\omega}$).

The remaining implications (vi) $\Longrightarrow$ (v), and (iii) $\Longleftrightarrow$ (viii) are implicit in earlier results or proofs, but we will make them explicit. To prove (vi) $\Longrightarrow$ (v), let $\mathscr{T}=\langle T,\prec\rangle$ be a κ-tree. We may assume $T\subseteq\kappa$. Let $\mathbb{A}=\langle H(\kappa),\in,T,\prec,\mathrm{lev}\rangle$. We can code up all of T, $\prec$, lev into one $R\subseteq H(\kappa)$ so, by assumption (vi), there is a proper elementary end extension $\mathfrak{B}=\langle B,E,T',\prec',\mathrm{lev}'\rangle$ of $\mathbb{A}$. Let $b\in B$ be an ordinal, $b\notin A$. Let $x\in T$ satisfy

$$\mathfrak{B}\models x\in\mathrm{lev}'(b).$$

Then $\{y\in A;\, y\prec' x\}$ is a branch through T. To prove (iii) $\Longrightarrow$ (viii), let $\Phi(x,\mathsf{R})=\forall\mathsf{S}\,\varphi(x,\mathsf{R},\mathsf{S})$ be a s-Π^1_1 formula involving an extra relation symbol R. For any R, $\langle H(\kappa),\in,\mathscr{P},R\rangle$ satisfies one of the below iff it satisfies all:

$$\Phi(x,\mathsf{R}),$$

$$\forall S\,\varphi(x,\mathsf{R},\mathsf{S}),$$

$$\exists a\,[\mathrm{Tran}(a)\wedge x\in a\wedge\forall\mathsf{S}\subseteq a\,\varphi^{(a)}(x,\mathsf{R},\mathsf{S})]\quad\text{(by (iii))},$$

$$\exists a\,\exists b\,[\mathrm{Tran}(a)\wedge x\in a\wedge b=\mathscr{P}(a)\wedge\forall\mathsf{S}\in b\,\varphi^{(a)}(x,\mathsf{R},\mathsf{S})].$$

The last line gives us a Σ_1 formula $\psi(x,\mathscr{P},\mathsf{R})$ equivalent to $\Phi(x,\mathsf{R})$ for all R. To prove (viii) $\Longrightarrow$ (iii), notice that since κ is inaccessible, $H(\kappa)=\mathrm{V}(\kappa)$ and that $\mathbb{A}=\langle H(\kappa),\in,\mathscr{P},R\rangle$ is resolvable, since $H(\kappa)=\bigcup_{\alpha<\kappa}\mathrm{V}(\alpha)$. Thus if $\mathbb{A}$ satisfies s-$\Pi^1_1=\Sigma_1$ then $\mathbb{A}$ satisfies s-Π^1_1 Reflection by Corollary 4.9. ☐

Some further equivalences are given in the Exercises.

Looking at this summary, one can hardly fail to be struck by the equivalence of notions coming to us from model theory, set theory and recursion theory. The summary is slightly misleading, however, in that it hides many important considerations which go into its proof, considerations including supervalidity properties, resolvability, essential uncountability, $\mathbb{A}$-trees, and so forth. It is only by understanding the earlier results involving these notions that one sees the various forces at work in Theorem 9.10.

9.11—9.16 Exercises

9.11. Let $\mathbb{A}$ be an essentially uncountable admissible set and let T be a s-Π^1_1 set of sentences of $\mathsf{L}_{\mathbb{A}}$. Let

$$\mathrm{Cn}_{\mathscr{Wf}}(T)=\{\varphi\in\mathsf{L}_{\mathbb{A}}\colon \varphi\text{ is true in all \textit{well-founded} models of }T\}.$$

Show that $\mathrm{Cn}_{\mathscr{Wf}}(T)$ is s-Π^1_1.

9.12. Let κ be weakly compact, $\kappa > \omega$. Show that if $C \subseteq \kappa$ is c.u.b. then there is a Mahlo cardinal $\lambda < \kappa$, $\lambda \in C$.

9.13. Suppose that for every R, $\langle H(\kappa), \in, \mathscr{P}, R \rangle$ satisfies $s\text{-}\Pi_1^1 = \Sigma_1$. Show that $\langle H(\kappa), \in, \mathscr{P} \rangle$ satisfies $s\text{-}\Pi_1^1(R) = \Sigma_1(R)$, uniformly in R.

9.14. The definition of weakly compact cardinal is often given in terms of $\mathsf{L}_{\kappa\kappa}$. We sketch a proof that the two definitions are equivalent. We define $\mathsf{L}_{\infty\infty}$ to be the smallest collecting containing $\mathsf{L}_{\infty\omega}$ closed under $\neg$, $\bigwedge$, $\bigvee$ and

> if $\varphi \in \mathsf{L}_{\infty\omega}$ and V is a *set* of variables occurring in φ then $\exists V \varphi$ and $\forall V \varphi$ are in $\mathsf{L}_{\infty\infty}$.

For any κ, $\mathsf{L}_{\kappa\kappa} = L_{\infty\infty} \cap H(\kappa)$.

(i) Prove that $\mathsf{L}_{\kappa\kappa}$ consists of those $\varphi \in \mathsf{L}_{\infty\infty}$ with $< \kappa$ subformulas.

(ii) The following are sentences of $\mathsf{L}_{\omega_1\omega_1}$:

$$\forall \{v_1, \ldots, v_n, \ldots\} \bigvee_{n<\omega} \neg(v_{n+1} \mathsf{E} v_n),$$

$$\forall \{v_1, \ldots, v_n, \ldots\} \exists w \forall x \left[x \mathsf{E} w \leftrightarrow \bigvee_n x = v_n\right].$$

Give a formal definition of $\mathfrak{M} \models \varphi[s]$ for $\varphi \in \mathsf{L}_{\infty\infty}$ so that these sentences express well-foundedness and essential uncountability, respectively.

(iii) Show that every subformula of a sentence of $\mathsf{L}_{\kappa\kappa}$ has less than κ free variables.

(iv) Let κ be inaccessible and let $\varphi \in \mathsf{L}_{\kappa\kappa}$. Show that if φ has a model then it has one in $H(\kappa)$. Let $T \subseteq H(\kappa)$ be a set of sentences of $\mathsf{L}_{\kappa\kappa}$. Show that if T has a model then it has one of power κ. [Modify the usual Löwenheim-Skolem proof.]

(v) Let κ be weakly compact for $\mathsf{L}_{\kappa\omega}$. Show that κ is weakly compact for $\mathsf{L}_{\kappa\kappa}$. That is, let $T \subseteq \mathsf{L}_{\kappa\kappa}$ be a set of sentences such that every $T_0 \subseteq T$, $\text{card}(T_0) < \kappa$, has a model. Show that T has a model. [For $\kappa = \omega$ this is trivial. For $\kappa > \omega$ apply 9.10 (vii) to $\langle H(\kappa), \in, \mathscr{P}, T \rangle$. Use the fact that (iv) holds in $H(\kappa)$ and hence in any elementary end extension. Also use the fact that $H(\kappa)$ is closed under sequences of length $< \kappa$.]

9.15. Show that κ is weakly compact iff $\kappa \to (\kappa)_2^2$; that is, iff for every partition

$$[\kappa]^2 = P_0 \cup P_1$$

of $[\kappa]^2 = \{\{\alpha, \beta\} : \alpha < \beta < \kappa\}$ into two sets, there is a subset $C \subseteq \kappa$ such that $[C]^2 \subseteq P_i$ for $i = 0$ or $i = 1$. [It is probably easiest to prove that 9.10 (vii) implies $\kappa \to (\kappa)_2^2$ and to prove $\omega \to (\omega)_2^2$ separately. To prove the other half show that $\kappa \to (\kappa)_2^2$ implies 9.10 (v).]

9.16. The parts (vi) and (vii) of Theorem 9.10 do not have significant lightface versions; that is, versions without the "for all R" clause, as the following example of Kunen shows. *Let κ be the least inaccessible cardinal such that $\langle H(\kappa), \in \rangle$ has* *an elementary end extension. Show that it has no well-founded elementary end extension.*

9.17 Notes. The "weakly" in weakly compact derives from the following. A cardinal κ is *strongly compact* if $\langle \mathfrak{M}; H(\kappa)_{\mathfrak{M}}, \in, R\rangle$ is Σ_1 compact for every structure $\mathfrak{M} = \langle M, S\rangle$ and every $R \subseteq H(\kappa)_{\mathfrak{M}}$, regardless of the size of $\mathfrak{M}$ as compared to κ. We see no convincing argument that strongly compact cardinals $> \omega$ are a natural limit of existing cardinals and so we do not study them here.

The equivalence, for $\kappa > \omega$, of weakly compact with Π_1^1 indescribability is due to Hanf and Scott [1961]. Some authors take Π_1^1 indescribability as the definition of weakly compact, thus ruling out ω. This seems not only silly (to rule out the one concrete example) but positively misleading since, as the proof of 9.7 shows, a number of considerations besides compactness are involved in the proof of Π_1^1 indescribability. The equivalences (in 9.10) (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(v)$\Longleftrightarrow$(vi)$\Longleftrightarrow$(vii) are all well known. Similarly for the other equivalences given in the exercises. Corollary 9.8 and Exercise 9.12, which show that the first weakly compact $\kappa > \omega$ is much larger than the first inaccessible cardinal, are due to Hanf [1964]. The last equivalence ((i)$\Longleftrightarrow$(viii)) in 9.10 is a uniform version of a result in Kunen [1968].

The remarkable argument that strongly compact cardinals exist "by analogy with ω" always reminds me of the goofang, described in *The Book of Imaginary Beings*, by Jorge Luis Borges:

> The yarns and tall tales of the lumber camps of Wisconsin and Minnesota include some singular creatures, in which, surely, no one ever believed...
>
> There's another fish, the *Goofang*, that swims backward to keep the water out of its eyes. It's described as "about the size of a sunfish, only much bigger".