Part A

The Basic Theory

"Logic is logic. That's all I say."

Oliver Wendel Holmes
The Deacon's Masterpiece

Chapter I
Admissible Set Theory

Admissible sets are the intended models of a certain first order theory. In this chapter we discuss the theory itself and show how to develop a significant part of intuitive set theory within it.

1. The Role of Urelements

Our approach to admissible sets is unorthodox in several respects, the most obvious being that we allow admissible sets to contain urelements. Bluntly put, we consider admissible sets which are built up out of the stuff of mathematics, not just the sets built up from the empty set. To make this a little clearer, and to see why it is an obvious step to take, we begin by reviewing the development of ZF, Zermelo-Fraenkel set theory, as it is correctly presented (in, for example, § 9.1 of Shoenfield [1967]).

The fundamental tenet of set theory is that, given a collection M of mathematical objects, subcollections are themselves perfectly reasonable mathematical objects, as are collections of these new objects, and so on. We begin with a collection M of objects called *urelements* (sometimes called *points*, *atoms* or *individuals*, depending principally on our subject), which we think of as being given outright. The objects in M might be real numbers, elements of some group or even physical objects. We construct sets out of the objects of M in stages. At each stage α we are allowed to form sets out of urelements and the sets formed at earlier stages. An object is a *set on M* just in case it is formed at some stage in this construction; the collection of all sets on M is denoted by $\mathbb{V}_M$.

Now it turns out, and it must have been a surprising discovery, that if we allow strong enough principles of construction at each stage α, and if we assume that there are enough stages, then urelements become superfluous. All ordinary mathematical objects occur, up to isomorphism, in $\mathbb{V}$, *i.e.* in $\mathbb{V}_M$ where M is the empty collection. It is consistent with this that the extensionality axiom of ZF explicitly rule out the existence of objects which are not sets; the combination of the power set and replacement axioms is so strong as to make urelements unnecessary.

Set theory, as formalized in ZF, provides an extremely powerful and elegant way to organize existing mathematics. It is not without its drawbacks, nevertheless. While it is too weak to decide some questions (like the continuum hypothesis) which seem meaningful (even important), it is in some ways too strong. Some examples:

(1) The most obvious advantage of the axiomatic method is lost since ZF has so few recognizable models in which to interpret its theorems.

(2) Important distinctions on the nature of the sets asserted to exist are completely lost.

(3) The principle of parsimony, of established value throughout the mathematical ages, is violated at every turn.

(4) Large parts of mathematical practice are distorted by the demand that all mathematical objects be *realized as* sets (as opposed to being *isomorphic to* sets). If these objections are not too clear, they should becomes so as we investigate the theory of admissible sets. At any rate these considerations, and others familiar to anyone versed in generalized recursion theory, eventually dictate the study of set theories weaker than ZF, weaker in the principles of set existence which they attempt to formalize. The theory we have in mind here, of course, is the Kripke-Platek theory KP for admissible sets.

It is at this point that one is tempted to make a simplifying mistake. We have first thrown out urelements from ZF because ZF is so strong. When we then weaken ZF to KP we must remember to reexamine the justification for banning the urelements. Doing so, we discover that the justification has completely disappeared. In this book we readmit urelements by "weakening" KP to a theory KPU. The original KP will be equivalent to the theory

KPU+ "there are no urelements".

This approach has many advantages. The chief is that it allows us to form, for any structure $\mathfrak{M}=\langle M,R_1\ldots R_k\rangle$ a particularly important admissible set $\mathbb{HYP}_{\mathfrak{M}}$ above $\mathfrak{M}$, one which is of great use in the study of definability over $\mathfrak{M}$. The approach has no disadvantages since we can always restrict attention to the special case where there are no urelements.

1.1—1.4 Examples

1.1. The point made in (1) above becomes clearer when we recall that if ZF is consistent, so is

ZF+ "There is no transitive model of ZF".

(Prove this without using Gödel's Incompleteness Theorems!)

1.2. The observation in (2) is illustrated by considering, for example, an arbitrary abelian group $\mathfrak{G}=\langle G,+\rangle$. Consider the following subgroups of G:

$$pG=\{px\mid x\in G\},$$

$$T=\{x\mid nx=0 \text{ for some natural number } n>0\}$$
$$=\text{the torsion subgroup of } G,$$

$$D=\bigcup\{H\mid H \text{ is a divisible subgroup of } G\}$$
$$=\text{the divisible part of } G.$$

While these definitions are clearly increasing in logical complexity, there is no distinction to be made between them from ZF's point of view. We will return to this example in Chapter IV.

1.3. As an example of the way one is tempted to violate the principle of parsimony when working in ZF, one need only look in the average text on set theory. There you will find the power set axiom (a very strong axiom from our point of view) used to verify a simple fact like the existence of $a \times b$.

1.4. The point made in (4) above is illustrated by considering the real line. While we know how to construct something isomorphic to the real line in ZF (either by Cauchy sequences or by Dedekind cuts), in practise the mathematician is not interested in the details of this construction. For example, he would never think of worrying about what the elements of $\sqrt{2}$ happen to be.

1.5 Notes. The notes at the end of sections are used to collect historical remarks, credit for theorems (when possible) and various remarks which might otherwise have gone into footnotes.

In the early days of set theory, certainly in the work of Zermelo, urelements were an integral part of the subject. The rehabilitation of urelements in the context of admissible set theory is such a simple idea that it would be silly to assign credit for it to any one person. Probably everyone who has thought at all about infinitary logic and admissible sets has had a similar idea.

Karp [1968] suggests the study of nontransitive admissible sets. Kreisel [1971] points out that "the principal gap in the existing model theoretic [generalized recursion theory] ... is its preoccupation with *sets* (that is sets built up from the empty set by some cumulative operation ...); not even sets of individuals are treated." Barwise [1974] contains the first published treatment of admissible sets with urelements. This book grew out of that paper, to some extent. It is worth remembering that the defense of urelements given in § 1 would have been unnecessary not too long ago. Perhaps it will be equally pointless sometime in the future.

2. *The Axioms of* KPU

Let L be a first order language with equality, some relation, function and constant symbols and let $\mathfrak{M} = \langle M, \text{---} \rangle$ be a structure for this language L. We wish to form admissible sets which have M as a collection of urelements; these admissible sets are the intended models of a theory KPU which we begin to develop in this section.

The theory KPU is formulated in a language $\mathsf{L}^* = \mathsf{L}(\in, \ldots)$ which extends L by adding a membership symbol $\in$ and, possibly other function, relation and constant symbols. Rather than describe L^* precisely, we describe its class of structures, leaving it to the reader to formalize L^* in a way that suits his tastes.

2.1 Definition. A structure $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ for L^* consists of

(i) a structure $\mathfrak{M} = \langle M, \text{---} \rangle$ for the language L, where $M = \emptyset$ is kept open as a possibility (the members of M are the *urelements* of $\mathfrak{A}_{\mathfrak{M}}$);

(ii) a nonempty set A disjoint from M (the members of A are the *sets* of $\mathfrak{A}_{\mathfrak{M}}$);

(iii) a relation $E \subseteq (M \cup A) \times A$ (which interprets the *membership* symbol $\in$);

(iv) other functions, relations and constants on $M \cup A$ to interpret any other symbols in $\mathsf{L}(\in, \ldots)$ (that is the symbols in the list indicated by the three dots).

The equality symbol of L^* is always interpreted as the usual equality relation.

We use variables of L^* subject to the following conventions: Given a structure $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ for L^*,

$p, q, p_1, \ldots$	range over M (urelements),
$a, b, c, d, f, r, a_1, \ldots$	range over A (sets),
$x, y, z, \ldots$	range over $M \cup A$.

This notation gives us an easy way to assert that something holds of sets, or of urelements. For example, $\forall p\, \exists a\, \forall x\, (x \in a \leftrightarrow x = p)$ asserts that $\{p\}$ exists for any urelement p, whereas $\forall p\, \exists a\, \forall q\, (q \in a \leftrightarrow q = p)$ asserts that there is a set a whose intersection with the class of all urelements is $\{p\}$.

We sometimes use (e.g. in 2.2(iii)) u, v, w to denote any kind of variable.

The axioms of KPU are of three kinds. The axioms of extensionality and foundation concern the basic nature of sets. The axioms of pair, union and Δ_0 separation deal with the principles of set construction available to us. The most important axiom, Δ_0 collection, guarantees that there are enough stages in our construction process. In order to state the latter two axioms we need to define the notion of Δ_0 formula of $\mathsf{L}(\in, \ldots)$, of Lévy [1965].

2.2 Definition. The collection of Δ_0 *formulas* of a language $\mathsf{L}(\in, \ldots)$ is the smallest collection Y containing the atomic formulas of $\mathsf{L}(\in, \ldots)$ closed under:

(i) if φ is in Y, then so is $\neg\varphi$;

(ii) if φ, ψ are in Y, so are $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$;

(iii) if φ is in Y, then so are $\forall u \in v\, \varphi$ and $\exists u \in v\, \varphi$ for all variables u and v.

The importance of Δ_0 formulas rests in the metamathematical fact that any predicate defined by a Δ_0 formula is absolute (see 7.3), and the empirical fact (which we will verify) that many predicates occuring in nature can be defined by Δ_0 formulas (see Table 1).

2.3 Definition. The theory KPU (relative to a language $\mathsf{L}(\in, \ldots)$) consists of the universal closures of the following formulas:

Extensionality: $\forall x\, (x \in a \leftrightarrow x \in b) \rightarrow a = b$;

Foundation: $\exists x\, \varphi(x) \rightarrow \exists x [\varphi(x) \wedge \forall y \in x\, \neg\varphi(y)]$ for all formulas $\varphi(x)$ in which y does not occur free;

Pair: $\exists a\,(x\in a \wedge y\in a)$;

Union: $\exists b\,\forall y\in a\,\forall x\in y\,(x\in b)$;

Δ_0 *Separation:* $\exists b\,\forall x\,(x\in b\leftrightarrow x\in a\wedge\varphi(x))$ for all Δ_0 formulas in which b does not occur free;

Δ_0 *Collection:* $\forall x\in a\,\exists y\,\varphi(x,y)\rightarrow\exists b\,\forall x\in a\,\exists y\in b\,\varphi(x,y)$ for all Δ_0 formulas in which b does not occur free.

Note that the formulas $\varphi(x)$, $\varphi(x,y)$ used above may have other free variables.

2.4 Definition. KPU^+ is KPU plus the axiom:

$$\exists a\,\forall x\,[x\in a\leftrightarrow\exists p\,(x=p)],$$

which asserts that there is a set of all urelements.

2.5 Definition. KP is KPU plus the axiom:

$$\forall x\,\exists a\,(x=a),$$

which asserts that every object is a set, i.e. that there are no urelements.

2.6 A word of caution. There are some axioms built into our definition of structure for $\mathsf{L}(\in,\ldots)$. The following sentences make these conditions explicit and should be considered part of the axioms of KPU:

$\forall p\,\forall a\,(p\neq a)$ (cf. 2.1 (ii));

$\exists a\,(a=a)$ (expresses $A\neq\emptyset$ in 2.1 (ii));

$\forall p\,\forall x\,(x\notin p)$ (cf. 2.1 (iii)).

2.7 Notes. The notions of Δ_0 and Σ_1 are due to Lévy [1965]. The axioms of KP go back to Platek (the P in KP), in particular, to Platek [1966]. He defined an admissible set A to be a transitive, nonempty set closed under TC satisfying Δ_0 separation and Σ reflection. Kripke [1964] (the K in KP) had, independently, a similar notion with Σ reflection replaced by Σ replacement. (For the models Kripke had in mind (L_α's) they are equivalent; but in general it is Σ reflection which matters.) Both of these men were influenced by Kreisel [1959] and Kreisel [1965]. See, e.g. Kreisel [1965, p. 199(b)]. (For the notion of a Σ formula, see 4.1 below.)

3. Elementary Parts of Set Theory in KPU

In this section we show how to define some of the elementary concepts of intuitive set theory in KPU. We thus want to show that certain sentences of $\mathsf{L}(\in,\ldots)$ are logical consequences of KPU. We do this here by translating these sentences into English and then giving their proofs in English, being careful to use only axioms from KPU. For example, rather than state:

$$\mathrm{KPU} \vdash \forall x \forall y \exists! a \forall z\, [z \in a \leftrightarrow z = x \vee z = y],$$

we state:

Given x, y, there is a unique set $a = \{x, y\}$ with only x, y as members;

and then we give an informal proof of the latter. (Given x, y, there is a b with $x, y \in b$, by Pair. By Δ_0 separation there is an a with $z \in a \leftrightarrow z \in b \wedge [z = x \text{ or } z = y]$, the part in brackets being a Δ_0 formula. By Extensionality, there can be at most one such a.) Thus all results in this section are proved in KPU.

3.1 Proposition. (i) *There is a unique set* 0 *with no elements.*
(ii) *Given a, there is a unique set $b = \bigcup a$ such that $x \in b$ iff $\exists y \in a\, (x \in y)$.*
(iii) *Given a, b there is a unique set $c = a \cup b$ such that $x \in c$ iff $x \in a$ or $x \in b$.*
(iv) *Given a, b there is a unique set $c = a \cap b$ such that $x \in c$ iff $x \in a$ and $x \in b$.*

Proof. These are all routine. By 2.1(ii) there is a set b. For (i) we apply Δ_0 separation to b and the formula $x \neq x$. For (ii) use the union axiom to get a b' such that $\forall y \in a\, \forall x \in y\, (x \in b')$, and then form

$$b = \{x \in b' \mid \exists y \in a\, (x \in y)\}$$

by Δ_0 separation. For (iii), form $\bigcup\{a, b\}$. To prove (iv), let $c = \{x \in a \mid x \in b\}$, which exists by Δ_0 separation. In each case uniqueness follows from the axiom of extensionality. ▯

We define, as ususal, the ordered pair of x, y by

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$$

and prove that $\langle x, y\rangle = \langle z, w\rangle$ iff $x = z$ and $y = w$.

3.2 Proposition. *For all a, b there is a set $c = a \times b$, the Cartesian product of a and b, such that*

$$c = \{\langle x, y\rangle \mid x \in a \text{ and } y \in b\}.$$

Proof. By Table 1 the predicate of a, b, u:

u is an ordered pair $\langle x, y\rangle$ with $x \in a$ and $y \in b$

is Δ_0 so we can use Δ_0 separation once we know that there is a set c with $\langle x, y\rangle \in c$ for all $x \in a$, $y \in b$. This follows from Δ_0 collection as follows. Given any $x \in a$ we first show that there is a w_x such that $\langle x, y\rangle \in w_x$ for all $y \in b$. Why? Well, given $y \in b$ there is a set $d = \langle x, y\rangle$. So, by Δ_0 collection there is a set w_x such that $\langle x,y\rangle \in w_x$ for all $y \in b$. Now, apply Δ_0 collection again. We have

$$\forall x \in a\, \exists w\, \underbrace{\forall y \in b\, \exists d \in w\, (d = \langle x,y\rangle)}_{\Delta_0}$$

so there is a c_1 such that for all $x \in a$, $y \in b$, $\langle x,y\rangle \in w$ for some $w \in c_1$. Thus, if $c = \bigcup c_1$, then $\langle x,y\rangle \in c$ for all $x \in a$, $y \in b$. □

The above is a good example of the principle of parsimony. In ZF, where one has the power set axiom, the set c needed in the proof can be taken to be just $P(P(a \cup b))$, but this proof does not carry over to KPU.

We can define ordered n-tuples, for $n > 2$, as follows, by induction on n:

$$\langle x_1, \ldots, x_n\rangle = \langle x_1, \langle x_2, \ldots, x_n\rangle\rangle$$

and, similarly,

$$a_1 \times \cdots \times a_n = a_1 \times (a_2 \times \cdots \times a_n)\,.$$

Thus $a_1 \times \cdots \times a_n$ is the set of n-tuples $\langle x_1, \ldots, x_n\rangle$ with $x_i \in a_i$ for $i = 1, \ldots, n$.

Now that we have ordered pairs, we can give the usual definitions of intuitive notions like relation, function, etc., all by Δ_0 formulas as in Table 1.

A set a is *transitive*, written $\mathrm{Tran}(a)$, iff

$$\forall y \in x\, \forall z \in y\, (z \in a)\,,$$

so that $\mathrm{Tran}(a)$ is a Δ_0 formula. Urelements are not considered transitive. Every set of urelements is transitive. The empty set 0 is transitive.

3.3 Definition. Let $\mathscr{S}(a) = a \cup \{a\}$.

3.4 Exercise. Prove (by induction) that for each n,

$$\mathrm{KPU} \vdash \forall x_1, \ldots, \forall x_n\, \exists a\, (a = \{x_1, \ldots, x_n\})\,.$$

3.5 Exercise. Show that if a is a set of transitive sets, then $\bigcup a$ is transitive. Show that if a is transitive and $b \subseteq a$, then $a \cup \{b\}$ is transitive. In particular, if a is transitive, so is $\mathscr{S}(a)$.

3.6 Definition. An *ordinal* is a transitive set a such that every member x of a is also a transitive set. Thus, we may write this definition as:

$$\mathrm{Ord}(a) \leftrightarrow \mathrm{Tran}(a) \wedge \forall x \in a\, \mathrm{Tran}(x)\,.$$

We use $\alpha, \beta, \gamma, \ldots$ to range over ordinals. We write $\alpha < \beta$ for $\alpha \in \beta$. An ordinal α is a *natural number* if for all $\beta \leqslant \alpha$, if $\beta \neq 0$ then $\beta = \mathscr{S}(\gamma)$ for some γ. We use variables $n, m, \ldots$ over natural numbers.

3.7 Exercise. We assume that the reader has some familiarity with ordinal numbers. He should verify that all the usual things are provable in KPU:

(i) 0 is an ordinal;

(ii) If α is an ordinal so is $\mathscr{S}(\alpha)$, usually written $\alpha+1$.

(iii) If $\alpha \neq \beta$ then $\alpha<\beta$ or $\beta<\alpha$. (This uses the axiom of foundation!)

(iv) For all α, $\alpha \not< \alpha$.

(v) If a is a set of ordinals, then $\bigcup a$ is an ordinal β with $\alpha \leqslant \beta$ whenever $\alpha \in a$, and $\exists \alpha \in a\,(\gamma \leqslant \alpha)$ whenever $\gamma<\beta$. (Thus β is the supremum of a, and we write $\beta=\sup(a)$.)

(vi) If $\alpha<\beta$ then $\alpha+1 \leqslant \beta$.

(vii) Every nonempty set of ordinals has a smallest element.

3.8 Definition. A set a is *finite* if there is a one-one function f with $\mathrm{dom}(f)=a$ and range some natural number n. A set a is *countable* if there is a one-one function f with domain a such that $f(x)$ is a natural number for every $x \in a$.

3.9 Exercise. (i) Show that every member of an ordinal is an ordinal.

(ii) Show that a set is an ordinal iff it is transitive and its elements are linearly ordered by $\in$.

(iii) Show that an ordinal is finite iff it is a natural number.

Table 1. Some Δ_0 Predicates

Predicate	*Abbreviation*	Δ_0 *Definition*
$x \subseteq y$		$\forall z \in x\ (z \in y)$
$a=\{y,z\}$		$y \in a \wedge z \in a \wedge \forall x \in a\ (x=y \vee x=z)$
$a=\langle y,z\rangle$		$\exists b \in a\, \exists c \in a\ (b=\{y\} \wedge c=\{y,z\} \wedge a=\{b,c\})$
$a=\langle x,y\rangle$ *for some* y	$1^{\mathrm{st}}(a)=x$	$\exists c \in a\, \exists y \in c\ (a=\langle x,y\rangle)$
$a=\langle x,y\rangle$ *for some* x	$2^{\mathrm{nd}}(a)=y$	$\exists c \in a\, \exists x \in c\ (a=\langle x,y\rangle)$
$a=\langle x,y\rangle$ *for some* x,y	"*a is an ordered pair*"	$\exists c \in a\, \exists x \in c\, \exists y \in c\ (a=\langle x,y\rangle)$
a is a relation	$\mathrm{Reln}(a)$	$\forall x \in a$ "*x is an ordered pair*"
f is a function	$\mathrm{Fun}(f)$	$\mathrm{Reln}(f) \wedge \forall a \in f\, \forall b \in f\ (1^{\mathrm{st}}a=1^{\mathrm{st}}b \rightarrow 2^{\mathrm{nd}}a=2^{\mathrm{nd}}b)$
r is a relation with domain a	$\mathrm{dom}(r)=a$	$\mathrm{Reln}(r) \wedge \forall b \in r\,(1^{\mathrm{st}}b \in a) \wedge \forall x \in a\, \exists b \in r\,(1^{\mathrm{st}}b=x)$
r is a relation with range a	$\mathrm{rng}(r)=a$	$\mathrm{Reln}(r) \wedge \forall b \in r\,(2^{\mathrm{nd}}b \in a) \wedge \forall x \in a\, \exists b \in r\,(2^{\mathrm{nd}}b=x)$
r is a relation with field a	$\mathrm{field}(r)=a$	$a=\mathrm{dom}(r) \cup \mathrm{rng}(r)$
$y=f(x)$		$\mathrm{Fun}(f) \wedge \langle x,y\rangle \in f$
$a=\bigcup b$		$\forall x \in b\, \forall y \in x\ (y \in a) \wedge \forall y \in a\, \exists x \in b\ (y \in x)$

4. *Some Derivable Forms of Separation and Replacement*

Our development of set theory progressed smoothly as long as the predicates involved were definable by Δ_0 formulas. With the notions of finite and countable in 3.8 we hit the first examples of predicates which cannot be so expressed.

For example, if we write either of these out they take the form

$$\exists f\, \varphi(f,a)$$

where φ is Δ_0. A formula of the form $\exists u\, \varphi(u)$, where φ is Δ_0, is called a Σ_1 *formula.* It turns out that a wide class of formulas are equivalent to Σ_1 formulas and that we can use these formulas in various forms of separation, collection and replacement.

4.1 Definition. The class of Σ *formulas* is the smallest class Y containing the Δ_0 formulas and closed under conjunction and disjunction (2.2(ii)), bounded quantification (2.2(iii)) and satisfying:

(i) if φ is in Y so is $\exists u\, \varphi$ for all variables u.

The class of Π *formulas,* on the other hand, is the smallest class Y' containing the Δ_0 formulas closed under conjunction, disjunction, bounded quantification and satisfying:

(ii) if φ is in Y' so is $\forall u\, \varphi$, for all variables u.

For example, the two formulas:

$$\forall b \in a\, [b \text{ is countable}] \quad \text{and} \quad \forall x \in a\, \exists b\, [\mathrm{Tran}(b) \wedge x \in b],$$

are Σ but not Σ_1. Clearly the negation of any Σ formula is logically equivalent to a Π formula and vice versa. As a corollary to Theorem 4.3 we will see that for every Σ formula φ, there is a Σ_1 formula φ' such that

$$\mathrm{KPU} \vdash \varphi \leftrightarrow \varphi'.$$

Given a formula φ and a variable w not appearing in φ, we write $\varphi^{(w)}$ for the result of replacing each *unbounded* quantifier in φ by a *bounded* quantifier; that is we replace:

$$\exists u \quad \text{by} \quad \exists u \in w, \quad \text{and}$$

$$\forall u \quad \text{by} \quad \forall u \in w,$$

for all variables u. Thus $\varphi^{(w)}$ is a Δ_0 formula. If φ is Δ_0 then $\varphi^{(w)} = \varphi$, since there are no unbounded quantifiers in φ. We *always* assume that w *does not* already appear in φ.

4.2 Lemma. *For each Σ formula φ the following are logically valid (i.e., true in all structures $\mathfrak{A}_{\mathfrak{M}}$):*

(i) $\varphi^{(u)} \wedge u \subseteq v \rightarrow \varphi^{(v)}$,

(ii) $\varphi^{(u)} \rightarrow \varphi$, *where* $u \subseteq v$ *abbreviates the formula* $\forall x\,[x \in u \to x \in v]$. (Actually it is the universal closures of these formulas which are true in all $\mathfrak{A}_{\mathfrak{M}}$ since φ may have other free variables. We will not bother with this comment in the future.)

Proof. Both facts are proved by induction following the inductive definition 4.1 of Σ formula. Let us just prove the first, the second being similar. Fix a structure $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ and $x, y \in A \cup M$ so that $x \subseteq y$ is true in $\mathfrak{A}_{\mathfrak{M}}$. For Δ_0 formulas φ, we have, obviously, $\varphi = \varphi^{(x)} = \varphi^{(y)}$. Assume first that $(\varphi \wedge \psi)^{(x)}$ (i.e., assume it's true in $\mathfrak{A}_{\mathfrak{M}}$). Hence, $\varphi^{(x)}$ and $\psi^{(x)}$. By induction $\varphi^{(y)}$ and $\psi^{(y)}$, so $(\varphi \wedge \psi)^{(y)}$. Similarly for $(\varphi \vee \psi)^{(x)} \to (\varphi \vee \psi)^{(y)}$ and bounded quantifiers.

Now assume $(\exists w\,\varphi(w))^{(x)}$, so there is a $w \in x$ such that $\varphi(w)^{(x)}$. By induction $\varphi(w)^{(y)}$; and, since $x \subseteq y$, $\exists w \in y(\varphi(w)^{(y)})$; i.e., $(\exists w\,\varphi(w))^{(y)}$. ▯

4.3 Theorem. (The Σ Reflection Principle). *For all* Σ *formulas* φ *we have the following:*

$$\text{KPU} \vdash \varphi \leftrightarrow \exists a\,\varphi^{(a)}.$$

(Here a is any set variable not occurring in φ; we will not continue to make these annoying conditions on variables explicit.) *In particular, every* Σ *formula is equivalent to a* Σ_1 *formula in* KPU.

Proof. We know from the previous lemma that $\exists a\,\varphi^{(a)} \to \varphi$ is valid, so the axioms of KPU come in only in showing $\varphi \to \exists a\,\varphi^{(a)}$. The proof is by induction on φ, the case for Δ_0 formulas being trivial. We take the three most interesting cases, leaving the other two to the reader.

Case 1. φ is $\psi \wedge \theta$. Assume that

$$\text{KPU} \vdash \psi \leftrightarrow \exists a\,\psi^{(a)}, \quad \text{and}$$
$$\text{KPU} \vdash \theta \leftrightarrow \exists a\,\theta^{(a)},$$

as induction hypothesis, and prove that

$$\text{KPU} \vdash (\psi \wedge \theta) \to \exists a\,[\psi \wedge \theta]^{(a)}.$$

Let us work in KPU, assuming $\psi \wedge \theta$ and proving $\exists a\,[\psi^{(a)} \wedge \theta^{(a)}]$. Now there are a_1, a_2 such that $\psi^{(a_1)}, \theta^{(a_2)}$, so let $a = a_1 \cup a_2$. Then $\varphi^{(a)}$ and $\psi^{(a)}$ hold by the previous lemma.

Case 2. φ is $\forall u \in v\,\psi(u)$. Assume that

$$\text{KPU} \vdash \psi \leftrightarrow \exists a\,\psi^{(a)}.$$

Again, working in KPU, assume $\forall u \in v\,\psi(u)$ and prove $\exists a\,\forall u \in v\,\psi(u)^{(a)}$. For each $u \in v$ there is a b such that $\psi(u)^{(b)}$, so by Δ_0 collection there is an a_0 such that

$\forall u \in v\, \exists b \in a_0\, \psi(u)^{(b)}$. Let $a = \bigcup a_0$. Now, for every $u \in v$, we have $\exists b \subseteq a\, \psi(u)^{(b)}$; so $\forall u \in v\, \psi(u)^{(a)}$, by the previous lemma.

Case 3. φ is $\exists u\, \psi(u)$. Assume $\psi(u) \leftrightarrow \exists b\, \psi(u)^{(b)}$ proved and suppose $\exists u\, \psi(u)$ true. We need an a such that $\exists u \in a\, \psi(u)^{(a)}$. If $\psi(u)$ holds, pick b so that $\psi(u)^{(b)}$ and let $a = b \cup \{u\}$. Then $u \in a$ and $\psi(u)^{(a)}$, by the previous lemma. □

In Platek's original definition of admissible set he took the Σ reflection principle as basic. It is very powerful, as we'll see below. The Δ_0 collection axiom is easier to verify in particular structures, however, and is also more like the replacement axioms with which one is familiar from ZF.

4.4 Theorem. (The Σ Collection Principle). *For every Σ formula φ the following is a theorem of* KPU: *If $\forall x \in a\, \exists y\, \varphi(x,y)$ then there is a set b such that $\forall x \in a\, \exists y \in b\, \varphi(x,y)$ and $\forall y \in b\, \exists x \in a\, \varphi(x,y)$.*

Proof. Assume that

$$\forall x \in a\, \exists y\, \varphi(x,y).$$

By Σ reflection there is a set c such that

$$(1) \qquad \forall x \in a\, \exists y \in c\, \varphi^{(c)}(x,y).$$

Let

$$(2) \qquad b = \{y \in c \mid \exists x \in a\, \varphi^{(c)}(x,y)\},$$

by Δ_0 separation. Now since $\varphi^{(c)}(x,y) \to \varphi(x,y)$ by 4.2, (1) gives us:

$$\forall x \in a\, \exists y \in b\, \varphi(x,y);$$

whereas (2) gives us:

$$\forall y \in b\, \exists x \in a\, \varphi(x,y). \quad □$$

4.5 Theorem. (Δ Separation). *For any Σ formula $\varphi(x)$ and Π formula $\psi(x)$, the following is a theorem of* KPU: *If for all $x \in a$, $\varphi(x) \leftrightarrow \psi(x)$, then there is a set $b = \{x \in a \mid \varphi(x)\}$.*

Proof. Assume $\forall x \in a\, (\varphi(x) \leftrightarrow \psi(x))$. Then $\forall x \in a\, [\varphi(x) \vee \neg\psi(x)]$, which is equivalent to a Σ formula, so there is a c such that $\forall x \in a\, [\varphi^{(c)}(x) \vee \neg\psi^{(c)}(x)]$. Let, by Δ_0 separation, $b = \{x \in a \mid \varphi^{(c)}(x)\}$. Clearly every $x \in b$ satisfies $\varphi(x)$. If $x \in a$ and $\varphi(x)$ then $\psi(x)$, so $\psi^{(c)}(x)$ (since $\psi(x) \to \psi^{(c)}(x)$); so $\varphi^{(c)}(x)$. Thus $x \in b$. □

4.6 Theorem. (Σ Replacement). *For each Σ formula $\varphi(x,y)$ the following is a theorem of* KPU: *If $\forall x \in a\, \exists! y\, \varphi(x,y)$ then there is a function f, with* $\mathrm{dom}(f) = a$, *such that $\forall x \in a\, \varphi(x, f(x))$.*

Proof. By Σ Collection there is a set b such that $\forall x \in a\, \exists y \in b\, \varphi(x,y)$. Using Δ Separation there is an f such that

$$\begin{aligned} f &= \{\langle x,y\rangle \in a \times b \mid \varphi(x,y)\} \\ &= \{\langle x,y\rangle \in a \times b \mid \neg\exists z\, [\varphi(x,z) \wedge y \neq z]\}. \quad \square \end{aligned}$$

The above is sometimes unsuable because of the uniqueness requirement $\exists!$ in the hypothesis. In these situations it is usually 4.7 which comes to the rescue.

4.7 Theorem. (Strong Σ Replacement). *For each Σ formula $\varphi(x,y)$ the following is a theorem of* KPU: *If $\forall x \in a\, \exists y\, \varphi(x,y)$ then there is a function f with* $\mathrm{dom}(f) = a$ *such that*

(i) $\forall x \in a\, f(x) \neq 0$;

(ii) $\forall x \in a\, \forall y \in f(x)\, \varphi(x,y)$.

Proof. By Σ Collection there is a b such that $\forall x \in a\, \exists y \in b\, \varphi(x,y)$ and $\forall y \in b\, \exists x \in a\, \varphi(x,y)$. Hence there is a w, by 4.3, such that

$$\forall x \in a\, \exists y \in b\, \varphi^{(w)}(x,y), \quad \text{and} \quad \forall y \in b\, \exists x \in a\, \varphi^{(w)}(x,y).$$

For any fixed $x \in a$ there is a unique set c_x such that

$$c_x = \{y \in b \mid \varphi^{(w)}(x,y)\}$$

by Δ_0 Separation and Extensionality; so, by Σ Replacement, there is a function f with domain a such that $f(x) = c_x$ for each $x \in a$. $\square$

4.8—4.9 Exercises. There are a number of minor variations on the above.

4.8. For example, prove that, for each Σ formula φ,

$$\mathrm{KPU} \vdash \varphi \rightarrow \exists a\, (x_1 \in a \wedge \cdots \wedge x_n \in a \wedge \varphi^{(a)}).$$

4.9. Given a Σ formula φ let φ^{*a} denote the result of replacing some, but not necessarily all, existential quantifiers $\exists u$ by $\exists u \in a$ for some new set variable a. Show that: $\mathrm{KPU} \vdash \varphi \leftrightarrow \exists a\, \varphi^{*a}$.

5. *Adding Defined Symbols to* KPU

The introduction of defined relation and function symbols is a common practice, but it must be used with just a little care in KPU. In a theory like ZF one is able to take any formula $\varphi(x_1, \ldots, x_n)$, define a new relation symbol by

(R) $\qquad \forall x_1 \ldots \forall x_n\, [\mathsf{R}(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)]$, and then use R as an atomic formula in other formulas—even in the axiom of replacement. After all, one could always go back and replace R by φ. For KPU, however, where we must pay attention to the syntactic form our axioms take, a definition like (R) would work, at first glance, only if the φ in (R) were Δ_0. We have tacitly used this form of introducing new relation symbols repeatedly in § 3. Using the principles of § 4 we may allow ourselves a bit more freedom.

5.1 Definition. Let $\varphi(x_1, \ldots, x_n)$ be a Σ formula of L^* and $\psi(x_1, \ldots, x_n)$ be a Π formula of L^* such that

$$\mathrm{KPU} \vdash \varphi \leftrightarrow \psi .$$

Let R be a new n-ary relation symbol and define R by (R) above. R is then called a Δ *relation symbol* of KPU.

To be really precise it would be the triple $\mathsf{R}, \varphi, \psi$ such that the above hold which constitute a Δ *definition of the relation symbol* R, but we do not need to be this careful. The next lemma shows that we can treat Δ relation symbols as though they were atomic formulas of L^*. Here, and elsewhere, we abbreviate $x_1, \ldots, x_k$ by $\vec{x}$.

5.2 Lemma. *Let* KPU *be formulated in* L^* *and let* R *be a* Δ *relation symbol of* KPU. *Let* KPU′ *be* KPU *as formulated in* $\mathsf{L}^*(\mathsf{R})$, *plus the defining axiom* (R) *above.*

(i) *For every formula* $\theta(x_1, \ldots, x_k, \mathsf{R})$ *of* $\mathsf{L}^*(\mathsf{R})$, *there is a formula* $\theta_0(x_1, \ldots, x_k)$ *of* L^* *such that* KPU + (R) *implies*

$$\theta(\vec{x}, \mathsf{R}) \leftrightarrow \theta_0(\vec{x}) .$$

Moreover, if θ *is a* Σ *formula of* $\mathsf{L}^*(\mathsf{R})$ *then* θ_0 *is a* Σ *formula of* L^*.

(ii) *For every* Δ_0 *formula* $\theta(x_1, \ldots, x_k, \mathsf{R})$ *of* $\mathsf{L}^*(\mathsf{R})$ *there are* Σ *and* Π *formulas* $\theta_0(x_1, \ldots, x_k)$, $\theta_1(x_1, \ldots, x_k)$ *of* L^* *such that* KPU + (R) *implies*

$$\theta(\vec{x}, \mathsf{R}) \leftrightarrow \theta_0(\vec{x}), \quad \textit{and} \quad \theta(\vec{x}, \mathsf{R}) \leftrightarrow \theta_1(\vec{x}) .$$

(iii) KPU′ *is a conservative extension of* KPU. *That is, for any sentence* θ *of* L^*,

$$\mathrm{KPU}' \vdash \theta \quad \textit{iff} \quad \mathrm{KPU} \vdash \theta .$$

Proof. Let us suppose that R is defined by

$$\mathsf{R}(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n),$$

where φ is a Σ formula, and that

$$\mathrm{KPU} \vdash \varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n),$$

where ψ is a Π formula. The first sentence in (i) is obvious since we may replace R by its definition. It is to make the second sentence of (i) true that we need R to be a Δ relation symbol of L^*. Using de Morgan's laws, push all negations in θ inside as far as possible so that they only apply to atomic formulas. Now replace each positive (i.e., unnegated) occurrence of R in θ by φ, each occurrence $\neg\mathsf{R}$ by the Σ formula equivalent to $\neg\psi$. The result is called θ_0. Since

$$\mathrm{KPU}'\vdash \mathsf{R}\leftrightarrow\varphi \quad \text{and} \quad \mathrm{KPU}'\vdash\neg\mathsf{R}\leftrightarrow\neg\psi,$$

it is clear that

$$\mathrm{KPU}'\vdash\theta(x_1,\ldots,x_k,\mathsf{R})\leftrightarrow\theta_0(x_1,\ldots,x_k).$$

It is also clear that this transformation takes Σ formulas into Σ formulas. Note, however, that the transformation does not take Δ_0 formulas into Δ_0 formulas, but only into Σ formulas. However, since the Δ_0 formulas are closed under negation, (ii) immediately follows from (i). To prove (iii) it suffices to show that every axiom of KPU′ is turned into a theorem of KPU when R is replaced as above. For example, Δ_0 Separation of KPU′ becomes Δ Separation in KPU and Δ_0 Collection for KPU′ becomes a consequence of Σ Collection for KPU. □

Using this lemma we can clear up a point which may have been bothering the reader. One way of formalizing $\mathsf{L}^*=\mathsf{L}(\in,\ldots)$ is to make it a single sorted language with predicate symbols U for urelements and S for sets. In this way $\forall p(\ldots p\ldots)$ would stand for $\forall x(\mathsf{U}(x)\rightarrow(\ldots x\ldots))$, and $\forall a(\ldots)$ would stand for $\forall x(\mathsf{S}(x)\rightarrow(\ldots x\ldots))$, and "$x$ is an urelement" would be a Δ_0 formula, $\mathsf{U}(x)$. The other way of formalizing L^* is to have a many-sorted language with the three sorts of variables

$$p,q,\ldots\,,$$
$$a,b,\ldots\,, \quad \text{and}$$
$$x,y,\ldots\,.$$

The predicate "x is an urelement" is no longer Δ_0, but it is Δ. Our definition of L^* insures that

$$x \text{ is an urelement} \leftrightarrow \exists p\,(x=p), \quad \text{and}$$

$$x \text{ is not an urelement} \leftrightarrow \exists a\,(x=a)\,;$$

so the predicate and its negation are Σ_1. The lemma assures us that we can introduce a new symbol by:

$$\mathsf{U}(x)\leftrightarrow x \text{ is an urelement},$$

and use it in Δ_0 formulas without fear. Similarly we can introduce

$$\begin{aligned} \mathsf{S}(x) &\leftrightarrow \exists a\,(x=a) \\ &\leftrightarrow \neg\exists p\,(x=p) \end{aligned}$$

for "x is a set" and treat it as a Δ_0 formula.

A predicate of intuitive set theory is said to be a Δ *predicate of* KPU if it can be defined by a Δ relation symbol. Using the above lemma we see that we may treat Δ predicates as though they were defined by atomic formulas of L^*. Furthermore, the Δ predicates are closed under $\wedge$, $\vee$, $\forall u\in v$, $\exists u\in v$. Using these observations, we see that all the predicates listed in Table 2 are indeed Δ predicates.

The introduction of defined relation symbols is a convenience, but the introduction of defined function symbols is a practical necessity (though theoretically a luxury). The conditions necessary for us to be able to do this are given in the following definition.

5.3 Definition. Let $\varphi(x_1,\ldots,x_n,y)$ be a Σ formula of L^* such that

$$\mathrm{KPU}\vdash \forall x_1,\ldots,x_n\,\exists!y\,\varphi(x_1,\ldots,x_n,y)\,.$$

Let F be a new n-ary function symbol and define F by:

$$(\mathsf{F})\qquad \forall x_1,\ldots,x_n,y[\mathsf{F}(x_1,\ldots,x_n)=y\leftrightarrow\varphi(x_1,\ldots,x_n,y)]\,.$$

F is then called a Σ *function symbol* of KPU.

The next lemma lets us treat Σ function symbols as though they were atomic symbols of the basic language L^*.

5.4 Lemma. *Let* KPU *be formulated in* L^* *and let* F *be a* Σ *function symbol of* KPU. *Let* KPU′ *be* KPU *as formulated in* $\mathsf{L}^*(\mathsf{F})$, *plus the defining axiom* (F) *above.*

(i) *For every formula* $\theta(x_1,\ldots,x_k,\mathsf{F})$ *of* $\mathsf{L}^*(\mathsf{F})$ *there is a formula* $\theta_0(x_1,\ldots,x_k)$ *of* L^* *such that* KPU$+$(F) *implies*

$$\theta(\vec{x},\mathsf{F})\leftrightarrow\theta_0(\vec{x})\,.$$

Moreover, if θ *is a* Σ *formula of* $\mathsf{L}^*(\mathsf{F})$ *then* θ_0 *is a* Σ *formula of* L^*.

(ii) *For every* Δ_0 *formula* $\theta(x_1,\ldots,x_k,\mathsf{F})$ *of* $\mathsf{L}^*(\mathsf{F})$ *there are* Σ *and* Π *formulas* $\theta_0(x_1,\ldots,x_k)$, $\theta_1(x_1,\ldots,x_k)$ *of* L^* *such that* KPU$+$(F) *implies*

$$\theta(\vec{x},\mathsf{F})\leftrightarrow\theta_0(\vec{x})\,,\ \textit{and}$$
$$\theta(\vec{x},\mathsf{F})\leftrightarrow\theta_1(\vec{x})\,.$$

(iii) KPU′ *is a conservative extension of* KPU.

Table 2. Some Δ predicates

Predicate	*Abbreviation*	*Definition*
x is an urelement	$\mathsf{U}(x)$	$\exists p\,(x=p)$ (*or* $\forall a\,(x\neq a)$)
x is a set	$\mathsf{S}(x)$	$\exists a\,(x=a)$ (*or* $\forall p\,(x\neq p)$)
x is transitive	$\mathrm{Tran}(x)$	$\mathsf{S}(x)\wedge\forall y\in x\,\forall z\in y\,(z\in x)$
x is an ordinal	$\mathrm{Ord}(x)$	$\mathrm{Tran}(x)\wedge\forall y\in x\,\mathrm{Tran}(y)$
x is a limit ordinal	$\mathrm{Lim}(x)$	$\mathrm{Ord}(x)\wedge x\neq 0\wedge\forall y\in x\,\exists z\in x\,(z=y\cup\{y\})$
x is a natural number	$\mathrm{Nat\ No}(x)$	$\mathrm{Ord}(x)\wedge\forall y\in x\neg\mathrm{Lim}(y)\wedge\neg\mathrm{Lim}(x)$
less than for ordinals	$\alpha<\beta$	$\mathrm{Ord}(\alpha)\wedge\mathrm{Ord}(\beta)\wedge\alpha\in\beta$
less than or equal	$\alpha\leqslant\beta$	$\alpha<\beta\vee\alpha=\beta$.

Proof. Note that if $\varphi(x_1,\dots,x_n,y)$ is a Σ formula and if $\mathsf{F}(x_1,\dots,x_n)=y$ iff $\varphi(x_1,\dots,x_n,y)$, then we can get a Σ definition for $\mathsf{F}(x_1,\dots,x_n)\neq y$ by

(1) $$\mathsf{F}(x_1,\dots,x_n)\neq y \quad\text{iff}\quad \exists z\,[\varphi(x_1,\dots,x_n,z)\wedge y\neq z].$$

Thus the graph of F is a Δ predicate. The only complication, then, that can occur here but not in the previous lemma, is that F may occur in θ in complicated contexts like:

$$\mathsf{F}(\mathsf{G}(x))=\mathsf{H}(y) \quad\text{and}\quad \mathsf{R}(\mathsf{F}(x),y).$$

Call a formula *simple* if F only appears in simple contexts like:

$$\mathsf{F}(x_1,\dots,x_n)=y \quad\text{and}\quad \mathsf{F}(x_1,\dots,x_n)\neq y.$$

Repeated uses of the equivalences below allow us to transform every formula into an equivalent simple formula in such a way that Σ formulas transform into Σ formulas:

$$\mathsf{F}(\mathsf{G}(x),x_2,\dots,x_n))=y\leftrightarrow\exists z\,[\mathsf{G}(x)=z\wedge\mathsf{F}(z,x_2,\dots,x_n)=y],$$
$$\mathsf{F}(\mathsf{G}(x),x_2,\dots,x_n))\neq y\leftrightarrow\exists z\,[\mathsf{G}(x)=z\wedge\mathsf{F}(z,x_2,\dots,x_n)\neq y],$$
$$\mathsf{F}(x_1,\dots,x_n)=\mathsf{H}(y)\quad\leftrightarrow\exists z\,[\mathsf{H}(y)=z\wedge\mathsf{F}(x_1,\dots,x_n)=y],$$
$$\mathsf{F}(x_1,\dots,x_n)\neq\mathsf{H}(y)\quad\leftrightarrow\exists z\,[\mathsf{H}(y)=z\wedge\mathsf{F}(x_1,\dots,x_n)\neq z],$$
$$\varphi(\mathsf{F}(\vec{x}),\dots)\quad\leftrightarrow\exists z\,[z=\mathsf{F}(\vec{x})\wedge\varphi(z,\dots)]\quad(\varphi\text{ quantifier free}).$$

The proof now proceeds as in 5.2, replacing occurrences of $\mathsf{F}(x_1,\dots,x_n)=y$ by $\varphi(x_1,\dots,x_n,y)$, occurrences of $\mathsf{F}(x_1,\dots,x_n)\neq y$ by the Σ formula in (1). □

When we use 5.1 (or 5.3) to introduce a Δ relation symbol R (or Σ function symbol F) we often abuse notation by using KPU to denote the new theory KPU′ of 5.2 (or 5.4). The lemmas insure us that we can't get into trouble with this abuse of notation.

Table 3. Some Σ operations

Operation	*Domain*	*Abbre-viation*	*Σ Definition (the unique z such that)*
domain of f	all functions f	$\mathrm{dom}(f)$	see Table 1
range of f	all functions f	$\mathrm{rng}(f)$	see Table 1
the first coordinate of x	all ordered pairs x	$1^{\mathrm{st}}x$	see Table 1
the second coordinate of x	all ordered pairs x	$2^{\mathrm{nd}}x$	see Table 1
the restriction of f to a	all functions f and sets a	$f\restriction a$	$z=\{x\in f \mid 1^{\mathrm{st}}x\in a\}$
the image of f restricted to a	all functions f and sets a	$f''a$	$z=\{x\in\mathrm{rng}(f)\mid\exists y\in a(f(x)=y)\}$
successor	all sets x	$\mathscr{S}(x)$	$z=x\cup\{x\}$
ordinal successor	all ordinals α	$\alpha+1$	$z=\mathscr{S}(\alpha)$
supremum	sets of ordinals	$\sup(a)$	$z=\bigcup a$

An operation of intuitive set theory is a Σ *operation of* KPU if it can be defined by a Σ function symbol of KPU. The following exercises summarize some of the ways, in addition to 5.3, we have of defining Σ operations. The most important method, though, must wait for the next section.

5.5—5.7 Exercises

5.5. Every function symbol of L* is a Σ function symbol.

5.6. The Σ operations are closed under composition.

5.7. The Σ operations are closed under definition by cases. That is, if $\mathsf{G}_1,\ldots,\mathsf{G}_k$ are n-ary Σ operations and $\varphi_1(x_1,\ldots,x_n),\ldots,\varphi_k(x_1,\ldots,x_n)$ are Σ formulas such that

$$\mathrm{KPU}\vdash\forall x[\underline{\bigvee}_{i\leq k}\varphi_i(x_1,\ldots,x_n)]$$

$\underline{\bigvee}$ indicates exclusive or), then we may define a Σ operation F by:

$$\mathsf{F}(x_1,\ldots,x_n)=\begin{cases}\mathsf{G}_1(x_1,\ldots,x_n) & \text{if}\quad \varphi_1(x_1,\ldots,x_n)\,,\\ \vdots & \\ \mathsf{G}_k(x_1,\ldots,x_n) & \text{if}\quad \varphi_k(x_1,\ldots,x_n)\,.\end{cases}$$

Frequently we are interested in the value of a function symbol only for certain kinds of objects. For example, we want to define $1^{\mathrm{st}}a$ to be the first coordinate of a if a is an ordered pair, but we don't really care what $1^{\mathrm{st}}a$ means otherwise. To introduce $1^{\mathrm{st}}a$ as a function symbol then, we should, to be completely rigorous, first do something like prove: $\forall x\,\exists!y\,\varphi(x,y)$, where $\varphi(x,y)$ is:

x is an ordered pair with first coordinate y, or

x is not an ordered pair and y is the empty set,

and then define:

$$1^{\mathrm{st}}x=y \quad \text{iff} \quad \varphi(x,y)\,.$$

Similarly, we are interested in $\bigcup x$ only when x is a set. We will not bother with such details in the future, as long as it is clear that the intended domain of our new function symbol is Δ definable.

6. Definition by Σ Recursion

Definition by recursion is a powerful tool. It will allow us to introduce, in accordance with 5.3, operations such as ordinal addition, ordinal multiplication and the support function sp:

$$\mathrm{sp}(p) = \{p\},$$

$$\mathrm{sp}(a) = \bigcup_{x \in a} \mathrm{sp}(x),$$

which gives the set of urelements which go into the construction of a set a. Before showing how to justify such recursions we must first prove outright what is in effect a special case.

6.1 Theorem (Existence of Transitive Closure). *We can introduce a Σ function symbol* TC *into* KPU *so that the following becomes a theorem of* KPU: *For every* x, $\mathrm{TC}(x)$ *is a transitive set such that* $x \subseteq \mathrm{TC}(x)$; *and for any other transitive set* a, *if* $x \subseteq a$, *then* $\mathrm{TC}(x) \subseteq a$.

The axiom of foundation will be used in the proof of 6.1, in the form of *Proof by Induction over* $\in$. If one takes the contrapositive of foundation one gets the following scheme. For every formula φ the following is a theorem of KPU:

$$\forall x (\forall y \in x\, \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x\, \varphi(x).$$

Thus in proving $\forall x\, \varphi(x)$, we pick an arbitrary x and prove $\varphi(x)$ using, in the proof, $\varphi(y)$ for any $y \in x$. (Of coure if x is an urelement then there are no such $y \in x$.)

Proof of 6.1. If we had the ordinal ω at our disposal (we cannot prove it exists in KPU) we could use it to define

$$\mathrm{TC}(a) = a \cup (\bigcup a) \cup (\bigcup \bigcup a) \cup \cdots.$$

This definition should be kept in mind to understand the following proof. Define $Q(x, a)$ to be:

$$x \subseteq a \wedge \mathrm{Tran}(a) \wedge \forall b (x \subseteq b \wedge \mathrm{Tran}(b) \rightarrow a \subseteq b).$$

Thus Q is defined by a Π formula and $Q(x, a)$ iff a is the smallest transitive set containing x. It is clear that $Q(x, a) \wedge Q(x, a') \rightarrow a = a'$.

Now let $P(x,a)$ be the following Σ predicate:

x is an urelement $\wedge a=0$, or

x is a set, $x \subseteq a$, $\mathrm{Tran}(a) \wedge \forall z \in a\, \exists f\, [\mathrm{Fun}(f) \wedge \mathrm{dom}(f)$

is a natural number $n+1=\{0,\ldots,n\} \wedge z=f(0) \in f(1) \in \cdots \in f(n) \in x.]$

(This can be easily formalized without writing "$\cdots$"; so there is no hidden recursion.) A simple induction on natural numbers n shows that $P(x,a) \rightarrow Q(x,a)$. In particular, $P(x,a) \wedge P(x,a') \rightarrow a=a'$.

If we can prove that for every x there is an a such that $P(x,a)$ then we will be able to define a Σ function symbol TC by

$$\mathrm{TC}(x)=a \quad \text{iff} \quad P(x,a)$$

and $\mathrm{TC}(x)$ will have the desired property of the transitive closure of x. We still need to show that $\forall x\, \exists a\, P(x,a)$. If x is an urelement, take $a=0$. Thus, we need only prove $\forall b\, \exists a\, P(b,a)$, which we do by induction on $\in$. Given b, in proving $\exists a\, P(b,a)$ we may assume

$$\forall x \in b\, \exists c\, P(x,c)$$

and hence, by the above,

$$\forall x \in b\, \exists !c\, P(x,c).$$

By Σ replacement there is a function g with $\mathrm{dom}(g)=b$, such that $P(x,g(x))$ holds for all $x \in b$. Let

$$a = b \cup (\bigcup \mathrm{rng}(g))$$
$$= b \cup \bigcup_{x \in b} g(x).$$

It is clear that $b \subseteq a$ and it is not difficult to check that a is transitive. Let us verify the last clause of $P(b,a)$. Thus, let $z \in a$. If $z \in b$ then take $f=\{\langle 0,z\rangle\}$. Now assume $z \in \bigcup \mathrm{rng}(g)$, i.e. $z \in g(x)$ for some $x \in b$. But then there is an h such that $\mathrm{dom}(h)$ is an integer $n+1$, $h(0)=z$, $h(i) \in h(i+1)$ and $h(n) \in x$ since $P(x,g(x))$. Let $f=h \cup \{\langle n+1,x\rangle\}$. Then $f(0)=z \in f(1) \in f(2) \in \cdots \in f(n+1)=x \in b$ so $P(a,b)$. □

6.2 Exercise. Verify

(i) $\mathrm{TC}(p)=0$, and

(ii) $\mathrm{TC}(a)=a \cup \bigcup\{\mathrm{TC}(x) \mid x \in a\}$.

Once we have Theorem 6.4 we could use the equations in 6.2 to define TC; unfortunately we need 6.1 and 6.3 to state and prove 6.4. The following is a strengthening of the method of proof by induction over $\in$.

6.3 Theorem (Proof by Induction over TC). *For any formula $\varphi(x)$ the following is a theorem of* KPU: *If, for each x, $(\forall y \in \mathrm{TC}(x)\, \varphi(y))$ implies $\varphi(x)$, then $\forall x\, \varphi(x)$.*

Proof. We show, under the hypothesis, that $\forall x\, \forall y \in \mathrm{TC}(x)\, \varphi(y)$. This implies $\forall x\, \varphi(x)$, since $x \in \mathrm{TC}(\{x\})$. We may assume, by induction on $\in$, that for all $z \in x$

(1) $$\forall y \in \mathrm{TC}(z)\, \varphi(y)$$

in showing $\forall y \in \mathrm{TC}(x)\, \varphi(y)$. But by the hypothesis, (1) implies $\varphi(z)$ so we have $\varphi(y)$, for all $y \in x \cup \bigcup \{\mathrm{TC}(z) \mid z \in x\} = \mathrm{TC}(x)$. □

The following theorem is of central importance to all that follows.

6.4 Theorem (Definition by Σ Recursion). *Let* G *be an $n+2$-ary Σ function symbol, $n \geqslant 0$. It is possible to define a new Σ function symbol* F *so that the following is a theorem of* KPU (+ *the defining axiom* (F)): *for all $x_1, \ldots, x_n, y$,*

(i) $$\mathsf{F}(x_1, \ldots, x_n, y) = \mathsf{G}(x_1, \ldots, x_n, y, \{\langle z, \mathsf{F}(x_1, \ldots, x_n, z)\rangle \mid z \in \mathrm{TC}(y)\}).$$

Before turning to the rather tedious proof of 6.4, let us make some remarks on variations which follow from it. For example, we could replace 6.4(i) by:

$$\mathsf{F}(x_1, \ldots, x_n, y) = \mathsf{G}(x_1, \ldots, x_n, y, \{\langle z, \mathsf{F}(x_1, \ldots, x_n, z)\rangle \mid z \in y)\}.$$

(Let $\mathsf{G}'(\vec{x}, y, f) = \mathsf{G}(\vec{x}, y, f \upharpoonright y)$, and apply 6.4 to G'.) We could also start out with two functions G, H and define

$$\mathsf{F}(x_1, \ldots, x_n, p) = \mathsf{H}(x_1, \ldots, x_n, p),$$

$$\mathsf{F}(x_1, \ldots, x_n, a) = \mathsf{G}(x_1, \ldots, x_n, a, \{\langle z, \mathsf{F}(x_1, \ldots, x_n)\rangle : z \in \mathrm{TC}(a)\}).$$

This is the form we usually use. (Let $\mathsf{G}'(\vec{x}, y, f)$ be $\mathsf{H}(\vec{x}, y)$, if y is an urelement, otherwise $\mathsf{G}(\vec{x}, y, f)$ if y is a set. Then apply 6.4 to G'.)

Proof of 6.4. To be a little more formal, what we really want to prove about F, once we find a way of defining it, is that for all $x_1, \ldots, x_n, y$ there is an f such that

(1) $$f \text{ is a function} \wedge \mathrm{dom}(f) = \mathrm{TC}(y),$$

(2) $$\forall w \in \mathrm{dom}(f)(f(w) = \mathsf{F}(x_1, \ldots, x_n, w)), \quad \text{and}$$

(3) $$\mathsf{F}(x_1, \ldots, x_n, y) = \mathsf{G}(x_1, \ldots, x_n, y, f).$$

This suggests the correct defining formula for F. Let $n = 1$ to simplify notation. Let $P(x, y, z, f)$ be the Σ predicate given by:

f *is a function* $\wedge \operatorname{dom}(f)=\mathrm{TC}(y)$

$\wedge \forall w \in \mathrm{TC}(y)\,(f(w)=\mathsf{G}(x,w,f\restriction \mathrm{TC}(w)))$

$\wedge z=\mathsf{G}(x,y,f)$.

We will prove:

(4) $\forall x\, \forall y\, \exists! z\, \exists f\, P(x,y,z,f)$;

and so we can introduce a Σ function symbol F by:

(5) $\mathsf{F}(x,y)=z \leftrightarrow \exists f\, P(x,y,z,f)$,

where it is clear that the right-hand side of (5) is a Σ formula. In order to prove (4) it suffices to prove;

(6) $P(x,y,z,f) \wedge P(x,y,z',f') \rightarrow z=z' \wedge f=f'$, and

(7) $\forall y\, \exists z\, \exists f\, P(x,y,z,f)$.

We prove both (6) and (7) by induction on $\mathrm{TC}(y)$. We use, in these proofs, lines (8), (9) below which are obtained by inspecting the definition of P:

(8) $P(x,y,z,f) \rightarrow z=\mathsf{G}(x,y,f)$;

(9) $P(x,y,z,f) \wedge w \in \mathrm{TC}(y) \rightarrow P(x,w,f(w),f\restriction \mathrm{TC}(w))$.

We now prove (6) by induction on $\mathrm{TC}(y)$. Thus, we may assume that for $w \in \mathrm{TC}(y)$ there is at most one u and g with $P(x,w,u,g)$ and prove that $P(x,y,z,f) \wedge P(x,y,z',f') \rightarrow z=z' \wedge f=f'$. Since $z=\mathsf{G}(x,y,f)$ and $z'=\mathsf{G}(x,y,f')$, it suffices to prove $f=f'$. But f and f' are functions with common domain $\mathrm{TC}(y)$ so it suffices to show that $f(w)=f'(w)$ for all $w \in \mathrm{TC}(y)$. But by (9), $P(x,w,f(w),f\restriction \mathrm{TC}(w))$ and $P(x,w,f'(w),f'\restriction \mathrm{TC}(w))$; so $f(w)=f'(w)$ by the induction hypothesis. It remains to prove (7), and this is where Δ_0 Collection enters in the guise of Σ Replacement. We prove $\exists z\, \exists f\, P(x,y,z,f)$ assuming, by induction on TC, that $\forall w \in \mathrm{TC}(y)\, \exists u\, \exists g\, P(x,w,u,g)$; and hence, by (6), there is a unique u_w, g_w such that $P(x,w,u_w,g_w)$. By Σ Replacement the function

$$f=\{\langle w,u_w\rangle \mid w \in \mathrm{TC}(y)\}$$

exists. To prove (7) it suffices to prove $P(x,y,\mathsf{G}(x,y,f),f)$ and this will follow from $\forall z \in \mathrm{TC}(x)(f(z)=\mathsf{G}(x,z,f\restriction \mathrm{TC}(z)))$. Since we have $P(x,z,u_z,g_z)$ we have $f(z)=u_z=\mathsf{G}(x,y,g_z)$. Thus, all we have to show is $f\restriction \mathrm{TC}(z)=g_z$. For $w \in \operatorname{dom}(g_z) = \mathrm{TC}(z)$, (9) implies $P(x,w,g_z(w),g_z\restriction \mathrm{TC}(w))$. Thus by (6) we have $g_z(w)=u_w=f(w)$; so $g_z=f\restriction \mathrm{TC}(w)$ as desired. This proves (7). Now let us introduce F by line (5) and go back to prove 6.4(i). By (5) we have

$$\mathsf{F}(x,y)=\mathsf{G}(x,y,f) \quad \text{where} \quad P(x,y,\mathsf{G}(x,y,f),f),$$

so we need only show that

$$f=\{\langle z,\mathsf{F}(x,z)\rangle \mid z\in \mathrm{TC}(y)\}.$$

For $z\in\mathrm{TC}(y)$ we have, by (9), $P(x,z,f(z),f\restriction\mathrm{TC}(z))$ so, by (5), $\mathsf{F}(x,z)=f(z)$ as desired. □

6.5 Exercise. Prove that if two operations F_1, F_2 both satisfy 6.4(i) in place of F for all $x_1,\ldots,x_n,y$ then $F_1(x_1,\ldots,x_n,y)=F_2(x_1,\ldots,x_n,y)$, for all $x_1,\ldots,x_n,y$.

In applications of 6.4 one does not usually bother to introduce the explicit function symbols G,H first.

6.6 Corollary (Δ Predicates Defined by Recursion). *Let P,Q be Δ predicates of $n+1$, $n+2$ arguments respectively, $n\geqslant 0$. We can introduce a Δ predicate* R *by definition so that the following are provable in the resulting* KPU:

(i) $\mathsf{R}(x_1,\ldots,x_n,p)\leftrightarrow P(x_1,\ldots,x_n,p)$;

(ii) $\mathsf{R}(x_1,\ldots,x_n,a)\leftrightarrow Q(x_1,\ldots,x_n,a,\{b\in\mathrm{TC}(a)\mid \mathsf{R}(x_1,\ldots,x_n,b)\})$.

Proof. Introduce the characteristic functions G,H of P,Q respectively. Use Σ Recursion to define the characteristic function F of R and then note that

$$\begin{aligned}\mathsf{R}(x_1,\ldots,x_n,y) &\leftrightarrow \mathsf{F}(x_1,\ldots,x_n,y)=1\\ &\leftrightarrow \mathsf{F}(x_1,\ldots,x_n,y)\neq 0,\end{aligned}$$

so that R is shown to be Δ. □

In Table 4 we give some examples of operations defined by recursion. The reader not familiar with this type of thing should work through the following exercises.

6.7—6.9 Exercises

6.7. (The rank function). (i) Show how to make the definition of rk given in Table 4 fit into the form of Theorem 6.4.

(ii) Prove that $\mathrm{rk}(x)$ is an ordinal, $\mathrm{rk}(\alpha)=\alpha$ for ordinals α, and $\mathrm{rk}(y)<\mathrm{rk}(x)$ whenever $y\in\mathrm{TC}(x)$.

(iii) Prove that $\mathrm{rk}(a)=\{\mathrm{rk}(y)\mid y\in\mathrm{TC}(a)\}$. (This could be used to give a different recursive definition of rk.)

6.8. (The support function). (i) Show how to make the definition of sp given in Table 4 fit into the form demanded by 6.4.

(ii) Prove that $\mathrm{sp}(a) = \{x \in \mathrm{TC}(a) \mid x \text{ is an urelement}\}$.

6.9. (Ordinal addition). (i) Show how to make ordinal addition fit into the form of 6.4.

(ii) Prove:

$$\alpha + 0 = \alpha;$$
$$\alpha + (\beta + 1) = (\alpha + \beta) + 1; \quad \text{and}$$
$$\alpha + \lambda = \sup\{\alpha + \beta \mid \beta < \gamma\}, \quad \text{if} \quad \mathrm{Lim}(\lambda).$$

(iii) Prove:

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma;$$
$$\beta \leqslant \alpha + \beta;$$
$$0 < \beta \Rightarrow \alpha < \alpha + \beta;$$
$$\alpha < \beta \Rightarrow \exists! \gamma \leqslant \beta(\alpha + \gamma = \beta);$$
$$\beta < \gamma \Rightarrow \alpha + \beta < \alpha + \gamma;$$
$$\alpha \leqslant \beta \wedge \gamma \leqslant \delta \Rightarrow \alpha + \gamma \leqslant \beta + \delta.$$

To conclude this section we point out that, like much of axiomatic mathematics, the development of set theory in KPU is largely a matter of refining proofs from ZF. Among its rewards is the Σ recursion theorem (I.6.4). Since we end with a Σ operation symbol, the operation defined by recursion is absolute. The usual development in ZF completely looses track of this vital information. (This is relevant to the point we made in § 1, line (2).)

Table 4. Some Σ Operations Defined by Recursion

Operation	*Domain*	*Abbreviation*	*Recursive definitions*
rank function	everything	$\mathrm{rk}(x)$	$\mathrm{rk}(p) = 0$ $\mathrm{rk}(a) = \sup\{\mathrm{rk}(x) + 1 \mid x \in a\}$
support function	everything	$\mathrm{sp}(x)$	$\mathrm{sp}(p) = \{p\}$ $\mathrm{sp}(a) = \bigcup_{x \in a} \mathrm{sp}(x)$
ordinal addition	pairs of ordinals α, β	$\alpha + \beta$	$\alpha + \beta = \alpha \cup \sup\{(\alpha + \gamma) + 1 \mid \gamma < \beta\}$
ordinal multiplication	pairs of ordinals α, β	$\alpha\beta$	$\alpha\beta = \sup\{\alpha\gamma + \alpha \mid \gamma < \beta\}$
collapsing function (cf. I.7)	pairs a, x	$\mathrm{C}_a(x)$	$\mathrm{C}_a(p) = p$ $\mathrm{C}_a(b) = \{\mathrm{C}_a(x) \mid x \in a \cap b\}$
constructible sets (cf. II.5)	pairs a, α	$\mathrm{L}(a, \alpha)$	$\mathrm{L}(a, 0) = \mathrm{TC}(a)$ $\mathrm{L}(a, \alpha + 1) = \mathscr{D}(\mathscr{S}(\mathrm{L}(a, \alpha)))$ $\mathrm{L}(a, \lambda) = \bigcup_{\alpha < \lambda} \mathrm{L}(a, \alpha)$, for limit λ.

7. The Collapsing Lemma

We return to the development of set theory in KPU to discuss an important operation C of two arguments; we write $C_x(y)$ instead of $C(x,y)$. The operation is defined in KPU using Σ Recursion by the equations:

$$C_x(p) = p\,;$$
$$C_x(a) = \{C_x(y) \mid y \in a \cap x\}\,.$$

(This falls under the second variation on Theorem 6.4.) C will be called Mostowski's *collapsing function.* We shall compute $C_x(y)$ for some specific values of x and y after we have a lemma to aid us. In this section we will only be interested $C_x(y)$ for $y \in x$.

7.1 Lemma. (i) $C_p(a) = 0$, *for all* p, a.
(ii) *If* $a \subseteq b$ *and* a *is transitive, then* $C_b(x) = x$ *for all* $x \in a$.
(iii) *For any* b *the set* $\{C_b(x) \mid x \in b\} = C_b(b)$ *is transitive.*

Proof. (i) is obvious. We prove (ii) by $\in$-induction. Thus, given $x \in a$ we suppose that $C_b(y) = y$ for all $y \in x$. But since a is transitive, $x \subseteq a \subseteq b$, so we have

$$\begin{aligned} C_b(x) &= \{C_b(y) \mid y \in x \cap b\} \\ &= \{C_b(y) \mid y \in x\} \\ &= \{y \mid y \in x\} \\ &= x\,. \end{aligned}$$

To prove (iii), let $a = \{C_b(x) \mid x \in b\}$. We must show that a is transitive. Let $z \in y \in a$. Thus $y = C_b(x)$ for some $x \in b$, hence $z \in \{C_b(x') \mid x' \in x \cap b\}$; so $z = C_b(x')$ for some $x' \in b$. Hence $z \in a$. ▯

7.2 Example. Let $b = \{0, 1, 2, 4, \{1, 3, 4\}, \{1, 4\}\}$. If we let $a = 3 = \{0, 1, 2\}$ then 7.1 (ii) applies to give:

$$C_b(0) = 0\,,$$
$$C_b(1) = 1\,,$$
$$C_b(2) = 2\,.$$

Let us compute $C_b(4)$:

$$\begin{aligned} C_b(4) &= \{C_b(x) \mid x \in b,\, x \in 4\} \\ &= \{C_b(x) \mid x = 0, 1, 2\} \\ &= \{0, 1, 2\} \\ &= 3\,. \end{aligned}$$

Thus C_b "collapses" 4 to 3 since 3 wasn't in b. Now let us compute $C_b(\{1,3,4\})$ and $C_b(\{1,4\})$:

$$\begin{aligned} C_b(\{1,3,4\}) &= \{C_b(x) \mid x \in \{1,3,4\} \cap b\} \\ &= \{C_b(1), C_b(4)\} \\ &= \{1,3\}; \end{aligned}$$

$$\begin{aligned} C_b(\{1,4\}) &= \{C_b(x) \colon x \in \{1,4\} \cap b\} \\ &= \{C_b(1), C_b(4)\} \\ &= \{1,3\}. \end{aligned}$$

Thus both the sets $\{1,3,4\}$ and $\{1,4\}$ are collapsed to $\{1,3\}$, all because 3 was left out of b. Note that

$$\{C_b(x) \colon x \in b\} = \{0,1,2,3,\{1,3\}\},$$

which is a transitive set, just as 7.1 (iii) foretold.

7.3 Definition. For any set b let c_b denote the restriction of $C_b(\cdot)$ to b; i.e. $c_b = \{\langle x, C_b(x)\rangle \colon x \in b\}$, and let

$$\text{clpse}(b) = \text{rng}(c_b) = \{C_b(x) \colon x \in b\} = C_b(b).$$

Note that the function c_b exists (as a set) by Σ replacement and that clpse(b) is a transitive set by 7.1 (iii).

A set b is *extensional* if for every two distinct sets $a_1, a_2 \in b$ there is an $x \in b$ such that x is in one of a_1, a_2 but not both; in other symbols,

$$\forall x \in b\, (x \in a_1 \leftrightarrow x \in a_2) \rightarrow a_1 = a_2.$$

We would like to say that b is extensional if

$$\langle b, \in \cap\, b^2 \rangle \models \text{"Extensionality"},$$

but we cannot do this because we have not yet defined syntax and semantics (say $\models$) in KPU. So, what we have done is simply to write this out in full.

In Example 7.2, b was not extensional because of the two sets

$$a_1 = \{1,3,4\}, \quad a_2 = \{1,4\}.$$

Any transitive set is extensional, as is any set of ordinals. The next lemma shows that any extensional set is isomorphic to a transitive set.

7.4 Theorem (The Collapsing Lemma). *If a is extensional then c_a maps a one-one onto the transitive set* clpse(a). *Furthermore, for all $x, y \in a$*

(i) $x \in y$ *iff* $c_a(x) \in c_a(y)$.

In other words, c_a is an isomorphism of $\langle a, \in \cap a^2 \rangle$ onto $\langle \text{clpse}(a), \in \cap \text{clpse}(a)^2 \rangle$.

Proof. We need to show c_a is one-one and that $c_a(x) \in c_a(y)$ implies $x \in y$. We prove both of these by proving $\forall x \forall y\, P(x, y)$ where $P(x, y)$ is the conjunction of:

$$x, y \in a \wedge c_a(x) = c_a(y) \rightarrow x = y\,,$$
$$x, y \in a \wedge c_a(x) \in c_a(y) \rightarrow x \in y\,, \quad \text{and}$$
$$x, y \in a \wedge c_a(y) \in c_a(x) \rightarrow y \in x\,.$$

Given an x_0 we can assume, by induction on $\in$,

(1) $\forall x \in x_0 \forall y\, P(x, y)$

in our proof of $\forall y\, P(x_0, y)$. Given an arbitrary y_0 we can assume

(2) $\forall y \in y_0\, P(x_0, y)$

in our proof of $P(x_0, y_0)$, again using $\in$-induction. Thus, suppose $x_0, y_0 \in a$.

Case 1. $c_a(x_0) = c_a(y_0)$. Suppose $x_0 \neq y_0$. We see that both x_0, y_0 must be sets since $c_a(p) = p$. But then, since a is extensional there is a $z \in a$ with $z \in (x_0 \cup y_0) - (x_0 \cap y_0)$. Suppose $z \in x_0 - y_0$, the other possibility being similar. Then $c_a(z) \in c_a(x_0) = c_a(y_0)$ but, by (1), $P(z, y_0)$ so $z \in y_0$, a contradiction.

Case 2. $c_a(x_0) \in c_a(y_0)$. But then $c_a(x_0) = c_a(z)$, for some $z \in y_0$, but $P(x_0, z)$ by (2), so $x_0 = z$ and $x_0 \in y_0$.

Case 3. $c_a(y) \in c_a(x_0)$. Similar to Case 2. ☐

We hint at some of the types of applications of the collapsing lemma in the exercises.

7.5—7.8 Exercises

7.5. Show that if a is finite so is $C_b(a)$. [Hint: Use induction on natural numbers.]

7.6. Show in KPU that a set a of ordinals is finite iff clpse(a) is a natural number. This shows that the predicate "a is a finite set of ordinals" is a Δ predicate in KPU. (For contrast see the remarks in 9.1.)

7.7. Assuming intuitive set theory, or ZF, use the collapsing lemma and the Löwenheim-Skolem theorem to show that for every transitive A there is a countable transitive set B such that $\langle A,\in\rangle \equiv \langle B,\in\rangle$. ($\equiv$ denotes elementary equivalence; we use $\langle A,\in\rangle$ for $\langle A,\in\cap A^2\rangle$ when A is transitive.) Show that $\equiv$ cannot in general be replaced by $\prec$ (elementary substructure).

7.8. Let a,b be transitive sets, f an isomorphism of $\langle a,\in\rangle$ and $\langle b,\in\rangle$. Show that if $f(p)=p$ for all urelements $p\in a$ then $f(x)=x$ for all $x\in a$ and hence $a=b$.

7.9 Notes. The collapsing lemma is due to Mostowski [1949] and is one of the standard tools of the set-theorist. (See also the notes to § 9.) The value $\mathrm{C}_b(x)$ of the collapsing function is of interest even when $x\notin b$. For example if b is countable one can use $\mathrm{C}_b(x)$ as a kind of countable approximation to x. Using a notion of "almost all" due to Kueker and Jech, one can prove that if P is a Σ predicate and $P(x)$ holds, then $P(\mathrm{C}_b(x))$ holds for almost all countable sets b. For more on this see Kueker [1972], Jech [1973] and Barwise [1974].

8. Persistent and Absolute Predicates

In this section we discuss the reason for the restriction to Δ_0 formulas in the axioms of separation and collection. The rationale behind this restriction rests in one of the basic notions of the subject, that of absoluteness.

Recall the discussion of $\mathbb{V}_M$ from § 1. The sets in $\mathbb{V}_M$ come in stages and separation tells us what principles are allowed in forming the sets at each stage. The content of Δ_0 Separation is that we allow ourselves to form the set $b=\{x\in a \mid \varphi(x,y)\}$ at stage α if we already have formed a and y, but only if the meaning of $\varphi(x,y)$ is completely (or absolutely) determined solely on the basis of the sets formed before stage α. In other words, when we come to a later stage β and form $\{x\in a \mid \varphi(x,y)\}$ we want to get the same set b, even though there are now more sets around which might conceivably affect the truth of $\varphi(x,y)$ by altering the range of any unbounded quantifiers in φ.

Similar considerations apply to collection. Suppose that, in the process of building $\mathbb{V}_M$, we suddenly notice that $\forall x\in a\, \exists y\, \varphi(x,y)$ is true. We want to be able to form at the next stage a set b for which $\forall x\in a\, \exists y\in b\, \varphi(x,y)$ is true, and remains true. But what if the introduction of this very set b destroyed the truth of $\varphi(x,y)$ for some $x\in a$? This can happen if φ has unbounded universal quantifiers in it. If we want this stability, we must apply collection only if $\varphi(x,y)$ cannot become false when we add new sets to our universe of set theory. That is, $\varphi(x,y)$ should persist.

The aim of this section is to extract formal consequences from these ideas.

8.1 Definition. Let $\mathfrak{A}_{\mathfrak{M}}=(\mathfrak{M}; A, E, \ldots)$ be a structure for L^*. For $a\in A$ we define

$$a_E=\{y\in M\cup A \mid yEa\}.$$

Note that the value of a_E in 8.1 depends on $\mathfrak{A}_{\mathfrak{M}}$, and a. The import of 8.1 is clear. Speaking very loosely, the set a_E is "the set that a believes itself to be". In the natural intended structures a_E will just be a itself.

The usual notion of substructures has an obvious generalization to L^*. We say that $\mathfrak{B}_{\mathfrak{N}}$ is an *extension* of $\mathfrak{A}_{\mathfrak{M}}$, and write $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$ (where $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ and $\mathfrak{B}_{\mathfrak{N}} = (\mathfrak{N}; B, E', \ldots)$) if $\mathfrak{M} \subseteq \mathfrak{N}$ (as L-structures), if $A \subseteq B$, and if the interpretations $E, \ldots$ are just the restrictions to $M \cup A$ of the interpretations $E', \ldots$.

A moment's reflection shows that $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$ is not really the natural notion of extension when one is thinking of models of set theory. For suppose $a \in A$. The trouble is that a may be "schizophrenic" in its role as a set in $\mathfrak{A}_{\mathfrak{M}}$ and as a set in $\mathfrak{B}_{\mathfrak{N}}$. The relation $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$ guarantees that $a_E \subseteq a_{E'}$ but it does not rule out the possibility that for some $x \in B - A$, $x \in (a_{E'} - a_E)$. This is clearly a chaotic situation (since a set is supposed to be determined by its members), so we introduce a stronger notion of extension suitable for the study of set theory.

8.2 Definition. Given structures $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ and $\mathfrak{B}_{\mathfrak{N}} = (\mathfrak{N}; B, E', \ldots)$ for L^*, we say that $\mathfrak{B}_{\mathfrak{N}}$ is an *end extension* of $\mathfrak{A}_{\mathfrak{M}}$, written either as:

$$\mathfrak{A}_{\mathfrak{M}} \subseteq_{\mathrm{end}} \mathfrak{B}_{\mathfrak{N}} \quad \text{or} \quad \mathfrak{B}_{\mathfrak{N}} \supseteq_{\mathrm{end}} \mathfrak{A}_{\mathfrak{M}},$$

if $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$ and if for each $a \in A$, $a_E = a_{E'}$. One sometimes reads, aloud, $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\mathrm{end}} \mathfrak{B}_{\mathfrak{N}}$ as "$\mathfrak{A}_{\mathfrak{M}}$ is an *initial substructure* of $\mathfrak{B}_{\mathfrak{N}}$".

8.3 Example. If A is a transitive set, $B \supseteq A$, $E = \in \cap A^2$, and $E' = \in \cap B^2$, then

$$(\mathfrak{M}; A, E) \subseteq_{\mathrm{end}} (\mathfrak{M}; B, E');$$

for in both structures any $a \in A$ has $a_E = a_{E'} = a$. If A were not transitive, however, this could fail.

8.4 Lemma. *Let $\mathfrak{A}_{\mathfrak{M}}, \mathfrak{B}_{\mathfrak{N}}$ be structures for L^*, $\mathfrak{B}_{\mathfrak{N}} \supseteq_{\mathrm{end}} \mathfrak{A}_{\mathfrak{M}}$. If φ is a Σ formula of L^* then for any $x_1, \ldots, x_n \in \mathfrak{A}_{\mathfrak{M}}$, $\mathfrak{A}_{\mathfrak{M}} \models \varphi[x_1, \ldots, x_n]$ implies $\mathfrak{B}_{\mathfrak{N}} \models \varphi[x_1, \ldots, x_n]$.*

Proof. This just repeats the proof of Lemma 4.2 proceeding by induction on φ. The end extension hypothesis is used to assure that $\forall x \in a$ has the same meaning in $\mathfrak{A}_{\mathfrak{M}}$ and $\mathfrak{B}_{\mathfrak{N}}$. ▯

8.5 Definition. A formula $\varphi(u_1, \ldots, u_n)$ of L^* is said to be *persistent* relative to a theory T of L^* if for all models $\mathfrak{A}_{\mathfrak{M}}, \mathfrak{B}_{\mathfrak{N}}$ of T with $\mathfrak{B}_{\mathfrak{N}} \supseteq_{\mathrm{end}} \mathfrak{A}_{\mathfrak{M}}$, and all $x_1, \ldots, x_n$ in $\mathfrak{A}_{\mathfrak{M}}$:

$$\mathfrak{A}_{\mathfrak{M}} \models \varphi[x_1, \ldots, x_n] \quad \text{implies} \quad \mathfrak{B}_{\mathfrak{N}} \models \varphi[x_1, \ldots, x_n].$$

The formula φ is *absolute* relative to T if for all $\mathfrak{A}_{\mathfrak{M}}, \mathfrak{B}_{\mathfrak{N}}, x_1, \ldots, x_n$ as above:

$$\mathfrak{A}_{\mathfrak{M}} \models \varphi[x_1, \ldots, x_n] \quad \text{iff} \quad \mathfrak{B}_{\mathfrak{M}} \models \varphi[x_1, \ldots, x_n].$$

The significance of 8.5 should be clear enough. Absolute formulas don't shift their meaning on us as we move from $\mathfrak{A}_{\mathfrak{M}}$ to its end extension $\mathfrak{B}_{\mathfrak{N}}$ and back again. Absoluteness is a precious attribute.

8.6 Corollary. *All Σ formulas are persistent and all Δ_0 formulas are absolute (relative to any theory* T*).*

Proof. By 8.4 all Σ formulas are persistent, hence all Δ_0 formulas are persistent. But the Δ_0 formulas are closed under negation and φ is absolute iff φ and $\neg\varphi$ are both persistent. □

8.7 Example. Let $\mathfrak{A}_{\mathfrak{M}}$ and $\mathfrak{B}_{\mathfrak{N}}$ be models of KPU, $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\text{end}} \mathfrak{B}_{\mathfrak{N}}$. We can interpret all the definitions and theorems of KPU in these two models. For example, let $a \in \mathfrak{A}_{\mathfrak{M}}$. Since $\mathrm{Ord}(x)$ is a Δ_0 formula,

$$\mathfrak{A}_{\mathfrak{M}} \models \mathrm{Ord}(a) \quad \text{iff} \quad \mathfrak{B}_{\mathfrak{M}} \models \mathrm{Ord}(a).$$

Now let us return to consider the rationale behind the Δ_0 in Δ_0 Separation and Δ_0 Collection. We see from Corollary 8.6 that we have asserted separation and collection for absolute formulas, at least some of them. For example, if we form the set $b = \{x \in a \mid \varphi(x)\}$ in $\mathfrak{A}_{\mathfrak{M}}$, a model of KPU, (with φ a Δ_0 formula), then in any $\mathfrak{B}_{\mathfrak{N}} \supseteq_{\text{end}} \mathfrak{A}_{\mathfrak{M}}$, the equation for b will remain true.

Have we asserted separation and collection for all absolute formulas? Yes, but not explicitly. There are formulas $\varphi(x, y)$ which are absolute relative to KPU which are not Δ_0; separation for such φ is not an axiom of KPU. It is a *theorem* of KPU, though, as we see from the following result of Feferman-Kreisel [1966].

8.8 Theorem. *For any theory* T *of* L*, *if* $\varphi(x_1, \ldots, x_n)$ *is persistent relative to* T *then there is a* Σ *formula* $\psi(x_1, \ldots, x_n)$ *such that*

$$\mathrm{T} \vdash \forall x_1, \ldots, x_n\, [\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)].$$

Hence, if φ *is absolute relative to* T, *there are* Σ *and* Π *formulas* $\psi(x_1, \ldots, x_n)$, $\theta(x_1, \ldots, x_n)$ *such that*

$$\mathrm{T} \vdash \forall x_1, \ldots, x_n\, [\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n) \wedge \varphi(x_1, \ldots, x_n) \leftrightarrow \theta(x_1, \ldots, x_n)].$$

From the results in § 4 it follows that we can prove separation in KPU for all formulas absolute relative to KPU and collection for all formulas persistent relative to KPU. Furthermore, if we later extend KPU to a stronger theory T (and we will from time to time) then we'll still have separation for all φ absolute relative to T and collection for all φ persistent relative to T. (If T is stronger then it has fewer models so, in general, it is easier for a formula to be persistent or absolute.) These results are not used in the actual study of KPU but they are reassuring.

We conclude this section with a lemma which will prove useful later on. We include it here so that the student can become familiar with the concept of absoluteness. First some remarks.

If $\mathfrak{B}_{\mathfrak{N}} \models \mathrm{KPU}$ and we use a phrase like "b is an ordinal of $\mathfrak{B}_{\mathfrak{N}}$", what we mean, of course, is that $b \in \mathfrak{B}_{\mathfrak{N}}$ and $\mathfrak{B}_{\mathfrak{N}} \models \mathrm{Ord}(b)$. The object b need not be a real ordinal at all. If $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\mathrm{end}} \mathfrak{B}_{\mathfrak{N}}$ and $a \in \mathfrak{A}_{\mathfrak{M}}$ then, as we saw in Example 8.7, a is an ordinal of $\mathfrak{A}_{\mathfrak{M}}$ iff a is an ordinal of $\mathfrak{B}_{\mathfrak{N}}$. Furthermore, since $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\mathrm{end}} \mathfrak{B}_{\mathfrak{N}}$ the ordinals of $\mathfrak{A}_{\mathfrak{M}}$ form an initial segment of the ordinals of $\mathfrak{B}_{\mathfrak{N}}$. (Why?) This initial segment may or may not exhaust the ordinals of $\mathfrak{B}_{\mathfrak{N}}$, even though $\mathfrak{A}_{\mathfrak{M}} \neq \mathfrak{B}_{\mathfrak{N}}$. In the case where it is a proper initial segment there need not be any ordinal b of $\mathfrak{B}_{\mathfrak{N}}$ which is the least upper bound of this segment.

8.9 Lemma. *Let* $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\mathrm{end}} \mathfrak{B}_{\mathfrak{M}}$ *where* $\mathfrak{B}_{\mathfrak{M}} \models \mathrm{KPU}$. *Suppose that whenever* $\mathfrak{B}_{\mathfrak{M}} \models \mathrm{rk}(a) = \alpha$ *we have* $a \in A$ *iff* $\alpha \in A$. *Suppose further that there is no ordinal* β *of* $\mathfrak{B}_{\mathfrak{M}}$ *which is the least upper bound of the ordinals of* $\mathfrak{A}_{\mathfrak{M}}$. *Then, with the possible exception of foundation, all the axioms of* KPU *hold in* $\mathfrak{A}_{\mathfrak{M}}$.

Proof. We check three axioms and trust the student to verify the other two, *Extensionality* and *Union.*

Pair: Suppose, $x, y \in \mathfrak{A}_{\mathfrak{M}}$, let $\alpha, \beta \in A$ be such that

$$\mathfrak{B}_{\mathfrak{M}} \models \alpha = \mathrm{rk}(x) \wedge \beta = \mathrm{rk}(y).$$

Then, if $\mathfrak{B}_{\mathfrak{M}} \models \gamma = (\alpha+1) \cup (\beta+1)$, we have $\gamma \in A$ since otherwise γ would be the least upper bound of the ordinals of $\mathfrak{A}_{\mathfrak{M}}$. Thus if we choose $b \in \mathfrak{B}_{\mathfrak{M}}$ with $\mathfrak{B}_{\mathfrak{M}} \models b = \{x, y\}$ so that $\mathfrak{B}_{\mathfrak{M}} \models \mathrm{rk}(b) = \gamma$, then $b \in A$ and $\mathfrak{A}_{\mathfrak{M}} \models b = \{x, y\}$ by absoluteness of the formula $b = \{x, y\}$, from Table 1.

Δ_0 *Separation:* Suppose $a, y \in \mathfrak{A}_{\mathfrak{M}}$. Let $\varphi(x, y)$ be Δ_0. We want to find $a, b \in \mathfrak{A}_{\mathfrak{M}}$ such that

(1) $b = \{x \in a \mid \varphi(x, y)\}$

holds in $\mathfrak{A}_{\mathfrak{M}}$. Let $b \in \mathfrak{B}_{\mathfrak{M}}$ be such that (1) holds in $\mathfrak{B}_{\mathfrak{M}}$, using Δ_0 Separation in $\mathfrak{B}_{\mathfrak{M}}$. But since

$$\mathfrak{B}_{\mathfrak{M}} \models \mathrm{rk}(b) \leqslant \mathrm{rk}(a)$$

the set b is in $\mathfrak{A}_{\mathfrak{M}}$. It still satisfies (1) in $\mathfrak{A}_{\mathfrak{M}}$ by absoluteness.

Δ_0 *Collection:* Suppose that $a \in \mathfrak{A}_{\mathfrak{M}}$, the formula $\varphi(x, y)$ is Δ_0 with parameters from $\mathfrak{A}_{\mathfrak{M}}$ and that $\forall x \in a\, \exists y\, \varphi(x, y)$ holds in $\mathfrak{A}_{\mathfrak{M}}$. Then we have:

(2) for each $x \in a$ there is a $y \in A$ and $\alpha \in A$ such that $\mathfrak{A}_{\mathfrak{M}} \models \varphi(x, y)$ and $\mathfrak{B}_{\mathfrak{M}} \models \mathrm{rk}(y) = \alpha$, and hence, by absoluteness $\mathfrak{B}_{\mathfrak{M}} \models \varphi(x, y) \wedge \mathrm{rk}(y) = \alpha$.

Thus in $\mathfrak{B}_{\mathfrak{M}}$ we have $\forall x \in a\, \exists \alpha\, \exists y [\mathrm{rk}(y) = \alpha \wedge \varphi(x, y)]$. So, by Σ Reflection in $\mathfrak{B}_{\mathfrak{M}}$, there is an ordinal $\beta \in \mathfrak{B}_{\mathfrak{M}}$ such that

$$\forall x \in a\, \exists \alpha < \beta\, \exists y\, [\mathrm{rk}(y) = \alpha \wedge \varphi(x, y)],$$

and hence:

(3) $\forall x \in a\, \exists y\, [\mathrm{rk}(y) < \beta \wedge \varphi(x,y)]$

holds in $\mathfrak{B}_{\mathfrak{M}}$. In $\mathfrak{B}_{\mathfrak{M}}$ pick the least ordinal β satisfying (3): it exists by foundation. By (2), β is a sup of ordinals $\alpha \in \mathfrak{A}_{\mathfrak{M}}$, so $\beta \in \mathfrak{A}_{\mathfrak{M}}$. Apply Δ_0 Collection in $\mathfrak{B}_{\mathfrak{M}}$ to (3) to find a set $b \in \mathfrak{B}_{\mathfrak{M}}$ such that

$$\forall x \in a\, \exists y \in b\, [\varphi(x,y) \wedge \mathrm{rk}(y) < \beta]$$

holds in $\mathfrak{B}_{\mathfrak{M}}$. Since $\mathfrak{B}_{\mathfrak{M}} \models \mathrm{rk}(b) \leqslant \beta$, $b \in \mathfrak{A}_{\mathfrak{M}}$. But then the formula

$$\forall x \in a\, \exists y \in b\, \varphi(x,y)$$

is Δ_0, it holds in $\mathfrak{B}_{\mathfrak{M}}$, and it has all its parameters in $\mathfrak{A}_{\mathfrak{M}}$. Hence by absoluteness, it holds in $\mathfrak{A}_{\mathfrak{M}}$. ▯

8.10—8.12 Exercises

8.10. Given $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$, we write $\mathfrak{A}_{\mathfrak{M}} \prec_1 \mathfrak{B}_{\mathfrak{N}}$ if for all Σ_1 formulas $\varphi(x_1,\ldots,x_n)$ of L* and all $x_1,\ldots,x_n$ in $\mathfrak{A}_{\mathfrak{M}}$:

$$\mathfrak{A}_{\mathfrak{M}} \models \varphi[x_1,\ldots,x_n] \quad \text{iff} \quad \mathfrak{B}_{\mathfrak{N}} \models \varphi[x_1,\ldots,x_n].$$

Show that if $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\mathrm{end}} \mathfrak{B}_{\mathfrak{N}}$, $\mathfrak{A}_{\mathfrak{M}} \prec_1 \mathfrak{B}_{\mathfrak{N}}$, and $\mathfrak{B}_{\mathfrak{N}} \models \mathrm{KPU}$ then, with the possible exception of foundation, all the axioms of KPU hold in $\mathfrak{A}_{\mathfrak{M}}$. (The end extension hypothesis is used to insure that Σ formulas persist from $\mathfrak{A}_{\mathfrak{M}}$ to $\mathfrak{B}_{\mathfrak{N}}$. Without this, the exercise is false.)

8.11. Given $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{B}_{\mathfrak{N}}$, show that $\mathfrak{A}_{\mathfrak{M}} \prec_1 \mathfrak{B}_{\mathfrak{N}}$ iff for every Δ_0 formula $\varphi(v_1,\ldots,v_n,v_{n+1})$ and all $x_1,\ldots,x_n \in \mathfrak{A}_{\mathfrak{M}}$,

$$\mathfrak{B}_{\mathfrak{N}} \models \exists v_{n+1}\, \varphi[x_1,\ldots,x_n] \quad \text{implies} \quad \exists x_{n+1} \in \mathfrak{A}_{\mathfrak{M}} (\mathfrak{B}_{\mathfrak{N}} \models \varphi[x_1,\ldots,x_{n+1}]).$$

(We are not assuming $\mathfrak{A}_{\mathfrak{M}} \subseteq_{\mathrm{end}} \mathfrak{B}_{\mathfrak{N}}$!)

8.12 (Schlipf). Find an example of two structures $\mathfrak{A}_{\mathfrak{M}}$ and $\mathfrak{B}_{\mathfrak{M}}$ satisfying the hypotheses of Lemma 8.9 but where $\mathfrak{A}_{\mathfrak{M}}$ fails to satisfy the axiom of foundation. [Let $\mathfrak{B}_{\mathfrak{M}}$ be a proper elementary extension of $\mathbb{HF}_{\mathfrak{M}}$. (Cf. § II.2.)]

8.13 Notes. The considerations involved in the choice of Δ_0 Separation and Δ_0 Collection are suggested by the informal notion of "predicative". Kripke, in fact, called his axioms for admissible sets PZF, for predicative ZF. As an explication of the intuitive idea of predicativity, however, KP has certain debatable features. See, for example, Feferman [1975] for a discussion and examples of set theories which are predicative in a more stringent sense.

9. Additional Axioms

There are certain extensions of KPU which surface from time to time. We have already defined KPU^+ and KP in § 2. We catalogue some of the others here.

9.1 Definition. *The axiom of infinity*, or *Infinity*, is the axiom:

$$\exists\alpha\, \mathrm{Lim}(\alpha)$$

where $\mathrm{Lim}(\alpha)$ is defined in Table 2. We use ω as a symbol for the first limit ordinal.

Note that $\neg$ (Infinity) asserts only that all ordinals are finite, not that all sets are finite.

The axiom of infinity is often used to form sets by taking $a=\bigcup_{n<\omega} b_n$ where b_n is defined by recursion on n. We saw one example where this would have been convenient in the proof of 6.1. For another example, define (in KPU)

$$F(a,0)=\{0\},$$
$$F(a,n+1)=\{b\cup\{x\}: b\in F(a,n),\, x\in a,\, x\notin b\},$$

by Σ recursion. We find that $F(a,n)$ is the set of n-element subsets of a. In KPU+(Infinity) we can introduce a new Σ operation symbol P_ω by

$$\mathsf{P}_\omega(a)=\bigcup_{n<\omega} F(a,n),$$

as the student should verify. We can use P_ω to convert quantifiers over finite subsets of a to bounded quantifiers:

$$\forall b\,[b\subseteq a \wedge b \text{ finite} \rightarrow (\ldots b\ldots)]$$

becomes

$$\forall b\in \mathsf{P}_\omega(a)\,(\ldots b\ldots)$$

in KPU+(Infinity). Since a is finite iff $a\in\mathsf{P}_\omega(a)$, we see that "a is finite" is Δ_1 in KPU+(Infinity) (whereas it is only Σ_1 in KPU).

The remaining axioms will be of secondary importance for our study.

9.2 Definition. Σ_1 *Separation* is the set of axioms of the form

(i) $\exists b\,\forall x\,(x\in b \leftrightarrow x\in a \wedge \varphi(x))$,

where φ is a Σ_1 formula of L^*.

9.3 Definition. *Full separation* asserts 9.2 (i) for *all* formulas φ of L^*.

9.4 Definition. *Full collection* asserts the collection scheme

$$\forall x \in a\, \exists y\, \varphi(x,y) \rightarrow \exists b\, \forall x \in a\, \exists y \in b\, \varphi(x,y)$$

for all formulas φ of L*.

9.5 Definition. *The axiom Beta.* A relation r is *well founded on a* if

$$\forall b\, [b \subseteq a \wedge b \neq 0 \rightarrow \exists x \in b\, \forall y \in b\, (\langle y,x\rangle \notin r)].$$

If $r \subseteq a \times a$ and r is well founded on a then we say that r is *well founded.* (If r is well founded on a, then $r \cap (a \times a)$ is well founded, but r itself may have some funny things going on outside the set a.) The axiom *Beta* asserts: for every well-founded relation $r \subseteq a \times a$ on a set a there is a function f, $\mathrm{dom}(f) = a$, satisfying:

(i) $f(x) = \{f(y) : y \in a \wedge \langle y,x\rangle \in r\}$,

for all $x \in a$. The function f is said to be *collapsing* for r.

The axiom Beta has the effect of making the Π_1 predicate "r is well founded on a" a Δ_1 predicate since it becomes equivalent to:

$$\exists f\, [\mathrm{dom}(f) = a \wedge f \text{ is collapsing for } r].$$

(See 9.8 (ii) (b).) Beta is not provable in KPU but it is provable if we add Σ_1 Separation.

9.6 Theorem. *Beta is provable in* KPU + (Σ_1 Separation).

Sketch of proof. Let us work in KPU+(Σ_1 Separation). Let r be well founded on a, and write $x \prec y$ for $\langle x,y\rangle \in r$. Define a operation F on the ordinals by Σ recursion:

$$F(\alpha) = \{x \in a \mid \forall y \in a\, (y \prec x \rightarrow \exists \beta < \alpha\; y \in F(\beta))\}$$
$$= \text{the set of all } x \in a \text{ such that } \{y \in a \mid y \prec x\} \subseteq \textstyle\bigcup_{\beta<\alpha} F(\beta).$$

Note that $\alpha \leqslant \beta$ implies $F(\alpha) \subseteq F(\beta)$. Let us show that every $x \in a$ is in some $F(\alpha)$. If not, then the set $b = a - b_0$ is non empty, where

$$b_0 = \{x \in a \mid \exists \alpha\, (x \in F(\alpha))\},$$

this being the place where we need Σ_1 Separation. Let $x \in b$ be such that for all $y \in b$, we have $y \nprec x$ (using the well-foundedness of r). Then

$$\forall y \in a\, [y \prec x \rightarrow \exists \beta\, (y \in F(\beta))]$$

so by Σ Reflection there is an α such that

$$\forall y \in a\,[y \prec x \rightarrow \exists \beta < \alpha\,(y \in F(\beta))], \text{ and hence}$$

$$\{y \in a : y \prec x\} \subseteq \bigcup_{\beta<\alpha} F(\beta).$$

So $x \in F(\alpha)$ by the definition of $F(\alpha)$, which contradicts $x \notin b_0$. Now since $a = \bigcup_\alpha F(\alpha)$ there is, by Σ Reflection, a γ such that $a = \bigcup_{\alpha<\gamma} F(\alpha)$.

The rest of the proof is easy. Define f_α, for $\alpha \leqslant \gamma$, by recursion on α: f_α is the function with domain $F(\alpha)$ such that

$$f_\alpha(x) = \{f_\beta(y) \mid \beta < \alpha \wedge y \in F(\beta) \wedge y \prec x\},$$

for all $x \in F(\alpha)$. These f_α are increasing ($\beta \leqslant \alpha$ implies $f_\beta \subseteq f_\alpha$, by induction on α), and $f = f_\gamma = \bigcup_{\alpha<\gamma} f_\alpha$ is the desired function satisfying

$$f(x) = \{f(y) : y \in a \wedge y \prec x\}$$

for all $x \in a$. □

9.7 Definition. *The power set axiom.* We think of the power set operation as a primitive operation. When we use the power set axiom we will assume $\mathsf{L}^* = \mathsf{L}(\in, \mathsf{P}, \ldots)$ where P is a 1-place operation symbol. The power set axiom asserts

$$\forall x \forall y\,[x \in \mathsf{P}(y) \leftrightarrow (\mathsf{S}(x) \wedge x \subseteq y)]$$

where, as in Table 2, $\mathsf{S}(x)$ means "x is a set".

9.8—9.12 Exercises

9.8. Prove in KP (not KPU) that every set of finite rank is finite. Hence KP + $\neg$(Infinity) implies that every set is finite. This greatly limits KP as opposed to KPU, as we'll see in later chapters.

9.9. Let $r \subseteq a \times a$ and let f be a function with $\text{dom}(f) = a$ which is collapsing for r. Prove the following in KPU:

(i) r is well founded;

(ii) $\text{rng}(f)$ is transitive and has no urelements in it;

(iii) If g is a function with $\text{dom}(g) = a$ and g is collapsing for r, then $f = g$. [Hint: Prove $\forall b\, \forall x \in a\,(f(x) = b \leftrightarrow g(x) = b)$ by ε-induction on b]

(iv) If for all $x, y \in a$, $x \neq y$, there is a $z \in a$ with $\neg(\langle z,x\rangle \in r \leftrightarrow \langle z,y\rangle \in r)$, then f is one-one and hence is an isomorphism of $\langle a, r\rangle$ with a transitive set $\langle b, \in \cap b^2\rangle$.

9.10. Show that in KPU + (Beta) we can introduce a Σ operation symbol B such that

$$\mathsf{B}(r,a) = 0 \quad \text{iff } r \text{ is not well founded on } a,$$

but if r is well founded on a, then $\mathsf{B}(r,a)$ is a (the) function f with $\mathrm{dom}(f)=a$ such that $f(x)=\{f(y): y\in a \wedge \langle y,x\rangle \in r\}$ for all $x\in a$. [Use 9.9.]

9.11. A relation $r\subseteq a\times a$ *well orders* a if it orders a linearly and is well founded. Show in KPU+(Beta), that if r well orders a, then there is a (unique) ordinal α such that $\langle a,r\rangle \cong \langle \alpha,\in\rangle$.

9.12. Show that adding Σ_1 Separation to KPU has the same effect (i.e. same theorems) as adding all the following axioms, where φ is Δ_0:

$$\exists b\, \forall x\in a\, [\exists y\, \varphi(x,y) \rightarrow \exists y\in b\, \varphi(x,y)].$$

9.13 Notes. In a theory like ZF containing Σ_1 Separation, Beta becomes a theorem and the collapsing lemma of § 7 is a consequence of it. In such theories Beta itself is often called *The Collapsing Lemma.* It is due to Mostowski [1949]. In KPU we must separate the two aspects since one is provable and the other is not. Beta is so named because Mostowski [1961] used the terminology "β-model" (with "β" for *bon ordre*) for models where well-orderings were absolute.