Part C

Towards a General Theory

"The *sensible* practical man realizes that the questions which he dismisses may be the key to a theory. Further, since he doesn't have a good theoretical analysis of familiar matters, sometimes not even the concepts needed to frame one, he will not be surprised if a novel situation turns out to be genuinely problematic."

G. Kreisel
Observations on Popular Discussions of Foundations

Chapter VII
More about $\mathsf{L}_{\infty\omega}$

In this chapter we resume the discussion of $\mathsf{L}_{\infty\omega}$ where we left it in Chapter III. This time, however, we do not restrict our attention to countable fragments but develop the beginning of a general theory. In this way we can gain insight into the countable case by seeing what principles are involved in the general case.

The most useful result, both for model-theoretic applications and for applications to generalized recursion theory, is the Weak Model Existence Theorem of § 2. Its model theoretic applications are discussed in §§ 3 and 4. The applications to definability theory can be found in Chapter VIII.

§§ 5, 6 and 7 are concerned with Scott sentences of $\mathsf{L}_{\infty\omega}$ and their approximations. These sections are independent of most of the rest of the book but they do illustrate the importance of $\mathsf{L}_{\infty\omega}$ and some uses of admissible sets in studying them.

1. Some Definitions and Examples

Once the hypothesis of countability is removed, all the major theorems of Chapter III fail dramatically. This section consists largely of "counter" examples to these statements. It also contains a number of definitions which will be important in our study.

1.1 Definition. An admissible set $\mathbb{A}$ is Σ_1 *compact* if for each admissible fragment of the form $\mathsf{L}_{\mathbb{A}}$ and each Σ_1 theory T of $\mathsf{L}_{\mathbb{A}}$, if every subset T_0 of T which is a element of $\mathbb{A}$ has a model, then T has a model.

The Compactness Theorem of § III.5 states that every countable, admissible set is Σ_1 compact.

1.2 Definition. An admissible set $\mathbb{A}$ is *self-definable* if for some language L containing the language of $\mathbb{A}$ there is a Σ_1 theory T of $\mathsf{L}_{\mathbb{A}}$ such that

(i) some expansion $(\mathbb{A}, \ldots)$ of $\mathbb{A}$ to an L-structure is a model of T.

(ii) if $(\mathfrak{B}, \ldots)$ is any model of T then $\mathfrak{B} \cong \mathbb{A}$.

If T can be chosen to be a single sentence of $\mathsf{L}_{\mathbb{A}}$ then $\mathbb{A}$ is called *strongly self-definable.*

We obtain a host of counter-examples to Σ_1 compactness by means of 1.3 and 1.4. The first is a trivial exercise in compactness.

1.3 Proposition. *If $\mathbb{A}$ is Σ_1 compact then $\mathbb{A}$ is not self-definable.* □

1.4 Proposition. *For all $\alpha \geqslant 0$, $H(\aleph_{\alpha+1})$ is self-definable.*

Proof. Let $A = H(\aleph_{\alpha+1})$ and let T be the theory consisting of the following sentences:

$$\mathrm{KP},$$

$$\forall x(x \in \bar{\mathsf{a}} \leftrightarrow \bigvee_{b\in a} x = \bar{\mathsf{b}}) \quad \text{for all} \quad a \in A,$$

$$\forall x \exists \beta \exists f [\beta \leqslant \bar{\omega}_\alpha \wedge f \text{ maps TC}(x) \text{ one-one onto } \beta].$$

With the obvious interpretation of the constant symbols, A is a model of T. Suppose $\langle B, E\rangle$ is some other model of T. The infinitary sentences of T insure that we can assume

$$\langle A, \in\rangle \subseteq_{\text{end}} \langle B, E\rangle .$$

Let $x \in B$ and suppose $y \in B$ is such that

$$\langle B, E\rangle \models \text{“TC}(x) = y\text{”} .$$

Pick $\beta \leqslant \aleph_\alpha$ such that

$$\langle B, E\rangle \models \exists f [f \text{ maps } y \text{ one-one onto } \beta]$$

by the last axiom of T. Then there is some $F \subseteq \beta \times \beta$ such that $\langle B, E\rangle$ is a model of $\langle y, E{\restriction}y\rangle \cong \langle \beta, F\rangle$ and hence “$\langle \beta, F\rangle$ is well founded” is true in $\langle B, E\rangle$. The crucial step in the proof is to verify that

(1) $\langle \beta, F\rangle$ really is well founded.

Suppose that $\langle \beta, F\rangle$ is not well founded and let $X \subseteq \beta$ have no F-minimal member. But $\text{card}(X) < \aleph_{\alpha+1}$, so $X \in A \subseteq B$, and hence $\langle B, E\rangle$ is a model of “X has no F-minimal element”, which is a contradiction. Thus (1) is established. But then the transitive set isomorphic to $\langle \beta, F\rangle$ is, on the one hand, $\langle y, E{\restriction}y\rangle$ and, on the other, in $H(\aleph_{\alpha+1})$. Thus $y \in H(\aleph_{\alpha+1})$ so $x \in H(\aleph_{\alpha+1})$. In other words $\langle A, \in\rangle = \langle B, E\rangle$. □

A strengthening of 1.4 is given in Exercise 1.12.

If we had wanted only to prove that $H(\aleph_{\alpha+1})$ is not Σ_1 compact, we could have come up with much simpler examples. A good example does more than just refute (the function of a *counterexample*), it makes almost explicit some of the ideas needed for understanding and generalizing existing results. Most of the examples in this section are good examples.

To understand the above example, the student should consider what happens to the proof of 1.4 if we replace $\langle H(\aleph_{\alpha+1}),\in\rangle$ by some countable, transitive set $\langle A,\in\rangle$ elementarily equivalent to it. Something must go wrong since A is Σ_1 compact. If he works through the proof he will see that the only step that fails is the proof of (1). This suggests the following proposition.

1.5 Proposition. *Let $\mathbb{A}$ be admissible.*

(i) *If $\mathbb{A}$ is self-definable then there is a Σ_1 theory $T(<)$ of $\mathsf{L}_{\mathbb{A}}$ which pins down ordinals greater than those in $\mathbb{A}$.*

(ii) *If $\mathbb{A}$ is strongly self-definable then there is a single sentence $\varphi(<)$ of $\mathsf{L}_{\mathbb{A}}$ which pins down ordinals greater than those in $\mathbb{A}$.*

Proof. We prove (i); the proof of (ii) is the same. Let T_0 be a theory which self-defines $\mathbb{A}$ and let $T = T_0 +$ "$< = \in\restriction$ ordinals". Then every model $\mathfrak{M}$ of T has $<^{\mathfrak{M}}$ of order type $o(\mathbb{A})$. $\square$

Thus, self-definable admissible sets show that the theorems of § III.7 on the ordinals pinned down by Σ_1 theories of $\mathsf{L}_{\mathbb{A}}$ cannot go through in general; for example, they fail when $\mathbb{A} = H(\aleph_1)$. To get an example where a single sentence pins down large ordinals, we need some strongly self-definable admissible sets.

A set $\mathbb{A}$ is *essentially uncountable* if every countable subset $X \subseteq \mathbb{A}$ is an element of $\mathbb{A}$.

1.6 Proposition. *Let $\mathbb{A}$ be an essentially uncountable admissible set and let $\mathbb{B} = \mathbb{HYP}(\mathbb{A})$. Then $\mathbb{B}$ is strongly self-definable.*

Proof. Let ψ be the conjunction of the following:

$$\bigwedge_{a\in A\cup\{A\}} \forall v[v\in \bar{a} \leftrightarrow \bigvee_{x\in a} v = \bar{x}],$$

$$\bigwedge \mathrm{KPU},$$

$$\forall v\,\exists\alpha[x\in \mathrm{L}(\bar{\mathrm{A}},\alpha)],$$

$$\forall\alpha \bigvee_{\varphi\in KP} \neg\varphi^{\mathrm{L}(\bar{\mathrm{A}},\alpha)},$$

$$\forall\alpha\,\exists r[r\subset \bar{\mathrm{A}} \wedge r \text{ is a pre-wellordering of type } \alpha].$$

Since $\mathbb{HYP}(\mathbb{A})$ is projectible into $\mathbb{A}$, $\mathbb{HYP}(\mathbb{A})$ is a model of the last conjunct and hence of ψ. The well founded models of the first four conjuncts are isomorphic to $\mathbb{HYP}(\mathbb{A})$ so it remains to see that all models of ψ are well founded. Using the rank function we see that if $\langle B',E\rangle$ is a non-wellfounded model of ψ then there is a descending sequence of ordinals in $\langle B',E\rangle$ so it suffices to see that the ordinals of $\langle B',E\rangle$ are wellfounded. Let $a\in B'$ be an "ordinal" of $\langle B',E\rangle$. Apply the last conjunct of ψ to get an $r\subseteq A$ such that

$$\langle B',E\rangle \models \text{"}r \text{ has order type } a\text{"}.$$

We need to see that r really is well ordered. Suppose

$$\ldots r x_{n+1} r x_n r \ldots r x_1$$

is an r-descending sequence. Let $b=\{x_n | n<\omega\}$. Since b is a countable subset of A, $b \in A$. But then $b \in B'$ and b has no r-minimal element, contradicting

$$\langle B', E \rangle \models \text{“}r \text{ is well ordered”}. \quad \square$$

For example, if $\mathrm{cf}(\kappa)>\omega$ then $\mathbb{A}=\mathbb{H}\mathrm{YP}(H(\kappa))$ is strongly self-definable. Hence $\mathsf{L}_{\mathbb{A}}$ is not Σ_1 compact and there is a single sentence of $\mathsf{L}_{\mathbb{A}}$ which pins down $o(\mathbb{A})$.

Our next examples have to do with attempts to generalize the Completeness and Extended Completeness Theorems of § III.5 to arbitrary admissible fragments.

1.7 Definition. Let $\mathbb{A}$ be an admissible set.

(i) $\mathbb{A}$ is *validity admissible* if the set of valid infinitary sentence of $\mathbb{A}$ is Σ_1 on $\mathbb{A}$.

(ii) $\mathbb{A}$ is Σ_1 *complete* if, for every Σ_1 theory T of $\mathsf{L}_{\mathbb{A}}$, the set

$$\mathrm{Cn}(T)=\{\varphi \in \mathsf{L}_{\mathbb{A}} | T \models \varphi\}$$

is Σ_1 on $\mathbb{A}$.

Don't forget, in reading 1.7, that the extra relations which may be part of $\mathbb{A}$ count in the definition of Σ_1. It is also important to notice that Σ_1 completeness implies validity admissibility.

1.8 Proposition. *Let $\mathbb{A}$ be admissible.*

(i) *If $\mathbb{A}$ is self-definable then $\mathbb{A}$ is not Σ_1 complete.*

(ii) *If $\mathbb{A}$ is a strongly self-definable pure admissible set then $\mathbb{A}$ is not even validity admissible.*

Proof. Recall, from § V.1, that there is a Π_1 subset of $\mathbb{A}$ which is not Σ_1. Hence, there is certainly a Π_1^1 subset of $\mathbb{A}$ which is not Σ_1. Thus the result follows from the following lemma. $\square$

1.9 Lemma. *Let $\mathbb{A}$ be admissible, let T be the theory which self-defines $\mathbb{A}$ in* 1.8 *and let $X \subseteq \mathbb{A}$ be Π_1^1 on $\mathbb{A}$. There is an $\mathbb{A}$-recursive function f such that for every $x \in \mathbb{A}$ we have $x \in X$* iff *$f(x) \in \mathrm{Cn}(T)$.*

Proof. Suppose

$$x \in X \quad \text{iff} \quad \mathbb{A} \models \forall \mathsf{R}\, \varphi(\mathsf{R}, x),$$

where R is a symbol not in the language of T. In case (i) of 1.8 we may assume that T contains the diagram of $\mathbb{A}$. Then $x \in X$ iff $\varphi(\mathsf{R}, \overline{\mathsf{x}}) \in \mathrm{Cn}(T)$.

In case (ii) we settle the question “$x \in X$?” by checking whether the conjunction of T and the diagram of $\mathrm{TC}(\{x\})$ implies $\varphi(\mathsf{R}, \overline{\mathsf{x}})$. $\square$

1.10 Corollary. *If A is pure and strongly self-definable then there are valid sentences of L_A which are not provable by the axioms and rules of Chapter III.*

Proof. The set of provable sentences is a Σ_1 set. ☐

Thus, $H(\aleph_{\alpha+1})$ is never Σ_1 complete, even if $\alpha=0$, and $\mathbb{HYP}(H(\aleph_{\alpha+1}))$ is never validity admissible.

We conclude this section with a counterexample to the interpolation theorem. It has a rather different flavor and will not be used in the following sections.

1.11 Proposition. *Let $\mathbb{A}$ be an admissible set with an uncountable element and $o(\mathbb{A})>\omega$. The interpolation theorem fails for L_A.*

Proof. Let $\varphi(<)$ characterize $\langle\omega,<\rangle$ up to isomorphism and let ψ be

$$\bigwedge_{\substack{x,y\in a\\ x\neq y}} \overline{\mathsf{x}}\neq\overline{\mathsf{y}}$$

where $a\in A$ is uncountable. (All we reed about ψ is that it has only uncountable models and has no symbols in common with φ.) Then $\varphi,\psi\in\mathbb{A}$ and $\models\varphi\to\neg\psi$. If the interpolation theorem held for $\mathsf{L}_{\mathbb{A}}$ then there would be a sentence θ involving only equality such that $\models\varphi\to\theta$ and $\models\psi\to\neg\theta$. Thus θ is true in all countable infinite structures since such structures can always be turned into models of φ. Similarly, $\neg\theta$ is true in all structures of power $\geqslant\mathrm{card}(a)$. But this contradicts:

(2) A sentence $\theta\in\mathsf{L}_{\infty\omega}$ involving only equality is true in all infinite structures or in none.

The proof of (2) is easy, given some notation and results of § 5, which we assume. Let $\mathfrak{M}=\langle M,=\rangle$, $\mathfrak{N}=\langle N,=\rangle$ be infinite. Let I be the set of all finite one-one maps from $M_0\subseteq M$ onto $N_0\subseteq N$. Then

$$I\colon\mathfrak{M}\cong_p\mathfrak{N}$$

so $\mathfrak{M}\equiv\mathfrak{N}(\mathsf{L}_{\infty\omega})$. Thus $\mathfrak{M}\models\theta$ iff $\mathfrak{N}\models\theta$. ☐

1.12—1.17 Exercises

1.12. Suppose $0<\alpha<\aleph_\alpha$ and $\mathrm{card}(\mathfrak{M})<\aleph_\alpha$. Show that $H(\aleph_\alpha)_{\mathfrak{M}}$ is self-definable. This includes 1.4 and $H(\aleph_\omega)$ as special cases.

1.13. A sentence $\varphi(<)$ (or theory $T(<)$) *pins down α exactly* if φ has models and every model $\mathfrak{M}$ of φ has $<^{\mathfrak{M}}$ of order type exactly α.

(i) Prove that if $\mathbb{A}$ is self-definable (strongly self-definable) then there is a Σ_1 theory T of $\mathsf{L}_{\mathbb{A}}$ (sentence φ of $\mathsf{L}_{\mathbb{A}}$) which pins down $o(\mathbb{A})$ exactly.

(ii) Let $\mathbb{A}$ be a resolvable admissible set and let T be a Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$ which pins down $o(\mathbb{A})$ exactly. Show that $\mathbb{A}$ is self-definable.

1.14. Let $\mathbb{A} = \mathbb{H}\mathrm{YP}(H(\aleph_{\alpha+1}))$. Show that there is a sentence of $\mathsf{L}_{\mathbb{A}}$ which pins down $\aleph_{\alpha+2}$.

1.15. Show that the results of § IV.1 fail in the uncountable case.

1.16. Show that if $\mathbb{A}$ is essentially uncountable then every inductive relation on $\mathbb{A}$ is Δ_1^1. Conclude that not every Π_1^1 relation on $\mathbb{A}$ is inductive on $\mathbb{A}$, for $\mathbb{A}$ essentially uncountable.

1.17. Improve 1.8 (ii) by allowing $\mathbb{A}_{\mathfrak{M}}$ admissible above $\mathfrak{M}$.

1.18 Notes. Counterexamples to compactness go back to Hanf [1964] and earlier unpublished work of Tarski. Karp [1967] showed that, for $\mathrm{cf}(\alpha) > \omega$, the set $H(\aleph_\alpha)$ is not validity admissible. The results on pinning down large ordinals (1.14 for example) are due to Chang [1968]. The counterexample to interpolation is due to Malitz [1971]. We have tried to unify the various examples by centering them on the notion of self-definable, admissible set. Our notion is suggested by, and equivalent to, that of Kunen [1968].

Kreisel [1968] has observed that the counterexample to interpolation has the defect that it might disappear by some reasonable strengthening of the logic $\mathsf{L}_{\mathbb{A}}$ or $\mathsf{L}_{\infty\omega}$. The other examples of this section do not have this defect. The situation with compactness, say, could only get worse if we were to increase the expression power of the logic by introducing some new quantifier or connective. Rather than strengthen $\mathsf{L}_{\mathbb{A}}$ we must look for strengthenings of the notion of admissibility which coincides with the old notion in the countable case. This is taken up in Chapter VIII.

2. A Weak Completeness Theorem for Arbitrary Fragments

The model theory of second-order logic is totally unmanageable and seems destined to remain so. Infinitary logic is an attempt to dent second-order logic by studying logics which have greater expressive power than $\mathsf{L}_{\omega\omega}$ but still have a workable model theory. The examples of § 1 show that uncountable fragments behave more like second-order logic than do countable fragments. This makes the problem of developing a theory which handles arbitrary admissible fragments very intriguing.

In spite of, or because of, the "counter"-examples, the model theory of arbitrary admissible fragments is becoming a rich subject. In this section we present some basic tools for studying these logics. In particular, we prove an analogue of the Extended Completeness Theorem of § III.5. Recall our line of attack on the problem of completeness in Chapter III:

(1) We defined the notion: *validity property* for $\mathsf{L}_{\mathbb{A}}$.

(2) We proved that if $\mathsf{L}_{\mathbb{A}}$ is countable then a sentence $\varphi \in \mathsf{L}_{\mathbb{A}}$ is valid iff φ is in every validity property.

(3) We showed that if $\mathsf{L}_{\mathbb{A}}$ is an admissible fragment then the intersection of all validity properties is a validity property which is $\mathbb{A}$-r.e., that is, Σ_1 on $\mathbb{A}$.

When we drop the assumption that $\mathsf{L}_{\mathbb{A}}$ is countable step (2) breaks down. In general, a sentence may be true in all models without being in every validity property (i.e., without being a theorem of $\mathsf{L}_{\mathbb{A}}$) as Corollary 1.10 shows. In this section we attack the problem of completeness as follows:

(1′) We define a stronger notion: *supervalidity property* for $\mathsf{L}_{\mathbb{A}}$.

(2′) We prove that a sentence $\varphi \in \mathsf{L}_{\mathbb{A}}$ is valid iff φ is in every supervalidity property.

(3′) In Chapter VIII we will introduce a semantic notion of r.e., called strict Π_1^1, and show that the intersection of all supervalidity properties for $\mathsf{L}_{\mathbb{A}}$ is a strict Π_1^1 set. When $\mathbb{A}$ is countable the notion of strict Π_1^1 reduces to Σ_1 on $\mathbb{A}$.

It is convenient in this part of the theory to work with sufficiently rich fragments, so-called Skolem fragments with constants.

2.1 Definition. Let $\mathsf{L}_{\mathbb{A}}$ be a fragment of $\mathsf{L}_{\infty\omega}$ and let C be a (possibly empty) set of constant symbols of L such that every formula of $\mathsf{L}_{\mathbb{A}}$ contains at most a finite number of constants from C.

(i) $\mathsf{L}_{\mathbb{A}}$ is a *Skolem fragment with constants* C if there is a one-one function which assigns to each formula of $\mathsf{L}_{\mathbb{A}}$ of the form

$\exists x\, \varphi(x, y_1, \ldots, y_n)$, where

φ contains no constants from C and

$y_1, \ldots, y_n$ are not bound in φ

an n-ary function symbol

$$\mathsf{F}_{\exists x \varphi}$$

of L not occuring in φ; it is called the *Skolem function symbol* for $\exists x\, \varphi(x, y_1, \ldots, y_n)$. If $C = 0$ we just say that $\mathsf{L}_{\mathbb{A}}$ is a *Skolem fragment.*

(ii) Let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment with constants C. The *Skolem theory for* $\mathsf{L}_{\mathbb{A}}$, denoted by T_{Skolem}, consists of all sentences of $\mathsf{L}_{\mathbb{A}}$ of the form

$$\forall y_1, \ldots, y_n [\exists x\, \varphi(x, y_1, \ldots, y_n) \to \varphi(\mathsf{F}_{\exists x \varphi}(y_1, \ldots, y_n), y_1, \ldots, y_n)]$$

for all formulas $\exists x\, \varphi(x, y_1, \ldots, y_n)$ as in (i). An L-structure $\mathfrak{M}$ is a *Skolem structure for* $\mathsf{L}_{\mathbb{A}}$ if

$$\mathfrak{M} \models T_{\text{Skolem}}.$$

The extra freedom permitted by the set C of constant symbols is crucial for many applications. For now we can barely hint at their use by the following lemmas.

2.2 Lemma. *Let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment with constants C and let $\mathscr{D}$ be any validity property for $\mathsf{L}_{\mathbb{A}}$ with $T_{\text{Skolem}} \subseteq \mathscr{D}$. Then for any formula*

$$\exists x\, \varphi(x, y_1, \ldots, y_n, \mathsf{c}_1, \ldots, \mathsf{c}_k)$$

of $\mathsf{L}_{\mathbb{A}}$ the sentence

$$\forall y_1, \ldots, y_n[\exists x\, \varphi(x, \vec{y}, \vec{\mathsf{c}}) \rightarrow \varphi(\mathsf{F}(\vec{y}, \vec{c}), \vec{y}, \vec{\mathsf{c}})]$$

is in $\mathscr{D}$, where F is the Skolem function symbol for

$$\exists x\, \varphi(x, y_1, \ldots, y_n, y_{n+1}, \ldots, y_{n+k}).$$

Proof. By the definition of T_{Skolem},

$$\forall y_1, \ldots, y_{n+k}[\exists x\, \varphi(x, y_1, \ldots, y_{n+k}) \rightarrow \varphi(\mathsf{F}(y_1, \ldots, y_{n+k}), y_1, \ldots, y_{n+k})]$$

is in $T_{\text{Skolem}} \subseteq \mathscr{D}$. Using the axioms for $\forall$ and modus ponens shows that the desired sentence is in $\mathscr{D}$. ☐

If $\mathsf{L}_{\mathbb{A}}$ is a fragment and C is a set of new constant symbols we use $\mathsf{L}_{\mathbb{A}}(C)$ to denote the fragment which consists of all substitution instances of formulas in $\mathsf{L}_{\mathbb{A}}$ by means of a *finite* number of constants from C. If $C = \{\mathsf{c}_1, \ldots, \mathsf{c}_n\}$ we sometimes use $\mathsf{L}_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n)$ for $\mathsf{L}_{\mathbb{A}}(C)$.

2.3 Lemma. *Let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment with constants C_0 and let C be a set of new constant symbols. Then $\mathsf{L}_{\mathbb{A}}(C)$ is a Skolem fragment with constants $C_0 \cup C$.*

Proof. Immediate from the definition. ☐

The next result shows us that we lose nothing (we gain a lot) by restricting ourselves to Skolem fragments and Skolem structures as far as the existence of models is concerned.

2.4 Proposition. *Let $\mathsf{L}_{\mathbb{A}}$ be a fragment of $\mathsf{L}_{\infty\omega}$. There is an expansion L' of L by new function symbols with the following properties:*

(i) *Let $\mathsf{L}'_{\mathbb{A}}$ be the set of formulas which result from a formula of $\mathsf{L}_{\mathbb{A}}$ by substituting a finite number of terms from L'. Then $\mathsf{L}'_{\mathbb{A}}$ is a Skolem fragment. Furthermore,* $\operatorname{card}(\mathsf{L}'_{\mathbb{A}}) = \operatorname{card}(\mathsf{L}_{\mathbb{A}})$ *and every Skolem function symbol is in $\mathsf{L}' - \mathsf{L}$.*

(ii) *Every L-structure $\mathfrak{M}$ has an expansion $\mathfrak{M}' = (\mathfrak{M}, \ldots)$ to a Skolem structure for $\mathsf{L}'_{\mathbb{A}}$.*

(iii) *If $\mathsf{L}_{\mathbb{A}}$ is an admissible fragment then we can define L' so that $\mathsf{L}'_{\mathbb{A}}$ is Δ_1 on $\mathbb{A}$ and such that the symbol $\mathsf{F}_{\exists x\varphi}$ is an $\mathbb{A}$-recursive function of $\exists x\, \varphi(x, y_1, \ldots, y_m)$. In particular, T_{Skolem} is then an $\mathbb{A}$-recursive set of sentences of $\mathsf{L}'_{\mathbb{A}}$.*

Proof. Let $\mathsf{L}^0 = \mathsf{L}$, $\mathsf{L}^0_{\mathbb{A}} = \mathsf{L}_{\mathbb{A}}$. For each formula

$$\exists x\, \varphi(x, y_1, \ldots, y_m)$$

of $\mathsf{L}^n_{\mathbb{A}}$ in which $y_1,\dots,y_m$ are not bound, add a new function symbol

$$\mathsf{F}_{\exists x\varphi}$$

to L^n and let $\mathsf{L}^{n+1}_{\mathbb{A}}$ be the resulting fragment. Let $\mathsf{L}'=\bigcup_n \mathsf{L}^n$ so that $\mathsf{L}'_{\mathbb{A}}=\bigcup_n \mathsf{L}^n_{\mathbb{A}}$. Part (ii) is obvious from thus construction. (See Lecture 13 of Keisler [1971] for more details, if necessary.) Part (iii) is obvious if we just code up $\mathsf{F}_{\exists x\varphi}$ by something like $\langle 17, \exists x\varphi\rangle$. ☐

We now come to the notion of supervalidity property.

2.5 Definition. Let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment with constants C. A validity property $\mathscr{D}$ for $\mathsf{L}_{\mathbb{A}}$ is a *supervalidity property* (s.v.p.) for $\mathsf{L}_{\mathbb{A}}$ (more precisely, for $(\mathsf{L}_{\mathbb{A}},C)$) if $T_{\mathrm{Skolem}}\subseteq\mathscr{D}$ and the following $\bigvee$-*rule* holds.

$\bigvee$-*Rule*: If $\bigvee\Phi$ is a SENTENCE of $\mathsf{L}_{\mathbb{A}}$ and $\bigvee\Phi\in\mathscr{D}$ then there is some $\varphi\in\Phi$ such that $\varphi\in\mathscr{D}$.

The $\bigvee$-rule causes supervalidity properties to behave in quite a different manner than ordinary validity properties. For example, it prevents the intersection of all supervalidity properties for $\mathsf{L}_{\mathbb{A}}$ from being an s.v.p. The next lemma shows just how strong the $\bigvee$-rule is.

2.6 Lemma. Let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment with constants and let $\mathscr{D}$ be a validity property for $\mathsf{L}_{\mathbb{A}}$ with $T_{\mathrm{Skolem}}\subseteq\mathscr{D}$. Then $\mathscr{D}$ is an s.v.p. iff $\mathscr{D}$ is complete, that is, iff for each sentence $\psi\in\mathsf{L}_{\mathbb{A}}$

$$\psi\in\mathscr{D} \quad\text{or}\quad (\neg\psi)\in\mathscr{D}.$$

Proof. Assume $\mathscr{D}$ is an s.v.p. Since all axioms of $\mathsf{L}_{\mathbb{A}}$ are in $\mathscr{D}$,

$$(\psi\vee\neg\psi)\in\mathscr{D}$$

so the conclusion follows by the $\bigvee$-rule. Now assume $\mathscr{D}$ is complete, $\bigvee\Phi$ a sentence of $\mathsf{L}_{\mathbb{A}}$, $\bigvee\Phi\in\mathscr{D}$. If for each $\varphi\in\Phi$, $\varphi\notin\mathscr{D}$, then, for each $\varphi\in\Phi$, $\neg\varphi\in\mathscr{D}$; so, by the $\bigwedge$-rule $R3$,

$$\bigwedge\{\neg\varphi\mid\varphi\in\Phi\}\in\mathscr{D}.$$

But this sentence is just $\sim\bigvee\Phi$. Since $\mathscr{D}$ is a validity property it cannot have both $\bigvee\Phi$ and $\sim\bigvee\Phi$ as members, so $\varphi\in\mathscr{D}$ for some $\varphi\in\Phi$. ☐

Note that if $\mathscr{D}$ is an s.v.p. for $\mathsf{L}_{\mathbb{A}}$ and $\varphi(v_1,\dots,v_n)\in\mathsf{L}_{\mathbb{A}}$ then

$$\varphi(v_1,\dots,v_n)\in\mathscr{D} \quad\text{iff}\quad \forall v_1,\dots,v_n\,\varphi(v_1,\dots,v_n)\in\mathscr{D}$$

so that $\mathscr{D}$ is determined by its sentences. We say that an L-structure $\mathfrak{M}$ is a *model of* $\mathscr{D}$ if $\mathfrak{M}$ is a model of all sentences in $\mathscr{D}$.

2.7 Definition. Let $\mathfrak{M}$ be a Skolem structure for the Skolem fragment $L_{\mathbb{A}}$ (with constants). The supervalidity property given by $\mathfrak{M}$, denoted by $\mathscr{D}_{\mathfrak{M}}$, is the set of all $\varphi(v_1, \ldots, v_n) \in L_{\mathbb{A}}$ such that

$$\mathfrak{M} \models \forall v_1, \ldots, v_n \varphi(v_1, \ldots, v_n).$$

In the notation of III.4.2, $\mathscr{D}_{\mathfrak{M}} = \Gamma_{\mathfrak{M}}$. It is clear that $\mathscr{D}_{\mathfrak{M}}$ is an s.v.p. for L_A. If a sentence $\varphi \in L_A$ is in all supervalidity properties then it is in all $\mathscr{D}_{\mathfrak{M}}$; hence it is true in all Skolem structures for $L_{\mathbb{A}}$. This gives the trivial half of the next theorem.

2.8 Theorem (Weak Completeness Theorem for Arbitrary Skolem Fragments). *Let $L_{\mathbb{A}}$ be a Skolem fragment with constants C.*

(i) *A sentence φ of $L_{\mathbb{A}}$ is true in all Skolem structures for $L_{\mathbb{A}}$ iff φ is in every supervalidity property.*

(ii) *Let T be a theory of $L_{\mathbb{A}}$, φ a sentence of $L_{\mathbb{A}}$. Then φ is true in every Skolem structure $\mathfrak{M}$ which is a model of T iff φ is in every s.v.p. $\mathscr{D}$ with $T \subseteq \mathscr{D}$.*

Proof. (i) is the special case of (ii) where $T = \emptyset$. The proof of ($\Leftarrow$) in (ii) is immediate by the remarks following Definition 2.7. Most of the work for proving ($\Rightarrow$) was done back in the proof of the model existence theorem. We break its proof up in two lemmas to make this clear and because we need one of the lemmas (2.9) later.

Compare the next lemma with the definition of consistency property on p. 85.

2.9 Lemma (Weak Model Existence Theorem). *Let L have at least one constant symbol and let $L_{\mathbb{A}}$ be any fragment of $L_{\infty\omega}$. Any set S of sentences of $L_{\mathbb{A}}$ which satisfies the following rules has a model.*

Consistency rule: If φ is atomic and $\varphi \in S$ then $(\neg\varphi) \notin S$.

$\neg$-rule: If $(\neg\varphi) \in S$ then $(\sim\varphi) \in S$.

$\bigwedge$-rule: If $\bigwedge \Phi \in S$ then for all $\varphi \in \Phi$, $\varphi \in S$.

$\forall$-rule: If $(\forall v \varphi(v)) \in S$ then for each closed term t of L, $\varphi(t/v) \in S$.

$\bigvee$-rule: If $\bigvee \Phi \in S$ then for some $\varphi \in \Phi$, $\varphi \in S$.

$\exists$-rule: If $(\exists v \varphi(v)) \in S$ then for some closed term t of L, $\varphi(t(v)) \in S$.

Equality rules: For all closed terms t_1, t_2 of L:

$$\text{if } (t_1 \equiv t_2) \in S \quad \text{then } (t_2 \equiv t_1) \in S, \text{ and}$$

$$\text{if } \varphi(t_1), (t_1 \equiv t_2) \in S \quad \text{then } \varphi(t_2) \in S.$$

Proof. The proof of the Model Existence Theorem was in two stages. We first showed how to construct a set s_ω of sentences having the above properties (plus some others involving constants from C) and then showed how to construct a model from such a set. The second stage of that proof constitutes the proof of this lemma. ▯

2.10 Lemma (Alternate form of Weak Completeness Theorem). *Let* $\mathsf{L}_{\mathbb{A}}$ *be a Skolem fragment with constants* C. *Let* $\mathcal{D}$ *be an s.v.p. for* $(\mathsf{L}_{\mathbb{A}}, C)$ *and let* S *be the set of sentences in* $\mathcal{D}$. *Then* S *is true in some Skolem structure for* L_A; *i.e.,* $\mathcal{D}$ *has a model.*

Proof. Since $T_{\text{Skolem}} \subseteq \mathcal{D}$, any model of S will be a Skolem structure for $\mathsf{L}_{\mathbb{A}}$. We need only prove that S satisfies the rules of Lemma 2.9. Since $\mathcal{D}$ contains the axioms (A 1)—(A 7) and is closed under (R 1)—(R 3), these are all routine except for the $\bigvee$ and $\exists$ rules. The $\bigvee$-rule for S follows from the $\bigvee$-rule for $\mathcal{D}$. To check the $\exists$-rule, suppose

$$\exists x\, \varphi(x, \mathsf{c}_1, \ldots, \mathsf{c}_n) \in S.$$

By Lemma 2.2,

$$[\exists x\, \varphi(x, \mathsf{c}_1, \ldots, \mathsf{c}_n) \rightarrow \varphi(\mathsf{F}(\mathsf{c}_1, \ldots, \mathsf{c}_n), \mathsf{c}_1, \ldots, \mathsf{c}_n)] \in S$$

for the appropriate function symbol F. Thus,

$$\varphi(\mathsf{F}(\mathsf{c}_1, \ldots, \mathsf{c}_n), \mathsf{c}_1, \ldots, \mathsf{c}_n) \in S$$

as demanded by the $\exists$-rule. □

Proof of Theorem 2.8 (ii) ($\Rightarrow$). Suppose $T \cup T_{\text{Skolem}} \models \varphi$. We need to see that if $\mathcal{D}$ is an s.v.p. with $T \subseteq \mathcal{D}$ then $\varphi \in \mathcal{D}$. If not, then $\neg\varphi \in \mathcal{D}$ by Lemma 2.6. Then, appying Lemma 2.10 we would get a Skolem model of $T \cup \{\neg\varphi\}$, a contradiction. □

We conclude this section with a result which allows us to construct interesting supervalidity properties and hence, by Weak Completeness, interesting models. It often gives us the effect of the ordinary Compactness Theorem for $\mathsf{L}_{\omega\omega}$. Given a Skolem fragment $\mathsf{L}_{\mathbb{A}}$ with constants C_0 and a Skolem fragment $\mathsf{K}_{\mathbb{B}}$ with constants C_1 we write

$$(\mathsf{L}_{\mathbb{A}}, C_0) \subseteq (\mathsf{K}_{\mathbb{B}}, C_1)$$

if $\mathsf{L}_{\mathbb{A}} \subseteq \mathsf{K}_{\mathbb{B}}$, $C_0 \subseteq C_1$, and if $\mathsf{F}_{\exists x\varphi}$ is the Skolem function symbol assigned to $\exists x\, \varphi(x, y_1, \ldots, y_n)$ by $\mathsf{L}_{\mathbb{A}}$, then it is also the one assigned to $\exists x\, \varphi(x, y_1, \ldots, y_n)$ by $\mathsf{K}_{\mathbb{B}}$.

2.11 Union of Chain Lemma. *Let* I *be a lineary ordered index set and suppose that, for each* $i \in I$, $\mathsf{L}_{\mathbb{A}}^{(i)}$ *is a Skolem fragment with constants* C_i *and* $\mathcal{D}_i$ *is a supervalidity property for* $(\mathsf{L}_{\mathbb{A}}^{(i)}, C_i)$. *Suppose, further, that for all* $i, j \in I$, *with* $i < j$,

$$(\mathsf{L}_{\mathbb{A}}^{(i)}, C_i) \subseteq (\mathsf{L}_{\mathbb{A}}^{(j)}, C_j) \quad \textit{and} \quad \mathcal{D}_i \subseteq \mathcal{D}_j.$$

Let $\mathsf{K}_{\mathbb{B}} = \bigcup_{i \in I} \mathsf{L}_{\mathbb{A}}^{(i)}$, $C_\infty = \bigcup_{i \in I} C_i$, $\mathcal{D}_\infty = \bigcup_{i \in I} \mathcal{D}_i$. *Then* $\mathsf{K}_{\mathbb{B}}$ *is a Skolem fragment with constants* C_∞, *and* $\mathcal{D}_\infty$ *is a supervalidity property for* $(\mathsf{K}_{\mathbb{B}}, C_\infty)$.

Proof. Simple checking of the definition shows that $\mathsf{K}_{\mathbb{B}}$ is a Skolem fragment with constants C_∞. The Skolem theory for $(\mathsf{K}_{\mathbb{B}}, C_\infty)$ is the union of the Skolem theories for the various $(\mathsf{L}_{\mathbb{A}}^{(i)}, C_i)$ so the Skolem theory for $(\mathsf{K}_{\mathbb{B}}, C_\infty)$ is contained in $\mathscr{D}_\infty$. Similarly, the axioms (A 1)—(A 7) for $\mathsf{K}_{\mathbb{B}}$ are all in $\mathscr{D}_\infty$. It is a trivial matter to check (R 1), (R 2) and the $\bigvee$-rule. This time it is the $\bigwedge$-rule which requires a moment's thought. Suppose $\bigwedge\Phi\in\mathsf{K}_{\mathbb{B}}$ and that, for each $\varphi\in\Phi$, $\varphi\in\mathscr{D}_\infty$. We need to check that $\bigwedge\Phi\in\mathscr{D}_\infty$. Choose i so that $\bigwedge\Phi\in\mathsf{L}_{\mathbb{A}}^{(i)}$. We claim that, for each $\varphi\in\Phi$, $\varphi\in\mathscr{D}_i$ (so that $\bigwedge\Phi\in\mathscr{D}_i\subseteq\mathscr{D}_\infty$). Otherwise, suppose $\varphi=\varphi(v_1,\dots,v_n)\in\Phi$ but that $\varphi\notin\mathscr{D}_i$. Then

$$\forall v_1,\dots,v_n\,\varphi(v_1,\dots,v_n)\notin\mathscr{D}_i.$$

By completeness (Lemma 2.6),

$$\neg\forall v_1,\dots,v_n\,\varphi(v_1,\dots,v_n)\in\mathscr{D}_i.$$

But $\varphi(v_1,\dots,v_n)\in\mathscr{D}_\infty$ so for some $j>i$, $\varphi(v_1,\dots,v_n)\in\mathscr{D}_j$. Hence

$$\forall v_1,\dots,v_n\,\varphi(v_1,\dots,v_n)\in\mathscr{D}_j.$$

But since $\mathscr{D}_i\subseteq\mathscr{D}_j$, this contradicts the consistency requirement for the validity property $\mathscr{D}_j$. ▯

All known applications of 2.11 follow from the following very special case. It exhibits the role of constants in our notion of Skolem fragment.

2.12 Union of Chain Lemma (Special form). *Let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment. Let $C=\{\mathsf{c}_n\mid 0<n<\omega\}$ be a countable set of new constant symbols. Suppose that for each n, $\mathscr{D}_n$ is an s.v.p. for $\mathsf{L}_{\mathbb{A}}(\mathsf{c}_1,\dots,\mathsf{c}_n)$ and that $\mathscr{D}_n\subseteq\mathscr{D}_m$ for $n\leqslant m$. Let $\mathscr{D}_\infty=\bigcup_n\mathscr{D}_n$. Then $\mathscr{D}_\infty$ is an s.v.p. for $\mathsf{L}_{\mathbb{A}}(C)$.*

Proof. $(\mathsf{L}_{\mathbb{A}}(\mathsf{c}_1,\dots,\mathsf{c}_n),\{\mathsf{c}_1,\dots,\mathsf{c}_n\})\subseteq(\mathsf{L}_{\mathbb{A}}(\mathsf{c}_1,\dots,\mathsf{c}_m),\{\mathsf{c}_1,\dots,\mathsf{c}_m\})$ for $n\leqslant m$ so the result follows at once from 2.11. ▯

Applications of the results of this section appear in the next two sections as well as in Chapter VIII.

2.13—2.16 Exercises

2.13. Let $\mathsf{L}_{\mathbb{A}}$ be a fragment if $\mathsf{L}_{\infty\omega}$ and let $\mathfrak{M},\mathfrak{N}$ be L-structures. $\mathfrak{M}$ is an $\mathsf{L}_{\mathbb{A}}$-*elementary substructure* of $\mathfrak{N}$, written

$$\mathfrak{M}\prec\mathfrak{N}\,(\mathsf{L}_{\mathbb{A}})$$

if $\mathfrak{M}\subseteq\mathfrak{N}$ and for every $\varphi(v_1,\dots,v_n)\in\mathsf{L}_{\mathbb{A}}$ and every $a_1,\dots,a_n\in\mathfrak{M}$

$$\mathfrak{M}\models\varphi[a_1,\dots,a_n]\quad\text{iff}\quad\mathfrak{N}\models\varphi[a_1,\dots,a_n].$$

(i) Prove that if $\mathfrak{M} \subseteq \mathfrak{N}$ then $\mathfrak{M} \prec \mathfrak{N}\,(\mathsf{L}_{\mathbb{A}})$ iff for every formula $\exists x\, \varphi(x, y_1, \ldots, y_n) \in \mathsf{L}_{\mathbb{A}}$ and every $a_1, \ldots, a_n \in \mathfrak{M}$, if

$$\mathfrak{N} \models \exists x\, \varphi(x, a_1, \ldots, a_n)$$

then there is a $b \in \mathfrak{M}$ such that

$$\mathfrak{N} \models \varphi(b, a_1, \ldots, a_n).$$

(ii) Prove that if

$$\mathfrak{M}_\alpha \prec \mathfrak{M}_\beta\,(\mathsf{L}_{\mathbb{A}})$$

for $\alpha < \beta < \gamma$ and $\mathfrak{M} = \bigcup_{\beta < \gamma} \mathfrak{M}_\beta$, then

$$\mathfrak{M}_\beta \prec \mathfrak{M}\,(\mathsf{L}_{\mathbb{A}})$$

for all $\beta < \gamma$.

2.14. Let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment with constants and let $\mathfrak{M}$, $\mathfrak{N}$ be Skolem structures for $\mathsf{L}_{\mathbb{A}}$. Show that if $\mathfrak{M} \subseteq \mathfrak{N}$ then $\mathfrak{M} \prec \mathfrak{N}\,(\mathsf{L}_{\mathbb{A}})$. [Use 2.13 (i).]

2.15 (Downward Lowenheim-Skolem-Tarski Theorem). Let $\mathsf{L}_{\mathbb{A}}$ be a fragment of $\mathsf{L}_{\infty\omega}$ and let $\kappa \geqslant \mathrm{card}(\mathsf{L}_{\mathbb{A}})$. Let $\mathfrak{M}$ be an L-structure, $X \subseteq \mathfrak{M}$, $\kappa \leqslant \mathrm{card}(\mathfrak{M})$, $\mathrm{card}(X) \leqslant \kappa$. Prove that there is an $\mathfrak{N}$ with

$$\mathfrak{N} \prec \mathfrak{M}\,(\mathsf{L}_{\mathbb{A}}), \quad \mathrm{card}(\mathfrak{N}) = \kappa, \quad \text{and} \quad X \subseteq \mathfrak{N}.$$

[By 2.4 you may assume $\mathsf{L}_{\mathbb{A}}$ is a Skolem fragment and that $\mathfrak{M}$ is a Skolem structure for $\mathsf{L}_{\mathbb{A}}$.]

2.16. If $\mathfrak{M}$ is an L-structure and $X \subseteq M$ then $\mathrm{Hull}_{\mathfrak{M}}(X)$ is the smallest substructure of $\mathfrak{M}$ containing X.

(i) Prove

$$\mathrm{card}\,(\mathrm{Hull}_{\mathfrak{M}}(X)) = \max\,\{\aleph_0, \mathrm{card}(\mathsf{L}), \mathrm{card}\,(X)\}.$$

(ii) Prove that if $\mathfrak{M}$ is a Skolem structure for a Skolem fragment $\mathsf{L}_{\mathbb{A}}$ and $X \subseteq \mathfrak{M}$ then

$$\mathrm{Hull}_{\mathfrak{M}}(X) \prec \mathfrak{M}(\mathsf{L}_{\mathbb{A}}).$$

2.17 Notes. The essential content of the Weak Completeness Theorem is as old as the Henkin [1949] proof of the completeness theorem for $\mathsf{L}_{\omega\omega}$. As we have tried to suggest in 2.9, it is implicit in the Model Existence Theorem. Only recently, however, has it become clear that the result is useful enough to deserve to be called a Weak Completeness Theorem. (The perjorative "weak" is there for the same reason as in § III.4; there is no nice notion of provability to go along with it.)

The first explicit statement of the Weak Completeness Theorem appears as Lemma 1.5 in Barwise-Kunen [1971], where it was used to attack the model theory of uncountable fragment.

Our treatment of Skolem fragments is a modification of that contained in Lecture 13 of Keisler [1971]. In particular, the exercises are proven there (in the countable case).

3. Pinning Down Ordinals: the General Case

Several of the examples in § 1 hinge on our ability to pin down ordinals larger then $o(\mathbb{A})$ by a Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$, for certain uncountable admissible sets $\mathbb{A}$. We will see, in fact, that a good deal of the model theory of uncountable, admissible fragments revolves about this question of pinning down ordinals. For this reason we choose it as the first application of the Weak Completeness Theorem.

The proof of the next theorem proves more than we state. In fact, it will allow us to compute exactly the ordinals pinned down by theories, once we develop some recursion theoretic machinery in the next chapter. For now we content ourselves with a crude statement of the result.

3.1 Theorem. *Let $T = T(<, \ldots)$ be a set of sentences of $\mathsf{L}_{\infty\omega}$. If T pins down ordinals then there is a ξ such that all ordinals pinned down by T are less than ξ.*

Proof. We may assume that T has models since otherwise $\xi = 0$ will do. We may also assume that if T pins down α and $\beta < \alpha$ the T pins down β, by a remark in § III.7. By 2.4 we may assume that $T \subseteq \mathsf{L}_{\mathbb{A}}$ where $\mathsf{L}_{\mathbb{A}}$ is a Skolem fragment and that $T_{\text{Skolem}} \subseteq T$. We want to set things up to apply the special form of 2.12, the Union of Chain Lemma, so let $C = \{\mathsf{c}_n \mid 0 < n < \omega\}$ be a set of new constant symbols. Let $\mathfrak{S}_n$ be the set of all supervalidity properties $\mathscr{D}$ for $\mathsf{L}_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n)$ (this is just $\mathsf{L}_{\mathbb{A}}$ if $n = 0$) such that

$$T \subseteq \mathscr{D} \quad \text{and} \quad (\mathsf{c}_2 < \mathsf{c}_1) \in \mathscr{D}, \ldots, (\mathsf{c}_n < \mathsf{c}_{n-1}) \in \mathscr{D}.$$

(For $n = 0, 1$, none of the sentences involving the c_i occur.) Since T has a model $\mathfrak{M}$, the s.v.p. $\mathscr{D}_{\mathfrak{M}}$ given by $\mathfrak{M}$ is in $\mathfrak{S}_0$, so $\mathfrak{S}_0 \neq \emptyset$. Let

$$\mathfrak{S} = \bigcup\nolimits_{0 \leqslant n < \omega} \mathfrak{S}_n$$

and put an ordering $\prec$ on $\mathfrak{S}$ by

$$\mathscr{D}' \prec \mathscr{D}$$

if $\mathscr{D} \subseteq \mathscr{D}'$ and the (unique) n such that $\mathscr{D}' \in \mathfrak{S}_n$ is greater than the unique m such that $\mathscr{D} \in \mathfrak{S}_m$. (Note that for $\mathscr{D} \in \mathfrak{S}$, we can tell which n has $\mathscr{D} \in \mathfrak{S}_n$ by just seeing what the largest n is such that $(\mathsf{c}_n = \mathsf{c}_n) \in \mathscr{D}$.) We claim that

(1) $\langle \mathfrak{S}, \prec \rangle$ is well founded.

For suppose

$$\mathscr{D}_0 \succ \mathscr{D}_1 \succ \mathscr{D}_2 \succ \cdots.$$

Let $\mathscr{D}_\infty = \bigcup_n \mathscr{D}_n$. By the union of chain lemma, $\mathscr{D}_\infty$ is an s.v.p. and hence, by the Weak Completeness Theorem, there is a model

$$(\mathfrak{M}, a_1, a_2, \ldots)$$

of $\mathscr{D}_\infty$, where a_n is the interpretation of c_n. But then $\mathfrak{M} \models T$, and $a_{n+1} < a_n$ for all $n < \omega$ which contradicts the hypothesis that T pins down ordinals. This proves (1).

Using (1) it is easy to get an upper bound for the ordinals pinned down by T. By (1), each $\mathscr{D} \in \mathfrak{S}$ has an ordinal rank $\rho(\mathscr{D})$,

$$\rho(\mathscr{D}) = \sup\{\rho(\mathscr{D}') + 1 \mid \mathscr{D}' \in \mathfrak{S}, \mathscr{D}' \prec \mathscr{D}\},$$

and $\langle \mathfrak{S}, \prec \rangle$ has a rank

$$\xi = \sup\{\rho(\mathscr{D}) + 1 \mid \mathscr{D} \in \mathfrak{S}\}.$$

We will prove that

(2) if $\mathscr{D} \in \mathfrak{S}_n$ and $(\mathfrak{M}, a_1, \ldots, a_n) \models \mathscr{D}$ then the $<^{\mathfrak{M}}$ predecessors of a_n have order type $\leqslant \rho(\mathscr{D})$ when $n > 0$; if $n = 0$ then $<^{\mathfrak{M}}$ has order type $\leqslant \rho(\mathscr{D})$.

Since every $\mathfrak{M} \models T$ is a model of $\mathscr{D}_{\mathfrak{M}} \in S_0$, and $\rho(\mathscr{D}_{\mathfrak{M}}) < \xi$, (2) gives us:

(3) *every model $\mathfrak{M}$ of T has $<^{\mathfrak{M}}$ of order type less than ξ,*

which proves the theorem. We prove (2) by induction on $\rho(\mathscr{D})$. Suppose $\mathscr{D} \in \mathfrak{S}_n$, $\alpha = \rho(\mathscr{D})$, $(\mathfrak{M}, a_1, \ldots, a_n) \models \mathscr{D}$ but that the predecessors of a_n have order type $> \alpha$. (The case $n = 0$ is essentially the same.) Let a_{n+1} be the α^{th} member of the field of $<^{\mathfrak{M}}$ as ordered by $<^{\mathfrak{M}}$ and let $\mathscr{D}'$ be the s.v.p. given by

$$\mathfrak{M}' = (\mathfrak{M}, a_1, \ldots, a_{n+1}).$$

Then $\mathscr{D}' \in \mathfrak{S}_{n+1}$, and $\mathscr{D} \subseteq \mathscr{D}'$ so $\mathscr{D}' \prec \mathscr{D}$ and hence $\rho(\mathscr{D}') < \alpha$. But $\mathfrak{M}'$ is a model of $\mathscr{D}'$ with the precedessors of a_{n+1} of order type $\alpha > \rho(\mathscr{D}')$, contradicting the inductive hypothesis. $\square$

Without Theorem 3.1 we could not be sure that the next definition made sense.

3.2 Definition. Let $\mathbb{A}$ be an admissible set.

(i) $h(\mathbb{A})$ is the least ordinal not pinned down by some sentence $\varphi(<, \ldots)$ in some admissible fragment $\mathsf{L}_{\mathbb{A}}$.

(ii) $h_\Sigma(\mathbb{A})$ is the least ordinal not pinned down by some Σ_1 theory of some admissible fragment $\mathsf{L}_{\mathbb{A}}$.

In the next chapter we will determine exact recursion-theoretic descriptions of $h_\Sigma(\mathbb{A})$ and, in most cases, of $h(\mathbb{A})$.

Let us collect together remarks made at various places.

3.3 Proposition. *Let* $\mathbb{A}$ *be admissible.*

(i) $h_\Sigma(\mathbb{A})$ *is the* sup *of the ordinals pinned down by* Σ_1 *theories of* $\mathsf{L}_\mathbb{A}$; *similarly,* $h(\mathbb{A})$ *is the* sup *of the ordinals pinned down by single sentences of* $L_\mathbb{A}$.

(ii) $h_\Sigma(\mathbb{A}) \geqslant h(\mathbb{A}) \geqslant o(\mathbb{A})$.

(iii) *If* $\mathbb{A}$ *is countable then*

$$h_\Sigma(\mathbb{A}) = h(\mathbb{A}) = o(\mathbb{A}).$$

(iv) *If* $\mathbb{A}$ *is* Σ_1 *compact then*

$$h_\Sigma(\mathbb{A}) = h(\mathbb{A}).$$

Proof. Only (iv) needs proving. Suppose $\mathbb{A}$ is Σ_1 compact but that $h_\Sigma(\mathbb{A}) > h(\mathbb{A})$. Let $T(<)$ be a Σ_1 theory which pins down some $\beta \geqslant h(\mathbb{A})$. Add new constant symbols $\mathsf{c}_1, \ldots, \mathsf{c}_n, \ldots$ and let T' be T plus the axioms

$$\mathsf{c}_{n+1} < \mathsf{c}_n \quad (\text{all } n < \omega).$$

Since $\beta \geqslant h(\mathbb{A})$, every $\mathbb{A}$-finite subset of T has a model which is not well founded so every $\mathbb{A}$-finite subset of T' has a model. Thus, by Σ_1 compactness, T' has a model, a contradiction. □

$H(\omega_1)$ is an example of a set $\mathbb{A}$ with $h_\Sigma(\mathbb{A}) > h(\mathbb{A}) = o(\mathbb{A})$. $\mathbb{HYP}(H(\omega_1))$ is an example with $h_\Sigma(\mathbb{A}) = h(\mathbb{A}) > o(\mathbb{A})$.

The next theorem is extremely useful in computations which involve $h_\Sigma(A)$ and $h(A)$.

3.4 Theorem. *Let* $\mathbb{A}$ *be admissible and let* $F\colon \mathrm{Ord}^n \to \mathrm{Ord}$ *be an n-ary function on ordinals which is* Σ_1 *definable in* KPU.

(i) $\alpha_1, \ldots, \alpha_n < h_\Sigma(\mathbb{A})$ *implies* $F(\alpha_1, \ldots, \alpha_n) < h_\Sigma(\mathbb{A})$.

(ii) $\alpha_1, \ldots, \alpha_n < h(\mathbb{A})$ *implies* $F(\alpha_1, \ldots, \alpha_n) < h(\mathbb{A})$.

Proof. We first prove (i) in case $n=2$. The case for $n \neq 2$ is similar. Let

$$F\colon \mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$$

be Σ_1 definable in KPU, hence in the stronger KP, say by the Σ_1 formula $\sigma(x_1, x_2, y)$:

(4) $\mathrm{KP} \vdash \forall x_1 x_2\, \exists!\, y\, \sigma(x_1, x_2, y)$,

(5) for all α_1, α_2 $\mathbb{V} \models \sigma(\alpha_1, \alpha_2, F(\alpha_1, \alpha_2))$.

Suppose $\alpha_1, \alpha_2 < h_\Sigma(\mathbb{A})$ and let $\beta = F(\alpha_1, \alpha_2)$. We need to prove that $\beta < h_\Sigma(\mathbb{A})$. Let $T_1(<_1, \mathsf{R}_1)$, $T_2(<_2, \mathsf{R}_2)$ be Σ_1 theories which pin down α_1, α_2 respectively, the case with more relation symbols being similar. We will define a Σ_1 theory $T(<)$ which pins down β, but first let us exhibit its intended model $\mathfrak{M}$, the one with $<^{\mathfrak{M}}$ of type β. Let κ be a regular cardinal, $\alpha_1, \alpha_2 < \kappa$, so that $\beta < \kappa$. Let

$$\mathfrak{M}_1 = \langle M_1, <_1, R_1 \rangle \models T_1, \quad <_1 \text{ of order type } \alpha_1,$$

$$\mathfrak{M}_2 = \langle M_2, <_2, R_2 \rangle \models T_2, \quad <_2 \text{ of order type } \alpha_2.$$

By the downward Löwenheim-Skolem Theorem (Exercise 2.15) (and the fact that isomorphic models satisfy the same sentences) we may assume

$$\alpha_i \subseteq M_i \subseteq \kappa \quad \text{and} \quad <_i = \in \upharpoonright \alpha_i .$$

Now let

$$\mathfrak{M} = \langle H(<), \in, <, M_1, <_1, R_1, M_2, <_2, R_2, \alpha_1, \alpha_2, \beta \rangle$$

where $< = \in \upharpoonright \beta$ and α_1, α_2 and β are treated as elements, not as subsets. Then $\mathfrak{M}$ is clearly a model of the following set of sentences, where U_i is interpreted as M_i, c_i is interpreted as α_i and d as β.

$\varphi^{(\mathsf{U}_1)}$ for all $\varphi \in T_1$,

$\varphi^{(\mathsf{U}_2)}$ for all $\varphi \in T_2$,

KP,

$\mathsf{c}_1, \mathsf{c}_2, \mathsf{d}$ are ordinals,

"$<_1 = \in \upharpoonright \mathsf{c}_1$",

"$<_2 = \in \upharpoonright \mathsf{c}_2$",

"$< \; = \in \upharpoonright \mathsf{d}$",

$\sigma(\mathsf{c}_1, \mathsf{c}_2, \mathsf{d})$.

If we call the above set of sentences $T(<, \ldots)$, then $\mathfrak{M}$ is a model of T with $<^{\mathfrak{M}}$ of order type β. We need to prove that every model $\mathfrak{M}$ of T has $<^{\mathfrak{M}}$ well ordered. Thus, let

$$\mathfrak{M} = \langle M, E, <, U_1, <_1, R_1, U_2, <_2, R_2, a_1, a_2, d \rangle$$

be any model of T. Identify the well-founded part of $\langle M, E \rangle$ with an admissible set $\langle B, \in \rangle$ by the Truncation Lemma. Since, for $i = 1, 2$

$$\langle U_i, <_i, R_i \rangle \models T_i,$$

$<_i$ is well ordered, so a_1 and a_2 are real ordinals and $a_1, a_2 \in B$. By (4),

$$\langle B, \in \rangle \models \exists ! y \, \sigma(a_1, a_2, y).$$

By (5), and the persistence of Σ formulas,

$$\langle B,\in\rangle \models \sigma(a_1,a_2,F(a_1,a_2))$$

and, since

$$\langle B,\in\rangle \subseteq_{\mathbf{end}} \langle M,E\rangle,$$

we have by persistence,

$$\langle M,E\rangle \models \sigma(a_1,a_2,F(a_1,a_2))$$

so that $b=F(a_1,a_2)$. Since $b=F(a_1,a_2)\in B$, and $<\,=\,\in\upharpoonright b$, $<$ is a real well-ordering. This proves (i).

The proof of (ii) is exactly the same when $o(\mathbb{A})>\omega$, since then we may form $\bigwedge$KP and the rest as a single sentence of $\mathsf{L}_{\mathbb{A}}$. If $o(\mathbb{A})=\omega$ we must replace KP by a single sentence θ of ZF-Power (and hence true in $H(\kappa)$ since κ is regular) strong enough to insure that the standard part of any model of θ is an admissible set. We leave this to the student. $\square$

All we will actually need of Theorem 3.4 is the following special case.

3.5 Corollary. *Let $\mathbb{A}$ be admissible. Then $h_\Sigma(\mathbb{A})$ and $h(\mathbb{A})$ are closed under ordinal successor, ordinal addition, multiplication, and exponentiation.*

Proof. We have shown that all these functions are Σ_1 definable in KP. $\square$

The final result of this section seems almost obvious, but it needs proof.

3.6 Theorem. *Let $\mathbb{A}$ be admissible.*

(i) *If T is a Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$ which pins down ordinals then there is a $\xi<h_\Sigma(\mathbb{A})$ such that every ordinal pinned down by T is less than ξ.*

(ii) *If φ is a sentence of $\mathsf{L}_{\mathbb{A}}$ which pins down ordinals then there is a $\xi<h(\mathbb{A})$ which is greater than all ordinals pinned down by φ.*

Proof. This is a typical example of a proof in soft model theory since the proof works for any logic. We prove (ii). We may assume that the sentence $\varphi(<)$ pins down an initial segment $\{\beta\mid\beta<\xi\}=\xi$ of ordinals. We show that some other sentence $\psi(\prec,\ldots)$ pins down ξ. As before, before writing down ψ, we describe its intended model $\mathfrak{M}$, the one with $\prec^{\mathfrak{M}}$ of type ξ. To simplify matters we assume $\varphi=\varphi(<,\mathsf{R})$, where R is binary, contains no other symbols. For each $\beta<\xi$, let

$$\mathfrak{M}_\beta=\langle M_\beta,<_\beta,R_\beta\rangle,\quad \mathfrak{M}\models\varphi \quad\text{and}\quad <_\beta \text{ have order type } \beta.$$

Since isomorphic structures satisfy the same sentences, we can rearrange $\mathfrak{M}_\beta$ a bit and assume $\beta\subseteq\mathfrak{M}_\beta$ and $<_\beta=\in\upharpoonright\beta$.

Define $\mathfrak{M} = \langle M, U, \prec, N, S_1, S_2 \rangle$ where

$$\begin{array}{lll} M = \bigcup_{\beta<\xi} M_\beta, & & U = \xi \subseteq M, \\ N(\beta, x) & \text{iff} & \beta < \xi \wedge x \in M_\beta, \\ \beta \prec \gamma & \text{iff} & \beta \in \gamma \in \xi, \\ S_1(\beta, y, z) & \text{iff} & \beta < \xi \wedge y <_\beta z, \\ S_2(\beta, y, z) & \text{iff} & \beta < \xi \wedge R_\beta(y, z). \end{array}$$

Thus $\mathfrak{M}$ is a structure where $\prec^{\mathfrak{M}}$ has order type ξ. Let $\psi(\prec, \ldots)$ be the sentence described as follows. Let $\theta(x, \mathsf{N}, \mathsf{S}_1, \mathsf{S}_2)$ result from $\varphi(<, \mathsf{R})$ by replacing

$$\begin{array}{lll} y<z & \text{by} & \mathsf{S}_1(x,y,z), \\ \mathsf{R}(y,z) & \text{by} & \mathsf{S}_2(x,y,z), \\ \forall y\,(\ldots) & \text{by} & \forall y\,(\mathsf{N}(x,y) \rightarrow \cdots), \quad \text{and} \\ \exists y\,(\ldots) & \text{by} & \exists y\,(\mathsf{N}(x,y) \wedge \cdots) \end{array}$$

taking care to avoid clashes of variables. Let ψ be the conjunction of:

(6) $\forall x\,[\mathsf{U}(x) \rightarrow \theta(x, \mathsf{N}, \mathsf{S}_1, \mathsf{S}_2)]$;

(7) "U is linearly ordered by $\prec$";

(8) $\forall x\,[\mathsf{U}(x) \rightarrow \forall y, z (y \prec z \prec x \leftrightarrow \mathsf{S}_1(x, y, z)]$.

It is clear that $\mathfrak{M}$ is a model of ψ since (6) just asserts that each $\mathfrak{M}_\beta$ is a model of θ. We need to show that any other model

$$\mathfrak{M} = \langle M, U, \prec, N, S_1, S_2 \rangle$$

of ψ has $\prec$ well ordered. To do this it suffices to prove that for any $x \in U$, the $\prec$ predecessors of x are well ordered. Let

$$\mathfrak{M}_x = \langle M_x, <_x, R_x \rangle$$

where $M_x = \{y \mid N(x, y)\}$, $y <_x z$ iff $S_1(x, y, z)$ and $R_x(y, z)$ iff $S_2(x, y, z)$. By (6), $\mathfrak{M}_x \models \varphi$, so $<_x$ is a well-ordering and $<_x$ agrees with $\prec$ on the predecessors of x. Thus ψ does pin down ordinals, ξ among them. □

3.7—3.8 Exercises

3.7. Let $\mathbb{A}$ be admissible, $o(\mathbb{A}) = \omega$, where $\mathbb{A}$ is Σ_1 compact. Show that

$$h(\mathbb{A}) = \omega.$$

3.8. Let $\mathbb{A}$ be Σ_1 compact and suppose that $\alpha = o(\mathbb{A}) > \omega$ is such that for some $x \in \mathbb{A}$,

$$\alpha = \text{least}\,\beta\ (\mathrm{L}(x,\beta) \text{ is admissible}).$$

Prove that

$$h_\Sigma(\mathbb{A}) = o(\mathbb{A}).$$

3.9 Notes. Theorem 3.1 is due to Lopez-Escobar [1966]. His proof, however, was by way of Hanf numbers and gave no clue as to the exact description of $h(\mathbb{A})$ or $h_\Sigma(\mathbb{A})$, even for $\mathbb{A} = H(\aleph_{\alpha+1})$. The proof given here is taken from Barwise-Kunen [1971]. Theorem 3.4 is also taken from there.

There are, by the way, admissible sets which are Σ_1 compact but such that $h_\Sigma(\mathbb{A}) > o(\mathbb{A})$. This follows from Theorem VIII.8.3. It is known that $h(\mathbb{A})$ need not be admissible. It is not known whether $h_\Sigma(\mathbb{A})$ is always admissible, though it seems unlikely.

4. Indiscernibles and Upward Löwenheim-Skolem Theorems

In this section we show how to use the Weak Completeness Theorem and the ordinal $h_\Sigma(A)$ to tackle some model theoretic problems for L_A. The material in this section is not used elsewhere in this book.

The simplest result to state is the following theorem, stated in terms of the Beth sequence. Given a cardinal κ, define the cardinal $\beth_\alpha(\kappa)$ by induction on α:

$$\beth_0(\kappa) = \kappa,$$

$$\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)},$$

$$\beth_\lambda(\kappa) = \sup_{\alpha<\lambda} \beth_\alpha(\kappa).$$

We write $\beth_\alpha$ for $\beth_\alpha(0)$, but warn the reader that some authors use $\beth_\alpha$ for $\beth_\alpha(\aleph_0)$. With our definition, $\beth_\alpha = \text{card}(V_\alpha)$.

4.1 Theorem. *Let $\mathbb{A}$ be an admissible set, let $\kappa = \text{card}(\mathbb{A})$ and $\alpha = h_\Sigma(\mathbb{A})$. Let T be a Σ_1 theory of $\mathsf{L}_\mathbb{A}$. If, for each $\beta < \alpha$, T has a model of power $\geqslant \beth_\beta(\kappa)$, then for any $\lambda \geqslant \kappa$, T has a model of power λ.*

The proof of 4.1 is given in 4.13 below. Actually the proof of this theorem is no more complicated for uncountable L_A; it is just that for countable $\mathbb{A}$ we know that $h_\Sigma(\mathbb{A}) = o(\mathbb{A})$. Thus 4.1 gives us the following corollary.

4.2 Corollary. *Let $\mathbb{A}$ be a countable, admissible set and let T be a Σ_1 theory of $\mathsf{L}_\mathbb{A}$. If, for each $\beta < \alpha = o(\mathbb{A})$, T has a model of power $\geqslant \beth_\beta(\aleph_0)$, then T has a model of each infinite power.* $\square$

If $\mathbb{A}_{\mathfrak{M}}$ is not $\mathbb{HF}_{\mathfrak{M}}$ then it is easy to show that for each $\beta \in A_{\mathfrak{M}}$ there is a sentence of $\mathsf{L}_{\mathbb{A}}$ which has a model of power $\beth_\beta(\aleph_0)$ but none larger (see Exercise 4.18), so 4.2 is best possible for $\mathbb{A}_{\mathfrak{M}} \neq \mathbb{HF}_{\mathfrak{M}}$. For $\mathbb{A}_{\mathfrak{M}} = \mathbb{HF}_{\mathfrak{M}}$, $\mathsf{L}_{\mathbb{A}} = \mathsf{L}_{\omega\omega}$ so we know a better result.

For applications, there are more useful upward Löwenheim-Skolem Theorems in terms of two cardinal models.

Assume our language L has a unary symbol U. A model $\mathfrak{M} = \langle M, U, \ldots \rangle$ for L is a *model of type* (κ, λ) if

$$\mathrm{card}(M) = \kappa,$$

$$\mathrm{card}(U) = \lambda.$$

A set T of sentences of $\mathsf{L}_{\infty\omega}$ is said to *admit* (κ, λ) if T has a model $\mathfrak{M}$ of type (κ, λ).

4.3 Theorem. *Let $\mathsf{L}_{\mathbb{A}}$ be an admissible fragment, let $\kappa = \mathrm{card}(\mathbb{A})$, $\alpha = h_\Sigma(A)$. Let T be a Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$. If for each $\beta < \alpha$ there is a $\lambda \geqslant \kappa$ such that T admits $(\beth_\beta(\lambda), \lambda)$, then T admits (δ, κ) for all cardinals $\delta \geqslant \kappa$.*

Theorem 4.1 is an easy consequence of 4.3 by adding a new symbol U to L without mentioning it in the theory T of 4.1. On the other hand, a direct proof of 4.1 is a bit simpler than the proof of 4.3, and since the student may be interested in 4.1, we will also give a direct proof of it.

4.4 Corollary. *Let T be a Σ_1 theory of a countable admissible fragment $\mathsf{L}_{\mathbb{A}}$. Suppose that for each $\beta < \alpha = o(\mathbb{A})$, there is a $\lambda \geqslant \omega$ such that T admits $(\beth_\beta(\lambda), \lambda)$. Then T admits (λ, ω) for all $\lambda \geqslant \omega$.*

Proof. Immediate from 4.3 since $h_\Sigma(\mathbb{A}) = o(\mathbb{A})$. □

4.5 Corollary (Morley's Two Cardinal Theorem). *Let T be a countable theory of $\mathsf{L}_{\omega_1\omega}$. Suppose that for each $\alpha < \omega_1$ there is a $\lambda \geqslant \omega$ such that T admits $(\beth_\alpha(\lambda), \lambda)$. Then T admits (λ, ω) for all $\lambda \geqslant \omega$.*

Proof. Immediate from 4.4 by putting T in some countable admissible fragment. □

The reader of Keisler [1971] will have discovered many applications of Corollary 4.5. Some of these have routine generalizations using 4.3.

Two-cardinal models are extremely natural when one is working with models of set theory of urelements. How many times have we written a typical model of KPU as a single sorted structure

$$\mathfrak{A}_{\mathcal{M}} = \langle A \cup M, M, \ldots \rangle ?$$

In fact, we can use such models to prove that Theorem 4.3 is an optimal result of its type, except for trivial generalizations using downward Löwenheim-Skolem arguments.

4.6 Example. *Let $\mathbb{A}$ be an admissible set with $\kappa = \text{card}(\mathbb{A})$, $\alpha = h_\Sigma(\mathbb{A})$. For any $\beta < \alpha$ one can find a Σ_1 theory $T = T(\mathsf{U}, \ldots)$ of $\mathsf{L}_{\mathbb{A}}$ and a ξ, $\beta < \xi < h_\Sigma(\mathbb{A})$ such that*

(i) *T has a model of type $(\beth_\beta(\kappa), \kappa)$.*

(ii) *If $\mathfrak{M}$ is a model of T of type (λ, δ) then $\lambda \leqslant \beth_\xi(\delta)$. In particular, T has no model of type $(\beth_\alpha(\kappa), \kappa)$.*

Proof. Let $T_0 = T_0(<)$ be a Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$ which pins down β. Let $\xi < h_\Sigma(\mathbb{A})$ be greater than all ordinals pinned down by T_0, by Theorem 3.6. Before describing T we describe its intended model, the one of type $(\beth_\beta(\kappa), \kappa)$. Let M be a set of urelements of power κ. Let

$$\mathfrak{M}_0 = (M_0, <, \ldots)$$

be a model of T_0 where $<$ has order type β. By the Downward Lowenheim-Skolem theorem we may assume $\text{card}(\mathfrak{M}_0) \leqslant \max(\kappa, \text{card}(\beta))$ so we may as well assume $M_0 \subseteq M \cup \beta$. Now let

$$\mathfrak{M} = (M \cup \mathrm{V}_{\mathfrak{M}}(\beta), M, \in, F, M_0, <, \ldots)$$

where, by definition,

$$F_0(a) = \text{rank of } a \text{ in } \mathbb{V}_M,$$

$$F(a) = \text{the } F_0(a)\text{-th member of } <.$$

The theory T is defined as follows. For each $x \in A$ let c_x be a constant symbol, so there are κ of them. T consists of

$$\mathsf{c}_x \neq \mathsf{c}_y \quad \text{for all} \quad x, y \in A,\ x \neq y,$$

$$\mathsf{U}(\mathsf{c}_x) \quad \text{for all} \quad x \in A,$$

$$\text{Extensionality (as in KPU)},$$

$$\forall x\, \forall y\, [x \in y \rightarrow \mathsf{F}(x) < \mathsf{F}(y)],$$

$$\varphi^{(\mathsf{U}_0)} \quad \text{for all} \quad \varphi \in T_0.$$

Here U and U_0 are new unary symbols. The theory T clearly holds in M. On the other hand, if $\mathfrak{M} = \langle A, U, E, F, U_0, <, \ldots \rangle$ is another model of T then $\langle U_0, <, \ldots \rangle \models T_0$, so $<$ is well ordered of order type $< \xi$. But then F insures that E is well founded and of rank $< \xi$ so $\langle A, U, E \rangle$ is isomorphic to a submodel of $\mathrm{V}_U(\xi)$ and hence has $\text{card}(A) \leqslant \beth_\xi(\text{card}(U))$. $\square$

We now turn to the tools for the proofs of these theorems. Anyone familiar with the model theory of $\mathsf{L}_{\omega\omega}$ is aware of the importance of the Ehrenfeucht-Mostowski method of indiscernibles. It plays an even more important role in the model theory of $\mathsf{L}_{\infty\omega}$.

4.7 Definition. Let $\mathsf{L}_{\mathbb{A}}$ be a fragment of $\mathsf{L}_{\infty\omega}$, $\mathfrak{M}$ be an L-structure and let $\langle X, < \rangle$ be a linearly ordered set with $X \subseteq \mathfrak{M}$. We say that $\langle X, < \rangle$ is a *set of indiscernibles* (for $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{M}$) if for every n and any two increasing n-tuples from $\langle X, < \rangle$,

$$x_1 < \cdots < x_n, \qquad y_1 < \cdots < y_n$$

we have

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv (\mathfrak{M}, y_1, \ldots, y_n) \quad (\mathsf{L}_{\mathbb{A}}),$$

i.e. the n-tuples $\langle x_1, \ldots, x_n \rangle$, $\langle y_1, \ldots, y_n \rangle$ satisfy the same formulas $\varphi(v_1, \ldots, v_n)$ of $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{M}$. If $\mathfrak{M} = \langle M, U, \ldots \rangle$ then we say that $\langle X, < \rangle$ is a *set of indiscernibles over* U if, for every finite set $u_1, \ldots, u_m \in U$ and all increasing n-typles from $\langle X, < \rangle$

$$x_1 < \cdots < x_n, \qquad y_1 < \cdots < y_n$$

we have

$$(\mathfrak{M}, u_1, \ldots, u_m, x_1, \ldots, x_n) \equiv (\mathfrak{M}, u_1, \ldots, u_m, y_1, \ldots, y_n) \quad (\mathsf{L}_{\mathbb{A}}).$$

The $<$ relation on X need not be definable on $\mathfrak{M}$ in the above definition.

The latter notion is really a special case of the first, for let $\mathfrak{M} = \langle M, U, \ldots \rangle$ be a structure for $\mathsf{L}_{\mathbb{A}}$, let $C = \{\mathsf{c}_u \mid u \in U\}$ be a set of new constant symbols, and let $\mathfrak{M}' = (\mathfrak{M}, u)_{u \in U}$ be the canonical expansion of $\mathfrak{M}$ to a model for $\mathsf{L}_{\mathbb{A}}(C)$. (The language $\mathsf{L}_{\mathbb{A}}(C)$ is defined in § 2.) Then $\langle X, < \rangle$ is a set of indiscernibles over U for $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{M}$ iff $\langle X, < \rangle$ is a set of indiscernibles for $\mathsf{L}_{\mathbb{A}}(C)$ in $\mathfrak{M}'$.

Indiscernibles help us build large models, and hence prove our theorems by means of the following Stretching Theorem.

4.8 Stretching Theorem. *Let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment with constants and let $\mathfrak{M}$ be a Skolem structure for $\mathsf{L}_{\mathbb{A}}$. Let $\langle X, < \rangle$ be an infinite set of indiscernibles for $\mathsf{L}_{\mathbb{A}}$. For any infinite linearly ordered set $\langle Y, < \rangle$ there is a Skolem structure $\mathfrak{N}$ for $\mathsf{L}_{\mathbb{A}}$ such that:*

(i) *$\langle Y, < \rangle$ is a set of indiscernibles for $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{N}$;*

(ii) *If $x_1 < \cdots < x_n$ in $\langle X, < \rangle$ and $\langle y_1 < \cdots < y_n \rangle$ in $\langle Y, < \rangle$*

$$\textit{then} \quad (\mathfrak{M}, x_1, \ldots, x_n) \equiv (\mathfrak{N}, y_1, \ldots, y_n) \quad (\mathsf{L}_{\mathbb{A}});$$

(iii) *In particular,* $\operatorname{card}(\mathfrak{N}) \geqslant \operatorname{card}(Y)$ *and* $\mathfrak{M} \equiv \mathfrak{N}$ $(\mathsf{L}_{\mathbb{A}})$.

Proof. Part (iii) is just part (ii) with $n=0$. Since the distinguished constants of $\mathsf{L}_{\mathbb{A}}$ do not play any role in this proof we simply assume $\mathsf{L}_{\mathbb{A}}$ is a Skolem fragment. Let

$$C = \{\mathsf{c}_y \mid y \in Y\}$$

be a set of new constant symbols and form $\mathsf{L}_{\mathbb{A}}(C)$ as described in § 2. Then $\mathsf{L}_{\mathbb{A}}(C)$ is a Skolem fragment with constants C. We define a set $\mathscr{D}$ of formulas of $\mathsf{L}_{\mathbb{A}}(C)$

as follows. Any formula of $\mathsf{L}_{\mathbb{A}}$ can be written in the form

(1) $\varphi(v_1,\ldots,v_n, \mathsf{c}_{y_1}/v_{n+1},\ldots,\mathsf{c}_{y_m}/v_{n+m})$

where

$$y_1<\cdots<y_m \quad \text{in} \quad \langle Y,<\rangle.$$

Put the formula (1) into $\mathscr{D}$ just in case

(2) $(\mathfrak{M},x_1,\ldots,x_m)\models\forall v_1,\ldots,v_n\, \varphi(\vec{v},\mathsf{c}_{y_1},\ldots,\mathsf{c}_{y_m})$

for some increasing sequence

$$x_1<\cdots<x_m \quad \text{in} \quad \langle X,<\rangle,$$

where x_i interprets c_{y_i}, of course. We claim that

(3) $\mathscr{D}$ is a supervalidity property for $\mathsf{L}_{\mathbb{A}}(C)$.

If (1) is a logical axiom, then (2) certainly holds, so (1)$\in\mathscr{D}$. We need to see that if $\varphi(\vec{v},\vec{\mathsf{c}})\in\mathscr{D}$ then $(\neg\varphi(\vec{v},\vec{\mathsf{c}}))\notin\mathscr{D}$. If not, then we would have

$$(\mathfrak{M},x_1,\ldots,x_m)\models\forall v_1,\ldots,v_n\, \varphi(\vec{v},\mathsf{c}_1,\ldots,\mathsf{c}_m),$$
$$(\mathfrak{M},x'_1,\ldots,x'_m)\models\forall v_1,\ldots,v_n\, \neg\varphi(\vec{v},\mathsf{c}_1,\ldots,\mathsf{c}_m)$$

where $x_1<\cdots<x_m$, $x'_1<\cdots<x'_m$ in $\langle X,<\rangle$. But this contradicts the indiscernibility of $\langle X,<\rangle$. The other clauses are equally trivial. We check the $\bigvee$-rule and leave the other three to the student. Suppose $\psi(\mathsf{c}_1,\ldots,\mathsf{c}_m)=\bigvee\Phi$ is a sentence of $\mathsf{L}_{\mathbb{A}}(C)$ and $\psi(\mathsf{c}_1,\ldots,\mathsf{c}_m)\in\mathscr{D}$. Then

$$(\mathfrak{M},x_1,\ldots,x_m)\models\bigvee\Phi$$

so, for some $\varphi\in\Phi$,

$$(\mathfrak{M},x_1,\ldots,x_m)\models\varphi$$

so $\varphi\in\mathscr{D}$. Thus $\mathscr{D}$ is a supervalidity property.

(4) If $\varphi(v_1,\ldots,v_n)\in\mathsf{L}$, $y_1<\cdots<y_n$, $y'_1<\cdots<y'_n$ in $\langle Y,<\rangle$ then the following $\mathsf{L}_{\mathbb{A}}(C)$ sentence is in $\mathscr{D}$:

(*) $$\varphi(\mathsf{c}_{y_1},\ldots,\mathsf{c}_{y_n})\leftrightarrow\varphi(\mathsf{c}_{y'_1},\ldots,\mathsf{c}_{y'_n}).$$

To see what is going on here, suppose φ is $\varphi(v_1,v_2,v_3)$ and that

$$y_1<y_2<y_3 \quad \text{and} \quad y'_1<y'_2<y'_3.$$

To see that the sentence (*) in question is in $\mathscr{D}$ we must first arrange these elements of $\langle Y,<\rangle$ in order. Suppose, for example, that

$$y_1 < y_1' < y_2' = y_2 < y_3 .$$

Thus there are only five elements in this case. Let $\psi(v_1,\ldots,v_5)$ be

$$\varphi(v_1,v_4,v_5) \leftrightarrow \varphi(v_2,v_3,v_4) .$$

The definition of $\mathscr{D}$ says that (*) is in $\mathscr{D}$ iff

$$\mathfrak{M} \models \psi[x_1,x_2,x_3,x_4,x_5]$$

whenever $x_1 < x_2 < \cdots < x_5$. That is, just in case

$$\mathfrak{M} \models \varphi[x_1,x_4,x_5] \quad \text{iff} \quad \mathfrak{M} \models \varphi[x_2,x_3,x_4]$$

whenever $x_1 < \cdots < x_5$. This is obvious from the indiscernibility of $\langle X,<\rangle$, so this proves (a typical example of) (4). Apply the Weak Completeness Theorem to get a model $(\mathfrak{N},a_y)_{y\in Y}$ of $\mathscr{D}$. Since $(\mathsf{c}_y \neq \mathsf{c}_{y'}) \in \mathscr{D}$ for $y \neq y'$, we can identify a_y with y. Then $\mathfrak{N}$ has properties (i), (ii) of the theorem. ☐

Using the Stretching Theorem we can reduce our theorems to proving the existence of models with indiscernibles, as in the next lemma.

4.9 Lemma. *Let $\mathsf{L}_{\mathbb{A}}$ be a Skolem fragment with constants and let T be a theory of $\mathsf{L}_{\mathbb{A}}$, $T_{\text{Skolem}} \subseteq T$. Let $\kappa = \text{card}(\mathsf{L}_{\mathbb{A}})$.*

(i) *If T has a model with an infinite set of indiscernibles for $\mathsf{L}_{\mathbb{A}}$ then T has a model of any power $\geqslant \kappa$.*

(ii) *If $T = T(\mathsf{U},\ldots)$ has a model $\mathfrak{M} = \langle M,U,\ldots\rangle$ with $\langle X,<\rangle$ an infinite set of indiscernibles over U for $\mathsf{L}_{\mathbb{A}}$ then T admits $(\lambda, \text{card}(U))$ for all $\lambda \geqslant \kappa + \text{card}(U)$.*

Proof. (i) is immediate from 4.8 (iii) and the Downward Löwenheim-Skolem Theorem for $\mathsf{L}_{\mathbb{A}}$. To prove (ii) let $\mathfrak{M}$ have $\langle X,<\rangle$ an infinite set of indiscernibles over U. Let

$$C = \{\mathsf{c}_u \mid u \in U\},$$

$$\mathfrak{M}' = (\mathfrak{M},u)_{u\in U}$$

be as usual. Thus, $\langle X,<\rangle$ is a set of indiscernibles for $\mathsf{L}_{\mathbb{A}}(C)$ in $\mathfrak{M}'$. Given $\lambda \geqslant \kappa$, let $\langle Y,<\rangle$ be a linearly ordered set of power λ and let

$$\mathfrak{N}' = (\mathfrak{N},u)_{u\in U}$$

be as given by 4.8, the Stretching Theorem. By Exercise 2.16, we may assume

$$\mathfrak{N}' = \text{Hull}_{\mathfrak{N}'}(Y),$$

since this Hull also has properties (i), (ii) of 4.8. Write $\mathfrak{N}$ as $\mathfrak{N}=\langle N, U', \ldots\rangle$. We claim that $U=U'$. For suppose $a\in U'$. Then

$$a=t(y_1,\ldots,y_n,u_1,\ldots,u_m)$$

for some term t of $\mathsf{L}_{\mathbb{A}}$, some $u_1,\ldots,u_m\in U$ and some $y_1<\cdots<y_n$ in $\langle Y,<\rangle$. But, then,

$$\mathfrak{N}\models \mathsf{U}(t(y_1,\ldots,y_n,u_1,\ldots,u_m))$$

so, by (ii) of 4.8,

$$\mathfrak{M}\models \mathsf{U}(t(x_1,\ldots,x_n,u_1,\ldots,u_m))$$

whenever $x_1<\cdots<x_n$ in $\langle X,<\rangle$. Pick such a sequence of x's. Then there is a $u\in U$ such that

$$\mathfrak{M}\models u=t(x_1,\ldots,x_n,u_1,\ldots,u_m)$$

and, hence by (ii) of 4.8,

$$\mathfrak{N}\models u=t(y_1,\ldots,y_n,u_1,\ldots,u_m)$$

so

$$\mathfrak{N}\models u=a.$$

In other words, every member of U' is one of the original members of U. Thus, $\operatorname{card}(U')=\operatorname{card}(U)$ but

$$\begin{aligned}\operatorname{card}(\mathfrak{N})&=\operatorname{card}(\mathsf{L}_{\mathbb{A}}(C))+\operatorname{card}(Y)\\&=\kappa+\operatorname{card}(C)+\lambda\\&=\kappa+\operatorname{card}(U)+\lambda\\&=\lambda.\quad\square\end{aligned}$$

To construct a model with an infinite set of indiscernibles, we use the Erdős-Rado theorem of cardinal arithmetic (Lemma 4.10) to construct "coherent sets of k-variable indiscernibles" and the Weak Completeness Theorem to piece them together to get a model with a set of indiscernibles.

We use $[X]^n$ to denote the set

$$\{x\subseteq X \mid \operatorname{card}(x)=n\}.$$

4.10 Lemma (Erdős-Rado Theorem). *Let κ be an infinite cardinal and let $0<n<\omega$. Let X be a set with $\operatorname{card}(X)>\beth_{n-1}(\kappa)$ and suppose $[X]^n$ is partitioned into $\leqslant\kappa$ subsets, say $[X]^n=\bigcup_{i\in I}C_i$ where $\operatorname{card}(I)\leqslant\kappa$. There is an $X_0\subseteq X$ and an $i_0\in I$ such that*

$$\operatorname{card}(X_0)>\kappa \quad \textit{and} \quad [X_0]^n\subseteq C_{i_0}.$$

Proof. If the reader is not familiar with this result, he can find a proof in most advanced books on set theory, in Keisler [1971] or in Chang-Keisler [1973]. ▯

Let $\mathfrak{M}$ be a structure for L and let $\langle X, <\rangle$ be linearly ordered with $X \subseteq \mathfrak{M}$. Let $k<\omega$ be fixed. We say that $\langle X, <\rangle$ is a set of *k-variable indiscernibles* for $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{M}$ if, for all increasing k-tuples

$$x_1 < \cdots < x_k, \qquad y_1 < \cdots < y_k$$

in $\langle X, <\rangle$, we have

$$(\mathfrak{M}, x_1, \ldots, x_k) \equiv (\mathfrak{M}, y_1, \ldots, y_k).$$

Thus $\langle X, <\rangle$ is a set of indiscernibles iff it is a set of k-variable indiscernibles for each $k<\omega$. Also note that if $\langle X, <\rangle$ is a set of k-variable indiscernibles then $\langle X, <\rangle$ is a set of l-variable indiscernibles for all $l<k$. Any linearly ordered $\langle X, <\rangle$ with $X \subseteq \mathfrak{M}$ is a set of 0-variable indiscernibles. The notion of *set of k-variable indiscernibles over U* (when $\mathfrak{M} = \langle M, U, \ldots\rangle$) is defined in the same way.

As a first simple use of the Erdős-Rado Theorem we can prove a result which is useful when $h_\Sigma(\mathbb{A}) = \omega$.

4.11 Proposition. *Let $\mathsf{L}_{\mathbb{A}}$ be a fragment of $\mathsf{L}_{\infty\omega}$ with $\mathrm{card}(\mathsf{L}_{\mathbb{A}}) = \kappa$. Let $0<k<\omega$ be fixed and let $\mathfrak{M}$ be a structure for L.*

(i) *If $\mathrm{card}(\mathfrak{M}) > \beth_k(\kappa)$ then there is an infinite set $\langle X, <\rangle$ of k-variable indiscernibles for $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{M}$.*

(ii) *If $\mathfrak{M} = \langle M, U, \ldots\rangle$ where $\mathrm{card}(U) \geq \kappa$ and $\mathrm{card}(M) > \beth_k(\mathrm{card}(U))$, then there is an infinite set $\langle X, <\rangle$ of k-variable indiscernibles over U for $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{M}$.*

Proof. (i) Let $<$ be a linear ordering of M and, for each k-tuple $\vec{x} = x_1 < \cdots < x_k$ from M, let

$$T_{\vec{x}} = \{\varphi(v_1, \ldots, v_k) \mid \mathfrak{M} \models \varphi[x_1, \ldots, x_k]\}.$$

This partitions $[M]^k$ up into $\leq 2^\kappa$ distinct sets, since there are $\leq 2^\kappa$ different sets of formulas of $\mathsf{L}_{\mathbb{A}}$. Since

$$\mathrm{card}(M) > \beth_k(\kappa) = \beth_{k-1}(2^\kappa),$$

the Erdős-Rado Theorem tells us that there is an $X \subseteq M$ (of power $> 2^\kappa > \aleph_0$) such that every element of $[X]^k$ is in one fixed member of the partition. That is, $T_{\vec{x}} = T_{\vec{y}}$ whenever $\vec{x} = x_1 < \cdots < x_k$, $\vec{y} = y_1 < \cdots < y_k$ and $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$. Thus $\langle X, < \restriction X\rangle$ is a set of k-variable indiscernibles in $\mathfrak{M}$. To prove (ii), let $C = \{\mathsf{c}_u \mid u \in U\}$ and apply (i) to $\mathsf{L}_{\mathbb{A}}(C)$ and $\mathfrak{M}' = (\mathfrak{M}, u)_{u \in U}$ with κ replaced by $\mathrm{card}(\mathsf{L}_{\mathbb{A}}(C))$. ▯

Theorem 4.1 follows easily from Lemma 4.9 (i) and the following theorem.

4.12 Theorem. *Let $L_{\mathbb{A}}$ be an admissible fragment, let* card$(\mathbb{A})$, $\alpha = h_\Sigma(\mathbb{A})$. *Let T be a Σ_1 theory of $L_{\mathbb{A}}$. If for each $\beta < \alpha$, T has model of power $\geqslant \beth_\beta(\kappa)$, then T has a model with an infinite set of indiscernibles.*

Proof. We may assume by 2.4 that $L_{\mathbb{A}}$ is a Skolem fragment, that $T_{\text{Skolem}} \subseteq T$ and that $L_{\mathbb{A}}$ is Δ_1 on $\mathbb{A}$. We assume that T has models but no model with a set of indiscernibles for $L_{\mathbb{A}}$ and prove that, for some $\beta < \alpha$, T has no models of power $\geqslant \beth_\beta(\kappa)$. Let $L'_{\mathbb{A}}$ be a Skolem fragment containing $L_{\mathbb{A}}$ and two new symbols X, $<$. Let

$$C = \{\mathsf{c}_n \mid 0 < n < \omega\}$$

be a set of new constant symbols. We will be concerned with all the languages

$$L_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n), \qquad L'_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n),$$
$$L_{\mathbb{A}}(C), \qquad L'_{\mathbb{A}}(C).$$

These are all Skolem fragments with constants. For $n \geqslant 0$ define $\mathfrak{S}_n$ to be the set of all supervalidity properties $\mathscr{D}$ for $L'_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n)$ with the following properties:

(a) $T \subseteq \mathscr{D}$;

(b) "X is linearly ordered by $<$ and has no last element" $\in \mathscr{D}$;

(c) "$\mathsf{c}_i \in X \wedge \mathsf{c}_i < \mathsf{c}_{i+1}$" $\in \mathscr{D}$ for $0 < i < n$;

(d) $\forall x_1, \ldots, x_n \in \mathsf{X}\, [x_1 < \cdots < x_n \to (\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi(\mathsf{c}_1, \ldots, \mathsf{c}_n))] \in \mathscr{D}$ for each $\varphi(v_1, \ldots, v_n) \in L_{\mathbb{A}}$, when $n > 0$.

It follows immediately from the Weak Completeness Theorem that

(1) $\begin{cases} \mathscr{D} \in \mathfrak{S}_n \text{ iff } \mathscr{D} \text{ is an s.v.p. for } L'_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n) \text{ given by some structure} \\ \quad (\mathfrak{M}, X, <, a_1, \ldots, a_n) \\ \text{where } \mathfrak{M} \models T,\ \langle X, < \rangle \text{ is an infinite set of } n\text{-variable indiscernibles for} \\ L_{\mathbb{A}} \text{ in } \mathfrak{M} \text{ and } a_1 < \cdots < a_n \text{ in } \langle X, < \rangle. \end{cases}$

Let $\mathfrak{S} = \bigcup_n \mathfrak{S}_n$. Note that each $\mathscr{D} \in \mathfrak{S}$ is in exactly one $\mathfrak{S}_n$ for $n \geqslant 0$; this n is called the *level of* $\mathscr{D}$ and we can determine the level n of $\mathscr{D}$ by seeing whether $(\mathsf{c}_n = \mathsf{c}_n) \in \mathscr{D}$ but $(\mathsf{c}_{n+1} = \mathsf{c}_{n+1}) \notin \mathscr{D}$. Let $l(\mathscr{D})$ be the level of $\mathscr{D}$. We define an order $\prec$ on $\mathfrak{S}$ by

$$\mathscr{D}' \prec \mathscr{D} \quad \text{iff} \quad l(\mathscr{D}') > l(\mathscr{D}) \quad \text{and} \quad \mathscr{D} \cap L_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_{l(\mathscr{D})}) \subseteq \mathscr{D}'.$$

Thus, if $\mathscr{D}' \prec \mathscr{D}$ then $\mathscr{D}$ and $\mathscr{D}'$ contain exactly the same formulas from the language $L_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n)$, $n = l(\mathscr{D})$, but not necessarily from $L'_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n)$.

The crucial step in the proof is to realize that

(2) $\langle \mathfrak{S}, \prec \rangle$ *is well founded.*

Suppose that it were not well founded and let

$$\cdots \prec \mathscr{D}_{n+1} \prec \mathscr{D}_n \prec \cdots \prec \mathscr{D}_1 \prec \mathscr{D}_0$$

be an infinite descending chain. If $\mathscr{D} \in \mathfrak{S}_n$ and $n > m$ then $\mathscr{D} \cap \mathsf{L}'_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_m) \in \mathfrak{S}_m$ so we may suppose that the level of $\mathscr{D}_n$ is n. Let $\mathscr{D}_n^0 = \mathscr{D}_n \cap \mathsf{L}_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_n)$ and let $\mathscr{D}_\infty^0 = \bigcup_n \mathscr{D}_n^0$. By the union of chain lemma, $\mathscr{D}_\infty^0$ is an s.v.p. for $\mathsf{L}_{\mathbb{A}}(C)$. Let $(\mathfrak{M}, a_1, \ldots, a_n, \ldots)$ be a model of $\mathscr{D}_\infty^0$, by the Weak Completeness Theorem. Then $\mathfrak{M} \models T$ and $X = \{a_1, a_2, \ldots\}$ is an infinite set of indiscernibles for $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{M}$ when ordered by $a_i < a_j$ if $i < j$. This proves (2).

Using (2), we can define the usual rank function on $\mathfrak{S}$:

$$\rho(\mathscr{D}) = \sup\{\rho(\mathscr{D}') + 1 \mid \mathscr{D}' \prec \mathscr{D}\},$$
$$\rho(\mathfrak{S}) = \sup\{\rho(\mathscr{D}) + 1 \mid \mathscr{D} \in \mathfrak{S}\}.$$

Since $\mathfrak{S}_0 \neq \emptyset$, $\rho(\mathfrak{S}) > 0$. We will prove later that $\rho(\mathfrak{S}) < h_\Sigma(\mathbb{A})$.

(3) *Assume $\rho(\mathfrak{S}) = n < \omega$. Then no model $\mathfrak{M}$ of T has an infinite set of n-variable indiscernibles.*

For suppose $\mathfrak{M} \models T$ and $\langle X, < \rangle$ is an infinite set of n-variable indiscernibles. Let, for $0 \leqslant m \leqslant n$,

$$\mathfrak{M}_m = (\mathfrak{M}, X, <, a_1, \ldots, a_m)$$

and let $\mathscr{D}_m$ be the s.v.p. for $\mathsf{L}'_{\mathbb{A}}(\mathsf{c}_1, \ldots, \mathsf{c}_m)$ given by $\mathfrak{M}_m$. Then $\mathscr{D}_m \in \mathfrak{S}_m$ and

$$\rho(\mathscr{D}_0) > \rho(\mathscr{D}_1) > \rho(\mathscr{D}_2) > \cdots > \rho(\mathscr{D}_n) \geqslant 0$$

so $\rho(\mathscr{D}_0) \geqslant n$ and hence $\rho(\mathfrak{S}) > n$, contrary to hypothesis.

From (3) and Proposition 4.11 (i), we immediately obtain

(4) *If $\rho(\mathfrak{S}) = n < \omega$ then T has no model of power $> \beth_n(\kappa)$.*

If $\rho(\mathfrak{S}) \geqslant \omega$ then we cannot put such an *a priori* upper bound on the "size" n of a set $\langle X, < \rangle$ of n-variables indiscernibles, but we can put a bound on $\operatorname{card}(X)$.

(5) *Suppose $\mathfrak{M} \models T$, $\langle X, < \rangle$ is a set of n-variable indiscernibles for $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{M}$ and that $a_1 < \cdots < a_n$ in $\langle X, < \rangle$. Let $\mathscr{D}$ be the s.v.p. in $\mathfrak{S}_n$ given by $(\mathfrak{M}, X, <, a_1, \ldots, a_n)$. If $\beta = \rho(\mathscr{D})$ then* $\operatorname{card}(X) < \beth_{\omega(\beta+1)}(\kappa)$.

We prove (5) by induction on β using the Erdős-Rado Theorem as in 4.11 (i). So suppose we know the result for ordinals $\gamma < \beta$ $(\beta > 0)$ and suppose

$\text{card}(X) \geqslant \beth_{\omega(\beta+1)}(\kappa)$. For each increasing $n+1$ tuple $\vec{x} = x_1 < \cdots < x_n < x_{n+1}$ from $\langle X, < \rangle$, let

$$T_{\vec{x}} = \{\varphi(v_1, \ldots, v_{n+1}) \in \mathsf{L}_{\mathbb{A}} \mid \mathfrak{M} \models \varphi[x_1, \ldots, x_{n+1}]\}.$$

This partitions $[X]^{n+1}$ into $\leqslant 2^\kappa$ sets. Since $2^\kappa \leqslant \beth_{\omega\beta+1}(\kappa)$ and

$$\begin{aligned} \text{card}(X) &\geqslant \beth_{\omega(\beta+1)}(\kappa) \\ &= \beth_{\omega(\beta+1)}(2^\kappa) \\ &> \beth_{\omega\beta+n}(2^\kappa) \\ &= \beth_n(\beth_{\omega\beta}(\kappa)) \end{aligned}$$

we can apply the Erdős-Rado Theorem to find an $X_0 \subseteq X$ with $\text{card}(X_0) > \beth_{\omega\beta}(\kappa)$ such that every member of $[X_0]^{n+1}$ lies in one member of the partition. That is, for $n+1$-tuples $x_1 < \cdots < x_{n+1}$ from X_0,

$$T_{\vec{x}} = T_{\vec{y}}$$

so that $\langle X_0, < \rangle$ forms a set of $(n+1)$-variable indiscernibles in $\mathfrak{M}$. Let $a_1 < \cdots < a_{n+1}$ be chosen from X_0 and let $\mathscr{D}_0$ be the s.v.p. given by

$$\mathfrak{M}_0 = (\mathfrak{M}, X_0, < \restriction X_0, a_1, \ldots, a_{n+1}).$$

Then $\mathscr{D}_0 \prec \mathscr{D}$ so $\rho(\mathscr{D}_0) < \beta$. But then $\mathfrak{M}_0$ contradicts the inductive hypothesis since $\text{card}(X_0) > \beth_{\omega\beta}(\kappa) \geqslant \beth_{\omega(\gamma+1)}(\kappa)$ where $\gamma = \rho(\mathscr{D}_0)$. This contradiction proves (5) for $\beta > 0$. The case for $\beta = 0$ is easier and is left to the ideal student.

From (5) we get at once:

(6) *Every model* $\mathfrak{M}$ *of* T *has power* $< \beth_{\omega\beta}(\kappa)$, *where* $\beta = \rho(\mathfrak{S})$.

For let $X = M$ and let $<$ be any linear ordering of X. Recall that $\langle X, < \rangle$ is a set of 0-ary indiscernibles for $\mathfrak{M}$. Then, if $\mathscr{D}$ is the s.v.p. for $\mathsf{L}'_{\mathbb{A}}$ given by

$$(\mathfrak{M}, X, <)$$

then $\rho(\mathscr{D}) < \beta$ and $\text{card}(\mathfrak{M}) = \text{card}(X) < \beth_{\omega(\rho(\mathscr{D})+1)}(\kappa)$ which is $\leqslant \beth_{\omega\beta}(\kappa)$.

Finally, we claim that

(7) $\rho(\mathfrak{S}) < h_\Sigma(\mathbb{A})$.

To see that this concludes the proof, we see that if $h_\Sigma(\mathbb{A}) = \omega$ then the result follows from (4). If $\rho(\mathfrak{S}) = \beta$ and $h_\Sigma(\mathbb{A}) > \omega$ then $\omega\beta < h_\Sigma(\mathbb{A})$ by Corollary 3.5, so the conclusion follows from (6). (This is the only use of anything remotely approaching admissibility in the entire proof.)

It remains only to prove (7). We will see in § VIII.6 that $\langle \mathfrak{S}, \prec \rangle$ is a Π definable well-founded tree of subsets of A and that every such tree has rank less than $h_\Sigma(\mathbb{A})$. That is probably the simplest proof of (7). It's good for the soul, though, and gives added appreciation of the machinery developed in § VIII.6, to give a direct proof. We present a sketch to be filled in by the student.

Our goal then is to write down a Σ_1 theory $T'(<)$ of $\mathsf{L}_\mathbb{A}$ which pins down $\beta=\rho(\mathfrak{S})$. As is our custom, we first describe the intended model $\mathfrak{M}$ of $T'(<)$, the one where $<^{\mathfrak{M}}$ has order type β. Let $\mathfrak{M}$ be the following structure:

$$\langle M;\beta,<;\mathbb{A};\mathrm{Power}(\mathbb{A}),E;\mathfrak{S},\prec,F,G,x\rangle_{x\in\mathbb{A}}$$

where

$$M=\beta\cup\mathbb{A}\cup\mathrm{Power}(\mathbb{A}),$$
$$< \;=\; \in\restriction\beta,$$
$$G(\mathscr{D})=\text{level of }\mathscr{D}\text{ for }\mathscr{D}\in\mathfrak{S}$$
$$=\text{some constant }\notin\omega,\text{ otherwise},$$
$$F(\mathscr{D})=\rho(\mathscr{D})\text{ if }\mathscr{D}\in\mathfrak{S}$$
$$=\text{some constant }\notin\beta\text{ otherwise},$$
$$E=\in\cap(\mathbb{A}\times\mathrm{Power}(\mathbb{A})).$$

Now suppose that

$$\mathfrak{M}'=\langle M';B,<',\mathfrak{A}';P,E',\mathfrak{S}',\prec',F',G',x\rangle_{x\in A_N}$$

satisfies all the finitary first order sentences true in M and that

$$\mathbb{A}\subseteq_{\mathrm{end}}\mathfrak{A}'.$$

We will show that $\langle B,<'\rangle$ is well ordered. The proof will show that the set of finitary sentences we actually use is Σ_1 on $\mathbb{A}$ so that will conclude the proof.

By the axiom of Extensionality for $\mathrm{Power}(\mathbb{A})$, we may assume that

$$P\subseteq\mathrm{Power}(\mathfrak{A}'),\quad E'=\in\cap(\mathfrak{A}'\times P),\quad\text{and}\quad\mathfrak{S}'\subseteq P.$$

Now suppose that the linear ordering $\langle B,<'\rangle$ is not well ordered, so that there is a subset $B_0\subseteq B$ with no $<'$-minimal element. Let

$$\mathfrak{S}_0'=\{\mathscr{D}\in\mathfrak{S}'\mid F'(\mathscr{D})\in B_0\}$$

and let

$$\mathfrak{S}_0''=\{\mathscr{D}\cap\mathsf{L}'_\mathbb{A}(\mathsf{c}_1,\ldots,\mathsf{c}_m)\mid\mathscr{D}\in\mathfrak{S}_0',m<\omega,m\leqslant G'(\mathscr{D})\},$$

where we must remember that $G'(\mathscr{D})$ might be a nonstandard integer. It is not difficult, though tedious, to see that $\mathfrak{S}_0''\subseteq\mathfrak{S}$, since each $\mathscr{D}\in\mathfrak{S}'$ claims to be an s.v.p. for $\mathsf{L}'_\mathbb{A}(\mathsf{c}_1,\ldots,\mathsf{c}_{G(\mathscr{D})})$ of the appropriate kind, and the relevant quantifiers are all universal. So $\mathfrak{S}_0''$ must have a minimal element $\mathscr{D}$. By chasing $\mathscr{D}$ back into B, a contradiction easily results by considering the cases $G'(\mathscr{D})$ standard and $G'(\mathscr{D})$ nonstandard separately. ☐

4.13 Proof of Theorem 4.1. Again, using 2.4 we may assume $\mathsf{L}_{\mathbb{A}}$ is a Skolem fragment and that $T_{\text{Skolem}} \subseteq T$. Then 4.1 follows from 4.12 and 4.9 (i). ☐

4.14 Corollary. *Let $\mathsf{L}_{\mathbb{A}}$ be an admissible Skolem fragment with $h_{\Sigma}(\mathbb{A}) = \omega$. Let T be a Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$. If for each $k < \omega$, T has a Skolem model with an infinite set of k-variable indiscernibles, then T has a Skolem model with an infinite set of indiscernibles for $\mathsf{L}_{\mathbb{A}}$.*

Proof. See line (3) of the proof of Theorem 4.12. ☐

We next turn to the analogous theorem for two cardinal models.

4.15 Theorem. *Let $\mathsf{L}_{\mathbb{A}}$ be an admissible fragment with $\kappa = \text{card}(\mathbb{A})$, $\alpha = h_{\Sigma}(\mathbb{A})$. Let $T = T(\mathsf{U}, \dots)$ be a Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$. If for each $\beta < \alpha$, there is a $\lambda \geqslant \kappa$ such that T admits $(\beth_{\beta}(\lambda), \lambda)$, then T has a model $\mathfrak{M} = \langle M, U, \dots \rangle$ with an infinite set of indiscernibles over U for $\mathsf{L}_{\mathbb{A}}$.*

Proof. We indicate the changes necessary in the proof of Theorem 4.12. We may again assume that $\mathsf{L}_{\mathbb{A}}$ is a Skolem fragment and that $T_{\text{Skolem}} \subseteq T$. We may also assume (by adding κ new constant symbols and some axioms of the form $\mathsf{U}(\mathsf{c}_x)$, $\mathsf{c}_x \neq \mathsf{c}_y$ to T) that every model $\mathfrak{M}$ of T has $\text{card}(U) \geqslant \kappa$.

Let $\mathsf{L}'_{\mathbb{A}}(\mathsf{c}_1, \dots, \mathsf{c}_n)$ be as before and let $\mathscr{D} \in \mathfrak{S}_n$ iff $\mathscr{D}$ is an s.v.p. for $\mathsf{L}'_{\mathbb{A}}(\mathsf{c}_1, \dots, \mathsf{c}_n)$ with properties (a), (b), (c), (d) as before plus

(e) $\mathsf{U}(t(\mathsf{c}_1, \dots, \mathsf{c}_n)) \to \forall x_1, \dots, x_n \in X\, [x_1 < \cdots < x_n \to t(x_1, \dots, x_n) = t(\mathsf{c}_1, \dots, \mathsf{c}_n)]$ for all terms $t(v_1, \dots, v_n)$ of $\mathsf{L}_{\mathbb{A}}$.

The analogue of (1) is the one way result:

(1′) *$\mathscr{D} \in \mathfrak{S}_n$ if $\mathscr{D}$ is the s.v.p. for $\mathsf{L}'_{\mathbb{A}}(\mathsf{c}_1, \dots, \mathsf{c}_n)$ given by some $(\mathfrak{M}, X, <, a_1, \dots, a_n)$ where $\langle X, < \rangle$ is a set of n-variable indiscernibles over $\mathsf{U}^{\mathfrak{M}}$ for $\mathsf{L}_{\mathbb{A}}$ and $a_1 < \cdots < a_n$ in $\langle X, < \rangle$.*

Luckily, we never really used the other half of (1).

The relation $\prec$ on $\mathfrak{S} = \bigcup_n \mathfrak{S}_n$ is defined just as before. Again we have, assuming T does not have a model $\mathfrak{M}$ with a set of indiscernibles over $\mathsf{U}^{\mathfrak{M}}$,

(2′) *$\langle \mathfrak{S}, \prec \rangle$ is well founded.*

This is just a bit trickier than (2). Suppose

$$\cdots \prec \mathscr{D}_{n+1} \prec \mathscr{D}_n \prec \cdots \prec \mathscr{D}_1 \prec \mathscr{D}_0 .$$

Again, we may assume that each $\mathscr{D}_n$ has level n. Let $\mathscr{D}_n^0 = \mathscr{D}_n \cap \mathsf{L}_{\mathbb{A}}(\mathsf{c}_1,\dots,\mathsf{c}_n)$ and let $\mathscr{D}_\infty^0 = \bigcup_n \mathscr{D}_n^0$. By the Union of Chain Lemma, $\mathscr{D}_\infty^0$ is an s.v.p. for $\mathsf{L}_{\mathbb{A}}(C)$. Let $(\mathfrak{M}_0, a_1, a_2, \dots, a_n, \dots)$ be a model for $\mathscr{D}_\infty^0$ let $X = \{a_1, a_2, \dots\}$, $a_i < a_j$ iff $i < j$. Let $\mathfrak{M} = \mathrm{Hull}_{\mathfrak{M}_0}(X)$. By Exercise 2.16, $\mathfrak{M} \models \mathscr{D}_\infty^0$. Thus, $\mathfrak{M}$ is a model for T and $\langle X, < \rangle$ is a set of indiscernibles for $\mathsf{L}_{\mathbb{A}}$ in $\mathfrak{M}$. We need to see that $\langle X, < \rangle$ is a set of indiscernibles *over* $\mathsf{U}^{\mathfrak{M}}$. Thus suppose $u \in \mathsf{U}^{\mathfrak{M}}$. We need to see that increasing n-tuples from $\langle X, < \rangle$ satisfy the same formulas in $(\mathfrak{M}, u)$. (The case with more that one u is similar.) Since $\mathfrak{M} = \mathrm{Hull}(X)$, there is a term $t(v_1, \dots, v_m)$ such that

$$\mathfrak{M} \models u = t(a_1, \dots, a_m).$$

Then, by (e)

$$\mathfrak{M} \models u = t(x_1, \dots, x_m)$$

whenever $x_1 < \dots < x_m$ in $\langle X, < \rangle$. Now suppose $n < \omega$, $x_1 < \dots < x_n$, $y_1 < \dots < y_n$ in $\langle X, < \rangle$. We need to see that for all formulas $\varphi(v_1, \dots, v_n, v_{n+1})$, if $\mathfrak{M} \models \varphi[x_1, \dots, x_n, u]$ then $\mathfrak{M} \models \varphi[y_1, \dots, y_n, u]$. Pick an increasing m-tuple $w_1 < \dots < w_m$ such that $w_1 > x_n$, $w_1 > y_n$. Now consider the formula $\psi(v_1, \dots, v_n, v_{n+1}, \dots, v_{n+m})$ given by

$$\varphi(v_1, \dots, v_n, t(v_{n+1}, \dots, v_{n+m})/v_{n+1}).$$

Then, since $u = t(w_1, \dots, w_m)$,

$$\mathfrak{M} \models \psi[x_1, \dots, x_n, w_1, \dots, w_m]$$

and hence,

$$\mathfrak{M} \models \psi[y_1, \dots, y_n, w_1, \dots, w_m]$$

by the indiscernibility of $\langle X, < \rangle$ in $\mathfrak{M}$. Thus

$$\mathfrak{M} \models \varphi[y_1, \dots, y_n, u].$$

Thus $\langle X, < \rangle$ is indiscernible over U, proving (2).

Define $\rho(\mathscr{D})$, $\rho(\mathfrak{S})$ as before.

(3′) *Assume $\rho(\mathfrak{S}) = n < \omega$. Then no model $\mathfrak{M}$ of T has an infinite set of n-variable indiscernibles over $\mathsf{U}^{\mathfrak{M}}$.*

The proof of (3′) is just like the proof of (3).

Using (3′) and 4.11 (ii), we get

(4′) *If $\rho(\mathfrak{S}) = n < \omega$ then T has no models of type $(\beth_{n+1}(\lambda), \lambda)$ for any λ.*

Corresponding to (5) we have

$$(5')\begin{cases} \textit{Suppose } M \models T, \langle X, < \rangle \textit{ is a set of n-variable indiscernibles over } U(=\mathsf{U}^{\mathfrak{M}}) \\ \textit{and that } a_1 < \cdots < a_n \textit{ in } \langle X, < \rangle. \textit{ Let } \mathscr{D} \textit{ be the s.v.p. in } \mathfrak{S}_n \textit{ given by} \\ (\mathfrak{M}, X, <, a_1, \ldots, a_n) \textit{ and let } \beta = \rho(\mathscr{D}). \textit{ Then } \operatorname{card}(X) < \beth_{\omega(\beta+1)}(\operatorname{card}(U)). \end{cases}$$

The proof is by induction on β and uses the Erdős-Rado Theorem. The proof it too similar to the proof of (5) to present. From (5′) we get

(6′) *If* $\mathfrak{M} \models T$ *then* $\operatorname{card}(\mathfrak{M}) < \beth_{\omega\beta}(\operatorname{card}(\mathsf{U}^{\mathfrak{M}}))$ *where* $\beta = \rho(\mathfrak{S})$.

The proof is concluded by showing that

(7′) $\rho(\mathfrak{S}) < h_\Sigma(\mathbb{A})$.

The proof of (7′) is just like the proof of (7). □

Theorem 4.3 follows from 4.15 just as Theorem 4.1 followed from 4.12.

4.16—4.20 Exercises

4.16. Prove that if α is admissible then

$$\begin{aligned} \beth_\alpha(\aleph_0) &= \beth_\alpha \quad \text{if } \alpha > \omega, \\ \beth_\alpha(\aleph_0) &= \beth_\omega(\aleph_0) \\ &= \beth_{\omega+\omega} \quad \text{if } \alpha = \omega. \end{aligned}$$

4.17. Let $\mathbb{A}_{\mathfrak{N}}$ be admissible above $\mathfrak{N}$, $\kappa_0 = \operatorname{card}(\mathfrak{N})$, $\alpha = o(\mathbb{A})$. Prove that

$$\operatorname{card}(\mathbb{A}_{\mathfrak{N}}) \leqslant \beth_\alpha(\kappa_0).$$

Let $\kappa_1 = \operatorname{card}(\mathbb{A}_{\mathfrak{N}})$. Prove that if $h_\Sigma(\mathbb{A}) = \beta > \alpha$ then

$$\beth_\beta(\kappa_1) = \beth_\beta(\kappa_0).$$

4.18. Let $\mathbb{A}$ be an admissible set, $\alpha = h(\mathbb{A})$. Prove that the Hanf number for single sentences of $\mathsf{L}_{\mathbb{A}}$ is at least

$$\lambda = \sup\{\beth_\alpha(\kappa) \mid \kappa = \operatorname{card}(X) \text{ for some } X \in A\}.$$

That is, show that for $\lambda_0 < \lambda$ there is a sentence φ of $\mathsf{L}_{\mathbb{A}}$ which has models of power $\geqslant \lambda_0$ but none of power $\geqslant \lambda$. [Given $X \in A$, $\beta < h(\mathbb{A})$, formalize $\mathsf{V}_X(\beta)$.] Prove that the Hanf number is always of the form $\beth_\lambda$ for some limit ordinal λ.

4.19. Let $\mathbb{A}$ be an admissible set with $o(\mathbb{A})>\omega$.

(i) Prove that each $\varphi\in\mathbb{A}$ can be put in a Skolem fragment $\mathsf{L}_B\in\mathbb{A}$ in such a way that every model of φ (not just those in $\mathbb{A}$) can be expanded to a model of T_{Skolem}. [Use Infinity to carry out the proof of 2.4 inside $\mathbb{A}$.]

(ii) Prove that the Hanf number for single sentences of $\mathsf{L}_{\mathbb{A}}$ is

$$\lambda=\sup\{\beth_\alpha(\kappa)\,|\,\kappa=\operatorname{card}(X),\, X\in A\}$$

where $\alpha=h(\mathbb{A})$. That is, prove that if $\varphi\in\mathsf{L}_{\mathbb{A}}$ does not have a model of every power $\geqslant\operatorname{card}(\mathbb{A})$ then there is an $X\in\mathbb{A}$ and a $\beta<h(\mathbb{A})$ such that φ has no model of power $\geqslant\beth_\beta(\operatorname{card}(X))$. [The set X will be the L_B of (i). Modify the proof of 4.12.]

(iii) Prove that if $\mathbb{A}$ is a pure admissible set then the Hanf number for single sentences of $\mathsf{L}_{\mathbb{A}}$ is $\beth_{h(\mathbb{A})}$, even if $o(\mathbb{A})=\omega$.

4.20. Let $\mathbb{A}$ be an admissible set, let $\alpha=h_\Sigma(\mathbb{A})$ and let

$$\lambda_0=\sup\{\beth_\alpha(\kappa)\,|\,\kappa=\operatorname{card}(X)\text{ some }X\in\mathbb{A}\},$$

$$\lambda_1=\beth_\alpha(\operatorname{card}(\mathbb{A})).$$

Theorem 4.12 states that the Hanf number for Σ_1 theories of $\mathsf{L}_{\mathbb{A}}$ is $\leqslant\lambda_1$.

(i) Prove that this Hanf number is $\geqslant\lambda_0$.

(ii) Prove that if $\mathbb{A}$ is countable and $\neq\mathbb{HF}_{\mathfrak{M}}$, or if $h_\Sigma(\mathbb{A})>o(\mathbb{A})$, then $\lambda_0=\lambda_1$. It is an open problem to describe this Hanf number in general. Is it λ_0 or λ_1 or something in between?

4.21 Notes. Morley [1965] shows that the Hanf number for single sentences of $\mathsf{L}_{\omega_1\omega}$ was $\beth_{\omega_1}$. (This follows from 4.2.) Morley [1967] showed that the Hanf number for single sentences of ω-logic was $\beth_\alpha$ where $\alpha=\omega_1^c$. (The hard half of this follows from 4.2 with $\mathbb{A}=\mathrm{L}(\alpha)$.) Barwise [1967] generalized this to obtain the Hanf number for any countable, admissible fragment. This was generalized in Barwise-Kunen [1971] to obtain 4.19(iii). The theorems of this section are a reworking of the ideas from Barwise-Kunen [1971] so that they apply to theories, not just single sentences. Theorem 4.3 is a generalization of Morely's Two Cardinal Theorem of Morley [1965]. The student should consult lectures 16 and 17 of Keisler [1971] for a different proof of the countable versions of these results.

The student should be aware of a difference between the results of this section and those in Chapter III. The use of admissible sets was absolutely essential in Chapter III to obtain our results. Here they provide a convenient setting but weaker assumptions would do. Of course we need to know that the countable set $\mathbb{A}$ is admissible to know that $h_\Sigma(\mathbb{A})=o(\mathbb{A})$.

5. Partially Isomorphic Structures

Having seen in the previous sections that the model theory of uncountable fragments is not completely beyond our control, even if it is less tractable than for countable fragments, we now investigate some uses of uncountable sentences.

One way to appreciate $L_{\infty\omega}$ is to see the role it plays in algebra, but this is not the book to discuss such topics. We can only give a few exercises. The topics we discuss are of a more logical nature. These final sections are completely independent of the first half of the chapter. Admissible sets will not appear in an essential way until § 7.

A *partial isomorphism* f from $\mathfrak{M}$ into $\mathfrak{N}$ is simply an isomorphism

$$f: \mathfrak{M}_0 \cong \mathfrak{N}_0$$

where $\mathfrak{M}_0, \mathfrak{N}_0$ are substructures of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. A set I of partial isomorphisms from $\mathfrak{M}$ into $\mathfrak{N}$ has the *back and forth property* if

(1) for every $f \in I$ and every $x \in \mathfrak{M}$ (or $y \in \mathfrak{N}$) there is a $g \in I$ with $f \subseteq g$ and $x \in \mathrm{dom}(g)$ (or $y \in \mathrm{rng}(g)$, resp.).

We write

$$I: \mathfrak{M} \cong_p \mathfrak{N}$$

if I is a nonempty set of partial isomorphisms and I has the back and forth property. If there is an I such that $I: \mathfrak{M} \cong_p \mathfrak{N}$ then we say that $\mathfrak{M}, \mathfrak{N}$ are *partially isomorphic* and write $\mathfrak{M} \cong_p \mathfrak{N}$. (Some authors prefer the more picturesque terminology *potentially isomorphic*, to suggest that $\mathfrak{M}$ and $\mathfrak{N}$ would become isomorphic if only they were to become countable, say in some larger universe of set theory.) Note that if $f: \mathfrak{M} \cong \mathfrak{N}$, then $\{f\}: \mathfrak{M} \cong_p \mathfrak{N}$.

5.1 Examples. (i) The canonical example is given by two dense linear orderings $\mathfrak{M} = \langle M, < \rangle$ nd $\mathfrak{N} = \langle N, < \rangle$ without end-points. Let I be the set of all *finite* partial isomorphisms from $\mathfrak{M}$ into $\mathfrak{N}$. Then

$$I: \mathfrak{M} \cong_p \mathfrak{N}$$

regardless of the cardinalities of $\mathfrak{M}$ and $\mathfrak{N}$. This is quite easy to verify. Combined with Theorem 5.2, this shows that the theory of dense linear orderings without end points is $\aleph_0$-categorical, i. e., that all its countable models are isomorphic.

(ii) If $\mathfrak{M}, \mathfrak{N}$ are dense linear orderings with first elements x_0, y_0 respectively, but without last elements, then $\mathfrak{M} \cong_p \mathfrak{N}$ but the set I used in (i) no longer has the back and forth property. Let

$$I_0 = \{f \in I \mid x_0 \in \mathrm{dom}(f), f(x_0) = y_0\}.$$

Then $I_0: \mathfrak{M} \cong_p \mathfrak{N}$.

(iii) We can generalize (i), (ii) as follows. Let $\mathsf{L}_{\mathbb{A}}$ be a countable fragment and let T be an $\aleph_0$-categorical theory of $\mathsf{L}_{\mathbb{A}}$. Then for any two infinite models $\mathfrak{M}$, $\mathfrak{N}$ of T,

$$\mathfrak{M} \cong_p \mathfrak{N}.$$

A proof of this will be given in 5.5 below.

(iv) We can get a different generalization of (i) and (ii) by looking at $\aleph_0$-saturated structures $\mathfrak{M}$ and $\mathfrak{N}$. If $\mathfrak{M} \equiv \mathfrak{N}$ $(\mathsf{L}_{\omega\omega})$ then $\mathfrak{M} \cong_p \mathfrak{N}$. The set I is defined as follows: Consider those partial isomorphisms

$$f: \mathfrak{M}_0 \cong \mathfrak{N}_0$$

where $\mathfrak{M}_0$ is finitely generated by some $a_1, \ldots, a_n$. We will let $f \in I$ iff

$$(\mathfrak{M}, a_1, \ldots, a_n) \equiv (\mathfrak{N}, f(a_1), \ldots, f(a_n)) \quad (\mathsf{L}_{\omega\omega}).$$

A simple use of $\aleph_0$-saturation shows that I has the back and forth property.

Traditionally, the back and forth property has been used for constructing isomorphisms of countable structures.

5.2 Theorem. *Let $\mathfrak{M}, \mathfrak{N}$ be countable structures for the same language and let $I: \mathfrak{M} \cong_p \mathfrak{N}$. For every $f_0 \in I$ there is an isomorphism*

$$f: \mathfrak{M} \cong \mathfrak{N}$$

with $f_0 \subseteq f$.

Proof. Enumerate $\mathfrak{M} = \{x_1, x_2, \ldots\}$, $\mathfrak{N} = \{y_1, y_2, \ldots\}$. Define

$$f_{2n+1} = \text{some } g \in I \quad \text{with} \quad f_{2n} \subseteq g,\ x_n \in \text{dom}(g),$$

$$f_{2n+2} = \text{some } g \in I \quad \text{with} \quad f_{2n+1} \subseteq g,\ y_n \in \text{rng}(g)$$

by using the back and forth property (1). Let $f = \bigcup_n f_n$. Then f maps $\mathfrak{M}$ onto $\mathfrak{N}$ and preserves atomic and negated atomic formulas so $f: \mathfrak{M} \cong \mathfrak{N}$.

The examples and Theorem 5.2 should suggest to the student of the previous chapter that $\cong_p$ could be the absolute version of $\cong$. After all, they agree on countable structures and $\cong_p$ does not seem to depend on cardinality. At first glance, though, it is not obvious that $\cong_p$ is absolute, but merely that it is Σ_1:

$$\mathfrak{M} \cong_p \mathfrak{N} \quad \text{iff} \quad \exists I\, [I: \mathfrak{M} \cong_p \mathfrak{N}]$$

where the part within brackets is Δ_0. This is no better that $\cong$, itself a Σ_1 notion. The Π_1 equivalent of $\cong_p$ is given by the next result. There is, of course, no Π_1 equivalent of $\cong$. This result as well as 5.7 appear in Karp [1965].

5.3 Karp's Theorem. *If* $\mathfrak{M}, \mathfrak{N}$ *are structures for the language* L, *then* $\mathfrak{M} \cong_p \mathfrak{N}$ *iff* $\mathfrak{M} \equiv \mathfrak{N}$ $(\mathsf{L}_{\infty\omega})$.

Proof. We first prove ($\Rightarrow$). Let $I: \mathfrak{M} \cong_p \mathfrak{N}$. We prove, by induction on formulas $\varphi(v_1,\ldots,v_n)$ of $\mathsf{L}_{\infty\omega}$ that if $f \in I$, $x_1,\ldots,x_n \in \mathrm{dom}(f)$ then

$$\mathfrak{M} \models \varphi[x_1,\ldots,x_n] \quad \text{iff} \quad \mathfrak{N} \models \varphi[f(x_1),\ldots,f(x_n)].$$

(The theorem follows by considering those $\varphi \in \mathsf{L}_{\infty\omega}$ which are sentences.) If φ is atomic, the result follows from the fact that each $f \in I$ is a partial isomorphism and so preserves atomic and negated atomic formulas. The case where φ is a propositional combination of simpler formulas is immediate by the induction hypothesis. The back and forth property (1) comes into play only in getting past quantifiers. Suppose φ is $\exists v_{n+1}\, \psi(v_1,\ldots,v_{n+1})$. Let $f, x_1,\ldots,x_n$ be given. We assume $\mathfrak{M} \models \varphi[x_1,\ldots,x_n]$ and prove $\mathfrak{N} \models \varphi[f(x_1),\ldots,f(x_n)]$, the other half being similar. Thus, there is a $y \in \mathfrak{M}$ so that

$$\mathfrak{M} \models \psi[x_1,\ldots,x_n,y].$$

Use (1) to get a $g \in I$ with $f \subseteq g$, $y \in \mathrm{dom}(g)$. Then, by the induction hypothesis,

$$\mathfrak{N} \models \psi[g(x_1),\ldots,g(x_n),g(y)]$$

so

$$\mathfrak{N} \models \exists v_{n+1}\, \psi[g(x_1),\ldots,g(x_n),v_{n+1})]$$

and $g(x_i) = f(x_i)$ so

$$\mathfrak{N} \models \varphi[f(x_1),\ldots,f(x_n)],$$

as desired. Since $\forall v_n\, \psi \leftrightarrow \neg\exists v_n \neg\psi$, we need not treat $\forall$ separately.

Now assume $\mathfrak{M} \equiv \mathfrak{N}$ $(\mathsf{L}_{\infty\omega})$. What should our set I be? The proof of the first half of the theorem tells use. Let $f \in I$ iff

$$f: \mathfrak{M}_0 \cong \mathfrak{N}_0, \qquad \mathfrak{M}_0 \subseteq \mathfrak{M}, \qquad \mathfrak{N}_0 \subseteq \mathfrak{N}$$

where $\mathfrak{M}_0$ is finitely generated by some $x_1,\ldots,x_n$ and

$$(\mathfrak{M}, x_1,\ldots,x_n) \equiv_{\infty\omega} (\mathfrak{N}, f(x_1),\ldots,f(x_n))$$

by which we mean that $x_1,\ldots,x_n$ satisfies the same formula of $\mathsf{L}_{\infty\omega}$ in M that $f(x_1),\ldots,f(x_n)$ satisfy in $\mathfrak{N}$. (Note that we need Π_1 Separation to define I so that we cannot carry out this proof in KPU.) Since $\mathfrak{M} \equiv \mathfrak{N}$ $(\mathsf{L}_{\infty\omega})$, the trivial partial isomorphism is in I. We claim that I has the back and forth property. Let $f \in I$ be as above and let x_{n+1} be a new element which we need to add to the domain of f. It suffices to find a $y \in \mathfrak{N}$ so that

$$(\mathfrak{M}, x_1, \ldots, x_n, x_{n+1}) \equiv_{\infty\omega} (\mathfrak{N}, f(x_1), \ldots, f(x_n), y)$$

for then we may set $g(x_{n+1})=y$ and extend to the substructure generated by $x_1, \ldots, x_{n+1}$ in the canonical fashion. So suppose there is no such y. Then, for every $y \in \mathfrak{N}$ there is a formula $\varphi_y(v_1, \ldots, v_{n+1})$ such that

$$\mathfrak{M} \models \varphi_y[x_1, \ldots, x_n, x_{n+1}],$$
$$\mathfrak{N} \models \neg\varphi_y[f(x_1), \ldots, f(x_n), y]$$

Let $\psi(v_1, \ldots, v_n)$ be

$$\exists v_{n+1} \bigwedge_{y \in N} \varphi_y(v_1, \ldots, v_n, v_{n+1}).$$

Then $\mathfrak{M} \models \psi[x_1, \ldots, x_n]$ by letting $v_{n+1} = x_{n+1}$ but

$$\mathfrak{N} \models \neg\psi[f(x_1), \ldots, f(x_n)].$$

This contradicts $f \in I$. ▯

This theorem has a number of important uses. Here we state those having to do with absoluteness.

5.4 Corollary. $\cong_p$ *is the absolute version of* $\cong$.

Proof. $\mathfrak{M} \equiv \mathfrak{N}(\mathsf{L}_{\infty\omega})$ is a Π_1 predicate of $\mathfrak{M}, \mathfrak{N}$, by the results of § III.1, so $\cong_p$ is Δ_1. It agrees with $\cong$ on countable structures by Theorem 5.2. ▯

5.5 Corollary. *Example* 5.1 (iii) *is true.*

Proof. Let T, L_A be as in 5.1 (iii). We need to show that

$$\forall \mathfrak{M} \forall \mathfrak{N} [\mathfrak{M}, \mathfrak{N} \text{ infinite} \wedge \mathfrak{M} \models T \wedge \mathfrak{N} \models T \rightarrow \mathfrak{M} \cong_p \mathfrak{N}].$$

By 5.4, the part within brackets is absolute (in the countable parameter T), so we need only verify the result for $\mathfrak{M}$, $\mathfrak{N}$ countable. But for such $\mathfrak{M}$, $\mathfrak{N}$, the result follows from the hypothesis that T is $\aleph_0$-categorical. ▯

This result (5.5) shows us that if a countable theory T is $\aleph_0$-categorical, then we should be able to prove this by a back and forth argument.

5.6 Corollary. *Let* $\mathfrak{M}, \mathfrak{N}$ *be partially isomorphic structures for a finite language* L.

(i) *For all* α, $L(\alpha)_{\mathfrak{M}} \cong_p L(\alpha)_{\mathfrak{N}}$.

(ii) *For all* α, α *is* $\mathfrak{M}$*-admissible iff* α *is* $\mathfrak{N}$*-admissible.*

(iii) $o(\mathbb{HYP}_{\mathfrak{M}}) = o(\mathbb{HYP}_{\mathfrak{N}})$.

Proof. (i) This is a Π_1 condition on $\mathfrak{M}, \mathfrak{N}$ which clearly holds when $\mathfrak{M}$, $\mathfrak{N}$ are countable since then they are isomorphic. Part (ii) follows immediately from (i) since α is $\mathfrak{M}$-admissible iff $L(\alpha)_{\mathfrak{M}} \models \mathrm{KPU}^+$. Part (iii) follows from (ii). ▯

One of the advantages of Theorem 5.3 is that it allows us to approximate the relation $\mathfrak{M} \cong_p \mathfrak{N}$ by approximating

$$\mathfrak{M} \equiv \mathfrak{N}(\mathsf{L}_{\infty\omega}).$$

Define the *quantifier rank* of a formula φ of $\mathsf{L}_{\infty\omega}$ recursively as follows:

$$\mathrm{qr}(\varphi)=0 \text{ if } \varphi \text{ is atomic},$$
$$\mathrm{qr}(\exists v\varphi)=\mathrm{qr}(\forall v\varphi)=\mathrm{qr}(\varphi)+1,$$
$$\mathrm{qr}(\neg\varphi)=\mathrm{qr}(\varphi),$$
$$\mathrm{qr}(\bigwedge\Phi)=\mathrm{qr}(\bigvee\Phi)=\sup\{\mathrm{qr}(\varphi)\mid\varphi\in\Phi\}.$$

Thus $\mathrm{qr}(\varphi)$ is an ordinal number. Since qr is defined by Σ Recursion in KPU, we have $\mathrm{qr}(\varphi)<o(\mathbb{A})$ whenever φ is in the admissible fragment $\mathsf{L}_{\mathbb{A}}$.

We write

$$\mathfrak{M} \equiv^{\alpha} \mathfrak{N}$$

if for all sentences φ of $\mathsf{L}_{\infty\omega}$ with $\mathrm{qr}(\varphi)\leqslant\alpha$,

$$\mathfrak{M}\models\varphi \quad \text{iff} \quad \mathfrak{N}\models\varphi.$$

Thus $\mathfrak{M}\equiv\mathfrak{N}(\mathsf{L}_{\infty\omega})$ iff for all α, $\mathfrak{M}\equiv^{\alpha}\mathfrak{N}$.

The following is a refinement of Karp's Theorem also due to Karp [1965].

5.7 Theorem. *Given structures* $\mathfrak{M}$, $\mathfrak{N}$ *for* L, $\mathfrak{M}\equiv^{\alpha}\mathfrak{N}$ *iff the following condition holds: There is a sequence*

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_\beta \supseteq \cdots \supseteq I_\alpha \qquad (\beta\leqslant\alpha)$$

where each I_β *is a nonempty set of partial isomorphisms from* $\mathfrak{M}$ *into* $\mathfrak{N}$ *and such that whenever* $\beta+1\leqslant\alpha$, $f\in I_{\beta+1}$ *and* $x\in\mathfrak{M}$ (or $y\in\mathfrak{N}$) *there is a* $g\in I_\beta$ *such that* $f\subseteq g$ *and* $x\in\mathrm{dom}(g)$ (*resp.*, $y\in\mathrm{rng}(g)$).

Proof. The proof is a routine refinement of the proof of Karp's Theorem. To prove ($\Leftarrow$), one shows that if

$$\mathrm{qr}(\varphi(v_1,\ldots,v_n))\leqslant\beta,\ f\in I_\beta,\ x_1,\ldots,x_n\in\mathrm{dom}(f)$$

then

$$\mathfrak{M}\models\varphi[x_1,\ldots,x_n] \quad \text{iff} \quad \mathfrak{N}\models\varphi[f(x_1),\ldots,f(x_n)].$$

To prove ($\Rightarrow$), let I_β be the set of those finitely generated partial isomorphisms f which preserved satisfaction of formulas φ with $\mathrm{qr}(\varphi)\leqslant\beta$. $\square$

5.8—5.12 Exercises

5.8. Prove that if a theory T of $\mathsf{L}_{\omega\omega}$ is $\aleph_0$-categorical then every model of T is $\aleph_0$-saturated. [Use 5.1 (iii), Theorem 5.3 and the fact that $\aleph_0$-saturation can be defined by a conjunction of sentences from $\mathsf{L}_{\omega_1\omega}$.]

5.9. Let $\mathfrak{M}$, $\mathfrak{N}$ be partially isomorphic structures for a finite language. Show that for every α, the pure sets in $L(\alpha)_{\mathfrak{M}}$ and $L(\alpha)_{\mathfrak{N}}$ are the same.

5.10. Let λ be a limit ordinal. Prove that if $\mathfrak{M} \equiv^{\beta} \mathfrak{N}$ for all $\beta < \lambda$ then $\mathfrak{M} \equiv^{\lambda} \mathfrak{N}$. [Each sentence of quantifier rank λ is a propositional combination of sentences of smaller quantifier rank.]

5.11. Show that the following notions are definable by a single sentence of $\mathsf{L}_{\infty\omega}$.
(i) G is an $\aleph_1$-free group.
(ii) G is an abelian p-group of length $\leqslant \alpha$ (for any ordinal α).

5.12. (i) Show that if G is a reduced abelian p-group and $G \equiv H$ ($\mathsf{L}_{\infty\omega}$) then H is a reduced abelian p-group.

(ii) Show that the notion of a reduced abelian p-group is not definable by a single sentence of $\mathsf{L}_{\infty\omega}$. [Hint: There are reduced p-groups of every ordinal length. Show that if the notion were definable then there would be a sentence which pinned down all ordinals, contrary to Theorem 4.1.]

6. Scott Sentences and their Approximations

One of the tasks the mathematician sets for himself is the discovery of invariants which classify a structure $\mathfrak{M}$ up to isomorphism (homomorphism, homeomorphism, etc.) among similar structures. In this section we consider the problem of characterizing arbitrary structures up to $\cong_p$. We will associate with each structure $\mathfrak{M}$, in a reasonably effective manner, a canonical object $\sigma_{\mathfrak{M}}$ such that

$$\mathfrak{M} \cong_p \mathfrak{N} \quad \text{iff} \quad \sigma_{\mathfrak{M}} = \sigma_{\mathfrak{N}} .$$

Hence, if $\mathfrak{M}, \mathfrak{N}$ are countable we will have $\mathfrak{M} \cong \mathfrak{N}$ iff $\sigma_{\mathfrak{M}} = \sigma_{\mathfrak{N}}$. Our invariants will not be cardinal or ordinal numbers, though, as is often the case. Rather, they will be sentences of $\mathsf{L}_{\infty\omega}$ with the additional properties:

$$\mathfrak{M} \models \sigma_{\mathfrak{M}}, \quad \text{and}$$

$$\mathfrak{N} \models \sigma_{\mathfrak{M}} \quad \text{implies} \quad \mathfrak{M} \cong_p \mathfrak{N} .$$

The sentence $\sigma_{\mathfrak{M}}$ is called the *canonical Scott sentence* of $\mathfrak{M}$.

The canonical Scott sentence is built up from its approximations defined below. We use s to range over finite sequences $\langle x_1, \dots, x_n \rangle$ from $\mathfrak{M}$ and $s^\frown x$ to denote the extension $\langle x_1, \dots, x_n, x \rangle$ of s by x.

6.1 Definition. Let $\mathfrak{M}$ be a structure for a language L. For each ordinal α and each sequence $s=\langle x_1,\dots,x_n\rangle$ we define a formula $\sigma_s^\alpha(v_1,\dots,v_n)$, the *$\alpha$-characteristic of s in $\mathfrak{M}$*, by recursion on α:

(i) $\sigma_s^0(v_1,\dots,v_n)$ is

$$\bigwedge\{\varphi(v_1,\dots,v_n)\mid\varphi \text{ is atomic or negated atomic and } \mathfrak{M}\models\varphi[s]\}.$$

(ii) $\sigma_s^{\beta+1}(v_1,\dots,v_n)$ is the conjunction of the following three formulas

$$(1)\qquad \sigma_s^\beta(v_1,\dots,v_n);$$

$$(2)\qquad \forall v_{n+1}\bigvee\nolimits_{x\in\mathfrak{M}}\sigma_{s^\frown x}^\beta(v_1,\dots,v_n);$$

$$(3)\qquad \bigwedge\nolimits_{x\in\mathfrak{M}}\exists v_{n+1}\,\sigma_{s^\frown x}^\beta(v_1,\dots,v_n).$$

(iii) If $\lambda>0$ is a limit ordinal then $\sigma_s^\lambda(v_1,\dots,v_n)$ is

$$\bigwedge\nolimits_{\beta<\lambda}\sigma_s(v_1,\dots,v_n).$$

If we need to indicate the dependence on $\mathfrak{M}$ we write $\sigma_{(\mathfrak{M},s)}^\alpha$ for σ_s^α. If s is the empty sequence we write σ^α or $\sigma_{\mathfrak{M}}^\alpha$.

6.2 Lemma. *Fix $\mathfrak{M},\alpha$ and $s=\langle x_1,\dots,x_n\rangle$.*
(i) $\mathrm{qr}(\sigma_s^\alpha)=\alpha$.
(ii) $\mathfrak{M}\models\sigma_s^\alpha[s]$.
(iii) *If $\alpha\geqslant\beta$ then*

$$\models\forall v_1,\dots,v_n(\sigma_s^\alpha(v_1,\dots,v_n)\to\sigma_s^\beta(v_1,\dots,v_n)).$$

(iv) *If κ is an infinite cardinal and* $\mathrm{card}(\mathfrak{M})<\kappa$, $\mathrm{card}(\mathsf{L})<\kappa$ *and* $\alpha<\kappa$ *then* $\mathrm{card}(\mathrm{sub}(\sigma_s^\alpha))<\kappa$.

Proof. A simple induction on α proves all these facts. □

The crucial properties of the α-characteristics are given by the next result. In this section we write

$$(\mathfrak{M},x_1,\dots,x_n)\equiv_{\infty\omega}(\mathfrak{N},y_1,\dots,y_n)$$

(and

$$(\mathfrak{M},x_1,\dots,x_n)\equiv^\alpha(\mathfrak{N},y_1,\dots,y_n))$$

to indicate that all $\langle x_1,\dots,x_n\rangle$ satisfies the same formulas $\varphi(v_1,\dots,v_n)$ (of quantifier rank at most α) in $\mathfrak{M}$ that $\langle y_1,\dots,y_n\rangle$ satisfies in $\mathfrak{N}$.

6.3 Theorem. *Let $\mathfrak{M},\mathfrak{N}$ be L-structures, $s=\langle x_1,\dots,x_n\rangle$ a sequence from $\mathfrak{M}$, $t=\langle y_1,\dots,y_n\rangle$ a sequence from $\mathfrak{N}$. The following are equivalent:*
(i) $(\mathfrak{M},x_1,\dots,x_n)\equiv^\alpha(\mathfrak{N},y_1,\dots,y_n)$.
(ii) $\mathfrak{N}\models\sigma_{(\mathfrak{M},s)}^\alpha[t]$.
(iii) *The α-characteristic of s in $\mathfrak{M}$ is identical with the α-characteristic of t in $\mathfrak{N}$.*

Proof. The proofs of (i)⇒(ii) and (iii)⇒(ii) and both trivial. The first follows immediately form 6.2(i), (ii). The second implication also follows from 6.2(ii), since $\mathfrak{N} \models \sigma^{\alpha}_{(\mathfrak{N},t)}[t]$, so if $\sigma^{\alpha}_{(\mathfrak{M},s)} = \sigma^{\alpha}_{(\mathfrak{N},t)}$, then

$$\mathfrak{N} \models \sigma^{\alpha}_{(\mathfrak{M},s)}\,.$$

We are left with task of proving (ii)⇒(i) and (ii)⇒(iii). To prove (ii)⇒(i), we use Theorem 5.7. Assume

$$\mathfrak{N} \models \sigma^{\alpha}_{(\mathfrak{M},s)}[t]$$

and define, for $\beta \leqslant \alpha$, a set I_β as follows: $f \in I_\beta$ iff

$$f: \mathfrak{M}_0 \cong \mathfrak{N}_0\,, \qquad \mathfrak{M}_0 \subseteq \mathfrak{M}\,, \qquad \mathfrak{N}_0 \subseteq \mathfrak{N}\,, \quad \text{where}$$

$\mathfrak{M}_0$ is generated by some $z_1, \ldots, z_k$, and

$$\mathfrak{N} \models \sigma^{\beta}_{(\mathfrak{M},z_1,\ldots,z_k)}[f(z_1), \ldots, f(z_k)]\,.$$

The map f_0 generated by sending x_i to y_i $(i = 1, \ldots, n)$ is in I_α by hypothesis. By 6.2(iii), we have

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_\beta \supseteq \cdots \supseteq I_\alpha \qquad (\beta \leqslant \alpha)\,.$$

The final condition on this sequence, the one demanded by 5.7, follows immediately from the definition of $\sigma^{\beta+1}_{(\mathfrak{M},z_1,\ldots,z_k)}$.

Finally, we prove (ii)⇒(iii) by induction on α. The cases for $\alpha = 0$ and α a limit ordinal are trivial. So suppose

$$\mathfrak{N} \models \sigma^{\beta+1}_{(\mathfrak{M},s)}[t]\,.$$

By 6.1(ii), we need to prove that

(4) $\qquad \sigma^{\beta}_{(\mathfrak{M},s)}$ is $\sigma^{\beta}_{(\mathfrak{N},t)}$,

(5) for each $x \in \mathfrak{M}$ there is a $y \in \mathfrak{N}$ such that

$$\sigma^{\beta}_{(\mathfrak{M},s^\frown x)} \text{ is } \sigma^{\beta}_{(\mathfrak{N},t^\frown y)}\,,$$

and

(6) for each $y \in \mathfrak{N}$ there is an $x \in \mathfrak{M}$ such that

$$\sigma^{\beta}_{(\mathfrak{M},s^\frown x)} \text{ is } \sigma^{\beta}_{(\mathfrak{N},t^\frown y)}\,.$$

Now, by the induction hypothesis, (4) is true, (5) reduces to

(5′) $\qquad \mathfrak{N} \models \bigwedge_{x \in \mathfrak{M}} \exists v_{n+1}\, \sigma^{\beta}_{(\mathfrak{M},s^\frown x)}(v_{n+1})[t]$

and (6) reduces to

$$(6') \qquad \mathfrak{N} \models \forall v_{n+1} \bigvee_{x \in \mathfrak{M}} \sigma^{\beta}_{(\mathfrak{M}, s^\frown x)}(v_{n+1})[t] .$$

But (5′), (6′) are immediate consequences of

$$\mathfrak{N} \models \sigma^{\beta+1}_{(\mathfrak{M}, s)}$$

by lines (3), (2) respectively. □

If we apply 6.3 to the empty sequence, we obtain the following result.

6.4 Corollary. For all $\mathfrak{M}, \mathfrak{N}$, the following are equivalent:
(i) $\mathfrak{M} \equiv^{\alpha} \mathfrak{N}$;
(ii) $\mathfrak{N} \models \sigma^{\alpha}_{\mathfrak{M}}$;
(iii) $\sigma^{\alpha}_{\mathfrak{M}} = \sigma^{\alpha}_{\mathfrak{N}}$. □

6.5 Definition. The *Scott rank of a structure* $\mathfrak{M}$, $\mathrm{sr}(\mathfrak{M})$, is the least ordinal α such that for all finite sequences $x_1, \ldots, x_n, y_1, \ldots, y_n$ from $\mathfrak{M}$,

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv^{\alpha} (\mathfrak{M}, y_1, \ldots, y_n)$$

implies

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv^{\alpha+1} (\mathfrak{M}, y_1, \ldots, y_n) .$$

We will see, quite soon, that if $\alpha = \mathrm{sr}(\mathfrak{M})$ then

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv^{\alpha} (\mathfrak{M}, y_1, \ldots, y_n)$$

actually implies

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv_{\infty\omega} (\mathfrak{M}, y_1, \ldots, y_n) .$$

It is more convenient to use 6.5 as the definition, though, since then the next lemma becomes obvious.

6.6 Lemma. *If* κ *is an infinite cardinal and* $\mathrm{card}(\mathfrak{M}) < \kappa$ *then* $\mathrm{sr}(\mathfrak{M}) < \kappa$.

Proof. The proof is easy and we will get a much better bound in the next section, so we leave the proof to the student. □

6.7 Definition. Let $\mathfrak{M}$ be a structure for L, let $\mu = \mathrm{sr}(\mathfrak{M})$. The *canonical Scott theory* of $\mathfrak{M}$, $S_{\mathfrak{M}}$ consists of the sentences below:

$$\sigma^{\mu}_{\mathfrak{M}},$$

$$\forall v_1, \ldots, v_k [\sigma^{\mu}_{(\mathfrak{M}, s)}(v_1, \ldots, v_k) \rightarrow \sigma^{\mu+1}_{(\mathfrak{M}, s)}(v_1, \ldots, v_k)]$$

for all finite sequences $s=\langle x_1,\ldots,x_n\rangle$ from $\mathfrak{M}$. The *canonical Scott sentence* of $\mathfrak{M}$, $\sigma_{\mathfrak{M}}$, is the conjunction of the canonical Scott theory of $\mathfrak{M}$:

$$\sigma_{\mathfrak{M}}=\bigwedge S_{\mathfrak{M}}.$$

Note that $\mathrm{qr}(\sigma_{\mathfrak{M}})=\mathrm{sr}(\mathfrak{M})+\omega$. Also, from the definition of $\mathrm{sr}(\mathfrak{M})$ we see that

$$\mathfrak{M}\models\sigma_{\mathfrak{M}}.$$

We now come to the main theorem on Scott sentences.

6.8 Theorem. *Given structures $\mathfrak{M},\mathfrak{N}$ for a language L, the following are equivalent:*
(i) $\mathfrak{M}\cong_p\mathfrak{N}$;
(ii) $\mathfrak{N}\models\sigma_{\mathfrak{M}}$;
(iii) $\sigma_{\mathfrak{M}}=\sigma_{\mathfrak{N}}$.

Proof. We already know that $\mathfrak{M}\cong_p\mathfrak{N}$ iff $\mathfrak{M}\equiv_{\infty\omega}\mathfrak{N}$. Since $\mathfrak{M}\models\sigma_{\mathfrak{M}}$ we see that (i)⇒(ii) is immediate. Similarly, since $\mathfrak{N}\models\sigma_{\mathfrak{N}}$, (iii)⇒(ii) is immediate. To prove (ii)⇒(i) define I_β, for all β, just as in the proof of 6.3. The hypothesis that $\mathfrak{N}\models\sigma_{\mathfrak{M}}$ insures that $I_{\mu+1}=I_\mu$ so

$$I_\mu\colon\mathfrak{M}\cong_p\mathfrak{N}.$$

Finally, we prove that (i)⇒(iii). Assume that $\mathfrak{M}\equiv_{\infty\omega}\mathfrak{N}$. Then $\mathrm{sr}(\mathfrak{M})=\mathrm{sr}(\mathfrak{N})$. Let $\mu=\mathrm{sr}(\mathfrak{M})$. For each $x_1,\ldots,x_n\in\mathfrak{M}$ there is a sequence $y_1,\ldots,y_n\in\mathfrak{N}$ such that

$$\mathfrak{N}\models\sigma^{\mu}_{(\mathfrak{M},x_1,\ldots,x_n)}[y_1,\ldots,y_n]$$

and vice versa. Then, by 6.3, every $\sigma^{\mu}_{(\mathfrak{M},s)}$ is some $\sigma^{\mu}_{(\mathfrak{N},t)}$ and vice versa. Thus $S_{\mathfrak{M}}=S_{\mathfrak{N}}$ and $\sigma_{\mathfrak{M}}=\sigma_{\mathfrak{N}}$. ☐

The remainder of this section is devoted to corollaries of Theorem 6.8. First we have Scott's original result.

6.9 Corollary (Scott's Theorem). *Let L be a countable language and let $\mathfrak{M}$ be a countable structure for L. The Scott sentence $\sigma_{\mathfrak{M}}$ is a sentence of $\mathsf{L}_{\omega_1\omega}$ with the property that*

$$\mathfrak{M}\cong\mathfrak{N}\quad \textit{iff}\quad \mathfrak{N}\models\sigma_{\mathfrak{M}}$$

for all countable L-structures $\mathfrak{N}$.

Proof. $\sigma_{\mathfrak{M}}$ is in $\mathsf{L}_{\omega_1\omega}$ by Lemma 6.2(iv). The result is then an immediate consequence of Theorem 5.2 and 6.8. ☐

An n-ary relation P on $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ is *invariant* if for every automorphism f of $\mathfrak{M}$ and every $x_1,\ldots,x_n\in\mathfrak{M}$,

$$P(x_1,\ldots,x_n)\quad\text{iff}\quad P(f(x_1),\ldots,f(x_n)).$$

From now on (in this section) we assume L is countable. Whenever we refer to $\mathbb{HYP}_{\mathfrak{M}}$ we assume L is finite.

6.10 Corollary. *If $\mathfrak{M}$ is a countable structure for L and P is an n-ary relation on $\mathfrak{M}$, then P is invariant iff P is definable by some formula $\varphi(v_1, \ldots, v_n)$ of $\mathsf{L}_{\omega_1\omega}$ (without additional parameters):*

$$P(x_1, \ldots, x_n) \quad \textit{iff} \quad \mathfrak{M} \models \varphi[x_1, \ldots, x_n].$$

Proof. If P is defined by φ then P must be invariant since $f: \mathfrak{M} \cong \mathfrak{M}$ and $\mathfrak{M} \models \varphi[x_1, \ldots, x_n]$ implies $\mathfrak{M} \models \varphi[f(x_1), \ldots, f(x_n)]$. Now assume P is invariant. Let $\varphi(v_1, \ldots, v_n)$ be

$$\bigvee \{\sigma^{\mu}_{\langle x_1, \ldots, x_n \rangle}(v_1, \ldots, v_n) \mid P(x_1, \ldots, x_n)\},$$

where $\mu = \mathrm{sr}(\mathfrak{M})$. It is clear that $P(x_1, \ldots, x_n)$ implies $\mathfrak{M} \models \varphi[x_1, \ldots, x_n]$.

To prove the converse, suppose that $\mathfrak{M} \models \varphi[y_1, \ldots, y_n]$, so that $\mathfrak{M} \models \sigma^{\mu}_{\langle x_1, \ldots, x_n \rangle}[y_1, \ldots, y_n]$ for some $x_1, \ldots, x_n$ with $P(x_1, \ldots, x_n)$. Then

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv_{\infty\omega} (\mathfrak{M}, y_1, \ldots, y_n),$$

so that there is an automorphism f of $\mathfrak{M}$ with $f(x_i) = y_i$ by 5.2. Since P is invariant, $P(y_1, \ldots, y_n)$ holds. □

6.11 Corollary. *Let $\mathfrak{M}$ be a countable structure for L and let $x \in \mathfrak{M}$ be an element fixed by every automorphism of $\mathfrak{M}$. Then x is definable by a formula $\varphi(v)$ of $\mathsf{L}_{\omega_1\omega}$:*

$$\mathfrak{M} \models \exists! v \varphi(v),$$

$$\mathfrak{M} \models \varphi[x].$$

Conversely, a definable element of $\mathfrak{M}$ is fixed by every automorphism.

Proof. Apply 6.10 with $P = \{x\}$. □

A *rigid* structure is one with only one automorphism, the identity map.

6.12 Corollary. *If $\mathfrak{M}$ is a countable structure for L then $\mathfrak{M}$ is rigid iff every element x of $\mathfrak{M}$ is definable by a formula $\varphi(v)$ of $\mathsf{L}_{\omega_1\omega}$:*

$$\mathfrak{M} \models \exists! v \varphi(v),$$

$$\mathfrak{M} \models \varphi[x]. \quad □$$

These results will be improved in the next section.

6.13—6.14 Exercises

6.13. Let $\mathfrak{M}$ be a countable structure with $x_1, \ldots, x_n \in \mathfrak{M}$ such that $(\mathfrak{M}, x_1, \ldots, x_n)$ is rigid; i.e., no nontrivial automorphisms of $\mathfrak{M}$ fix, $x_1, \ldots, x_n$. Show that $\mathfrak{M}$ has $\leqslant \aleph_0$ automorphisms.

6.14. Let $\mathfrak{M}$ be a countable L-structure with $< 2^{\aleph_0}$ automorphisms.

(i) Prove that there is a finite sequence $x_1, \ldots, x_n$ from $\mathfrak{M}$ such that $(\mathfrak{M}, x_1, \ldots, x_n)$ is rigid. [Hint (P. M. Cohn): Let σ_n fix $x_1, \ldots, x_n$ but move, say, x_{n+1}. Let $\sigma = \ldots \sigma_n^{\varepsilon_n} \ldots \sigma_2^{\varepsilon_2} \sigma_1^{\varepsilon_1}$ where $\varepsilon_i = 0$ or $= 1$. Show that this gives $2^{\aleph_0}$ automorphisms.]

(ii) Show that for all $\mathfrak{N}$, $\mathfrak{N} \models \sigma_{\mathfrak{M}}$ implies $\mathfrak{M} \cong \mathfrak{N}$; i.e., that there are no uncountable $\mathfrak{N}$ with $\mathfrak{M} \equiv \mathfrak{N}$ $(\mathsf{L}_{\omega_1 \omega})$.

6.15. Show that if G is an $\aleph_1$-free abelian group then $G \cong_p H$ iff H is $\aleph_1$-free. Thus the notion of free group is not definable in $\mathsf{L}_{\infty\omega}$.

6.15 Notes. Scott's Theorem and Corollary 6.10 were announced in Scott [1965]. A proof, in the context of invariant Borel sets, appears in Scott [1964]. The Scott sentences used here are derived from Chang's proof of Scott's Theorem in Chang [1968]. The presentation follows that used in the survey article Barwise [1973]. Exercises 6.13, 6.14, 6.15 are due to Kueker. They are proved in Barwise [1973].

7. Scott Sentences and Admissible Sets

The first systematic study of the relationship between α-characteristics, canonical Scott sentences and admissible sets was undertaken by Nadel in his doctoral dissertation. His idea was to use α-characteristics and Scott sentences as approximations of models, asking to which admissible sets the formulas $\sigma_{\mathfrak{M}}, \sigma_{\mathfrak{M}}^{\alpha}$ belong as an alternative to asking to which admissible sets $\mathfrak{M}$ itself belongs. This has proven to be a fruitful idea. In this section we delve into the more elementary parts of the theory.

To simplify matters we assume the underlying language L of $\mathsf{L}_{\infty\omega}$ has no function symbols. Since function symbols can always be replaced by relation symbols, this is no essential loss. (The sole point in this restriction is that if L is an element of an admissible set $\mathbb{A}$ then the set of atomic and negated atomic formulas of the form

$$\varphi(v_1, \ldots, v_n)$$

(for fixed $n < \omega$) is a set in $\mathbb{A}$ *if* L has no function symbols, or *if* $o(\mathbb{A}) > \omega$, but *not* if L has a function symbol and $o(\mathbb{A}) = \omega$).

7.1 Proposition. *The formula*

$$\sigma^{\alpha}_{\mathfrak{M},s}(v_1, \ldots, v_n)$$

is definable in KPU *as a* Σ_1 *operation of* $\mathfrak{M}, s, \alpha$.

Proof. Consider sequences s as functions with dom(s) some $n<\omega$ and range $\subseteq \mathfrak{M}$. Let

$$F(\mathfrak{M}, s, \alpha) = \sigma^{\alpha}_{\mathfrak{M},s}(v_1, \ldots, v_n) .$$

If we write out the definition of F as given in 6.1 it takes the following form:

$$F(\mathfrak{M}, s, \alpha) = y \quad \text{iff} \quad \text{(i)} \vee \text{(ii)} \vee \text{(iii)}$$

where

(i) $\alpha = 0 \wedge \Delta_0(\mathfrak{M}, s, y)$ (a Δ_0 predicate of $\mathfrak{M}, s$ and y);

(ii) $\alpha = \beta + 1$ for some $\beta < \alpha$ and $y = \bigwedge\{\theta_1, \theta_2, \theta_3\}$ where

$\theta_1 = F(\mathfrak{M}, s, \beta)$,

θ_2 is $\forall v_{n+1} \bigvee \Phi$ where

$\forall x \in \mathfrak{M}\, \exists z \in \Phi\, F(\mathfrak{M}, s^\wedge x, \beta) = z$,

$\forall z \in \Phi\, \exists x \in \mathfrak{M}\, F(M, s^\wedge x, \beta) = z$, and

θ_3 is similar to θ_2.

(iii) $\mathrm{Lim}(\alpha) \wedge y = \bigwedge\{F(\mathfrak{M}, s, \beta) \mid \beta < \alpha\}$.

This definition clearly falls under the second recursion theorem. □

7.2 Corollary. *If* $\mathsf{L}_{\mathbb{A}}$ *is an admissible fragment and* $\mathfrak{M}$ *is an* L*-structure in the admissible set* $\mathbb{A}$ *then, for any* L*-structure* $\mathfrak{N}$,

$$\mathfrak{M} \equiv \mathfrak{N}\ (\mathsf{L}_{\mathbb{A}}) \quad \textit{implies} \quad \mathfrak{M} \equiv^{\alpha} \mathfrak{N}$$

where $\alpha = o(\mathbb{A})$.

Proof. By Exercise 5.10 it suffices to prove that

$$\mathfrak{M} \equiv^{\beta} \mathfrak{N}$$

for all $\beta < \alpha$. But for $\beta < \alpha$, $\sigma^{\beta}_{\mathfrak{M}} \in \mathsf{L}_{\mathbb{A}}$ by 7.1 and $\mathfrak{M} \models \sigma^{\beta}_{\mathfrak{M}}$ so $\mathfrak{N} \models \sigma^{\beta}_{\mathfrak{M}}$. But then $\mathfrak{M} \equiv^{\beta} \mathfrak{N}$ by Corollary 6.4. □

If $\sigma_{\mathfrak{M}}$ were definable as a Σ_1 operation of $\mathfrak{M}$ in KPU then we could extend 7.2 to read

$$\mathfrak{M} \equiv \mathfrak{N}\ (\mathsf{L}_{\mathbb{A}}) \quad \text{implies} \quad \mathfrak{M} \equiv_{\infty\omega} \mathfrak{N} ,$$

since then $\sigma_{\mathfrak{M}}$ would be in $\mathsf{L}_{\mathbb{A}}$. This, however, is not true. Unlike its approximations, *the canonical Scott sentence* $\sigma_{\mathfrak{M}}$ *is not definable in* KPU *as a* Σ_1 *operation of* $\mathfrak{M}$. The problem is that sr($\mathfrak{M}$) may be just a bit too big; that is, sr($\mathfrak{M}$) may equal $o(\mathbb{HYP}_{\mathfrak{M}})$. (See Exercise 7.13, 7.14.) This is as big as it can get, though, as we see in Corollary 7.4.

7.3 Theorem. *Let* $\mathsf{L}_{\mathbb{A}}$ *be an admissible fragment of* $\mathsf{L}_{\infty\omega}$ *and let* $\mathfrak{M}, \mathfrak{N}$ *be* L*-structures which are both elements of the admissible set* $\mathbb{A}$*. Then*

$$\mathfrak{M} \equiv \mathfrak{N}\ (\mathsf{L}_{\mathbb{A}}) \quad \textit{implies} \quad \mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}\,.$$

Proof. By 7.2 we see that $\mathfrak{M} \equiv^{\alpha} \mathfrak{N}$ where $\alpha = o(\mathbb{A})$. Let I be the set of finite partial isomorphisms $f = \{\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle\}$ $(0 \leqslant n < \omega)$ such that

$$(1) \qquad (\mathfrak{M}, x_1, \ldots, x_n) \equiv^{\alpha} (\mathfrak{N}, y_1, \ldots, y_n)\,.$$

Since $\mathfrak{M} \equiv^{\alpha} \mathfrak{N}$, the trivial map is in I so $I \neq \emptyset$. We will prove that

$$I : \mathfrak{M} \cong_p \mathfrak{N}\,.$$

Suppose (1) holds and that a new $x_{n+1} \in \mathfrak{M}$ is given. We need to find a $y_{n+1} \in \mathfrak{N}$ such that

$$(\mathfrak{M}, x_1, \ldots, x_n, x_{n+1}) \equiv^{\alpha} (\mathfrak{N}, y_1, \ldots, y_n, y_{n+1})\,.$$

By Exercise 5.10 it suffices to insure that

$$(\mathfrak{M}, x_1, \ldots, x_n, x_{n+1}) \equiv^{\beta} (\mathfrak{N}, y_1, \ldots, y_n, y_{n+1})$$

for each $\beta < \alpha$. Suppose that no such y_{n+1} exists. Then

$$\forall y_{n+1} \in \mathfrak{N}\, \exists \beta < \alpha (\mathfrak{N} \models \neg \sigma^{\beta}_{(\mathfrak{M}, s)}[y_1, \ldots, y_n, y_{n+1}])$$

where $s = \langle x_1, \ldots, x_{n+1}\rangle$. By Σ Reflection in $\mathbb{A}$, there is a $\gamma < \alpha$ such that

$$\forall y_{n+1} \in \mathfrak{N}\, \exists \beta < \gamma (\mathfrak{N} \models \neg \sigma^{\beta}_{\mathfrak{M}, s}[y_1, \ldots, y_n, y_{n+1}])$$

and hence

$$\mathfrak{N} \models \forall v_{n+1} \neg \sigma^{\gamma}_{\mathfrak{M}, s}(v_{n+1})\,[y_1, \ldots, y_n]$$

so

$$\mathfrak{N} \models \neg \sigma^{\gamma+1}_{(\mathfrak{M}, s)}[y_1, \ldots, y_n]$$

contradicting

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv^{\alpha} (\mathfrak{N}, y_1, \ldots, y_n)\,.$$

This establishes the "forth" half of the back and forth property; the "back" half follows from the symmetry of $\mathfrak{M}$ and $\mathfrak{N}$ in the theorem. □

Theorem 7.3 is sometimes called Nadel's Basis Theorem. The reason for calling it a basis theorem is seen by stating the converse of its conclusion: If there is a sentence φ of $\mathsf{L}_{\infty\omega}$ true in $\mathfrak{M}$ and false in $\mathfrak{N}$, then there is such a sentence in $\mathsf{L}_{\mathbb{A}}$.

Our first application of 7.3 is to get the best possible bound on sr($\mathfrak{M}$). Another proof of this can be given by means of inductive definitions.

7.4 Corollary. *Let $\mathfrak{M}$ be a structure in an admissible set $\mathbb{A}$. Then*

$$\mathrm{sr}(\mathfrak{M}) \leqslant o(\mathbb{A}).$$

Proof. Let $\alpha = o(\mathbb{A})$. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{M}$ be such that

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv^{\alpha} (\mathfrak{M}, y_1, \ldots, y_n).$$

But then

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv (\mathfrak{M}, y_1, \ldots, y_n)\,(\mathsf{L}_{\mathbb{A}})$$

so, by 7.3,

$$(\mathfrak{M}, x_1, \ldots, x_n) \equiv_{\infty\omega} (\mathfrak{M}, y_1, \ldots, y_n). \quad \square$$

The remainder of this section deals with uses of Nadel's Basis Theorem to improve the results of the previous section.

7.5 Theorem. *Let $\mathfrak{M}$ be an L-structure and let P be a relation on $\mathfrak{M}$ which is definable by some formula of $\mathsf{L}_{\infty\omega}$ without parameters. Let $\mathbb{A}$ be any admissible set with $(\mathfrak{M}, P) \in \mathbb{A}$. Then P is definable by a formula of $\mathsf{L}_{\mathbb{A}}$ without parameters.*

Proof. Let us suppose, for convenience, that P is unary. We assume that P is not definable by any formula of $\mathsf{L}_{\mathbb{A}}$. If we can find an x, y such that

$$P(x), \neg P(y), \quad \text{and} \quad (\mathfrak{M}, x) \equiv (\mathfrak{M}, y)\,(\mathsf{L}_{\mathbb{A}})$$

then, by 7.3,

$$(\mathfrak{M}, x) \equiv_{\infty\omega} (\mathfrak{M}, y)$$

so P is not definable by any formula of $\mathsf{L}_{\infty\omega}$. To find such an x, y we proceed as follows. Define, for $\beta < \alpha$, $\varphi_\beta(v)$ to be the formula

$$\bigvee \{\sigma_x^\beta(v) \mid x \in P\}.$$

Then $\varphi_\beta(v) \in \mathsf{L}_{\mathbb{A}}$ by 7.1 and

(2) $$\models \varphi_\beta(v) \rightarrow \varphi_\gamma(v)$$

for $\beta \geqslant \gamma$. Since $\mathfrak{M} \models \varphi_\beta[x]$ for all $x \in P$, and φ_β does not define P (nothing in $\mathsf{L}_{\mathbb{A}}$ does) there must be some $y \in M - P$ such that $\mathfrak{M} \models \varphi_\beta[y]$.

We claim that there is a fixed $y \in M - P$ which works for all $\beta < \alpha$:

(3) $$\exists y \in M - P \, \forall \beta < \alpha (\mathfrak{M} \models \varphi_\beta[y]) .$$

For otherwise we would have

$$\forall y \in M - P \, \exists \beta < \alpha \, (\mathfrak{M} \models \neg \varphi_\beta[y]) .$$

But then by Σ Reflection there is a $\gamma < \alpha$ such that for all $y \in M - P$

$$\mathfrak{M} \models \bigvee_{\beta < \gamma} \neg \varphi_\beta[y]$$

and hence by (2),

$$\forall y \in M - P \, (\mathfrak{M} \models \neg \varphi_\gamma[y]) ,$$

a contradiction. Thus (3) is established. Let y be as in (3). For each β there is an $x \in P$ such that

$$\mathfrak{M} \models \sigma_x^\beta[y]$$

by the definition of φ_β. By an argument entirely analogous to the proof of (3), we see that

$$\exists x \in P \, \forall \beta < \alpha \, (\mathfrak{M} \models \sigma_x^\beta[y]) .$$

For any such x we have $(\mathfrak{M}, x) \equiv^\alpha (\mathfrak{M}, y)$ and hence $(\mathfrak{M}, x) \equiv (\mathfrak{M}, y)(\mathsf{L}_{\mathbb{A}})$, as desired. ☐

7.6 Corollary. *Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a countable structure for* L. *A relation P on $\mathfrak{M}$ is invariant on $\mathfrak{M}$ iff it is definable by a formula in $\mathsf{L}_{\infty\omega} \cap \mathbb{HYP}_{(\mathfrak{M}, P)}$.*

Proof. Combine 6.10 with 7.5. ☐

7.7 Corollary. *Let $\mathsf{L}_{\mathbb{A}}$ be an admissible fragment of $\mathsf{L}_{\infty\omega}$. If $\mathfrak{M}$ is an* L*-structure, $\mathfrak{M} \in \mathbb{A}$, then every element of $\mathfrak{M}$ definable by some formula of $\mathsf{L}_{\infty\omega}$ is definable by a formula of $\mathsf{L}_{\mathbb{A}}$.*

Proof. Apply 7.5 with $P = \{x\}$. ☐

7.8 Corollary. *Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a countable* L*-structure. Then $\mathfrak{M}$ is rigid iff every element of $\mathfrak{M}$ is definable by a formula of $\mathsf{L}_{\infty\omega} \cap \mathbb{HYP}_{\mathfrak{M}}$.*

Proof. Combine 6.12 with 7.7. ☐

7.9 Corollary. *If* $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ *is a countable rigid structure then* $\mathrm{sr}(\mathfrak{M})<O(\mathfrak{M})$.

Proof. By 7.8 we know that

$$\forall x\in M\,\exists\beta\,\mathfrak{M}\models\exists!\,v\,\sigma_x^\beta(v)\,.$$

Let $\beta(x)$ be the least such β. Then $\sigma_x^{\beta(x)}(v)$ is a $\mathbb{HYP}_{\mathfrak{M}}$-recursive function of x so, by Σ Replacement,

$$\Phi(v)=\{\sigma_x^{\beta(x)}(v)\,|\,x\in M\}$$

is in $\mathbb{HYP}_{\mathfrak{M}}$ and every element of M is definable by some member of it. Let $\gamma=\sup\{\beta(x)\,|\,x\in M\}$. We claim that $\mathrm{sr}(\mathfrak{M})\leqslant\gamma$. For suppose

$$(\mathfrak{M},x_1,\ldots,x_n)\equiv^\gamma(\mathfrak{M},y_1,\ldots,y_n)\,.$$

Then

$$\mathfrak{M}\models\varphi_{x_i}^{\beta(x_i)}[y_i]$$

so $x_i=y_i$ for $i=1,\ldots,n$, and hence

$$(\mathfrak{M},x_1,\ldots,x_n)\equiv_{\infty\omega}(\mathfrak{M},y_1,\ldots,y_n)\,.\quad\square$$

We can improve 7.9 by replacing the requirement that $\mathfrak{M}$ is rigid by the requirement that $\mathfrak{M}$ have $<2^{\aleph_0}$ automorphisms. See Exercise 7.15.

We end this section by returning to our old favorite, recursively saturated structures, to see what some of our results say in this case.

7.10 Corollary. *Let* $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ *be a recursively saturated* L-*structure and let* P *be a relation on* $\mathfrak{M}$ *definable by some formula of* $\mathsf{L}_{\infty\omega}$. *Then* $(\mathfrak{M},P)$ *is recursively saturated iff* P *is definable by a finitary formula of* $\mathsf{L}_{\omega\omega}$.

Proof. The ($\Rightarrow$) half follows from 7.5 with $\mathbb{A}=\mathbb{HYP}_{(\mathfrak{M},P)}$. To prove the ($\Leftarrow$) half, note that if P is definable by a formula $\varphi\in\mathbb{HYP}_{\mathfrak{M}}$ then $P\in\mathbb{HYP}_{\mathfrak{M}}$ by Δ_1 Separation so $o(\mathbb{HYP}_{(\mathfrak{M},P)})=\omega$. $\square$

Note that if $\mathfrak{M}$ is recursively saturated then so is $(\mathfrak{M},\vec{x})$ for any $\vec{x}\in\mathfrak{M}$ so 7.10 also applies to relations definable by a fixed finite number of parameters. The same remark applies to the next result.

7.11 Corollary. *Let* $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ *be an infinite recursively saturated* L-*structure and let*

$$\mathscr{D}\!f(\mathfrak{M})=\{y\in M\,|\,y\text{ is definable by some formula }\varphi(v)\text{ of }\mathsf{L}_{\infty\omega}\text{ without parameters}\}\,.$$

Then we have the following:

(i) *Every element of* $\mathscr{D}f(\mathfrak{M})$ *is definable by a finitary formula of* $\mathsf{L}_{\omega\omega}$.

(ii) $\mathscr{D}f(\mathfrak{M})$ *is* Σ_1 *on* $\mathbb{HYP}_{\mathfrak{M}}$, *hence inductive* on* $\mathfrak{M}$.

(iii) *If* $\mathscr{D}f(\mathfrak{M})$ *is hyperelementary* on* $\mathfrak{M}$ (*i.e., if it is in* $\mathbb{HYP}_{\mathfrak{M}}$) *then* $\mathscr{D}f(\mathfrak{M})$ *is finite.*

(iv) $\mathfrak{M}-\mathscr{D}f(\mathfrak{M})$ *is infinite.*

Proof. (i) follows from 7.7 and (i)$\Rightarrow$(ii). To prove (iii) suppose that $\mathscr{D}f(\mathfrak{M})\in\mathbb{HYP}_{\mathfrak{M}}$. Let

$$\Phi=\{\sigma_x^n(v)\,|\,x\in\mathscr{D}f(\mathfrak{M}),\ \mathfrak{M}\models\exists!v\,\sigma_x^n(v)\text{ and }\mathfrak{M}\models\textstyle\bigwedge_{m<n}\neg\exists!v\,\varphi_x^m(v)\}.$$

Then, exactly as in the proof of 7.9, Φ is an element of $\mathbb{HYP}_{\mathfrak{M}}$. But Φ is a pure set and $o(\mathbb{HYP}_{\mathfrak{M}})=\omega$ so Φ is finite. Thus $\mathscr{D}f(\mathfrak{M})$ must also be finite, since every member is defined by a formula in Φ. Part (iv) is immediate from (iii), for if $\mathfrak{M}-\mathscr{D}f(\mathfrak{M})$ is finite then $\mathscr{D}f(\mathfrak{M})\in\mathbb{HYP}_{\mathfrak{M}}$. □

7.12. Example. Let $\mathscr{N}'$ be a nonstandard model of Peano Arithmetic and let $x\in N'$ be a nonstandard integer. Let $\mathscr{N}[x]$ be the submodel of $\mathscr{N}'$ with universe

$$\mathscr{D}f((\mathscr{N}',x)).$$

The axiom of induction insures that

$$\mathscr{N}[x]\prec\mathscr{N}'.$$

Corollary 7.11 (iv) (applied to $(\mathscr{N}[x],x)$) shows that models of the form $\mathscr{N}[x]$ can never be recursively saturated. Hence, the standard integers of $\mathscr{N}[x]$ form a hyperelementary subset of $\mathscr{N}[x]$ by VI.5.1 (ii). From this it follows that such models can never be expanded to a model of second order arithmetic, by Exercise IV.5.13.

7.13—7.18 Exercises

7.13. Let M be countable, α a countable admissible ordinal, $\alpha>\omega$, and let η be the order type of the rationals.

(i) Prove that if $<_1$ is a linear ordering of M of order type $\alpha(1+\eta)$ than, setting $\mathfrak{M}_1=\langle M,<_1\rangle$,

$$\mathbb{HYP}_{\mathfrak{M}_1}\models\text{"}<_1\text{ is well founded"},$$

$$\alpha=o(\mathbb{HYP}_{\mathfrak{M}_1}).$$

[See the proof of IV.6.1.]

(ii) Let $\mathfrak{M}_0=\mathscr{W}f(\mathfrak{M}_1)$. Let $\mathsf{L}_{\mathbb{A}}$ be the admissible fragment of $\mathsf{L}_{\infty\omega}$ given by $\mathbb{HYP}_{\mathfrak{M}_1}$, where $\mathsf{L}=\{<\}$. Prove that

$$\mathfrak{M}_0\prec\mathfrak{M}_1\quad[\mathsf{L}_{\mathbb{A}}].$$

[Use the Tarski Criterion for $\mathsf{L}_{\mathbb{A}}$ (Exercise 2.13) and the fact that any x in the non-wellfounded part of $<_1$ can be moved by an automorphism of $\mathfrak{M}_1$.]

(iii) Prove that

$$\mathfrak{M}_0 \equiv^{\alpha} \mathfrak{M}_1 .$$

(iv) Prove that $\mathrm{sr}(\mathfrak{M}_1)=\alpha$.

(v) Conclude that $\sigma_{\mathfrak{M}}$ is not definable in KPU as a Σ_1 operation of $\mathfrak{M}$.

7.14. Prove that $\mathrm{sr}(\mathfrak{M})$ and σ_M are Σ_1 definable in KPU + Infinity + Σ_1 Separation, as operations of $\mathfrak{M}$.

7.15. Use 6.14 (i) to improve 7.9 to the case where $\mathfrak{M}$ has $<2^{\aleph_0}$ automorphisms.

7.16. Prove that if $o(\mathbb{HYP}_{\mathfrak{M}})>\omega$ and $\mathrm{sr}(\mathfrak{M})<o(\mathbb{HYP}_{\mathfrak{M}})$ then $\sigma_{\mathfrak{M}}\in\mathbb{HYP}_{\mathfrak{M}}$.

7.17. Prove that the absolute version of

"P is invariant on $\mathfrak{M}$"

is

"P is definable by a formula of $\mathsf{L}_{\infty\omega}\cap\mathbb{HYP}_{(\mathfrak{M},P)}$".

7.18. Prove that the absolute version of "$\mathfrak{M}$ is rigid" is "Every element of $\mathfrak{M}$ is definable by a formula of $\mathsf{L}_{\infty\omega}\cap\mathbb{HYP}_{\mathfrak{M}}$".

7.19 Notes. There are a number of interesting and important results which could be gone into at this point, but they would take us too far afield. The student is urged to read Makkai [1975] and Nadel [1974].

Theorem 7.3 is from Nadel [1971] (and Nadel [1974]) as are Collaries 7.7 and 7.8. Theorem 7.5 is new here but it is a fairly routine generalization of Nadel's 7.7. The important example 7.13 is also taken from Nadel [1971]. The last sentence of Example 7.12 is a theorem of Ehrenfeucht and Kreisel. [Added in proof: A recent paper by Nadel and Stari called "The pure part of $\mathbb{HYP}$" (to appear in the Journal of Symbolic Logic) has a number of interesting and highly relevant results. In particular, they characterize the pure part of $\mathbb{HYP}_{\mathfrak{M}}$ in terms of the sentences $\sigma^{\beta}_{\mathfrak{M}}$ for $\beta<O(\mathfrak{M})$.]