Chapter XVIII
Compactness, Embeddings and Definability

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This chapter presents an overview of the author's joint work with S. Shelah in abstract model theory, which had started as early as 1972. It is mainly based on our papers (Makowsky–Shelah–Stavi [1976]; Makowsky–Shelah [1979, 1981, 1983]) and on an unpublished manuscript of S. Shelah (Shelah [198?e]) which he wrote while this chapter came into being. The present exposition, however, tries to give a more coherent picture by putting all our results into a single perspective together with results of M. Magidor, H. Mannila, D. Mundici, and J. Stavi.

The main theme of this chapter is abstract model theory proper, especially the relationship between various compactness, embedding, and definability properties which do not characterize first-order logic. More precisely, we look at various classes of logics defined axiomatically, such as compact logics, logics satisfying certain model existence or definability properties. The classes of logics are sometimes further specified by set-theoretic parameters, such as finitely generated, absolute, set presentable, bounds on the size function, or by set-theoretic assumptions such as large cardinal axioms. Within such classes of logics we want to explore which other properties of logics follow from the axiomatic description of the class. In Chapter III first-order logic was characterized in this way. In Chapter XVII the class of absolute logics was studied. Most of the other chapters (with the exception of Chapters XIX and XX) study families of logics which bear some inherent similarity which stems from the way they evolved, such as infinitary logics or logics based on cardinality quantifiers, and establish particular model-theoretic results for those logics. In this chapter we want to clarify the conceptual and metalogical relationship between these model theoretic properties. Success in this program can be achieved in three ways: by establishing non-trivial connections between these properties; by applying the former to gain new insight about particular logics previously studied; and by using this insight to construct new examples of logics, and ultimately, by showing, that our list of examples is, in some reasonable sense, exhaustive.

The chapter consists of four sections, in each of which one aspect of abstract model theory is developed to a certain depth.

Section 1 is devoted to compactness properties and is almost self-contained. Its main results are the abstract compactness theorem and the description of the compactness spectrum. Here a thorough understanding of various compactness phenomena is obtained and the theory is provided with new examples. Especially, the examples described in Section 1.6 play an important role in the successive sections as well.

Section 2 is devoted to the study of the dependence number. Its main result is the finite dependence theorem, the proof of which is given completely on the basis of three lemmas, which are only stated. The complete proof may be found in Makowsky–Shelah [1983]. The finite dependence theorem clarifies how little compactness is needed to ensure that a logic is equivalent to a logic which has the finite dependence property. In fact, assuming there are no uncountable measurable cardinals, $[\omega]$-compactness suffices. Finally, the dependence structure is introduced, a concept which appears here for the first time. It is the appropriate generalization of the dependence number, as the examples and the finite dependence structure theorem show.

Section 3 is devoted to various aspects of embeddings, whose existence is implied by the compactness theorem, such as proper extensions, amalgamation, and joint embeddings. Joint embeddings are also discussed in Chapter XIX and amalgamations in Chapter XX. The main result here is the connection between $[\omega]$-compactness and proper extensions and the abstract amalgamation theorem. Again, this section is rather self-contained. The abstract amalgamation theorem also leads to the discovery that various logics with cardinality quantifiers do not satisfy the amalgamation property. This solves a problem which had been stated explicitly in Malitz–Reinhardt [1972b].

Section 4 is devoted to definability properties, as introduced already in Section II.7, and to preservation properties. Preservation properties for sum-like operations already played an important role in Chapters XII and XIII. A common generalization of these two properties, the uniform reduction property, was introduced in Feferman [1974b]. The first two subsections are devoted to an exposition of those properties and their interrelations. The main results here are the equivalence of the uniform reduction property UR_1 with the interpolation property and the equivalence, for compact logics, of the pair preservation property and the uniform reduction property for pairs. The Robinson property and especially its weaker versions, the finite Robinson property and the weak finite Robinson property are the topic of the next three subsections. In Chapter XIX the Robinson property is studied further.

Our main results here are: The finite Robinson property together with the pair preservation property implies that a logic is ultimately compact, and therefore has the finite dependence property, provided that there are no uncountable measurable cardinals. The Beth property together with the tree preservation property implies the weak finite Robinson property and the Robinson property together with the pair preservation property implies the existence of models with arbitrarily large automorphism groups. The last subsection discusses more examples, in particular a compact logic which satisfies the Beth property, the pair preservation property, but not the interpolation property.

Measurable cardinals play an important role in our presentation. They are in some sense $\mathscr{L}$-compact cardinals, which is to say, if such a cardinal μ exists then every finitely generated logic is, stationary often, weakly compact below μ. The first cardinal for which a logic is $[\kappa]$-compact is always measurable (or ω). But measurable cardinals, of which the first could conceivably be as big as the first strongly compact cardinal, also appear frequently in the hypotheses of various of our theorems. They also appear in various examples and counterexamples and sometimes their existence turns out to be equivalent to certain assumptions in abstract model theory.

In the same sense, it turns out, Vopenka's principle is a compactness axiom: It is equivalent to the statement that every finitely generated logic is ultimately compact or, alternatively, that every finitely generated logic has a global Hanf number. We have not centered our presentation around this theme, but the reader will easily discern it throughout the chapter.

Finally, a word on future research. Some of the possible directions of future research in abstract model theory are outlined in Chapters XIX and XX. The purpose there is to get *away* from the syntactic aspects of logic completely and to study classes of structures more in the spirit of universal algebra. If we want to stay in the framework of abstract model theory and logics I can see three directions in which to pursue further research.

The first direction is to study, what we have rather neglected in this chapter, the impact of various axiomatizability and dependence properties of logics on their respective model theory. We know that axiomatizability implies recursive compactness. But we do not know, for instance, if there are any model-theoretic properties distinguishing axiomatizable logics from logics axiomatizable by a finite set of axiom schemas. Only recently, in Shelah–Steinhorn [1982], it is shown that the logic $\mathscr{L}_{\omega\omega}(Q_{\beth_\omega})$ is an axiomatizable logic which cannot be axiomatized by schemas. This was the first example of its kind. Similarly, we know that $[\omega]$-compactness implies the finite dependence property (assuming there are no uncountable measurable cardinals), but we have not investigated if other model-theoretic properties, such as Lowenheim or Hanf numbers, have similar effects. The same holds for the finite dependence structure and dependence filters, as discussed in Section 2.4.

The second direction is the search for more model-theoretic properties which fit into the abstract framework. In Section 4.5 an attempt in this direction is presented: the existence of models with large automorphism groups. Incidentally, this also gives us a new proof for the case of first-order logic. In Shelah [198?e] a host of new notions occur in his study of Beth closures of logics preserving compactness and preservation properties. There is a danger here of proving theorems which apply only to first-order logic, such as compactness and chain properties imply the Robinson property. Since it is open whether there are logics satisfying both the Robinson property and the pair preservation property, the results in Section 4.5 should be taken with a grain of salt.

The third direction consists in incorporating the theory of second-order quantifiers, as presented in Chapter XII, into the study of the model-theoretic properties as presented in this chapter. What are the compact second-order quantifiers, what are the second-order quantifiers satisfying preservation and definability properties, etc? I am convinced that abstract model theory will remain a fruitful area of active research for many years to come.

We have not included detailed historical notes. Most of the results presented in this chapter are taken from my joint papers with S. Shelah and from his unpublished manuscript mentioned above. Some of the theorems and corollaries are stated here for the first time as a result of reflection upon the material presented. Results which appear here for the first time in print are marked with an asterisk. Whenever possible, we refer to the other chapters in the book rather than to original papers.

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1. Compact Logics

1.1. $[\kappa, \lambda]$*-compactness*

In this section we will study compactness properties of abstract logics. Traditionally, one looks at a set Σ of sentences of cardinality κ such that every subset $\Sigma_0 \subset \Sigma$ of cardinality less than λ has a model and concludes that Σ has a model. This is called (κ, λ)-compactness. By abuse of notation we write (∞, κ)-compactness instead of $(<\infty, \kappa)$-compactness. We call $(<\infty, \omega)$-compactness just compactness.

In contrast to this we look at two different situations:

(*) Given two sets of sentences Δ and Σ with $\mathrm{card}(\Sigma) = \kappa$, $\mathrm{card}(\Delta)$ arbitrary and such that for every subset $\Sigma_0 \subset \Sigma$ of cardinality less than λ $\Sigma_0 \cup \Delta$ has model. Then $\Sigma \cup \Delta$ has a model.

(**) Given a family Γ_α $(\alpha < \kappa)$ of sets of sentences such that for every set $X \subset \kappa$ of cardinality less than λ the union $\bigcup_{\alpha \in X} \Gamma_\alpha$ has a model. Then $\bigcup_{\alpha < \kappa} \Gamma_\alpha$ has a model.

1.1.1 Proposition. *For a regular logic $\mathscr{L}$ properties* (*) *and* (**) *are equivalent.*

Proof. (*) → (**) Let P_α ($\alpha < \mu$) be unary predicates not in $\bigcup_{\alpha<\kappa} \Gamma_\alpha$ and let ψ_α be the formula $\exists x\, P_\alpha(x)$. Now we put

$$\Delta = \{\psi_\alpha \to \varphi : \alpha < \kappa, \varphi \in \Gamma_\alpha\},$$

and

$$\Sigma = \{\psi_\alpha : \alpha < \kappa\}.$$

Clearly $\Delta \cup \Sigma_0$ has a model iff $\bigcup_{\psi_\alpha \in \Sigma_0} \Gamma_\alpha$ has a model.

(**) → (*) Let $\{\psi_\alpha : \alpha < \kappa\}$ be an enumeration of the formulas of Σ and put

$$\Gamma_\alpha = \Delta \cup \{\psi_\alpha\}. \quad \square$$

1.1.2 Remark. (*) was first systematically studied in Makowsky–Shelah [1979b] and in Makowsky–Shelah [1983]. (**) was introduced for topological spaces in Alexandroff–Urysohn [1929], as was pointed out to us by H. Mannila. (*) was called first *relative* (κ, λ)-*compact* and then $(\kappa, \lambda)^*$-compact. (**) is called in the topological literature $[\kappa, \lambda]$-*compact*.

The motivation behind (*) stems from working with elementary extensions and with diagrams. Δ usually plays the role of a diagram, and Σ describes the properties the extension should have. A similar situation occurs in Chang–Keisler [1973, Exercise 4.3.22].

1.1.3 Definition. A regular logic $\mathscr{L}$ with property (*) or (**) is called $[\kappa, \lambda]$-*compact*. If $\kappa = \lambda$ we simply write $[\kappa]$-compact.

1.1.4 Examples. (i) $\mathscr{L}(Q_{\omega_1})$ is (ω, ω)-compact but not $[\omega]$-compact.

(ii) (Bell–Slomson [1969, Theorem 2.2, p. 263]). If κ is small for λ, then $\mathscr{L}(Q_\lambda)$ is $[\kappa, \omega]$-compact. In particular, ω is small for $(2^\omega)^+$.

Recall that κ is small for λ if for every family μ_i ($i < \kappa$) such that $\mu_i < \lambda$ $\prod_i \mu_i < \lambda$.

1.1.5 Definition. We write $[\kappa, \lambda] \to [\mu, \nu]$ whenever $[\kappa, \lambda]$-compactness implies $[\mu, \nu]$-compactness. Similarly for conjunctions of compactness properties implying other such properties.

The following lemma collects some simple but useful facts:

1.1.6 Lemma. (i) $[\kappa, \lambda] \to [\mu, \lambda]$ *for* $\mu < \kappa$.

(ii) $[\kappa, \lambda] \to [\kappa, \nu]$ *for* $\nu > \lambda$.

(iii) $[\mu] \wedge [\kappa, \mu^+] \to [\kappa, \mu]$.

(iv) $[\kappa^+] \wedge [\kappa, \mu] \to [\kappa^+, \mu]$.

(v) *If* $[\beta]$ *and for every* $\alpha < \beta$, $[\kappa_\alpha]$ *and* $[\kappa_\alpha, \mu]$ *then* $[\sum_{\alpha<\beta} \kappa_\alpha, \mu]$.

(vi) $[\mathrm{cf}(\kappa)] \to [\kappa]$.

Proof. Trivial for (i) and (ii).

(iii), (iv) and (v) follow from definition (*).

(vi) follows from definition (**). $\square$

1.1.7 Proposition. (i) *A logic $\mathscr{L}$ is $[\kappa, \lambda]$-compact iff $\mathscr{L}$ is $[\mu]$-compact for every μ, $\lambda \leq \mu \leq \kappa$.*
(ii) *A logic $\mathscr{L}$ is $[\infty, \kappa]$-compact iff $\mathscr{L}$ is (∞, κ)-comact.*

Proof. For (i) we use Lemma 1.1.6 and (ii) follows from the definition. ☐

Mannila [1982, 1983] has investigated what results from topology give us refinements of Theorem 1.1.7. He showed that results from Alexandroff–Urysohn [1929] and Vaughan [1975] can be translated into our framework and we obtain

1.1.8 Proposition. (i) *A logic $\mathscr{L}$ is $[\kappa, \omega]$-compact iff $\mathscr{L}$ is $[\mu]$-compact for every regular μ, $\omega \leq \mu \leq \kappa$.*
(ii) *Assume* $\mathrm{cf}(\kappa) \geq \lambda$. *A logic $\mathscr{L}$ is $[\kappa, \lambda]$-compact iff $\mathscr{L}$ is $[\mu, \lambda]$-compact for every regular μ, $\lambda \leq \mu \leq \kappa$.*

Proposition 1.1.8 was first stated in Makowsky–Shelah [1983], where it was derived from Lemma 1.1.6.

Using the methods developed in Sections 1.3 and 1.4 this can be sharpened to:

1.1.9 Theorem. *Let λ be a cardinal and $\mathscr{L}$ a logic. The following are equivalent:*

(i) *$\mathscr{L}$ is $[\mu]$-compact for every regular $\mu \geq \lambda$.*
(ii) *$\mathscr{L}$ is $[\mu]$-compact for every $\mu \geq \lambda$.*
(iii) *$\mathscr{L}$ is $[\infty, \lambda]$-compact.*
(iv) *$\mathscr{L}$ is (∞, λ)-compact.*

Proof. (ii) implies (iii) by Proposition 1.1.7(i); (iii) is equivalent to (iv) by Proposition 1.1.7(ii) and (iii) implies (i) by Lemma 1.1.6(i) and (ii). So we have to prove that (i) implies (ii). Assume (i) and that λ is singular. So $\mathscr{L}$ is $[\lambda^+]$-compact. Now we use the abstract compactness theorem (1.3.9(ii)) which gives us a uniform ultrafilter F on λ^+. By Lemma 1.3.11(i) F is $[\lambda^+, \lambda]$-regular, so by Theorem 1.3.9(i) $\mathscr{L}$ is $[\lambda^+, \lambda]$-compact, and therefore $[\lambda]$-compact. ☐

We have put this proof here, though it uses material from Section 1.3, to illustrate the power of the abstract compactness theorem, which gives rise to various transfer results. We shall see more transfer results in Section 1.5.

We shall call logics $\mathscr{L}$ satisfying any of equivalent properties above *ultimately compact.*

1.2. Cofinal Extensions

One useful tool for the study of $[\kappa]$-compactness is its characterization via the non-characterizability of certain ordered structures. In Chapter II, Proposition 5.2.4 we have seen the paradigm of this procedure: A logic $\mathscr{L}$ is (∞, ω)-compact iff its well-ordering number is ω. Here the well-ordering is replaced by the cofinality of some linear order.

1.2.1 Definition. (i) Let $\mathfrak{A}$ be an expansion (possibly with new sorts) of the structure $\langle \kappa, < \rangle$ and $\mathfrak{B}$ and $\mathscr{L}$-extension of A. $\mathfrak{B}$ *extends* $\mathfrak{A}$ *beyond* κ if there is an element $b \in B \cap \mathrm{dom}(<^B)$ such that for every $a \in A \cap \mathrm{dom}(<^A)$ $B \models a < b$. If there is no such element, we call $\mathfrak{B}$ a *confinal extension of* $\mathfrak{A}$.

(ii) Let $\mathscr{L}$ be a logic and κ a regular cardinal. $\mathscr{L}$ *confinally characterizes* κ or κ is *cofinally characterizable in* $\mathscr{L}$ if there exists an expansion $\mathfrak{A}$ (possibly many-sorted with additional sorts) of the structure $\langle \kappa, < \rangle$ such that every $\mathscr{L}$-extension $\mathfrak{B}$ of $\mathfrak{A}$ is a cofinal extension of $\mathfrak{A}$. In this case we also say that $\mathscr{L}$ *cofinally characterizes* κ *via* $\mathfrak{A}$.

1.2.2 Theorem. *Let κ be a regular cardinal. A logic $\mathscr{L}$ is $[\kappa]$-compact iff κ is not cofinally characterizable in $\mathscr{L}$.*

Proof. Like in Chapter II, Proposition 5.2.4. □

Theorem 1.2.2 gives a quick proof of Lemma 1.1.6(vi). It can be used, together with a classical result due to Rabin and Keisler (Keisler [1964]) (cf. also Chang–Keisler [1973, Theorem 6.4.5]), to study the existence of $\mathscr{L}$-maximal structures.

Recall that a *complete structure* $\mathfrak{A}$ is a one-sorted structure where every subset $X \subset A^n$ is the interpretation of some relation symbol R_X. In the case of many-sorted structures we have to allow also relations with mixed arities.

1.2.3 Theorem (Rabin–Keisler). *Let $\mathfrak{A}$ be a complete structure of cardinality $\lambda <$ first uncountable measurable cardinal, P^A be a countable infinite predicate of $\mathfrak{A}$ and $\mathfrak{B}$ be a proper $\mathscr{L}_{\omega\omega}$-extension of $\mathfrak{A}$. Then $P^A \subsetneq P^B$.*

One can now easily prove from Theorems 1.2.2 and 1.2.3 a generalization of a result of Malitz–Reinhardt [1972b] and independently (Shelah [1967]):

1.2.4 Proposition. *If a logic $\mathscr{L}$ is not $[\omega]$-compact then there are arbitrarily large $\mathscr{L}$-maximal structures of cardinality less than the first uncountable measurable cardinal.*

Recall that a structure is $\mathscr{L}$-maximal if it has no proper $\mathscr{L}$-extensions. $\mathscr{L}$-extensions are further studied in Section 3.

The following observations will be useful later:

1.2.5 Lemma (Mundici). *Let κ, λ be regular cardinals and $\mathscr{L}$ a logic. Let $\mathfrak{A}_\kappa$, $\mathfrak{A}_\lambda$ be expansions of $\langle \kappa, < \rangle$, $\langle \lambda, < \rangle$ to τ-structures such that $\mathscr{L}$ cofinally characterizes κ, (λ) via $\mathfrak{A}_\kappa$, $\mathfrak{A}_\lambda$, respectively. Then there exists no $\mathscr{L}$-embedding of $\mathfrak{A}_\kappa$ into $\mathfrak{A}_\lambda$.*

The proof is left to the reader.

1.2.6 Proposition*. *Let $\mathscr{L}$ be a logic which is not ultimately compact. Then there is a proper class of τ-structures $\mathfrak{C}$ such that for no two $\mathfrak{A}, \mathfrak{B} \in \mathfrak{C}$ there is an $\mathscr{L}$-embedding from $\mathfrak{A}$ into $\mathfrak{B}$.*

Proof. If $\mathscr{L}$ is not ultimately compact, there is a proper class $\mathfrak{C}_0$ of regular cardinals κ for which $\mathscr{L}$ is not $[\kappa]$-compact (use Lemma 1.1.6(vi)). So by Theorem 1.2.2 each $\kappa \in \mathfrak{C}_0$ is cofinally characterizable in $\mathscr{L}$ via some $\mathfrak{A}_\kappa$. We can arrange it that each $\mathfrak{A}_\kappa$ is a τ-structure for some countable τ. For this we code many n-ary relation symbol by one $(n + 1)$-ary relation symbol and the use of constants. Now put $\mathfrak{C} = \{\mathfrak{A}_\kappa : \kappa \in \mathfrak{C}_0\}$. By Lemma 1.2.5 $\mathfrak{C}$ has the required property. □

1.3. Ultrafilters, Ultrapowers and Compactness

In first-order logic compactness is intimately related to the ultrapower construction. One can turn this observation easily into a characterization theorem for $\mathscr{L}_{\omega\omega}$.

1.3.1 Definition. Let $\mathscr{L}$ be a logic. $\mathscr{L}$ is said to have the *Los property* if for every τ-structure $\mathfrak{A}$ and every ultrafilter F and every formula $\varphi \in \mathscr{L}[\tau]$ the ultrapower $\prod \mathfrak{A}_i/F \models \varphi$ iff $\{i \in I : \mathfrak{A}_i \models \varphi\} \in F$.

1.3.2 Theorem. *Let $\mathscr{L}$ be a regular logic which has the Los property. Then $L \equiv \mathscr{L}_{\omega\omega}$.*

Proof. By coding a family of structures in one structure and using the Keisler–Shelah theorem, that elementarily equivalent structures have isomorphic ultrapowers, the proof is straightforward. □

1.3.3 Remark. Theorem 1.3.2 was folklore already around 1972. A detailed version may be found in Sgro [1977] and Monk [1976, Exercise 25.53]. Sgro [1977] contains interesting additional material concerning maximal logics.

To study compactness for abstract logics we need a generalization of the Los property.

1.3.4 Definitions. (i) Let $\mathscr{L}$ be a logic and F be an ultrafilter over I. We say that F *relates to* $\mathscr{L}$ if for every τ and for every τ-structure $\mathfrak{A}$ there exists a τ-structure $\mathfrak{B}$ extending $\prod_I \mathfrak{A}/F$ such that for every formula $\varphi \in \mathscr{L}[\tau]$, $\varphi = \varphi(x_1, x_2, \ldots, x_i, \ldots)$, $i < \alpha$ with α many free variables and every $f_i \in A^I$, $i < \alpha$ we have:

$$\mathfrak{B} \models \varphi(f_1/F, f_2/F, \ldots, f_i/F, \ldots)$$

iff

$$\{j \in I : \mathfrak{A} \models \varphi(f_1(j), f_2(j), \ldots, f_i(j), \ldots)\} \in F.$$

(ii) We define UF($\mathscr{L}$) to be the class of ultrafilters F which are related to $\mathscr{L}$.

1.3.5 Remark. Note that $\mathfrak{B}$ is always an elementary extension of $\prod_I \mathfrak{A}/F$.

1.3.6 Examples. (i) Every ultrafilter is in $\mathrm{UF}(\mathscr{L}_{\omega\omega})$.

(ii) Let $\mathscr{L}$ be $\mathscr{L}_{\omega\omega}(Q_\kappa)$, i.e., first-order logic with the additional quantifier "there exist at least κ many." Then every ultrafilter on ω is related to $\mathscr{L}$, provided ω is small for κ.

1.3.7 Proposition. *$\mathscr{L}$ is compact iff every ultrafilter is related to $\mathscr{L}$.*

Proof. Let $\mathfrak{M}$ be a τ-structure and F an ultrafilter on a set I. For every $f \in M^I$ let c_f be a new constant symbol not in τ. Put

$$T = \{\varphi(c_{f_1}, c_{f_2}, \ldots): \varphi \in \mathscr{L}[\tau] \quad \text{and} \quad \{t \in I: \mathfrak{M} \models \varphi(f_1(t), f_2(t), \ldots)\} \in F\}.$$

If $\mathscr{L}(\tau)$ is a set, so is T and obviously every finite subset of T has a model: We just expand $\mathfrak{M}$ appropriately. So let $\mathfrak{N}$ be a model of T. Clearly

$$\prod_I \mathfrak{M}^I/F \subset \mathfrak{N},$$

and by the definition of T, $\mathfrak{N}$ satisfies the requirements for $F \in \mathrm{UF}(\mathscr{L})$. In the case $\mathscr{L}(\tau)$ is a proper class, we have to take a subclass $T_0 \subset T$ which is a set and still guarantees that

$$\prod_I \mathfrak{M}^I/F \subset \mathfrak{N},$$

and that $\mathfrak{N}$ satisfies the requirements for $F \in \mathrm{UF}(\mathscr{L})$. For this we observe that over the structure $\mathfrak{M}^I$ there are only set many inequivalent formulas with less than $\mathrm{card}(\mathfrak{M}^I)^+$-many free variables.

The converse is trivial. □

The next theorem connects the compactness spectrum $\mathrm{Comp}(\mathscr{L})$ with the filters in $\mathrm{UF}(\mathscr{L})$. To be more explicit, we need some more definitions.

1.3.8 Definitions. Let F be an ultrafilter on I, and λ, μ be cardinals with $\lambda \geq \mu$.

(i) F is said to be (λ, μ)-regular if there is a family $\{X_\alpha: \alpha < \lambda\}$, $X_\alpha \in F$ such that if $\{\alpha_i < \lambda: i < \mu\}$ is any enumeration of μ ordinals less than λ, then $\bigcap_{i<\mu} X_{\alpha_i} = \varnothing$. The family $\{X_\alpha, \alpha < \lambda\}$ is called a (λ, μ)-regular family.

(ii) A (λ, ω)-regular ultrafilter on λ is called regular.

(iii) F is λ-descendingly incomplete if there exists a family $\{X_\alpha: \alpha < \lambda\}$, $X_\alpha \in F$ with $X_\beta \subset X_\alpha$ for $\alpha < \beta < \lambda$ such that $\bigcap_{\alpha<\lambda} X_\alpha = \varnothing$.

(iv) F is uniform on λ if every $X \in F$ has cardinality λ.

1.3.9 Theorem (Abstract Compactness Theorem). *Let λ, μ be cardinals, $\lambda \geq \mu$, and let $\mathscr{L}$ be a logic.*

(i) *$\mathscr{L}$ is $[\lambda, \mu]$-compact iff there is a (λ, μ)-regular ultrafilter F on $I = P_{<\mu}(\lambda)$ in* $\mathrm{UF}(\mathscr{L})$.

(ii) *If $\lambda = \mu$ and μ regular, then $\mathscr{L}$ is $[\lambda]$-compact iff there is a uniform ultrafilter F on λ in* $\mathrm{UF}(\mathscr{L})$.

The proof of this theorem is delayed to Section 1.4.

Theorem 1.3.9 allows us to use known results from the theory of ultrafilters to understand $[\lambda, \mu]$-compactness. The following lemma collects some simple results from (but not due to) Comfort–Negrepontis [1974].

1.3.10 Lemma. (i) *If F is (λ, μ)-regular and $\mu \leq \mu_1 \leq \lambda_1 \leq \lambda$ then F is (λ_1, μ_1)-regular.*

(ii) *If λ is a regular cardinal and F is λ-descendingly incomplete, then F is (λ, λ)-regular.*

(iii) *If E is uniform on λ then F is (λ, λ)-regular.*

(iv) *If F is $(\mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$-regular then F is (λ, λ)-regular.*

The abstract compactness theorem and Lemma 1.3.10 give us immediately the corresponding statements in Lemma 1.1.6.

The next lemma collects some more sophisticated theorems from the literature on ultrafilters. For Lemma 1.3.11(ii) one may also consult Comfort–Negrepontis [1974, Theorem 8.36].

1.3.11 Lemma. (i) (Kanamori [1976]). *If F is uniform on λ^+ and λ is singular, then F is (λ^+, λ)-regular.*

(ii) (Kunen–Prikry [1971]; Cudnovskii–Cudnovskii [1971]). *If F is uniform on λ^+ and λ is regular, then F is λ-descendingly incomplete, and hence (λ, λ)-regular.*

This lemma, together with the abstract compactness theorem, is the key to the study of the compactness spectrum in Sections 1.5 and 1.6. It is also used in the proof of Theorem 1.1.9.

1.4. Proof of the Abstract Compactness Theorem

Before we prove the abstract compactness theorem we shall give a model-theoretic characterization of (λ, μ)-regular ultrafilters which will give us the link between $[\lambda, \mu]$-compactness and the existence of (λ, μ)-regular ultrafilters. This is implicitly in Keisler [1967b] (cf. also Comfort–Negrepontis [1974, Theorem 13.6]).

Let $H(\lambda)$ denote the set of sets hereditarily of cardinality $< \lambda$ and let $\mathfrak{H}(\lambda)$ be the structure $\langle H(\lambda), \in \rangle$ where $\in$ is the natural membership relation on $H(\lambda)$.

1.4.1 Lemma (Keisler). *For an ultrafilter F on a set I the following are equivalent:*

(i) *F is (λ, μ)-regular.*

(ii) *In the structure $\mathfrak{N} = \prod_I \mathfrak{H}(\lambda^+)/F$ there is an element $\mathfrak{b} = b/F$ where $b: I \to H(\lambda^+)$ is a function, such that $\mathfrak{N} \models \mathfrak{b} \subset \lambda^N$ and $\mathfrak{N} \models \mathrm{card}(\mathfrak{b}) < \mu^N$ but for every $\alpha < \lambda$ $\mathfrak{N} \models \alpha^N \in \mathfrak{b}$.*

Recall that for an ordinal $\alpha \leq \lambda$, α^N denotes the image of α under the natural embedding into $\mathfrak{N}$.

Proof. (i) → (ii) Define $b: I \to H(\lambda^+)$ by $b(t) = \{\alpha \in \lambda: t \in X_\alpha\}$ for $t \in I$ and $\{X_\alpha: \alpha \in \lambda\}$ a (λ, μ)-regular family. Now $X_\alpha = \{t \in I: \alpha \in b(t)\}$ so $\mathfrak{N} \models \alpha^N \in \mathfrak{b}$, since for each $\alpha \in \lambda$, $X_\alpha \in F$. But clearly, $b(t)$ has cardinality $< \mu$ for each $t \in I$, since $\{X_\alpha: \alpha \in \lambda\}$ is a (λ, μ)-regular family, so $\mathfrak{N} \models \text{card}(b) < \mu$. Trivially, we have also $\mathfrak{N} \models b \subset \lambda^N$.

(ii) → (i) Let $\mathfrak{b} = b/F$ be the required element in $\mathfrak{N}$. Define $\mathfrak{b}'$ by $b'(t) = b(t)$ if $b(t) \subset \lambda$ and $\text{card}(b(t)) < \mu$ and $b'(t) = \varnothing$ otherwise.

Obviously $\mathfrak{b}/F = \mathfrak{b}'/F$ since $\mathfrak{N} \models \mathfrak{b} \subset \lambda^N$. We want to construct a (λ, μ)-regular family. Put $X_\alpha = \{t \in I: \alpha \in b'(t)\}$ for each $\alpha \in \lambda$. Now suppose that for some $\{\alpha_i: i \in \mu\}$ the intersection $\bigcap_{i \in \mu} X_{\alpha_i} \neq \varnothing$. So there is a $t \in I$ such that for each $i \in \mu$, $\alpha_i \in b'(t)$, which contradicts the fact that $\text{card}(b'(t) < \mu$. □

1.4.2 Definition. Let F_i be ultrafilters on I_i $(i = 1, 2)$. *F_2 is a projection of F_1* if there is a map $f: I_1 \to I_2$ which is onto and such that $F_1 = \{f^{-1}(X): X \in F_2\}$.

Projections are closely related to the *Rudin–Keisler order* on ultrafilters over a fixed set I, cf. Comfort–Negrepontis [1974]. We use now Lemma 1.4.1 together with *complete expansions* (i.e., complete structures over their original universe, cf. Section 1.2), to get:

1.4.3 Lemma. *If λ is regular and F_1 is (λ, λ)-regular ultrafilter on I then there is a uniform ultrafilter F_2 on λ which is a projection of F_1.*

Proof. Let $\mathfrak{N}^\#$ be the complete expansion of $\mathfrak{N} = \prod_I \mathfrak{H}(\lambda^+)$ and $b: I \to H(\lambda^+)$ as in Lemma 1.4.1 and without loss of generality $b(t) \subset \lambda$ for all $t \in I$. Now put $c(t) = \sup(b(t))$ so $c(t) \in \lambda$ since λ is regular, and $\mathfrak{N} \models \mathfrak{b} \subset \mathfrak{c}$. Clearly $c: I \to \lambda$. We define now F_2 by $F_2 = \{S \subset \lambda: \mathfrak{N}^\# = \mathfrak{c} \in \mathrm{S}\}$ where S is the name of S in $\mathfrak{N}^\#$. It is now easy to verify that F_2 is a uniform ultrafilter on λ which is a projection of F_1. □

To prove the abstract compactness theorem we shall prove a slightly more elaborate statement:

1.4.4 Theorem (Abstract Compactness Theorem). *Let $\mathscr{L}$ be a logic, λ, μ be cardinals and $\lambda \geq \mu$.*

(i) *The following are equivalent:*

(a) *There is (λ, μ)-regular ultrafilter F on $I = P_{<\mu}(\lambda)$ which is in* $\text{UF}(\mathscr{L})$.
(b) *For every (relativized) expansion $\mathfrak{A}$ of $\mathfrak{H}(\lambda^+)$ there is an $\mathscr{L}$-extension $\mathfrak{B}$ and an element $b \in B$ such that $\mathfrak{B} \models \text{card}(b) < \mu^B$ but for every $\alpha < \lambda$ we have $\mathfrak{B} \models \alpha^B \in b$.*
(c) *$\mathscr{L}$ is $[\lambda, \mu]$-compact.*

(ii) *Furthermore, if λ is regular then the following are equivalent:*

(d) *There is a uniform ultrafilter F on λ which is in* $\text{UF}(\mathscr{L})$.
(e) *$\mathscr{L}$ is $[\lambda]$-compact.*

(iii) In particular, we have:

(f) If there is a (λ, μ)-regular ultrafilter F on any set I which is in $\text{UF}(\mathscr{L})$, then $\mathscr{L}$ is $[\lambda, \mu]$-compact.

Proof. (a) → (b) Let F be a (λ, μ)-regular ultrafilter in $\mathrm{UF}(\mathscr{L})$ and let $\mathfrak{M}$ be any expansion of $\langle H(\lambda^+), \in\rangle$. Put $\mathfrak{N}_0$ to be the ultrapower $\prod_I \mathfrak{M}/F$ and $\mathfrak{N}_1$ the extension of $\mathfrak{N}_0$ as required for $F \in \mathrm{UF}(\mathscr{L})$. First we observe that $\mathfrak{N}_0 < \mathfrak{N}_1$ $(\mathscr{L}_{\omega\omega})$ and, by Lemma 1.4.1 there is an element b in $\mathfrak{N}_0$ with the required properties. But then the same element b has the same properties also in $\mathfrak{N}_1$ since $\mathfrak{N}_0 < \mathfrak{N}_1(\mathscr{L}_{\omega\omega})$. But by the definition of $\mathfrak{N}_1$, $\mathfrak{M} < \mathfrak{N}_1(\mathscr{L})$, so we are done.

(b) → (c) Let Δ, Σ be sets of $\mathscr{L}[\tau]$-sentences satisfying the hypothesis of $[\lambda, \mu]$-compactness. We define an expansion $\mathfrak{M}(\Delta, \Sigma)$ of $\langle H(\lambda^+), \in\rangle$ to apply (b). For this purpose let $\{S_\alpha : \alpha < \lambda^{<\mu}\}$ be an enumeration of all the subsets of Σ of cardinality less than μ, $\mathfrak{A}_\alpha$ be a model of $\Delta \cup S_\alpha$ and $\{c_\alpha : \alpha < P_{<\mu}(\lambda)\}$ an enumeration of all the subsets of λ of cardinality less than μ. Finally we put $v = (\sup_\alpha(\mathrm{card}(\mathfrak{A}_\alpha)) + \lambda^+)$, and define $\lambda_\alpha = \mathrm{card}(\mathfrak{A}_\alpha)$. We now define $\mathfrak{M}(\Delta, \Sigma)$ to be $\langle H(v), d_\alpha, \in, R, P\rangle_{\alpha<\lambda^+, P\in\tau}$ such that d_α is the name of $\alpha < \lambda^+$, R is a binary predicate not in τ and the domain of R is λ. We arrange it such that for each $\alpha < \lambda$ the set $R_\alpha = \{x \in H(v) : (\alpha, x) \in R\}$ has cardinality λ_α and such that $\langle R_\alpha, P\rangle_{P\in\tau} \cong \mathfrak{A}_\alpha$. In other words we put all the models $\mathfrak{A}_\alpha$ into $\mathfrak{M}(\Delta, \Sigma)$ in way, that when we now apply (b) we shall get a model for $\Delta \cup \Sigma$. More precisely, we observe that for each formula $\phi \in \Delta$:

(1) $$\mathfrak{M}(\Delta, \Sigma) \models \mathrm{card}(b) < d_\mu \to \varphi^{R_b}$$

and for each $\beta < \lambda$ and for $\Sigma = \{\varphi_i : i < \lambda\}$ an enumeration of Σ we have

(2) $$\mathfrak{M}(\Delta, \Sigma) \models (d_\beta \in c \wedge \mathrm{card}(c) < d_\mu) \to \varphi_\beta^{R_b}.$$

Now let $\mathfrak{B}$, $b \in B$ be as in the conclusion of (b) for $\mathfrak{A} = \prod \mathfrak{M}(\Delta, \Sigma)/F$.

Claim. $\langle R_b, P\rangle_{P\in\tau} \models \Delta \cup \Sigma$.

This follows from the definition and from (1) and (2).

(c) → (a): So assume $\mathscr{L}$ is $[\lambda, \mu]$-compact but no (λ, μ)-regular ultrafilter F on $P_{<\mu}(\lambda)$ is related to $\mathscr{L}$. So for every such F there is an $\mathscr{L}_F$-structure $\mathfrak{A}_F$ exemplifying this.

We now proceed to construct an ultrafilter F_0 on λ which contradicts the choice of the $\mathfrak{A}_F$'s. For this we construct first a rich enough structure $\mathfrak{M}$ such that:

(1) for each $\mathfrak{A}_F$ there is a unary predicate P_F in $\mathfrak{M}$ with $\langle P_F, P\rangle_{P\in\tau} \cong \mathfrak{A}_F$;
(2) $\mathfrak{M}$ is a model of enough set theory to carry out the argument; and
(3) $\mathfrak{M}$ is an extension and expansion of $\langle H(\lambda^+), \in\rangle$ (or equivalently $\langle H(\lambda^+), \in\rangle$ is a relativized reduct of $\mathfrak{M}$).

Let $\mathfrak{M}^\#$ be the complete expansion of $\mathfrak{M}$ and put $\Delta = \mathrm{Th}_{\mathscr{L}}(\mathfrak{M}^\#)$, the first-order theory of $\mathfrak{M}^\#$ where $\mathscr{L}^\#$ is the vocabulary of $\mathfrak{M}^\#$. Furthermore, put

$$\Sigma = \{b \subset d_\lambda \wedge \mathrm{card}(b) < d_\mu \wedge d \ \in b : \alpha < \lambda\}.$$

Clearly Δ and Σ satisfy the hypothesis of $[\lambda, \mu]$-compactness using the model $\mathfrak{M}^\#$. So $\Delta \cup \Sigma$ has a model $\mathfrak{N}$. We want to use $\mathfrak{N}$ to construct our filter F_0. First we observe that $\mathfrak{M}^\# <_{\mathscr{L}} \mathfrak{N}$. Let a_b be the interpretation of b in $\mathfrak{N}$. We define F_0 on

$P_{<\mu}(\lambda)$ by $F_0 = \{R \in P_{<\mu}(\lambda): \mathfrak{N} \models a_b \in R^{\mathfrak{N}}\}$. This makes sense, since $\mathfrak{M}^{\#}$ is a complete expansion and hence every subset of λ of cardinality $<\mu$ corresponds to a predicate in $\mathfrak{M}^{\#}$ (remember $\langle H(\lambda^+), \in\rangle$ is present in $\mathfrak{M}^{\#}$).

To complete the proof we have to verify several claims:

Claim 1. *F_0 is ultrafilter.*

Obvious.

Claim 2. *F_0 is (λ, μ)-regular.*

Let $X_\alpha = \{t \in P_{<\mu}(\lambda): \alpha \in t\}_{\alpha<\lambda}$. Now $X_\alpha \in F_0$, for say X_α corresponds to R_α then $\mathfrak{N} \models a_b \in R_\alpha$ iff $\mathfrak{N} \models d_\alpha \in a_b$, which is true for all $\alpha < \lambda$ be definition of a_b. Now $\{X_{\alpha_i}: i < \mu\}$ be a subfamily of the X_α's. Clearly, $\bigcap_{i<\mu} X_{\alpha_i} = \emptyset$, since each t in some X_α has cardinality $<\mu$.

Now consider the ultraproduct $\prod \mathfrak{M}^{\#}/F_0 = \mathfrak{N}_0$. If g is an element of $\mathfrak{N}_0$ then g is an F_0-equivalence class of functions $g: P_{<\mu}(\lambda) \to \mathfrak{M}^{\#}$ so g corresponds to a function $\mathfrak{g}^{\mathfrak{M}}$ in $\mathfrak{M}^{\#}$ with name $\mathfrak{g}$ (since $\mathfrak{M}^{\#}$ is the complete expansion) and $a_b \in \mathrm{Dom}(\mathfrak{g}^{\mathfrak{N}})$. So we define an embedding $f: \mathfrak{N}_0 \to \mathfrak{N}$ by $f(g/F_0) = \mathfrak{g}^{\mathfrak{N}}(a_b)$.

Claim 3. *f is well defined and* 1–1.

Let $g/F_0 = g'/F_0$. We want to show that this is equivalent to $\mathfrak{N} \models \mathfrak{g}(a_c) = \mathfrak{g}'(a_c)$ iff $Y = \{t \in P_{<\mu}: g(t) = g'(t)\} \in F_0$. But the latter is true iff $a_b \in Y^{\mathfrak{N}}$ which is equivalent to $\mathfrak{g}(a_b) = \mathfrak{g}'(a_b)$.

So we have shown that f is an embedding of $\mathfrak{N}_0$ into $\mathfrak{N}$.

Now let $\bar{g} = \{g_i/F_0: i < \alpha\}$ be in $\mathfrak{N}_0$.

Claim 4. *For every $\mathscr{L}$-formula φ we have*

$$\mathfrak{N} \models \varphi(\bar{g}) \quad \textit{iff} \quad Y = \{t \in P_{<\mu}: \mathfrak{M} \models \varphi(g_1(t), g_2(t), \ldots)\} \in F_0.$$

Clear, since $Y \in F_0$ iff $Y^{\mathfrak{N}}$ contains a_c iff $\mathfrak{N} \models \phi(\mathfrak{g}_1(a_c), \mathfrak{g}_2(a_c), \ldots)$.

Now look at $\mathfrak{A}_{F_0}$. By assumption there is no $\mathfrak{N}'$ extending $\prod \mathfrak{A}_{F_0}/F_0$ satisfying Claim 4. But $\langle P_{F_0}^{\mathfrak{N}}, P\rangle_{P \in \tau_{F_0}}$ is such an $\mathfrak{N}'$ by construction. This completes the proof of (i).

(d) $\to$ (e) This follows from the above, since uniform ultrafilters on λ are $(\mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$-regular and λ is a regular cardinal by our hypothesis.

(e) $\to$ (d) Here we use Lemma 1.4.3 and (a) $\to$ (c). This completes the proof of (ii).

To prove (f) we just observe that in the proof of (a) $\to$ (c) we did not use that $I = P_{<\mu}(\lambda)$. This completes the proof of Theorem 1.3.9. □

1.5. The Compactness Spectrum

In this section we study the structure of the *compactness spectrum* $\mathrm{Comp}(\mathscr{L})$ and the *regular compactness spectrum* $\mathrm{RComp}(\mathscr{L})$ defined below.

1.5.1 Definition. For a logic $\mathscr{L}$ we define Comp($\mathscr{L}$), (RComp($\mathscr{L}$)) to be the class of all (regular) cardinals such that $\mathscr{L}$ is $[\kappa]$-compact.

1.5.2 Theorem. *The first cardinal* λ_0 *in* Comp($\mathscr{L}$) *is measurable* (*or* ω).

Proof. By Theorem 1.2.2(i) each regular $\lambda < \lambda_0$ is cofinally characterizable in $\mathscr{L}$ via a structure $\mathfrak{B}(\lambda)$ with κ_λ the cardinality of $\mathfrak{B}(\lambda)$. Let μ be defined by

$$\mu = (\sup\{\kappa_\lambda : \lambda < \lambda_0\}) + \lambda_0^+$$

and let $\mathfrak{B}$ be the complete expansion of the structure $\langle \mu, \varepsilon \rangle$. Therefore (*) in every $\mathscr{L}$-extension of $\mathfrak{B}$ all the ordinals smaller than λ_0 are standard. By $[\lambda_0]$-compactness $\mathfrak{B}$ has an $\mathscr{L}$-elementary extension $\mathfrak{C}$ with some $c \in C - B$ and such that $\mathfrak{C} \models c \in \lambda_0^B$. Since λ_0 is minimal we have for no $\lambda < \lambda_0$ that $\mathfrak{C} \models c \in \lambda^C$. We now define an ultrafilter F on λ_0 by

$$F = \{X \subset \lambda_0 : \mathfrak{C} \models c \in \mathbf{X}\},$$

where $\mathbf{X}$ is the name of the set X in $\mathfrak{B}$. Clearly F is an ultrafilter. We propose to show that F is λ_0-complete.

Let $\{X_\alpha : \alpha < \mu < \lambda_0\}$ be any family in F. The function f with $f(\alpha) = X_\alpha$ is a function in $\mathfrak{B}$ with name, say, $\mathbf{f}$. Put now $X = \bigcap_{\alpha<\mu} X_\alpha$. So $\mathfrak{B} \models \mathbf{X} = \bigcap_{\alpha<\mu} \mathbf{X}_\alpha$ and therefore

$$\mathfrak{B} \models \forall x(\forall i(i < \alpha \rightarrow x \in \mathbf{f}(i)) \rightarrow x \in \textstyle\bigcap_{i<\alpha} \mathbf{f}(i)).$$

But by (*) the ordinals $\alpha < \lambda_0$ in $\mathfrak{B}$ are the same as in $\mathfrak{C}$. So $\mathfrak{C} \models c \in \mathbf{X}$ since $\mathbf{f}$ is a function of $\mathfrak{C}$ with $\mathbf{f}^C \upharpoonright B = \mathbf{f}^B$. So $X \in F$ and therefore λ_0 is measurable. □

1.5.3 Example. If κ is a strongly compact cardinal, the logic $\mathscr{L}_{\kappa\kappa}$ is (∞, κ)-compact and therefore $[\kappa]$-compact. But the logic $\mathscr{L}_{\kappa\kappa}$ is not $[\lambda]$-compact for any $\lambda < \kappa$.

Note that, as a corollary, we get that strongly compact cardinals are measurable. By Magidor [1976] it is consistent that the first measurable and the first strongly compact cardinal coincide.

Our next aim is to study the structure of Comp($\mathscr{L}$). The main theorem here is

1.5.4 Theorem. *For every cardinal* λ *and every logic* $\mathscr{L}$, $\lambda^+ \in$ Comp($\mathscr{L}$) *implies* $\lambda \in$ Comp($\mathscr{L}$).

Proof. Use the abstract compactness theorem 1.4.4 and Lemma 1.3.11. □

For λ regular this was first proved in Makowsky–Shelah [1979] giving a direct proof by relating $[\lambda]$-compactness to descendingly incomplete ultrafilters. The general result was proved in Makowsky–Shelah [1983]. There the connection with ultrafilters was first recognized, on which the presentation here is based.

The next result concerns the structure of Comp($\mathscr{L}$). The following was proven in Makowsky–Shelah [1979, Lemma 6.4(ii)] by an extension of the argument for Theorem 1.5.2.

1.5.5 Lemma. *Let $\lambda > \mu$ be two regular cardinals and $\mathscr{L}$ be a logic such that $\lambda \in \mathrm{Comp}(\mathscr{L})$ but $\mu \notin \mathrm{Comp}(\mathscr{L})$. Then there is a uniform μ-descendingly complete ultrafilter on λ.*

Consider the following assumption $A(\lambda)$, where λ is an uncountable cardinal.

> $A(\lambda)$: "if $\mathfrak{F}$ is a uniform ultrafilter on λ, then $\mathfrak{F}$ is μ-descendingly incomplete for every $\mu \leq \lambda$."
> We denote by $A(\infty)$ the statement "for every infinite cardinal λ, $A(\lambda)$ holds."

Donder–Jensen–Koppelberg [1981] and Magidor [198?] have studied this assumption. The following theorem summarizes their results (with part (v) being Theorem 8.36 in Comfort–Negrepontis [1974], see also Lemma 1.3.11).

1.5.6 Theorem. (i) (Jensen–Koppelberg). *Assume $\neg O^{\#}$. Then for every regular cardinal λ we have $A(\lambda)$.*

(ii) (Donder). *Assume there is no inner model of* ZFC *with an uncountable measurable cardinal. Then $A(\infty)$ holds.*

(iii) *If $A(\infty)$ holds then there are no uncountable measurable cardinals.*

(iv)* (Woodin). *Assume there are uncountable measurable cardinals. Then it is consistent with* ZFC *that $A(\omega_\omega)$ fails.*
However, in ZFC *we already have*:

(v) (Kunen–Prikry and Cudnovskii–Cudnovskii). *For every $n \in \omega$, $A(\omega_n)$ holds.*

Magidor has informed us of the yet unpublished result of Theorem 1.5.6(iv) of Woodin. He had previously proved a similar result, where one has to replace the existence of an uncountable measurable cardinal in the hypothesis by the existence of a supercompact cardinal.

The assumption $A(\infty)$ is intimately connected with compactness properties: It implies that Comp($\mathscr{L}$) has no gaps. On the other hand, the existence of strongly compact cardinals allows us to construct logics where Comp($\mathscr{L}$) does have gaps. More precisely:

1.5.7 Theorem. (i) *Assume $A(\infty)$ holds. Then* Comp($\mathscr{L}$) *is an initial segment of the cardinals, i.e., $\lambda \in \mathrm{Comp}(\mathscr{L})$ and $\mu < \lambda$ implies that $\mu \in \mathrm{Comp}(\mathscr{L})$.*

(ii)* (Shelah). *Let $\mu_1 < \mu_2$ be two uncountable strongly compact cardinals. Then there is a logic $\mathscr{L}$ which is $[\kappa]$-compact iff $\kappa < \mu_1$ or $\kappa \geq \mu_2$.*

Proof. (i) Assume Comp(L) $\neq \varnothing$. Since $A(\infty)$ implies that there are no uncountable measurable cardinals, by Theorem 1.5.6(iii), the first cardinal in Comp($\mathscr{L}$) is

ω, by Theorem 1.5.2. Now, if $\omega < \lambda \in \mathrm{Comp}(\mathscr{L})$ and $\omega < \mu < \lambda$, $\mu \notin \mathrm{Comp}(\mathscr{L})$, μ regular, we apply Lemma 1.5.5 and get a contradiction to $A(\infty)$. If μ is singular, we apply Lemma 1.5.5 to $\mathrm{cf}(\mu)$ and then use Lemma 1.1.6(v).

(ii) will follow from Proposition 1.6.7. ☐

The question which remains, is whether $\mathrm{Comp}(\mathscr{L})$ is empty or not. Now clearly the logic $\mathscr{L}_{\infty\omega}$ is not compact in any sense, so $\mathrm{Comp}(\mathscr{L}_{\infty\omega})$ is empty. But if we assume that the logic $\mathscr{L}$ is bounded in some sense and have some very strong assumption on the existence of large cardinals we can get more specific results. For terminology and results on large cardinals we refer to Jech [1978].

1.5.8 Definition. A logic is *set presentable* if:

(i) there is a cardinal κ such that whenever a vocabulary $\tau \in H(\kappa)$ and $\Sigma \subset \mathscr{L}[\tau]$ has cardinality $<\kappa$ then $\Sigma \subset H(\kappa)$; and
(ii) for every $\varphi \in \mathscr{L}[\tau]$ $\mathrm{Mod}(\varphi)$ is a set-theoretically definable class of τ-structures.

(Recall that $H(\kappa)$ is the family of sets hereditarily of cardinality $<\kappa$.)

1.5.9 Example. Let $\mathscr{L} = \mathscr{L}^n_k$ be like nth-order logic except that we allow conjunctions and disjunctions of less than κ many formulas. Clearly $\mathscr{L}$ is set presentable and so is every sublogic of it.

1.5.10 Definition. Let $\mathrm{SComp}(\mathscr{L})$ be the class of cardinals κ such that $\mathscr{L}$ is (∞, κ)-compact and $\mathrm{WComp}(\mathscr{L})$ be the class of cardinals κ such that $\mathscr{L}$ is (κ, κ)-compact. Clearly we have $\mathrm{SComp}(\mathscr{L}) \subset \mathrm{Comp}(\mathscr{L}) \subset \mathrm{WComp}(\mathscr{L})$.

1.5.11 Proposition (Magidor [1971]). *If κ is an extendible cardinal then $\kappa \in \mathrm{SComp}(\mathscr{L}^n_k)$.*

1.5.12 Definition. The following statement is called *Vopenka's principle*:
Let $\mathbf{C}$ be a proper class of τ-structures for some finite vocabulary τ. Then there are two structures $\mathfrak{A}, \mathfrak{B} \in \mathbf{C}$ such that $\mathfrak{A}$ is (first-order) elementary embeddable into $\mathfrak{B}$.

Now Magidor [1971] also shows

1.5.13 Proposition. *If Vopenka's principle holds then the class of all extendible cardinals is closed unbounded.*

So Propositions 1.5.11 and 1.5.13 give us immediately:

1.5.14 Theorem (Magidor–Stavi). *Assume Vopenka's principle holds and that $\mathscr{L}$ is a set presentable logic. Then* $\mathrm{SComp}(\mathscr{L})$ *is a non-empty final segment of the cardinals (in other words, $\mathscr{L}$ is ultimately compact).*

For $\mathrm{WComp}(\mathscr{L})$ we do not need Vopenka's principle to prove an analogue of Theorem 1.5.14.

1.5.15 Theorem (Stavi [1978]). *Let μ be a uncountable measurable cardinal and F be a normal ultrafilter on μ and $\mathscr{L}$ be a sublogic of $\mathscr{L}^n_\mu$. Then* WComp($\mathscr{L}$) $\cap \mu \in F$.

Theorem 1.5.15 holds under much weaker assumptions (cf. Stavi [1978, Section 5]) and is also discussed and proved in Chapter XVII, Section 4.2.

The structure of Comp($\mathscr{L}$) definitely deserves further investigation. We combine the content of Lemma 1.1.6(v), and Theorems 1.5.2, 1.5.4, and 1.5.14 into the statement:

1.5.16 Theorem. *For a logic $\mathscr{L}$ we have*:

(i) $\mathrm{cf}(\kappa) \in \mathrm{Comp}(\mathscr{L}) \to \kappa \in \mathrm{Comp}(\mathscr{L})$.
(ii) $\kappa^+ \in \mathrm{Comp}(\mathscr{L}) \to \kappa \in \mathrm{Comp}(\mathscr{L})$.
(iii) *The first cardinal in* Comp($\mathscr{L}$) *is measurable* (*or* ω).
(iv) *If $\mathscr{L}$ is set presentable and Vopenka's principle holds, then* Comp($\mathscr{L}$) *contains a final segment of the class of all cardinals.*

Our last theorem illustrates that Vopenka's principle is the right large cardinal assumption in this context.

1.5.17 Theorem* (Makowsky). *The following are equivalent*:

(i) *Vopenka's principle.*
(ii) *For every logic $\mathscr{L}$* SComp($\mathscr{L}$) $\neq \varnothing$.
(iii) *For every finitely generated logic $\mathscr{L}$* SComp($\mathscr{L}$) $\neq \varnothing$.
(iv) *For every finitely generated logic $\mathscr{L}$* Comp($\mathscr{L}$) $\neq \varnothing$.

Proof. (i) → (ii) follows from Proposition 1.2.6. So we only have to prove (iv) → (i). Let **C** be a proper class of τ-structures and let Q_C be the Lindstrom quantifier defined by **C** and $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_C)$. Clearly **C** contains a proper subclass $\mathbf{C}_0$ of the form $\mathbf{C}_0 = \mathrm{Mod}(T)$ where T is a complete $\mathscr{L}[\tau]$-theory. Assume that $\kappa \in \mathrm{Comp}(\mathscr{L})$ and let $\mathfrak{A} \in \mathbf{C}_0$ be of cardinality $\geq \kappa$. Using $[\kappa]$-compactness we now find $\mathfrak{B} \models T$ which is an (first-order) elementary extension of $\mathfrak{A}$ and clearly $\mathfrak{B} \in \mathbf{C}$. ☐

1.6. Gaps in the Compactness Spectrum

In this section we want to study a family of examples of logics with gaps in the compactness spectrum. These examples will also be used in the subsequent sections to illustrate various phenomena concerning dependence numbers and amalgamation properties (see Example 2.2.5 and Section 3.5).

1.6.1 Example. Let κ be a cardinal and F be an ultrafilter on κ. We define a logic $\mathscr{L} = \mathscr{L}_{F\omega}$ by adding to first-order logic $\mathscr{L}_{\omega\omega}$ the following formation rule: If $\{\varphi_i : i < \kappa\}$ is an indexed family of $\mathscr{L}$-sentences, then $\bigcap_F \{\varphi_i : i \leq \kappa\}$ is an $\mathscr{L}$-sentence. We additionally assume that $\mathscr{L}$-formulas have $< \omega$ many free variables. Satisfaction for $\mathscr{L}$ is defined by the additional clause: If $\mathfrak{A}$ is a τ-structure then $\mathfrak{A} \models \bigcap_F \{\varphi_i : i \leq \kappa\}$ iff $\{i \leq \kappa : \mathfrak{A} \models \varphi_i\} \in F$.

1.6.2 Proposition. *Let μ be a measurable cardinal and F be a μ-complete non-principal ultrafilter on μ.*

(i) $\mathscr{L}_{F\omega} < \mathscr{L}_{(2^\mu)^+\omega}$.
(ii) *$\mathscr{L}_{F\omega}$ is not $[\mu]$-compact.*
(iii) *$\mathscr{L}_{F\omega}$ is $[\lambda]$-compact for every $\lambda < \mu$.*

Proof. (i) and (ii) are left to the reader. To prove (iii) we make use of the abstract compactness theorem (1.3.9) and we show that every ultrafilter D on λ is in $\mathrm{UF}(\mathscr{L})$. Let us spell this out precisely:

1.6.3 Lemma. *Let $\mathscr{L} = \mathscr{L}_{F\omega}$ and D be any ultrafilter on $\lambda < \mu$. Furthermore let $\{\mathfrak{A}_i : i < \lambda\}$ be a family of τ-structures, $\varphi \in \mathscr{L}[\tau]$ and $\{\mathbf{f}_j : j \leq \nu < \mu\}$ be a family of functions in $\prod_{i\in\lambda} \mathfrak{A}_i$. Then the following are equivalent:*

(i) $\prod_{i\in\lambda} \mathfrak{A}_i/D \models \varphi(\mathbf{f}_1\mathbf{f}_2, \ldots, \mathbf{f}_j, \ldots)_{j\leq\nu}$.
(ii) $X_\varphi = \{i \in \lambda : \mathfrak{A}_i \models \varphi(\mathbf{f}_1(i), \mathbf{f}_2(i), \ldots, \mathbf{f}_j(i), \ldots)_{j\leq\nu}\} \in D$.

Proof. Like Los' theorem for first-order logic. ☐

Example 1.6.1 can be still further extended:

1.6.4 Example*. Let $\mu_1 < \mu_2$ with μ_1 measurable and μ_2 strongly compact. Let $\mathfrak{F}$ be a μ_1-complete non-principal ultrafilter on μ_1. We define the logic $\mathscr{L}_{F,\mu_2}$ as above, but we allow existential quantification over sequences of variables $\{x_j : j < \alpha < \mu_2\}$.

1.6.5 Proposition* (Shelah). (i) $\mathscr{L}_{F,\mu_2} < \mathscr{L}_{\mu_2,\mu_2}$.
(ii) *The logic $\mathscr{L}_{F,\mu_2}$ is $[\kappa]$-compact for every $\kappa < \mu_1$ and $\kappa \geq \mu_2$.*

Proof. (i) Clearly, the operation $\bigcap_F$ can be expressed by conjunctions and disjunctions in $\mathscr{L}_{\mu_2,\mu_2}$, since μ_2 is a strong limit cardinal and $\mu_1 < \mu_2$.
(ii) For $\kappa < \mu_1$ this is similar to Lemma 1.6.3 and for $\kappa \geq \mu_2$ this follows from (i) and the fact that μ_2 is strongly compact. ☐

Clearly, in Proposition 1.6.5, $[\mu_1]$-compactness fails. But it is not clear, whether for any κ with $\mu_1 < \kappa < \mu_2$, we have $[\kappa]$-compactness. However, we can construct a more refined example:

1.6.6 Example* (Shelah). Let $D(\mu_1, \mu_2)$ be the set of μ_1-complete ultrafilter F on some set $I \subset \mu_2$ such that $\mu_1 \leq \mathrm{card}(I) < \mu_2$. Instead of allowing $\bigcap_F$ for one ultrafilter we can now form a logic $\mathscr{L}_{D(\mu_1,\mu_2),\mu_2}$ as follows: We close first-order logic $\mathscr{L}_{\omega_1\omega}$ under all the operations $\bigcap_F$ for $F \in D(\mu_1, \mu_2)$ as in the previous example. Additionally we close under existential quantification over strictly less than μ_2 many individual variables.

The next proposition is proved exactly as Proposition 1.6.2.

1.6.7 Proposition* (Shelah). *Let μ_1 be measurable and μ_2 be a strongly compact cardinal bigger than μ_1. Then*:

(i) $\mathscr{L}_{D(\mu_1,\mu_2),\mu_2} < \mathscr{L}_{\mu_2,\mu_2}$,
(ii) $\mathscr{L}_{D(\mu_1,\mu_2),\mu_2}$ *is* $[\kappa]$*-compact for every* $\kappa < \mu_1$ *and* $\kappa \geq \mu_2$; *and*
(iii) $\mathscr{L}_{D(\mu_1,\mu_2),\mu_2}$ *is not* $[\kappa]$*-compact for any* κ *with* $\mu_1 \leq \kappa < \mu_2$.

This also establishes Theorem 1.5.7(ii). Using the same type of examples we can actually find logics with a compactness spectrum containing various gaps. How far we can go with this, is described in the following theorem:

1.6.8 Theorem*. (i) *Assume there are arbitrarily large measurable cardinals. Then there is a* $[\omega]$*-compact logic* $\mathscr{L}$ *such that both* Comp($\mathscr{L}$) *and its complement are confinal in the class of all cardinals.*
(ii) *Assume there are arbitrarily large strongly compact cardinals. Then there is a* $[\omega]$*-compact logic* $\mathscr{L}$ *such that both* Comp($\mathscr{L}$) *and its complement are cofinal in the class of all cardinals and consist of intervals whose length is a strongly compact cardinal.*

Proof. Combine Examples 1.6.1 and 1.6.4, respectively. □

Note however, that for set-presentable logics $\mathscr{L}$, Vopenka's principle (Theorem 1.5.14) implies that Comp($\mathscr{L}$) is a final segment of all cardinals.

2. *The Dependence Number*

2.1. *Introduction*

In this section we develop further an idea mentioned briefly in Chapter II, Section 5.1, namely the meaning of the assertion that a formula $\varphi \in \mathscr{L}[\tau]$ depends only on a subset $\sigma \subset \tau$. We present the material of this section for one-sorted logics only. We leave it to the reader to adopt the definitions and results to the many-sorted case. Let us recall a definition:

2.1.1 Proposition. Let $\mathscr{L}$ be a logic and $\varphi \in \mathscr{L}[\tau]$.

(i) φ *depends* (*only*) on (the symbols in) σ, $\sigma \subset \tau$ if for all τ-structures $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \upharpoonright \sigma \cong \mathfrak{B} \upharpoonright \sigma$ we have $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.
(ii) A logic $\mathscr{L}$ is *weakly regular, if* $\mathscr{L}$ *satisfies the basic closure properties* (1.2.1) and the relativization property (1.2.2) of Chapter II.

The difference between weakly regular and regular is the absence of the substitution property (1.2.3) of Chapter II.

If $\varphi \in \mathscr{L}[\tau]$ does only depend on $\sigma \subset \tau$, one would generally expect, that there is a $\psi \in \mathscr{L}[\sigma]$ which is equivalent to φ. If this is the case, we say that the logic

$\mathscr{L}$ is *occurrence normal*. However, our definition of a weakly regular logic does not imply this. Nevertheless we have:

2.1.2 Proposition*. *For every weakly regular logic $\mathscr{L}$ there is a logic $\mathscr{L}_1$ such that:*

(i) $\mathscr{L} \equiv \mathscr{L}_1$; *and*
(ii) *if $\varphi \in \mathscr{L}_1[\tau]$, $\sigma \subset \tau$ and φ depends only on σ, then there is a $\psi \in \mathscr{L}_1[\sigma]$ such that for every σ-structure $\mathfrak{A}$, $\mathfrak{A} \models \psi$ iff every expansion of $\mathfrak{A}$ to a τ-structure $\mathfrak{A}_1$, $\mathfrak{A}_1 \models \varphi$.*

Proof. We just add new atomic formulas and consider them as being of the required vocabulary. □

Regular logics are closed under substitutions of formulas for atomic predicate letters. For one-sorted logics there is no problem in stating this directly, for many-sorted logics we have to be a bit careful about the sorts. Gaifman pointed out that the definition of a regular logic ensures that $\mathscr{L}_1$ actually is $\mathscr{L}$.

2.1.3 Proposition*. *Every regular logic is occurrence normal.*

Proof. One-sorted case: Assume φ, σ, and τ as in the definition of occurrence normal above. To construct ψ we first make use of the eliminability of function symbols (which follows from the substitution property, Definition II.1.2.3) and assume that $\tau - \sigma$ contains only relation symbols. Next we construct for every predicate symbol $R \in \tau - \sigma$ a formula of first-order logic ϑ_R with equality only and with free variables according to the specifications of the one-sorted arity of R. We now obtain ψ by substituting ϑ_R for every occurrence of R in φ, using the substitution property again. Note that we do not need the relativization property here.

In the case of many-sorted logics, the definition of the substitution property (1.2.3) from Chapter II has to be modified. There is no difficulty in doing this so that it implies occurrence normality. We leave this as an exercise to the reader. □

In the light of Propositions 2.1.2 and 2.1.3 we can restrict ourselves for the rest of this chapter to occurrence normal or regular logics. For such logics we can define the concept of a *dependence number* in a semantical way. In Chapter II (after Definition 1.2.3) a syntactic concept of occurrence property was introduced.

2.1.4 Definition. (i) Given a regular logic $\mathscr{L}$, we define a cardinal $o(\mathscr{L}) = \kappa$ to be the smallest cardinal such that every formula $\varphi \in \mathscr{L}[\tau]$ depends only on some subset $\tau_0 \subset \tau$ with $\text{card}(\tau_0) < \kappa$. If no such κ exists we write $o(\mathscr{L}) = \infty$. If $o(\mathscr{L}) = \omega$ we also say that $\mathscr{L}$ has *finite dependence* or has the *finite dependence property*.

(ii) Given a regular logic $\mathscr{L}$, we define a cardinal $OC(\mathscr{L}) = \kappa$ to be the smallest cardinal such that for every formula $\varphi \in \mathscr{L}[\tau]$ there is $\sigma \subset \tau$ with $\text{card}(\sigma) < OC(\mathscr{L})$ and $\varphi \in L(\sigma)$.

In Chapter II (Definition 6.1.3) the *finite* occurrence property was introduced, which is the syntactic counterpart of our finite dependence property. In our terminology the finite occurrence property is equivalent to $OC(\mathscr{L}) = \omega$. Using Proposition 2.1.3 one easily sees that every logic $\mathscr{L}$ which has the finite dependence property, contains a sublogic $\mathscr{L}_0$ equivalent to it which has the occurrence property in the syntactic sense. In fact, more generally we have:

2.1.5 Proposition*. *Let $\mathscr{L}$ be a regular logic with dependence number* $o(\mathscr{L})$. *Then there is a regular logic $\mathscr{L}_1$ with* $OC(\mathscr{L}) = o(\mathscr{L})$ *which is equivalent to $\mathscr{L}$.*

Proof. Similar to Proposition 2.1.2. ☐

The above proposition shows that up to equivalence of logics, the occurrence number and the dependence number coincide. In Makowsky–Shelah [1983] the dependence number is, indeed, called occurrence number. The change in terminology was motivated by the requirements of Chapter II and by the notion of the *dependence structure*, introduced in Section 2.4.

2.1.6 Examples. (i) In Chapter II, Proposition 5.1.3 shows that for a (κ, λ)-compact logic with $o(\mathscr{L}) \leq \kappa$ we actually have $o(\mathscr{L}) \leq \lambda$. This fact was first pointed out in H. Friedman [1970].

(ii) Let us look at the logic $\mathscr{L}_{F\omega}$ defined in Example 1.6.1. Obviously $o(\mathscr{L}) \leq \kappa^+$. But if $\varphi \in \mathscr{L}[\tau]$, $\text{card}(\tau) = \kappa$ then there is no smallest $\tau_0 \subset \tau$ such that φ depends exactly on the symbols in τ_0.

2.1.7 Substitutes for the Dependence Number. The dependence number is a concept which keeps the size of a logic limited. Other assumptions in this direction are:

(i) For every vocabulary τ with τ *a set* $\mathscr{L}[\tau]$ is also a set. We call such logics *small.* In Section 4.3 this concept will be used.

(ii) For every vocabulary τ, if τ is a set, $\text{card}(\mathscr{L}[\tau]) = \text{card}(\tau) + \kappa$ for some fixed cardinal κ. This gives us a special case of a *size function*, as defined in Section 4.3. There we also look at *tiny* logics, i.e., logics $\mathscr{L}$ such that whenever $\text{card}(\tau)$ is smaller than the first uncountable measurable cardinal μ_0, then $\text{card}(\mathscr{L}[\tau])$ is also smaller than μ_0.

(iii) The presence of a Lowenheim number $l_\kappa(\mathscr{L})$, as introduced in Section II.6.2.

For various theorems in abstract model theory such limiting assumptions are needed, as we shall see in the further course of this and the next chapter. Note that from the above properties (ii) → (i) and in the presence of an dependence number (iii) → (ii), up to equivalence of logics. In fact, we have the following:

2.1.8 Proposition. *Let $\mathscr{L}$ be a logic with* $o(\mathscr{L}) = \mu$ *and* $l_1(\mathscr{L}) = \kappa$ *and τ be a vocabulary with* $\text{card}(\tau) = \lambda$ *and* $\mu \leq \lambda \leq \kappa$. *Then there are, up to logical equivalence, only* $\leq 2^{2^\kappa}$ *many τ-sentences.*

The proof consists of a crude counting argument. Note that we do not get the stronger conclusion $\text{card}(\mathscr{L}[\tau]) \le 2^{2^\kappa}$, since there may be many equivalent formulas.

One would actually expect that if $l_1(\mathscr{L}) = \kappa$ then $o(\mathscr{L}) \le \kappa^+$ and one might add this to the definition of the Lowenheim number, but it is an open field to determine which model-theoretic properties have what impact on the size of the dependence numbers. The only exception is compactness and the rest of Section 2 is devoted to this.

2.2. *Compactness and Dependence Numbers*

This section is devoted to the statement of the finite dependence theorem and the discussion of several examples. The proof of the finite dependence theorem is discussed in the following section but for a technically complete exposition of the proof we refer the reader to Makowsky–Shelah [1983].

To simplify the statements of the following theorem and its corollaries, we denote by $\bar{\mu}$ the first uncountable measurable cardinal, if there is one, and ∞ otherwise. We stipulate further that if $\bar{\mu} = \infty$, then $\bar{\mu}^+ = \infty$.

2.2.1 Theorem (Finite Dependence Theorem). (i) (Global version). *Let $\mathscr{L}$ be a regular, $[\omega]$-compact logic with dependence number $o(\mathscr{L}) < \bar{\mu}$. Then $\mathscr{L}$ has the finite dependence property, i.e., $o(\mathscr{L}) = \omega$.*

(ii) (Local version). *Let $\mathscr{L}$ be a regular, $[\omega]$-compact logic, τ a vocabulary and $\varphi \in \mathscr{L}[\tau]$ a formula which depends only on some $\tau_0 \subset \tau$ with* $\text{card}(\tau_0)$ *less than the first uncountable measurable cardinal. Then there is a finite $\tau_1 \subset \tau_0$ such that φ depends only on τ_1.*

Clearly, (ii) implies (i). The proof of (ii) is presented in Section 2.3.

2.2.2 Corollary. *Let $\mathscr{L}$ be a regular, $[\kappa]$-compact logic, $\kappa < \bar{\mu}$ and $o(\mathscr{L}) \le \bar{\mu}^+$. Then $\mathscr{L}$ has the finite dependence property.*

Proof of Corollary. By Theorem 1.5.2 $\mathscr{L}$ is $[\omega]$-compact, so we can apply the finite dependence theorem. □

As a second corollary we get a representation theorem of some compact logics via Lindstrom quantifiers (cf. Section II.4). Let us recall a definition:

2.2.3 Definition. A logic $\mathscr{L}$ is a *Lindstrom logic* if $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_i)_{i \in I}$ for some indexed set of Lindstrom quantifiers Q_i $(i \in I)$. $\mathscr{L}$ is *finitely generated* if $\mathscr{L}$ is a Lindstrom logic and $\text{card}(I) < \omega$.

Note that by Theorem 4.1.3 of Chapter II every regular logic $\mathscr{L}$ which has the (syntactic) finite occurrence property is a Lindstrom logic.

2.2.4 Proposition*. (i) *Let $\mathscr{L}$ be a regular logic with* $\mathrm{o}(\mathscr{L}) = \omega$. *Then $\mathscr{L}$ is equivalent to a Lindstrom logic.*

(ii) *If a regular logic $\mathscr{L}$ is small,* $[\kappa]$*-compact and* $\mathrm{o}(\mathscr{L}) \leq \kappa < \bar{\mu}^+$ *then $\mathscr{L}$ is equivalent to a Lindstrom logic.*

Proof. Using Corollary 2.2.2 we can reduce (ii) to (i). So assume that $\mathscr{L}$ has finite dependence. Let τ be a finite vocabulary. We want to replace every $\varphi \in \mathscr{L}[\tau]$, which is not equivalent to a first-order formula, by a formula consisting of a new quantifier Q_φ applied to a sequence of atomic formulas. The problem is to keep the number of quantifiers so introduced small. But the type of the quantifier does not really depend on the vocabulary τ, but only on the similarity type, i.e., on the number and arities of the symbols τ. Now there is a countable universal vocabulary τ_∞ such that for every finite τ there is $\tau' \subset \tau_\infty$ which is of the same similarity type as τ. Therefore, every $\varphi \in \mathscr{L}[\tau]$ can be obtained from some $\psi \in \mathscr{L}[\tau_\infty]$ by an application of substitution. By our assumption, $\mathscr{L}[\tau_\infty]$ is a set. So writing every formula in $\mathscr{L}[\tau_\infty]$ as a Lindstrom quantifier, we complete the proof. ▯

Both the theorem and the corollaries have assumptions involving measurable cardinals. In the sequel we shall discuss examples which show, that these assumptions are necessary.

2.2.5 Examples. (i) Let μ be a strongly compact cardinal. So $\mathscr{L} = \mathscr{L}_{\mu\mu}$ is $[\mu]$-compact and $\mathrm{o}(\mathscr{L}) = \mu$. As noted before, it is consistent that the first strongly compact and the first measurable cardinal coincide, by Magidor [1976]. This shows that the assumption on measurable cardinals cannot be dropped in the corollaries.

(ii) Let μ be a measurable cardinal and F be a μ-complete non-principal ultrafilter on μ. We look again at the logic $\mathscr{L} = \mathscr{L}_{F\omega}$ from Example 1.6.1. By Proposition 1.6.2 this logic is $[\omega]$-compact, but clearly its dependence number is μ^+. This shows that the assumption on the measurable cardinal cannot be dropped in the finite dependence theorem.

2.3. *Proof of the Finite Dependence Theorem*

The proof of the finite dependence theorem uses three lemmas (Lemmas A, B, C). We do not prove these lemmas here and refer the reader to [Makowsky–Shelah [1983]. Instead, we present the three lemmas without proofs and show how the finite dependence theorem is proved from them. The reader will gain a rather transparent picture of the structure of the proof.

Let us fix a $[\lambda]$-compact logic $\mathscr{L}$, a vocabulary τ and a sentence $\varphi \in \mathscr{L}[\tau]$. We want to study subsets of τ on which φ does not depend. Each lemma introduces a new aspect of the notions involved: Lemma A uses compactness to construct a dummy subset of τ. Lemma B builds a function on the power set of τ which is used to apply Lemma C, which makes us conclude that $\mathrm{card}(\tau)$ was measurable.

Lemma A′ is an improvement of Theorem 5.1.2 in Chapter II, and its proof is very similar.

2.3.1 Lemma A′. (i) *For every* $\tau_1 \subset \tau$ *with* $\text{card}(\tau_1) \leq \lambda$ *there is a* $\tau_0 \subset \tau_1$ *with* $\text{card}(\tau_0) < \lambda$ *such that* φ *does not depend on* $\tau_1 - \tau_0$.
(ii) *There is a* $\mu < \lambda$ *such that for every* $\tau_1 \subset \tau$ *with* $\text{card}(\tau_1) \leq \lambda$ *there is a* $\tau_0 \subset \tau_1$ *with* $\text{card}(\tau_0) \leq \mu$ *such that* φ *does not depend on* $\tau_1 - \tau_0$.

Now Lemma A′ can be used to prove Lemma A.

2.3.2 Lemma A. There is a $\tau_1 \subset \tau$ *with* $\text{card}(\tau_1) < \lambda$ *such that for every* $\tau_0 \subset \tau - \tau_1$ *with* $\text{card}(\tau_0) \leq \lambda$ *does not depend on* τ_0.

The second lemma used in the proof of the finite dependence theorem gives us the connection to ultrafilters. Here we use some material from Section 1.3, in particular, the definition of $\text{UF}(\mathscr{L})$.

2.3.3 Lemma B. *Let* μ *be a cardinal,* $\mathscr{L}$ *be a logic and* φ *a* $\mathscr{L}[\tau]$*-sentence. If* $\tau_2 \subset \tau$ *but for each* $\tau_1 \subset \tau_2$ *with* $\text{card}(\tau_1) \leq \lambda$, φ *does not depend on* τ_1, *then there is a function* $f\colon P(\tau_2) \to \{0, 1\}$ *such that*:

(i) *f is non-constant.*
(ii) *For every* $\sigma_1, \sigma_2 \subset \tau_1$ *with* $\text{card}(\sigma_1 \Delta \sigma_2) \leq \lambda$ *we have* $f(\sigma_1) = f(\sigma_2)$.
(iii) *For every ultrafilter* $F \in \text{UF}(\mathscr{L})$ *(on* μ*) f is F-continuous.*

Recall that if F is an ultrafilter on μ, $\{\sigma_i\colon i < \mu\}$, σ are subsets of τ_2 then $\lim_F \sigma_i = \sigma$ iff for every $P \in \tau_2$ the set $I_P = \{i \in \mu : P \in \sigma_i \leftrightarrow P \in \sigma\} \in F$ and f is F-continuous iff $\sigma = \lim_F \sigma_i$ implies that $f(\sigma) = \lim_F f(\sigma_i)$.

The third lemma, used in the proof of the finite dependence theorem, gives us the connection to measurable cardinals:

2.3.4 Lemma C. *If F is a uniform ultrafilter on* ω *and* $f\colon P(k) \to \{0, 1\}$ *satisfies* (i)–(iii) *of the previous lemma, then there is a measurable cardinal* μ_0 *such that* $\omega < \mu_0 \leq \kappa$.

We are now in a position to prove the finite dependence theorem.

Proof of the Finite Dependence Theorem. Assume $\mathscr{L}$ is $[\omega]$-compact and $o(\mathscr{L}) > \omega$. Then there is an $\mathscr{L}[\tau]$-sentence φ which does not depend only on a finite subset of τ. So $\text{card}(\tau) \geq \omega$, and if $\text{card}(\tau) = \omega$ we are done by Theorem 5.1.2 of Chapter II. So $\text{card}(\tau) > \omega$. By Lemma A (for $\lambda = \omega$) we can assume that φ does not depend on any countable subset of τ. Now we apply Lemma B to construct the function f and by the abstract compactness theorem (1.3.9) and Lemma A we know that f is F-continuous for some uniform ultrafilter on ω. So by Lemma C we know that $\text{card}(\tau) \geq \mu_0$, the first uncountable measurable cardinal. But this shows that $o(\mathscr{L}) \geq \mu_0$, a contradiction. ☐

2.4. Dependence Filters

So far we have studied the concept of a formula depending on some subset of a vocabulary τ, and our main result was the finite dependence theorem. However, as the examples in Section 1.6 and their discussion in Examples 2.1.6 show, this need not be the appropriate notion. We are facing here a similar problem as in the analysis of compactness properties. There it turned out that the more appropriate tool to study compactness is the class of ultrafilters UF($\mathscr{L}$). Similarly here, we have to look at *dependence filters*.

2.4.1 Definition. Let τ be an infinite vocabulary and assume, for notational simplicity, that $\tau = \{R_i: i < \lambda\}$, where R_i are relation symbols. Let $\varphi \in \mathscr{L}[\tau]$ be a formula of some logic $\mathscr{L}$. If $X \subset \lambda$ we write τ_X for $\{R_i: i \in X\}$.

(i) Let F be an ultrafilter on λ. We say that φ *depends on F only*, if, given two τ-structures $\mathfrak{A} = \langle A, R_i^A \rangle_{i<\lambda}$ and $\mathfrak{B} = \langle B, R_i^B \rangle_{i<\lambda}$, and a set $X \in F$ such that $\mathfrak{A} \upharpoonright \tau_X \cong \mathfrak{B} \upharpoonright \tau_X$ then $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. We call F an *dependence filter for φ*.

(ii) Let $Y_0 \cup Y_1 \cup \cdots \cup Y_n$ be a finite partition of λ and F_k ($k = 0, 1, \ldots, n$) be ultrafilters on Y_k, respectively. We say that φ *depends on $F_0, F_1, \ldots, F_n$ only*, if, given two τ-structures $\mathfrak{A} = \langle A, R_i^A \rangle_{i<\lambda}$ and $\mathfrak{B} = \langle B, R_i^B \rangle_{i<\lambda}$, and sets $X_k \in F_k$ such that $\mathfrak{A} \upharpoonright \tau_X \cong \mathfrak{B} \upharpoonright \tau_X$, where $X = \bigcup_0^n X_i$, then $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. We call $F_0, F_1, \ldots, F_n$ a *finite dependence structure for φ*.

(iii) We can modify (ii) to allow infinite partitions. In this case we speak of *dependence structures for φ*.

2.4.2 Examples. (i) If a logic $\mathscr{L}$ has finite dependence, $\varphi \in \mathscr{L}[\tau]$, then φ has a principal dependence filter generated by the finite set $\tau_0 \subset \tau$ on which φ only depends.

(ii) Let us return to the logic $\mathscr{L}_{F\omega}$ from Example 2.1.6(ii), introduced in Example 1.6.1 Recall that F is an ultrafilter on some set I. Let R_i, $i \in I$ be relation symbols. The formula $\bigcap_F \{R_i: i \in I\}$ has among its dependence filters also the ultrafilter F. However, if $\tau = \{R_i: i \in I\} \cup \{S_i: i \in I\}$ then the dependences of the formula $\bigcap_F \{R_i: i \in I\} \wedge \bigcap_F \{R_i: i \in I\}$ has to be described by a finite partition of τ and a filter on each of the components, which in this case is F.

(iii) If we look at Example 1.6.5 it is easy to construct examples of sentences whose dependence is described by more complicated partitions and more complicated ultrafilters.

That those examples are more than accidental is shown by the following theorem from the treasure box (Shelah [198?e]).

2.4.3 Theorem* (Shelah's Finite Dependence Structure Theorem). *Let $\mathscr{L}$ be a $[\omega]$-compact logic, $\tau = \{R_i: i < \lambda\}$ a vocabulary and $\varphi \in \mathscr{L}[\tau]$. Then there is a finite partition $Y_0 \cup Y_1 \cup \cdots \cup Y_n$ of λ and countably complete ultrafilters F_k ($k = 0, 1, \ldots, n$) on Y_k, respectively, such that φ only depends on $F_0, F_1, \ldots, F_n$. In other words, every $\varphi \in \mathscr{L}[\tau]$ has finite dependence structure.*

The proof of the finite dependence structure theorem consists of elaborations of the Lemmas A, B, and C in Section 2.3. The finite dependence structure theorem opens new perspectives in the study of dependence phenomena for compact logics for the case that there are uncountable measurable cardinals.

3. $\mathscr{L}$-Extensions and Amalgamation

3.1. Basics

Given a logic $\mathscr{L}$, it is clear how to define the analogue of elementary equivalence of two structures of the same language τ: They have to satisfy the same τ-sentences. It is more problematic to generalize the notion of elementary embeddings, because already in the first-order case either free variables or new constant symbols are used in the definition and various definitions are equivalent only because of the finite occurrence (finite dependence) or even because of compactness. In the general case it is convenient to introduce a cardinal parameter.

Let us recall that the $\mathscr{L}$-diagram of an τ-structure $\mathfrak{A}$ is the set of $\mathscr{L}$ sentences true in the structure $\langle \mathfrak{A}, A \rangle$, i.e., the structure $\mathfrak{A}$ augmented with names for all its elements. We denote the $\mathscr{L}$-diagram of $\mathfrak{A}$ by $D_{\mathscr{L}}(\mathfrak{A})$.

3.1.1 Definitions. (i) A τ-structure $\mathfrak{B}$ is an $\mathscr{L}$-extension of a τ-structure $\mathfrak{A}$, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and the two structures $\langle \mathfrak{A}, A \rangle$ and $\langle \mathfrak{B}, A \rangle$ satisfy the same $\mathscr{L}$-sentences. In this case we write $\mathfrak{A} <_{\mathscr{L}} \mathfrak{B}$.

(ii) A τ-structure $\mathfrak{B}$ is a $(\kappa, \mathscr{L})$-extension of a τ-structure $\mathfrak{A}$, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and for every subset $A_0 \models A$ with $\mathrm{card}(A_0) < \kappa$ the two structures $\langle \mathfrak{A}, A_0 \rangle$ and $\langle \mathfrak{B}, A_0 \rangle$ are $\mathscr{L}$-equivalent. In this case we write $\mathfrak{A} <^{\kappa}_{\mathscr{L}} \mathfrak{B}$.

3.1.2 Examples. (i) For $\mathscr{L} = \mathscr{L}_{\infty\omega}$ without occurrence restrictions we have clearly $\mathfrak{A} <_{\mathscr{L}} \mathfrak{B}$ iff $\mathfrak{A} = \mathfrak{B}$. Using indiscernibles, it is easy to construct $\mathfrak{A}, \mathfrak{B}$ such that $\mathfrak{A} <^{\kappa}_{\mathscr{L}_{\infty\omega}} \mathfrak{B}$ for a given κ.

(ii) If $\mathrm{o}(\mathscr{L}) = \kappa$ then clearly every $(\kappa, \mathscr{L})$-extension is an $\mathscr{L}$-extension.

(iii) If $\mathscr{L}$ is a compact logic, then we have, by the finite dependence theorem of the previous section, that $\mathscr{L}$-extensions and $(\kappa, \mathscr{L})$-extensions coincide for every κ.

In model-theory extensions are studied extensively and the following three situations are characteristic:

(i) Do models have $(\kappa, \mathscr{L})$-extensions?

(ii) Given a chain of extensions, is the union an extension of each member of the chain?

(iii) Given three τ-structures $\mathfrak{A}_i$, $i = 0, 1, 2$ such that $\mathfrak{A}_0$ is an $\mathscr{L}$-substructure of both $\mathfrak{A}_1$ and $\mathfrak{A}_2$, does there exists an amalgamating extension $\mathfrak{A}_3$?

In fact, in Chapter XX we shall describe an approach to abstract model theory, which is entirely based on those aspects and not on the notion of formulas and logics. Here, however, we shall study logics which allow these constructions.

In this chapter we shall deal with logics which allow one of the above constructions (i)–(iii) universally.

3.1.3 Definition. (i) A logic $\mathscr{L}$ satisfies $\mathrm{EXT}(\mathscr{L})$ or has the *extension property*, if every infinite τ-structure $\mathfrak{A}$ has an $\mathscr{L}$-extension $\mathfrak{B}$.

(ii) A logic $\mathscr{L}$ satisfies REXT($\mathscr{L}$) or has the *relativized extension property*, if for every infinite definable set X in some τ-structure $\mathfrak{A}$ there is a τ-structure $\mathfrak{B}$ which is a $\mathscr{L}$-extension of $\mathfrak{A}$ which extends X properly.

Clearly, REXT($\mathscr{L}$) implies EXT($\mathscr{L}$) for every logic $\mathscr{L}$.

3.1.4 Example. Every compact logic $\mathscr{L}$ satisfies REXT($\mathscr{L}$).

In fact, the following proposition is easily proved by the reader:

3.1.5 Proposition. *If a logic $\mathscr{L}$ is $[\omega]$-compact then $\mathscr{L}$ satisfies* REXT($\mathscr{L}$).

We shall return to the study of EXT and REXT in Section 3.2.

3.1.6 Definitions. (i) A family of τ-structures $\mathfrak{A}_i$, $i < \kappa$ is an $\mathscr{L}$-chain if $\mathfrak{A}_i$ is an $\mathscr{L}$-extension of $\mathfrak{A}_j$ for every $j < i < \kappa$.

(ii) A logic $\mathscr{L}$ satisfies CHAIN(κ, $\mathscr{L}$) or *respects* chains of length κ, if given a $\mathscr{L}$-chain $\mathfrak{A}_i$, $i < \kappa$ then $\bigcup_{i<\kappa} \mathfrak{A}_i$ is an $\mathscr{L}$-extension of each of the $\mathfrak{A}_i$'s.

(iii) A logic $\mathscr{L}$ satisfies CHAIN($\mathscr{L}$) or has the *chain property*, if it satisfies CHAIN(κ, $\mathscr{L}$) for every κ.

3.1.7 Remark. CHAIN(ω, $\mathscr{L}$) was called in Chapter III the Tarski-union-property.

3.1.8 Examples. (i) $\mathscr{L}_{\kappa\omega}$ has the chain property.

(ii) If κ is regular than $\mathscr{L}_{\kappa\kappa}$ respects chains of length λ, $\mathrm{cf}(\lambda) > \kappa$.

In Chapter III (Theorem 2.2.2) the following result of Lindström [1973] was proved:

3.1.9 Theorem (Lindström). *If a logic $\mathscr{L}$ is compact and respects chains of length ω then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*

There are no logics known which are $[\omega]$-compact and satisfy CHAIN($\mathscr{L}$). It is open whether this is due to a theorem or simple ignorance of more examples. It would be interesting to explore more consequences of CHAIN-properties. In Tharp [1974] and Makowsky [1975] "continuous" or "securable" quantifiers are studied, which, if added to first-order logic, give us logics which do satisfy CHAIN($\mathscr{L}$). In Lindström [1973a, 1983] a variation of Theorem 3.1.9 is studied involving only (λ, ω)-compactness and a modification of the Tarski-union-property.

3.1.10 Definitions. (i) A logic $\mathscr{L}$ satisfies Am(κ, $\mathscr{L}$) or has the *κ-amalgamation property* if, given three τ-structures $\mathfrak{A}_i$, $i = 0, 1, 2$ such that $\mathfrak{A}_0 <^{\kappa}_{\mathscr{L}} \mathfrak{A}_j$, $j = 1, 2$ there is a τ-structure $\mathfrak{B}$ such that $\mathfrak{A}_i <^{\kappa}_{\mathscr{L}} \mathfrak{B}$, $i = 0, 1, 2$ and the diagram commutes.

(ii) A logic $\mathscr{L}$ satisfies Am($\mathscr{L}$) or has the *amalgamation property*, if Am(κ, $\mathscr{L}$) holds for every κ.

(iii) A logic $\mathscr{L}$ satisfies JEP($\mathscr{L}$) or has the *joint embedding property* if any two $\mathscr{L}$-equivalent τ-structures $\mathfrak{A}_i$, $i = 1, 2$ have a common $\mathscr{L}$-extension $\mathfrak{B}$.

One can also introduce cardinal parameters for $\mathscr{L}$-equivalence and the joint embedding property, but we shall not need this in our exposition.

3.1.11 Theorem. (i) *Every compact logic $\mathscr{L}$ has the joint embedding property.*
(ii) *If a logic $\mathscr{L}$ satisfies* JEP($\mathscr{L}$) *then it has the amalgamation property.*

Proof. (i) Since $\mathscr{L}$ is compact, $\mathscr{L}$ has finite dependence, by the finite dependence theorem. So we can use compactness again to show that $D_L(\mathfrak{A}_1) \cup D_L(\mathfrak{A}_2)$ has a model $\mathfrak{B}$ which is a $(\kappa, \mathscr{L})$-extension of both the $\mathfrak{A}_i$, $i = 1, 2$.

(ii) Let $\mathfrak{A}_i$, $i = 0, 1, 2$ be as in the hypothesis of the amalgamation property. Clearly the two structures $\langle \mathfrak{A}_1, A_0 \rangle$, $\langle \mathfrak{A}_2, A_0 \rangle$ are $\mathscr{L}$-equivalent, so let $\mathfrak{B}$ be an $\mathscr{L}$-extension of both of them. Clearly this $\mathfrak{B}$ satisfies the requirements of the amalgamation property. ☐

3.1.12 Examples. (i) If κ is a strongly compact cardinal, then $\mathscr{L}_{\kappa\kappa}$ satisfies the joint embedding property.
(ii) Let $\mathscr{L} = \mathscr{L}_{\infty\omega}$, but with finite occurrence. It is easy to see that $\mathscr{L}$ does not satisfy the amalgamation property.

3.1.13 Definition. A logic $\mathscr{L}$ has the *Robinson property* if whenever $\Sigma_i \subset \mathscr{L}[\tau_i]$, $i = 0, 1, 2$ are such that $\tau_0 = \tau_1 \cap \tau_2$ and Σ_0 is complete and $\Sigma_0 \cup \Sigma_j, j = 1, 2$ has a model, then $\bigcup_{i=0}^{i=2} \Sigma_i$ has a model. Recall that a set of sentences Σ is *complete* if any two models of Σ are $\mathscr{L}$-equivalent.

D. Mundici has studied various aspects of the Robinson property, cf. Mundici [1981d, 1981c]. The Robinson property is extensively discussed in Chapter XIX. Here we only note the following theorem:

3.1.14 Theorem. *Every logic $\mathscr{L}$, which has the Robinson property also has the amalgamation property.*

Proof. Let $\Sigma_i = D_{\mathscr{L}}(\mathfrak{A}_i)$ where the $\mathfrak{A}_i$ are as in the hypothesis of the amalgamation property. ☐

The amalgamation property is further studied in Sections 3.3 and 3.4.

Let us summarize here some rather unexpected consequences of the amalgamation property, as they follow from Theorem 3.2.1 and the abstract amalgamation theorem (3.3.1).

3.1.15 Theorem. *Let $\mathscr{L}$ be a regular logic with occurrence (dependence) number less than the first uncountable measurable cardinal.*

(i) *If $\mathscr{L}$ has the amalgamation property, then* REXT($\mathscr{L}$) *holds.*
(ii) *If* CHAIN(ω, $\mathscr{L}$) *holds and $\mathscr{L}$ has the amalgamation property then* $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.

This theorem stresses the connections between the more "algebraic" properties of logics, as they are at the core of Chapter XX. In our context the theorem is trivial. But then, the reader may try to prove (i) directly. The same challenge applies to Corollary 3.3.4.

3.2. $\mathscr{L}$-Extensions

In this section we prove a converse of Proposition 3.1.5 and explore further variations of extension properties.

3.2.1 Theorem. *A regular logic $\mathscr{L}$ satisfies* REXT($\mathscr{L}$) *iff $\mathscr{L}$ is $[\omega]$-compact.*

Proof. Assume REXT($\mathscr{L}$) and that $\mathscr{L}$ is not $[\omega]$-compact. So by Theorem 1.2.2 (or Chapter II, Proposition 5.2.4) ω is cofinally characterizable in $\mathscr{L}$ by some expansion $\mathfrak{A}$ of $\langle \kappa, < \rangle$. But clearly ω^A is a maximal definable subset of $\mathfrak{A}$, a contradiction. The other direction was Proposition 3.1.5. ☐

We next introduce a cardinal parameter into our extension properties:

3.2.2 Definition. A logic $\mathscr{L}$ satisfies EXT(κ, $\mathscr{L}$) if, whenever a τ-structure $\mathfrak{A}$ has no proper $\mathscr{L}$-extension then card($\mathfrak{A}$) $< \kappa$.

3.2.3 Proposition. *If a logic $\mathscr{L}$ is $[\lambda]$-compact then $\mathscr{L}$ satisfies* EXT(λ, $\mathscr{L}$).

The proof is left to the reader.

The next theorem is one of the least constructive theorems in logic: Its proof uses the replacement axiom very heavily. To test our assertion the reader should try to prove Theorem 3.2.4 below in ZC rather than in ZFC. (This problem was suggested by A. Dodd.)

3.2.4 Theorem. *Let λ_0 be an infinite cardinal and $\mathscr{L}$ satisfies* EXT(λ_0, $\mathscr{L}$) *then there is a cardinal κ such that $\mathscr{L}$ is $[\kappa]$-compact.*

Proof. We prove the contraposition: If $\mathscr{L}$ is not $[\kappa]$-compact for any cardinal κ then for every cardinal λ_0 there is a maximal structure $\mathfrak{B}$ with card($\mathfrak{B}$) $\geq \lambda_0$. (Recall that a structure is maximal for $\mathscr{L}$ if it has no proper $\mathscr{L}$-extensions.)

By Theorem 1.2.2 every regular cardinal λ is cofinally characterizable via some expansion $\mathfrak{B}_\lambda$ which we assume without loss of generality of minimal cardinality $g(\lambda)$.

Now let μ be the first cardinal such that:

(i) If $\nu < \mu$ then $g(\nu) \leq \mu$.
(ii) $\lambda_0 \leq \mu$.
(iii) $\mathrm{cf}(\mu) = \omega$.

Clearly such a cardinal exist, e.g., the ω-limit of the first fixed points of the function $g(\nu)$. (This is where the replacement axiom is used without control over the complexity of the set-theoretic formula involved.)

Let $\mathfrak{B}$ be the complete expansion of the structure $\langle \mu, \in \rangle$. We claim that $\mathfrak{B}$ is maximal. For otherwise, let $\mathfrak{C}$ be an $\mathscr{L}$-extension of $\mathfrak{B}$. If $\mathfrak{C}$ is proper there is a $c \in C - B$. Remember $\mathrm{cf}(\mu) = \omega$ and let $\{b_n : n \in \omega\}$ be a cofinal sequence in $\mathfrak{B}$. Since ω is cofinally characterizable in $\mathscr{L}$ via $\mathfrak{B}$, $g(\omega) \leq \mu$ and $\mathfrak{B}$ is a complete structure, $\{b_n : n \in \omega\}$ is also cofinal in $\mathfrak{C}$. So clearly, $\mathfrak{C} \models c \in b_k$ for some $k \in \omega$. Now let $d \in B$ be the smallest (with respect to $\in$) element in $\mathfrak{B}$ such that $\mathfrak{C} \models c \in d$. We note that d is an ordinal. Let $\delta = \mathrm{cf}(d)$ and $\{d_i : i < \delta\}$ be a sequence cofinal to d in $\mathfrak{B}$. Again, since $g(\delta) \leq \mu$ and δ is cofinally characterizable in $\mathscr{L}$ via $\mathfrak{B}$ $\{d_i : i < \delta\}$ is cofinal to d in $\mathfrak{C}$. So there is a $j < \delta$ with $\mathfrak{C} \models c \in d_j$, which contradicts the minimality of d. This establishes that $\mathfrak{B}$ is maximal. Clearly, $\mathrm{card}(\mathfrak{B}) > \lambda_0$ by our construction, which completes the proof. ▯

If there are no uncontable measurable cardinals, we get the following situation:

3.2.5 Theorem. *Assume there are no uncountable measurable cardinals and $\mathscr{L}$ is a regular logic. Then the following are equivalent:*

(i) *$\mathscr{L}$ is $[\omega]$-compact.*
(ii) *$\mathscr{L}$ satisfies* EXT($\mathscr{L}$).
(iii) *$\mathscr{L}$ satisfies* REXT($\mathscr{L}$).

Proof. (i) → (iii) was Proposition 3.1.5 and (iii) → (ii) follows from the definitions. To prove (ii) → (i) we apply Theorem 3.2.4 and then Theorem 1.5.2. ▯

Also the existence of uncountable measurable cardinals is closely related to our extension properties. Let us look at the following example:

3.2.6 Example. A logic $\mathscr{L}$ for which EXT($\mathscr{L}$) and REXT($\mathscr{L}$) do not coincide. Let $Q_{\lambda\kappa}$ be a quantifier of type $\langle 1, 1\rangle$ with satisfaction defined by

$$\mathfrak{A} \models Q_{\lambda\kappa} xy(\varphi(x), \psi(y)) \text{ iff } \mathrm{card}(\varphi^A) < \lambda \quad \text{and} \quad \mathrm{card}(\psi^A) > \kappa$$

3.2.7 Lemma. *Let $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_{\omega 2^{\mu_0}})$ where μ_0 is the first uncountable measurable cardinal.*

(i) *$\mathscr{L}$ is $[\mu_0]$-compact.*
(ii) *$\mathscr{L}$ satisfies* EXT($\mathscr{L}$).
(iii) *$\mathscr{L}$ does not satisfy* REXT($\mathscr{L}$) *and therefore is not $[\omega]$-compact.*

Proof. We prove (iii) first. For this we look at the structure $\mathfrak{A} = \langle (2^{\mu_0})^+, \in \rangle$. It is straightforward to find an expansion $\mathfrak{A}_1$ of $\mathfrak{A}$ in which $\langle \omega, \in \rangle$ is cofinally characterized in $\mathscr{L}$, so we apply Theorem 3.2.1 together with Theorem 1.2.2.

To prove (ii) we distinguish two cases: On structures $\mathfrak{A}$ with $\mathrm{card}(\mathfrak{A}) \leq 2^{\mu_0}$ $\mathscr{L}$ is equivalent to first-order logic, since the quantifier Q acts trivially, being always false, so first-order extensions will do. On structures $\mathfrak{A}$ with $\mathrm{card}(\mathfrak{A}) > 2^{\mu_0}$ we apply (i).

To prove (i) we use the abstract compactness theorem (1.3.9) and show that every μ_0-complete ultrafilter F on μ_0 is in UF($\mathscr{L}$). We need μ_0-completeness to see that finiteness is preserved under ultrapowers over F and we need that μ_0 is small for $(2^{\mu_0})^+$ to see that the other cardinality restriction is preserved under such ultrapowers. □

This example together with Theorem 3.2.5 gives us immediately the following characterization of the existence of uncountable measurable cardinals.

3.2.8 Theorem. *The following are equivalent:*

(i) *For every logic* $\mathscr{L}$ EXT($\mathscr{L}$) *holds iff* REXT($\mathscr{L}$) *holds.*
(ii) *There are no uncountable measurable cardinals.*

Finally, let us have a look at Hanf numbers. We shall draw some corollaries from results in the previous sections, giving links between existence of some new type of Hanf numbers and various forms of compactness. The existence of this new Hanf number for every finitely generated logic is, as it turns out, equivalent to Vopenka's principle. Let us first recall some definitions from Section II.6:

3.2.9 Definitions. Let $\mathscr{L}$ be a logic.

(i) Let $\Phi \subset \mathscr{L}[\tau]$ be a set of sentences and λ be a cardinal. Φ *pins down the cardinal* λ, iff Φ has a model of cardinality λ, but Φ has no models of arbitrary large cardinalities.
(ii) We define a function $h_\kappa(\mathscr{L})$ to be the supremum of all cardinals that can be pinned down by a set of $\mathscr{L}$-sentences of power $\leq \kappa$. $h_1(\mathscr{L}) = h(\mathscr{L})$ from Section II.6.
(iii) We define $h_\infty(\mathscr{L})$ to be the supremum of all $h_\kappa(\mathscr{L})$ if it exists, and otherwise we write $h_\infty(\mathscr{L}) = \infty$. We say that $\mathscr{L}$ *has a global Hanf number*, if $h_\infty(\mathscr{L}) < \infty$.

Global Hanf numbers do not necessarily exist, even for finitely generated logics. Clearly compact logics do have global Hanf number ω. The following clarifies the relationship between compactness and global Hanf numbers:

3.2.10 Proposition* (Makowsky). *Let* $\mathscr{L}$ *be a logic.*

(i) *If* $\mathscr{L}$ *is* (∞, λ)*-compact then* $h_\infty(\mathscr{L}) \leq \lambda$.
(ii) *If* $\mathscr{L}$ *is* $[\omega]$*-compact and* CHAIN($\mathscr{L}$) *holds, then* $h_\infty(\mathscr{L}) = \omega$.
(iii) *If* $\mathscr{L}$ *has a global Hanf number, then* Comp($\mathscr{L}$) $\neq \varnothing$.

Proof. (i) This is a standard application of the method of diagrams.

(ii) Using Proposition 3.1.5 we construct an $\mathscr{L}$-chain of proper $\mathscr{L}$-extensions. Now CHAIN($\mathscr{L}$) allows us to go as far as we want.

(iii) Let λ_0 be the global Hanf number of $\mathscr{L}$. Clearly, every structure of cardinality $\geq \lambda_0$ has a proper $\mathscr{L}$-extension, i.e., EXT(λ_0, $\mathscr{L}$) holds, so the result follows from Theorem 3.2.4. □

3.2.11 Corollary* (Makowsky). *Assume there are no uncountable measurable cardinals. If $\mathscr{L}$ is a logic which has a global Hanf number then $\mathscr{L}$ has finite dependence.*

Proof. By Proposition 3.2.10, Comp($\mathscr{L}$) $\neq \emptyset$, so by Theorem 1.5.2 and the assumption on measurable cardinals, $\mathscr{L}$ is $[\omega]$-compact. Now we apply the finite dependence theorem (2.2.1). □

The following is an improvement of Theorem 1.5.17.

3.2.12 Theorem* (Makowsky). *The following statements are equivalent:*

(i) *For every finitely generated logic $\mathscr{L}$* SComp($\mathscr{L}$) $\neq \emptyset$.
(ii) *Every finitely generated logic $\mathscr{L}$ has a global Hanf number.*
(iii) *For every finitely generated logic $\mathscr{L}$* Comp($\mathscr{L}$) $\neq \emptyset$.
(iv) *Vopenka's principle.*

Proof. (i) → (ii) This follows from Proposition 3.2.10(i) above.
(ii) → (iii) This follows from Proposition 3.2.10(iii) above.
(iii) → (iv) and (iv) → (i) both follow from Theorem 1.5.17. □

Theorem 3.2.12 tells us that there are logics which have no global Hanf number provided Vopenka's principle is false. Let us end this section with some examples:

3.2.13 Examples. (i) Let $\mathscr{L}$ be $\mathscr{L}_{\omega_1\omega}$. Let $\mathfrak{A}$ be a complete expansion of a structure of cardinality λ. If there are no uncountable measurable cardinals, $\mathfrak{A}$ has no proper $\mathscr{L}$-extensions (see Theorem 1.2.3), so the complete $\mathscr{L}$-theory of $\mathfrak{A}$ pins down λ. Hence, assuming there are no uncountable measurable cardinals, $\mathscr{L}$ has no global Hanf number.

(ii) Let $\mathscr{L}_0$ be the logic $\mathscr{L}_{\omega\omega}(Q_0)$ and $\mathscr{L}_1$ be $\mathscr{L}_{\omega\omega}(Q_1)$. In Malitz–Reinhardt [1972b] it is shown that $h_\omega(\mathscr{L}_i)$ ($i = 0, 1$) is bigger than the first uncountable measurable cardinal.

(iii) Let $\mathscr{L}$ be $\mathscr{L}^2_{\omega\omega}$, i.e., second-order logic. By Magidor [1971] $h_\infty(\mathscr{L})$ is smaller than the first extendible cardinal.

(iv) In Corollary XVII.4.5.12 it is shown that $h_1(\Delta_3(\mathscr{L}_A))$ is bigger than the first extendible cardinal.

3.3. The Amalgamation Property

In this section we present our main theorem in the analysis of the amalgamation properties:

3.3.1 Theorem (Abstract Amalgamation Theorem). *Let $\mathscr{L}$ be a logic with dependence number* $o(\mathscr{L}) = \lambda$ *and with the amalgamation property. Then $\mathscr{L}$ is ultimately compact. In fact it is* $[\infty, \lambda]$*-compact.*

The proof of this theorem will be outlined in Section 3.5. Here we mainly illustrate various consequences of this theorem and discuss examples and limitations.

For logics with finite dependence we immediately get:

3.3.2 Theorem. *For a logic $\mathscr{L}$ with finite dependence the following are equivalent*:

(i) *$\mathscr{L}$ is compact.*
(ii) *$\mathscr{L}$ has the amalgamation property.*
(iii) *$\mathscr{L}$ has the joint embedding property.*

Proof. We have seen in Theorem 3.1.11 that (i) implies (ii) and (iii), and that (iii) implies (ii). So let us assume (ii). From Theorem 3.3.1 we get immediately that $\mathscr{L}$ is $[\lambda]$-compact for every regular λ and therefore compact by Theorem 1.1.8 ☐

D. Mundici has studied the joint embedding property extensively, cf. Mundici [1982b, 1983a]. In general the joint embedding property is not known to be equivalent to the *amalgamation property*. In Chapter XIX some consequences of the joint embedding property are studied. Using more of the set-theoretic machinery we get

3.3.3 Theorem. *If $\mathscr{L}$ is a logic with* $o(\mathscr{L}) < \mu_0$, *where μ_0 is the first uncountable measurable cardinal, then the following are equivalent*:

(i) *$\mathscr{L}$ is compact.*
(ii) *$\mathscr{L}$ has the amalgamation property.*
(iii) *$\mathscr{L}$ has the joint embedding property.*

Proof. We only have to prove (ii) $\rightarrow$ (i): Using Theorem 3.3.1 we get $[\kappa]$-compactness for some $\kappa < \mu_0$, so by Theorem 1.5.2 we get $[\omega]$-compactness and therefore by Theorem 2.2.1, finite dependence. So now the results follows by another application of Theorem 3.3.1. ☐

3.3.4 Corollary. *Let $\mathscr{L}$ be a logic with* $o(\mathscr{L}) < \mu_0$, *where μ_0 is the first uncountable measurable cardinal.*

(i) *If $\mathscr{L}$ has the amalgamation property (joint embedding property) then every sublogic $\mathscr{L}_0 < \mathscr{L}$ has the amalgamation property (joint embedding property).*
(ii) *If $\mathscr{L}$ has the amalgamation property (joint embedding property) then $\Delta(\mathscr{L})$ also has the amalgamation property (joint embedding property).*

Proof. (i) This is clearly true for compactness, so by Theorem 3.3.3 also for the amalgamation property.

(ii) It is easy to see, that the Δ-closure of logics preserves compactness and finite dependence. ☐

The reader may try to prove this without using Theorem 3.3.3.

3.3.5 Corollary. *Let $\mathscr{L}$ be a logic with* $o(\mathscr{L}) < \mu_0$, *where μ_0 is the first uncountable measurable cardinal. If $\mathscr{L}$ has the Robinson property, then $\mathscr{L}$ is compact.*

Proof. Use Theorem 3.1.14 and Theorem 3.3.3. ☐

For logics with finite dependence we shall see in Chapter XIX another proof of Corollary 3.3.5 without using Theorem 3.3.1.

The rest of this section is devoted to examples and applications of the above theorems. The first example gives a real application of Theorem 3.3.2 for the following result was originally derived from it:

3.3.6 Example. Let $\mathscr{L}_{\omega\omega}(Q_\kappa)$ be the first-order logic with the additional quantifier "there exist at least κ many." Theorem 3.3.2 gives us immediately that this logic does not satisfy the amalgamation property for any cardinal κ. For $\kappa = \omega$ or ω_1 this was shown by Malitz–Reinhardt [1972b], the other cases were open till Theorem 3.3.2 was proven.

The next examples all show that the assumption on large cardinals cannot be dropped in any of the above statements.

3.3.7 Examples. (i) The logic $\mathscr{L}_{\infty\infty}$ has no occurrence number. Since this logic can describe any structure up to isomorphism, one easily verifies that the Robinson property and the amalgamation property hold trivially, but $\mathscr{L}_{\infty\infty}$ has no compactness whatsoever.

(ii) In Makowsky–Shelah [1983] it is shown that if κ is an extendible cardinal, then $\mathscr{L}^2_{\kappa\kappa}$, i.e., second-order logic with conjunctions, first-order and second-order quantification over $<\kappa$ many formulas or variables, satisfies the Robinson property, and hence the Amalgamation property and is $[\infty, \kappa]$-compact. Clearly, $\mathrm{o}(\mathscr{L}^2_{\kappa\kappa}) = \kappa$ and $\mathscr{L}^2_{\kappa\kappa}$ is not $[\lambda]$-compact for any $\lambda < \kappa$.

(iii) Now let us look at $\mathscr{L}_{\lambda\omega}$ with additionally the finite dependence property. It is easy to see, that for $\lambda > \omega$ the amalgamation property fails. But $\mathscr{L}_{\lambda\omega} < \mathscr{L}^2_{\kappa\kappa}$ for $\lambda < \kappa$, so Corollary 3.3.4 cannot be improved.

(iv) The logic $\mathscr{L}_{\infty\omega}$ satisfies the amalgamation property trivially, but does not satisfy the Robinson property, as pointed out in Makowsky–Shelah [1979].

(v) In Section 3.5 we present a $[\omega]$-compact logic $\mathscr{L}$ which has the amalgamation property, but for which $\mathrm{Comp}(\mathscr{L})$ has a large gap. This example presupposes the existence of strongly compact cardinals.

3.4. *Proof of the Abstract Amalgamation Theorem*

3.4.1 Synopsis. We first observe that by Theorem 1.1.9 it suffices to prove the following weaker theorem:

3.4.2 Theorem. *Let λ be a regular cardinal and $\mathscr{L}$ be a logic with dependence number $\mathrm{o}(\mathscr{L}) \le \lambda$ and with the amalgamation property. Then $\mathscr{L}$ is $[\lambda]$-compact.*

We give first an outline of the proof, to help the reader. We assume for contradiction that λ is regular and $\mathscr{L}$ is not $[\lambda]$-compact. Using Theorem 1.2.2 we construct a class K of linear orderings with additional predicates in which points of confinality λ are absolute. Inside K we show the existence of some sufficiently homogeneous structure $\mathfrak{N}$. In $\mathfrak{N}$ we shall find $\mathfrak{M}_i$ $(i = 0, 1, 2)$ being a counterexample to the amalgamation property for $\mathscr{L}$. The dependence number and the isomorphism axiom will be needed to show that $\mathfrak{M}_0 <_L \mathfrak{M}_i$ $(i = 1, 2)$ and the absoluteness of "cofinality λ" to show that there is no amalgamating structure.

The counterexample to amalgamation is patterned after the following example: Let K be the class of dense linear orderings with an additional unary predicate Red such that both Red and its complement are dense. Let $\mathfrak{A} <_K \mathfrak{B}$ hold if $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ and the universe of $\mathfrak{A}$ is a dense subset of the universe of $\mathfrak{B}$. We shall show that K with this notion of substructure $<_K$ does not allow amalgamation: For this let $\mathfrak{A}_0$ be the rationals properly coloured, and let $\mathfrak{A}_i$ $(i = 1, 2)$ the rationals augmented by one element (say π) coloured Red in $\mathfrak{A}_1$ and not coloured in $\mathfrak{A}_2$. Clearly, $\mathfrak{A}_0 <_K \mathfrak{A}_i$ $(i = 1, 2)$, but no amalgamating structure exists, since otherwise π is simultaneously coloured and not coloured.

3.4.3 The Structure $\mathfrak{M}$. Now, let $\lambda \geq \mathrm{OC}(\mathscr{L})$ be regular and $\mathscr{L}$ not $[\lambda]$-compact. By Theorem 1.2.2, λ is cofinally characterizable in $\mathscr{L}$ in a structure $\mathfrak{M}$. We need some more information on $\mathfrak{M}$:

Let Δ, $\Sigma_1 = \{\varphi_\alpha : \alpha < \lambda\}$ be the counterexample to $[\lambda]$-compactness. Put $\Sigma^{\alpha_1} = \{\varphi_\beta : \beta < \alpha\}$ and $\mathfrak{M}_\alpha \models \Delta \cup \Sigma^{\alpha_1}$. Without loss of generality the $\mathfrak{M}_\alpha$'s are structures of some countable vocabulary τ (coding more predicates with parameters), and have the same power $\mu \geq \lambda$, $\mathfrak{M}_\alpha = \langle M_\alpha, Q_n\ (n \in \omega)\rangle$.

We want to code all the $\mathfrak{M}_\alpha$'s into one structure. So we let $\mathfrak{M}$ be such that:

(1) $\mathfrak{M} = \langle M, <, \bar{Q}_n, c_j\ (n \in \omega, j \in \lambda)\rangle$.
(2) $\langle M, <\rangle$ is a linear order of cofinality λ such that every initial segment has power μ (of order type $\mu^* + \lambda$, for example).
(3) $\{c_j : j < \lambda\} \subset M$ is increasing and unbounded.
(4) If $x \leq c_j$ but $x > c_i$ for every $i < j$ then

$$\langle \{y \in M : y < x\}, \bar{Q}_n(x, -, -, \ldots, -) \rangle \cong \mathfrak{M}_\alpha.$$

Let $T = Th_{\mathscr{L}}(\mathfrak{M})$ for some fixed $\mathfrak{M}$ as described above.

Claim. Then T cofinally characterizes λ.

This is proved like Theorem 1.2.2.

3.4.4 The Class $K(\mathfrak{M})$. For the rest of this section $\mathfrak{M}$ is fixed. We now define a class of structures $K(\mathfrak{M})$:

The vocabulary of $K(\mathfrak{M})$ is that of $\mathfrak{M}$ without the constant symbols for c_j but with two additional unary predicate symbols P and R and one additional binary predicate symbol I. Actually our main focus is on the order together with P, R, and I is used to code copies of $\mathfrak{M}$, which we need to guarantee the absoluteness of cofinality λ.

A model in $K(\mathfrak{M})$ is of the form $\mathfrak{A} = \langle A, <, \bar{Q}_i, P, R, I \rangle$ with the requirements:

(K1) If $x \in P$ then the cofinality of x in $\langle A, < \rangle$ is λ with a witnessing sequence $\{c_j(x): j < \lambda\}$.
(K2) $(a, x) \in I$ implies that $a < x$.
(K3) $(a, x) \in I$ implies that $x \in P$ and $a \notin P$.
(K4) $P(x)$ implies that $I(c_j(x), x)$ for every $j \in \lambda$.

Put $J_A^x = \{a \in A: (a, x) \in I\}$ and $\mathfrak{J}_A^x$ be the substructure of $\langle A, <, \bar{Q}_i \rangle$ induced by J_A^x.

(K5) The structure $\langle \mathfrak{J}_A^x, c_j(x) \rangle$ is isomorphic to $\mathfrak{M}$.
(K6) $R \subset P$.

We call a structure in $K(\mathfrak{M})$ *pure* if additionally

(K7) $\bar{Q}_i$ is false where not defined by the previous requirements.

3.4.5 Comments. Note that if $\mathfrak{A} \in K(\mathfrak{M})$ is pure and P in $\mathfrak{A}$ is empty, then $\mathfrak{A}$ is just a linear ordering, i.e., all the other relations are empty, too, by (K7). If we add to $\mathfrak{M}$ one point at the end, say x and let $P = \{x\}$, we get a structure in $K(\mathfrak{M})$. We denote this structure by $\mathfrak{M}^{+1}$.

In general the structures in $K(\mathfrak{M})$ are linearly ordered structures where every point in P has a copy of $\mathfrak{M}$ attached to it in such a way that different points have almost disjoint copies of $\mathfrak{M}$, and $\mathfrak{M}$ cofinally reaches its point in P. The choice of R can be any subset of P. More precisely:

Fact 1. For every $\mathfrak{A} \in K(\mathfrak{M})$ and every $a, a' \in A$, $J_A^a \cap J_A^{a'}$ is bounded below both a, a'.

This is proved using the fact that $\mathfrak{M}$ is of order type $\mu^* + \lambda$. Note that this is first-order expressible and could have been stated also as an axiom among (K1–K7).

Fact 2. If $\mathfrak{A} \in K(\mathfrak{M})$ and $a \in P^A$ and we form $\mathfrak{A}'$ by changing the truth value of $a \in R^A$, but leaving everything else fixed, then $\mathfrak{A}' \in K(\mathfrak{M})$.

Next we define the notion of K-substructure, $\mathfrak{A} \subset_K \mathfrak{B}$ for, $\mathfrak{A}, \mathfrak{B} \in K(\mathfrak{M})$ by:

(K8) $\mathfrak{A} \subset \mathfrak{B}$.
(K9) If $x \in P^A$ then $J_B^x \subset A$.
(K10) If $x \in P^B - P^A$ then $\{a \in A: a < x\}$ is bounded below x in $\mathfrak{B}$, i.e., there is $b_x \in B$ such that $b_x < x$ and for each $a \in A$ with $a < x$ we have $a < b_x$.

The idea behind this is that in $\mathfrak{B}$ new points in P_B are added to P^A in a way that they are not limits of points from $\mathfrak{A}$, and that points in $\mathfrak{A}$ which are of cofinality λ, are also of cofinality λ in $\mathfrak{B}$ with the same copy of $\mathfrak{M}$ ensuring this as in $\mathfrak{A}$.

This ends the definition of $K(\mathfrak{M})$ and of K-substructures.

3.4.6 Some More Facts About $K(\mathfrak{M})$. Before we proceed with the proof of the theorem we collect some more facts:

Definition. If $\mathfrak{A}_1$. $\mathfrak{A}_2 \in K(\mathfrak{M})$ we define $\mathfrak{A}_1 + \mathfrak{A}_2$ to be the disjoint union of $\mathfrak{A}_1, \mathfrak{A}_2$ with the linear ordering of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ for their elements and $a_1 < a_2$ for every $a_1 \in A_1, a_2 \in A_2$. For the other relations we just take their unions.

Fact 3. If $\mathfrak{A}_1, \mathfrak{A}_2 \in K(\mathfrak{M})$ so $\mathfrak{A}_1 + \mathfrak{A}_2 \in K(\mathfrak{M})$ and $\mathfrak{A}_i \subset_K \mathfrak{A}_1 + \mathfrak{A}_2$ $(i = 1, 2)$.

This is clear from the definitions.

Definition. Denote by $L_A^x = \{a \in A : a < x\}$ and by $\mathfrak{L}_A^x$ the structure $\mathfrak{A} \upharpoonright L_A^x$. If $\mathfrak{B} \in K(\mathfrak{M})$ and $A \subset B$ we define a substructure $\mathfrak{C}(A)$ of $\mathfrak{B}$ by

$$\mathfrak{C}(A) = \mathfrak{B} \upharpoonright \bigcup_{a \in A} J_B^a \cup A.$$

This makes sense by Fact 1 and ensures that:

Fact 4. For every $\mathfrak{B} \in K(\mathfrak{M})$, $A \subset B$, $\mathfrak{C}(A) \subset_K \mathfrak{B}$, but in general $\mathfrak{C}(A)$ is not pure. Furthermore, if A is bounded in $\mathfrak{B}$ by b, i.e., there is $b \subset B$ with $A \subset L_B^b$, so $\mathfrak{C}(A) \subset \mathfrak{L}_B^b$ and $\mathfrak{C}(L_B^b) = \mathfrak{L}_B^b$.

Fact 5. If $\mathfrak{A} \in K(\mathfrak{M})$ and $d \in P^A$ then $\mathfrak{A} \upharpoonright L_A^d \subset_K \mathfrak{A}$.

Fact 6. If $\{\mathfrak{A}_i : i < \alpha\}$ is a sequence of structures in $K(\mathfrak{M})$ such that $\mathfrak{A}_i \subset_K \mathfrak{A}_{i+1}$ then $\mathfrak{A} = \bigcup_{i < \bar{a}} \mathfrak{A}_i \in K(\mathfrak{M})$ and $\mathfrak{A}_i \subset_K \mathfrak{A}$ for each $i < \alpha$.

Definition. If $\mathfrak{A}_1, \mathfrak{A}_2 \in K(\mathfrak{M})$, $\mathfrak{B}_i \subset_K \mathfrak{A}_i$ $(i = 1, 2)$ and $f: \mathfrak{B}_1 \cong \mathfrak{B}_2$ is an isomorphism, we define $\mathfrak{A}_1 +_f \mathfrak{A}_2$ in the following way: Form the disjoint union of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ modulo f (i.e., identify elements only via f). This makes it into a partially ordered structure where $a_i \in A_i$ $(i = 1, 2)$ are comparable only if one of them is in the range or domain of f, or there is b between a_1, a_2 which has been identified. For incomparable a_1, a_2 we extend the order on $\mathfrak{A}_1 +_f \mathfrak{A}_2$ setting $a_1 < a_2$.

Fact 7. If $\mathfrak{A}_1, \mathfrak{A}_2 \in K(\mathfrak{M})$ and $f: \mathfrak{B}_i \cong \mathfrak{B}_2$, $\mathfrak{B}_i \subset_K \mathfrak{A}_i$ $(i = 1, 2)$ then $\mathfrak{A}_1 +_f \mathfrak{A}_2 \in K(\mathfrak{M})$ and $\mathfrak{A}_i \subset_K \mathfrak{A}_1 +_f \mathfrak{A}_2$.

The proofs of the facts are left to the reader.

3.4.7 Two Lemmas. The next lemma is crucial for our construction:

Lemma 1. *If* $\mathfrak{A} \in K(\mathfrak{M})$ *and* $\mathfrak{B}$ *is an* $\mathscr{L}$*-extension of* $\mathfrak{A}$ *and* $\{d_j : j < \lambda\}$ *is confinal in* J_A^a *for* $a \in P^A$, *then* $\{d_j : j < \lambda\}$ *cofinal in* J_B^a.

Proof. Let $a \in P^A$, so $\mathfrak{J}_A^a \cong \mathfrak{M}$ by (K5) and by our assumption on $\mathscr{L}$ and $\mathfrak{M}$, $\mathscr{L}$ cofinally characterizes λ in $\mathfrak{M}$. Using relativization of $\mathscr{L}$ the structure $\mathfrak{J}_B^a$ is an $\mathscr{L}$-extension of $\mathfrak{M}$ so $\mathfrak{M}$ is confinal in $\mathfrak{J}_B^a$, hence $\{d_j : j < \lambda\}$ is cofinal in $\mathfrak{J}_B^a$ which proves the lemma. □

The next lemma is proved in a similar way as one usually proves the existence of homogeneous structures for Jonsson classes (cf. Chapter XX). We omit the proof here and show how one can now complete the proof of the theorem. A detailed proof of the lemma may be found in Makowsky–Shelah [1983].

Lemma 2. *There is a structure $\mathfrak{N}$ in $K(\mathfrak{M})$ and $d_1 < d_2 < d_3$ in $\mathfrak{N}$ with $d_i \in P^N$ ($i = 1, 2, 3$), $d_1 \in R^N$, $d_2 \notin R^N$ such that:*

(i) $\mathfrak{N} \restriction L_N^{d_1} \cong \mathfrak{N} \restriction L_N^{d_2} \cong \mathfrak{N} \restriction L_N^{d_3}$; *and*

(ii) *If* $\mathfrak{A} \subset_K \mathfrak{N} \restriction L_N^{d_i}$ ($i = 1, 2$) *is bounded in* $\mathfrak{N} \restriction L_N^{d_i}$ *then* $\mathfrak{N} \restriction L_N^{d_i} \cong \mathfrak{N} \restriction L_N^{d_3}$ *over* $\mathfrak{A}$ ($i = 1, 2$).

3.4.8 Proof of the Abstract Amalgamation Theorem. Put $\mathfrak{M}_i = \mathfrak{N} \restriction L_N^{d_i}$ ($i = 1, 2, 3$). We have to verify some claims:

Claim 1. $\mathfrak{M}_i <_{\mathscr{L}} \mathfrak{M}_3$ ($1 = 1, 2$).

Proof. Let φ be an $L[\tau(\mathfrak{M}_i)]$-sentence. Since the dependence number $o(\mathscr{L}) \leq \lambda$, φ depends on $< \lambda$ many constants, hence there is $a \in M_i$ and all the constants of φ are in $L_{M_i}^a$. So by Fact 4, $\mathfrak{M}_i \restriction L_{M_i}^a$ is a bounded K-substructure of both $\mathfrak{M}_i$ and $\mathfrak{M}_3$. So, by Lemma 2(ii) above, $\langle \mathfrak{M}_i, L_{M_i}^a \rangle$ is isomorphic to $\langle \mathfrak{M}_3, L_{M_3}^a \rangle$ hence by the basic isomorphism axiom,

$$\langle \mathfrak{M}_i, L_{M_i}^a \rangle \models \varphi \quad \text{iff} \quad \langle \mathfrak{M}_3, L_{M_3}^a \rangle \models \varphi.$$

Now let $f\colon \mathfrak{M}_1 \cong \mathfrak{M}_2$ be the isomorphism from Lemma 2(i) above, and $g_i\colon \mathfrak{M}_i \to \mathfrak{M}_3$ ($i = 1, 2$) the $\mathscr{L}$-embeddings from Claim 1.

Since $\mathscr{L}$ has AP, let $\mathfrak{A}$ be the amalgamation for $g_1\colon \mathfrak{M}_1 \to \mathfrak{M}_3, g_2 f\colon \mathfrak{M}_1 \to \mathfrak{M}_3$.

Claim 2. $\mathfrak{A} \models d_1 = d_2$.

Proof. $d_i \in P^{M_3}$ ($i = 1, 2$) are both of cofinality λ and $g_i(M_1)$ is cofinal in $\mathfrak{M}_3 \restriction L_{M_3}^{d_1}$, and $g_2 f(M_1)$ is cofinal in $\mathfrak{M}_3 \restriction L_{M_3}^{d_2}$, so by Lemma 1 above also in $\mathfrak{A} \restriction L_A^{d_1}$ and $\mathfrak{A} \restriction L_A^{d_2}$, hence $\mathfrak{A} \models d_1 = d_2$.

But Claim 2 contradicts our assumption of Lemma 2 above that $d_1 \in R^{\mathfrak{A}}$ and $d_2 \notin R^{\mathfrak{A}}$. This completes the proof of the abstract amalgamation theorem. □

In fact the same proof gives also the following versions of the abstract amalgamation theorem:

3.4.9 Theorem*. *Let κ be a regular cardinal and $\mathscr{L}$ be a logic such that:*

(i) *The Lowenheim number $l_\kappa(\mathscr{L})$ of $\mathscr{L}$ is κ.*

(ii) $\mathrm{Am}(\kappa, \mathscr{L})$ *holds.*

Then $\mathscr{L}$ is (κ, κ)-compact.

3.4.10 Theorem*. *Let $\mathscr{L}$ be a logic with dependence number* $o(\mathscr{L}) \leq \lambda$. *If* $\mathrm{Am}(\kappa, \mathscr{L})$ *holds for every $\kappa \geq \lambda$ then $\mathscr{L}$ is $[\infty, \lambda]$-compact.*

It is open whether the converse of Theorem 3.4.10 also holds. Note however that for λ smaller than the first uncountable measurable cardinal the converse does hold.

3.5. An Intriguing Example

Let us now look at logics which do have the amalgamation property, but have a large occurrence number. One naturally wonders if such a logic has to be an extension of $\mathscr{L}_{\kappa\kappa}$ for some uncountable κ, possibly bigger than the occurrence number. This is clearly not the case, provided the logic $\mathscr{L}$ is $[\omega]$-compact. The purpose of this section is to present an example of a logic $\mathscr{L}$ with occurrence number $\mathrm{OC}(\mathscr{L})$ bigger than the first uncountable measurable cardinal μ_0, which is still $[\lambda]$-compact for every $\lambda < \mu_0$, satisfies the amalgamation property, but is not compact. If, however, a logic $\mathscr{L}$ satisfies the amalgamation property but is not $[\omega]$-compact, then we know that its occurrence number is bigger than μ_0, and therefore, by Proposition 1.2.4, every τ-structure $\mathfrak{A}$ with $\operatorname{card}(\mathfrak{A}) < \mu_0$ has an $\mathscr{L}$-maximal expansion. This can be used to show that for every $\varphi \in \mathscr{L}_{\mu_0\omega}[\tau]$ there is τ', $\tau \subset \tau'$ and a set $\Sigma \subset \mathscr{L}[\tau']$ such that $\mathrm{Mod}_{\mathscr{L}_{\mu_0\omega}}(\varphi) = \mathrm{Mod}_{\mathscr{L}}(\Sigma) \restriction \tau$. In the presence of the Robinson property τ' can be assumed to be τ. We develop this idea further in Chapter XIX, Theorem 1.12.

3.5.1 Definitions. Let μ be a cardinal and $E \subset P(\mu)$ a family of subsets of μ.

(i) We say that E is $(<\kappa)$-closed, κ a cardinal, if for every $\lambda < \kappa$ and every ultrafilter F on λ the following holds: Given $\{A_i \subset \mu : i < \lambda\}$, then $\{i \in \mu : A_i \in E\} \in F$ implies that $\lim_F A_i = \{\alpha \in \mu : \{i \in \lambda : \alpha \in A_i\} \in F\} \in E$. We say that E is $(<\kappa)$-bi-closed if both E and $P(\mu) - E$ are $(<\kappa)$-closed.

(ii) If $\{\psi_i : i \in \mu\}$ is a family of $\mathscr{L}$-formulas, we define a connective $\bigwedge_{i\in\mu}^{E} \psi_i$ by $\bigvee_{A\in E}(\bigwedge_{i\in A} \psi_i \vee \bigwedge_{i\in\mu - A} \neg\, \psi_i)$.

3.5.2 Remarks. (i) If E is a κ-complete ultrafilter on μ then both E and $P(\mu) - E$ are $(<\kappa)$-closed.

(ii) The connective $\bigwedge_{i\in\mu}^{E} \psi_i$ is a generalization of the connective $\bigcap_F$ where F is some ultrafilter.

3.5.3 Definitions. (i) Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals. We denote by $E(\kappa_1, \kappa_2)$ the set of $(<\kappa_1)$-bi-closed families $E \subset P(\mu)$ with $\mu < \kappa_2$.

(ii) Let $\mathscr{L}_{E(\kappa_1,\kappa_2),\kappa_2}$ be the closure of first-order logic under all the infinitary operations $\bigwedge_{i\in\mu}^{E}$ for $E \in E(\kappa_1, \kappa_2)$.

(iii) Recall that $\mathscr{L} = \mathscr{L}_{D(\kappa_1,\kappa_2),\kappa_2}$ was defined in Example 1.6.6 in a similar way as (ii) above, but instead of $(<\kappa_1)$-bi-closed sets we only used κ_1-complete ultrafilters.

3.5.4 Proposition* (Shelah). *Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals and $\mathscr{L} = \mathscr{L}_{E(\kappa_1,\kappa_2),\kappa_2}$.*

(i) $\mathscr{L}_{D(\kappa_1,\kappa_2),\kappa_2} < \mathscr{L}$.

(ii) $\mathscr{L} < \mathscr{L}_{\kappa_2,\kappa_2}$.

(iii) *$\mathscr{L}$ is $[\infty, \kappa_2]$-compact.*

(iv) *Every ultrafilter F on $\mu < \kappa_1$ is in* UF($\mathscr{L}$), *i.e., is related to* $\mathscr{L}$.
(v) *For every cardinal $\mu < \kappa_1$ is the logic $\mathscr{L}$ $[\mu]$-compact.*
(vi) *For no cardinal μ, $\kappa_1 \leq \mu < \kappa_2$ is $\mathscr{L}$ $[\mu]$-compact.*

Proof. Essentially the same as in Section 1.6. □

3.5.5 Theorem* (Shelah). *Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals and $\mathscr{L} = \mathscr{L}_{E(\kappa_1,\kappa_2),\kappa_2}$. Then $\mathscr{L}$ satisfies the joint embedding property, and therefore the amalgamation property.*

Outline of Proof. Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be two disjoint τ-structures such that $\mathfrak{M}_1 \equiv_{\mathscr{L}} \mathfrak{M}_2$ and let $D_{\mathscr{L}}(\mathfrak{M}_i)$ $(i = 1, 2)$ be their $\mathscr{L}$-diagrams. We want to show that $D^* = D_{\mathscr{L}}(\mathfrak{M}_1) \cup D_{\mathscr{L}}(\mathfrak{M}_2)$ has a model. Since $\mathscr{L} < \mathscr{L}_{\kappa_2,\kappa_2}$ and κ_2 is strongly compact, it suffices to show that for every subset $\Gamma_1 \subset D_{\mathscr{L}}(\mathfrak{M}_1)$ and $\Gamma_2 \subset D_{\mathscr{L}}(\mathfrak{M}_2)$ with $\text{card}(\Gamma_l) < \kappa_2$, $\Gamma_1 \cup \Gamma_2$ has a model.

Let Γ_1, Γ_2 be given and assume $\Gamma_2 = \{\varphi_i(\bar{a}): i < \mu < \kappa_2\}$. Put

$$E_0 = \{A \subset \mu: \Gamma_1 \cup \{\varphi_i: i \in A\} \textit{ has a model}\}.$$

If $\mu \in E_0$ we are done. So assume, for contradiction that $\mu \notin E_0$. Clearly, $\varnothing \in E_0$, since $\mathfrak{M}_1$ can be expanded to a model of Γ_1.

Claim 1. *E_0 is $(<\kappa_1)$-closed.*

This can be established using Proposition 3.5.4(iv).

Claim 2. *If $E \subset P(\mu)$, $\mu < \kappa_2$ is $(<\kappa_1)$-closed and $\mu \notin E$, then there is $E_1 \subset P(\mu) - E$ with $\mu \in E_1$ such that E_1 is $(<\kappa_1)$-bi-closed.*

This is proved using a reduction to infinitary propositional calculus with conjunctions of length less than κ_1 and the fact that κ_1 is strongly compact.

Clearly, $\mathfrak{M}_2 \models \bigwedge_{i\in\mu} \varphi_i(\bar{a})$, and therefore, $\mathfrak{M}_2 \models \bigwedge^{E_1}_{i\in\mu} \varphi_i(\bar{a})$. Since $\mathscr{L}$ is closed under existential quantification of length less than κ_2, $\exists\bar{x} \bigwedge^{E_2}_{i\in\mu} \varphi_i(\bar{x})$ is an $\mathscr{L}$-sentence and $\mathfrak{M}_2 \models \exists\bar{x} \bigwedge^{E_1}_{i\in\mu} \varphi_i(\bar{x})$. So also $\mathfrak{M}_1 \models \exists\bar{x} \bigwedge^{E_1}_{i\in\mu} \varphi_i(\bar{x})$. Therefore there is $\bar{b}$ from $\mathfrak{M}_1$ and $A \in E_1$ such that $\mathfrak{M}_1 \models \bigwedge_{i\in A} \varphi_1(\bar{b})$ which shows that $\Gamma_1 \cup \{\varphi_i(a): i \in A\}$ has a model. From this we conclude that $A \in E_0$, contradicting $E_1 \subset P(\mu) - E_0$. □

Using the finite dependence structure theorem and the fact that $\mathscr{L}_{E(\kappa_1,\kappa_2),\kappa_2}$ is $[\omega]$-compact, we get now

3.5.6 Proposition* (Shelah). *Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals. Then the two logics $\mathscr{L}_{E(\kappa_1,\kappa_2),\kappa_2}$ and $\mathscr{L}_{D(\kappa_1,\kappa_2),\kappa_2}$ are equivalent.*

3.5.7 Corollary* (Shelah). *Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals. Then the logic $\mathscr{L}_{D(\kappa_1,\kappa_2),\kappa_2}$ has the joint embedding property, and therefore the amalgamation property.*

3.5.8 Remark. In Chapter XIX, Theorem 1.1 states that, if $\mathscr{L}$ is a small logic with $s(\omega) = \lambda$ (s the size function of $\mathscr{L}$) which satisfies the joint embedding property, then there are atmost 2^λ many regular cardinals μ such that $\mathscr{L}$ not $[\mu]$-compact. Theorem 3.5.4 shows that this is best possible.

4. *Definability*

4.1. *Preservation Theorems for Sum-like Operations*

In model theory one frequently builds new models from a set of given models and it is often very useful to know that the theory of the so-constructed model only depends on the theories of the models it was built from. Examples are the ultraproduct construction and various other product-like constructions, which mostly go back to the seminal papers (Mostowski [1952], Łos–Suszko [1957], Feferman–Vaught [1959], and Frayne–Morel–Scott [1962]). The possibilities of generalizations of the Łos lemma to logics in general are rather limited, as we have shown in Section 1. For simpler constructions, such as disjoint unions or ordered sums, the preservation properties are usually proved with the use of back-and-forth arguments, as they are generalized in Chapter XIX. The first to consider such properties in the context of abstract model theory was S. Feferman in his papers (Feferman [1972, 1974a, b, 1975]. The theme was then pursued in Shelah [1975], Makowsky [1978], and Makowsky–Shelah [1979].

In the context of abstract model theory, in contrast to specific examples of logics, only sum-like operations have played an independent role. They are also used heavily in Chapters XII and XIII. For this reason we restrict our exposition here to the description of sum-like operations as they are used in the following subsections, and as we think they are of interest for future research. Recent trends in theoretical computer science have shown that abstract model theory offers the appropriate framework to state problems and theorems dealing with specification of abstract data types (Goguen–Burstall [1983] and Mahr–Makowsky [1983a, b, 1984]), correctness of programs (Harel [1979, 1983], Makowsky [1980], and Manders–Daley [1982]) and data base theory (Makowsky [1984]). Especially sum-like operations on abstract data types have been recently investigated by Bergstra–Tucker [1984] to show that some of the concepts in program correctness are probably not stable enough to be transferred from one formalization to another.

4.1.1 Definitions. (i) (Pair of Two Structures). Let τ_1, τ_2 be two disjoint one-sorted vocabularies and $\mathfrak{A}_1, \mathfrak{A}_2$ be τ_i-structures, respectively. We define the *pair* $[\mathfrak{A}_1, \mathfrak{A}_2]$ to be the two-sorted $\tau_1 \cup \tau_2$-structure with universes A_1, A_2 and their respective relations, functions, and constants. If the vocabularies τ_1, τ_2 are not disjoint, we make them disjoint by a name changer and write nevertheless $[\tau_1, \tau_2]$.

(ii) (Pair Preservation Property). If $\mathscr{L}$ is a logic, we say that $\mathscr{L}$ satisfies the *pair preservation property* and write PPP($\mathscr{L}$), if whenever $\mathfrak{A}_{11}$, $\mathfrak{A}_{12}$, $\mathfrak{A}_{21}$, $\mathfrak{A}_{22}$ are structures such that $\mathfrak{A}_{i1} \equiv_{\mathscr{L}} \mathfrak{A}_{i2}$ then $[\mathfrak{A}_{11}, \mathfrak{A}_{21}] \equiv_{\mathscr{L}} [\mathfrak{A}_{12}, \mathfrak{A}_{22}]$.

To verify that a given logic satisfies PPP($\mathscr{L}$) it is often useful to use back-and-forth type arguments, as described in Chapter II and more generally in Chapter XIX. It should be possible to state a general theorem to the effect of when a back-and-forth property implies the pair preservation property, but this does not seem to be a very rewarding line of thought. For the traditional back-and-forth arguments for infinitary logics this analysis has been carried out in Feferman [1972].

4.1.2 Examples. (i) Both $\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_{\infty\omega}$ satisfy the pair preservation property.

(ii) $\mathscr{L}_{\omega_1\omega}$ does not satisfy the pair preservation property (Malitz [1971]). $\mathscr{L}_{\kappa\lambda}$ does satisfy the pair preservation property iff κ is strongly inaccessible (Malitz [1971]).

(iii) $\mathscr{L}_{\omega\omega}(Q_\kappa)$ satisfies the pair preservation property by Wojciechowska [1969].

(iv) For logics with second-order quantification, such as stationary logic $\mathscr{L}_{\omega\omega}(aa)$ we have to distinguish between the possibility that subsets range over the union of the universes, or that we have also two sorts of set variables. In the former case $\mathscr{L}_{\omega\omega}(aa)$ does not satisfy the pair preservation property (cf. Example IV.6.1.2), in the latter case it does (Makowsky–Shelah [1981]).

4.1.3 Definitions. (i) (Algebraic Operations). Let $n \in \omega$ and $\tau_1, \tau_2, \ldots, \tau_n, \sigma$ be vocabularies. Let $\mathbf{F}\colon \mathrm{Str}(\tau_1) \times \cdots \times \mathrm{Str}(\tau_n) \to \mathrm{Str}(\sigma)$ be a function. We say that $\mathbf{F}$ is an *n-ary algebraic operation of type* $\tau = [\tau_1, \tau_2, \ldots, \tau_n, \sigma]$, if $\mathfrak{A}_i$, $\mathfrak{B}_i$ are τ_i-structures and $\mathfrak{A}_i \cong \mathfrak{B}_i$ $(i = 1, \ldots, n)$ then

$$\mathbf{F}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n) \cong \mathbf{F}(\mathfrak{B}_1, \ldots, \mathfrak{B}_n).$$

(ii) ($\mathscr{L}$-Projective Operations). Let $\mathscr{L}$ be a logic. An algebraic operation $\mathbf{F}$ of type τ as above is an *$\mathscr{L}$-projective operation* if the graph of $\mathbf{F}$ is an $\mathscr{L}$-projective class.

(iii) (Preservation Property for Projective Operations). We say, a logic $\mathscr{L}$ has the *preservation property for projective operations* and write PPPO($\mathscr{L}$), if for every $\mathscr{L}$-projective operation $\mathbf{F}$ of type τ, if $\mathfrak{A}_i$, $\mathfrak{B}_i$ are τ_i-structures and $\mathfrak{A}_i \equiv_{\mathscr{L}} \mathfrak{B}_i$ $(i = 1, \ldots, n)$ then $\mathbf{F}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n) \equiv_{\mathscr{L}} \mathbf{F}(\mathfrak{B}_1, \ldots, \mathfrak{B}_n)$.

4.1.4 Examples. (i) First-order logic satisfies the PPPO by Feferman [1974].

(ii) The PPPO follows from the uniform reduction property UR_2 defined in Section 4.2.

(iii) The pair construction in Definition 4.1.1 is a first-order projective operation. Therefore PPP follows from PPPO for any regular logic.

(iv) Various other algebraic operations are studied in Gaifman [1967, 1974], Isbell [1973], Hodges [1974, 1975, 1980], and H. Friedman [1979c].

The preservation property for projective operations seems to be very rare. In fact, it is only known to hold for first-order logic, or for logics with uniform reduction (see Section 4.2). For many applications, however, we need much less. A construction somewhere between disjoint unions and general projective operations is enough to obtain interesting theorems in abstract model theory. In the spirit of this section, dealing with definability properties in logics, we give both an implicit and an explicit definition.

4.1.5 Definitions. (i) (Tree-like Structure). Let τ_{tree} be one-sorted and consist of one unary function symbol $\mathbf{f}$ and one constant symbol $\mathbf{c}$. A τ_{tree}-structure $\mathfrak{T} = \langle T, f, c\rangle$ is a *tree-like* structure, if the following hold.

(a) For every $x \in T, f(x) = x$ iff $x = c$, i.e., f is cycle-free but for its only fixed point c, the root of f.
(b) f is onto.
(c) For every $x \in T$ there is an $n \in \omega$ with $f^n(x) = c$.

For $x \in T$ we denote by T_x the set $f^{-1}(x) - \{x\}$.

(ii) (Augmented Tree-like Structure). Let τ_{aug} be $\tau_{\text{tree}} \cup \{\mathbf{P}\}$, where $\mathbf{P}$ is a unary predicate symbol. A τ_{aug}-structure $\mathfrak{T} = \langle T, f, c, P\rangle$ is an *augmented tree-like structure*, if $\mathfrak{T} \upharpoonright \tau_{\text{tree}}$ is a tree-like structure.

(iii) (Tree-like Sum, Implicit Version). Let τ be a vocabulary with a distinguished predicate symbol $\mathbf{P}$ and let $\mathfrak{A}$, $\mathfrak{B}$ two τ-structures. We now define two structures over the vocabulary $\tau \cup_{\text{disjoint}} \tau_{\text{tree}}$, $\mathfrak{N}^i = \text{Tree}^i_P(\mathfrak{A}, \mathfrak{B})$, $i = 0, 1$, the *tree-like sum over* $\mathbf{P}$, in the following way:

(a) $\mathfrak{N}^i \upharpoonright (\tau_{\text{tree}} \cup \{\mathbf{P}\})$ is an augmented tree-like structure. We write now N^{ix} for T_x above.
(b) For every $x \in N^i$ there is bijection $\varepsilon_x\colon \mathfrak{C} \to N^{ix}$ where $\mathfrak{C}$ is either $\mathfrak{A}$ or $\mathfrak{B}$. This bijection makes N^i_x naturally into a τ-structure which we denote by $\mathfrak{N}^i_x$.
(c) For each symbol $\mathbf{R} \in \tau$ let R_x be its interpretation in $\mathfrak{N}^i_x$. We require now that $R = \mathbf{R}^N = \bigcup_{x \in N} R_x$.
(d) We require further that $P = \mathbf{P}^{Ni}$ be defined by: If $x \in P$ then $\mathfrak{N}^i_x \cong \mathfrak{A}$ and $x \notin P$ then $\mathfrak{N}^i_x \cong \mathfrak{B}$.
(e) If $i = 1$ then $c \in P$ and if $i = 0$ then $c \notin P$.

(iv) (Tree-like Sum, Explicit Version). To make the definition of the tree-like sum $\mathfrak{N}^i = \text{Tree}^i_P(\mathfrak{A}, \mathfrak{B})$ explicit we proceed as follows: We let the universe of $\mathfrak{N}^i$ consist of the set of finite sequences $\langle a_k\colon k < n\rangle$ such that:

(a) $a_k \in A \cup B$;
(b) if $i = 0$ then $a_0 \in A$, but if $i = 1$ then $a_0 \in B$;
(c) $a_k \in P^A \cup P^B$ iff $a_{k+1} \in A$;

Next we define f, the interpretation of $\mathbf{f}$:

(d) For the empty sequence $\langle\ \rangle$ we put $f(\langle\ \rangle) = \langle\ \rangle$;
(e) $f(\langle a_k\colon k \le n\rangle) = \langle a_k\colon k < n\rangle$.

Finally, for every relation symbol $\mathbf{R} \in \tau$ we define its interpretation R by

(f) $(\langle a_k : k \leq n\rangle, \langle b_k : k \leq n\rangle) \in R$ iff $a_k = b_k$ for every $k < n$ and

$$(a_n, b_n) \in R^A \cup R^B.$$

(v) (Tree Preservation Property). Let $\mathscr{L}$ be a logic. We say that $\mathscr{L}$ has the *tree preservation property* and write TPP($\mathscr{L}$), if whenever $\mathfrak{A}$, $\mathfrak{B}$ are as above, $\tau = \tau_0 \cup \{\mathbf{P}\}$ and additionally $\mathfrak{A} \upharpoonright \tau_0 \equiv_{\mathscr{L}} \mathfrak{B} \upharpoonright \tau_0$ then

$$\mathrm{Tree}^0_P(\mathfrak{A}, \mathfrak{B}) \upharpoonright \tau_0 \cup \tau_{\mathrm{tree}} \equiv_{\mathscr{L}} \mathrm{Tree}^1_P(\mathfrak{A}, \mathfrak{B}) \upharpoonright \tau_0 \cup \tau_{\mathrm{tree}}.$$

4.1.6 Remarks. (i) The tree-like sum is not, in general, a projective operation, since Definition 4.1.5(c) is not first-order definable. However, if the logic $\mathscr{L}$ is such, that the structure $\langle \omega, < \rangle$ is $\mathrm{PC}_{\mathscr{L}}$-characterizable, then the tree-like sum is an $\mathscr{L}$-projective operation.

(ii) For regular logics $\mathscr{L}$ the tree preservation property implies the pair preservation property, since the pair can be constructed as a relativized reduct of the tree sum.

(iii) If the distinguished predicate $\mathbf{P}$ in the tree-like sum is not unary, we can still define a tree-like sum over $\mathbf{P}$. We just replace f by a function $s: T \to T^n$ and define s_1 to be s followed by a projection to the first coordinate. Then we express Definition 4.1.5(i)(a) and (b) with s and (c) with s_1.

The construction of the tree-like sum over a predicate P can sometimes be used to define the predicate P implicitly. The precise situation where this is possible is given in the following lemma from Makowsky–Shelah [1979b]. The idea goes back to S. Shelah.

4.1.7 Lemma. *Let $\mathscr{L}$ be a logic, $\tau_i = \tau_0 \cup_{\mathrm{disjoint}} \{\mathbf{P}_i\}$ $(i = 1, 2)$ vocabularies, and $\varphi_i \in \mathscr{L}[\tau_i]$ be sentences having a model, but such that $\varphi_1 \wedge \varphi_2$ has no model. Then there is a sentence $\psi \in \mathscr{L}[\tau_0 \cup_{\mathrm{disjoint}} \tau_{\mathrm{aug}}]$ such that:*

(i) *Every $\tau_0 \cup_{\mathrm{disjoint}} \tau_{\mathrm{tree}}$-structure $\mathfrak{A}$ has at most one expansion $\mathfrak{A}^* \models \psi$.*

(ii) *If $\mathfrak{A}_i (i = 1, 2)$ are τ_i-structures and $\mathfrak{A}_i \models \varphi_i$ then* $\mathrm{Tree}^i_P(\mathfrak{A}_1, \mathfrak{A}_2) \models \psi$ provided we substitute $\mathbf{P}$ for $\mathbf{P}_1, \mathbf{P}_2$, respectively.

Proof. Let $\psi = \psi_0 \wedge \psi_1 \wedge \psi_2$ with:

ψ_0 expresses Definition 4.1.5(i)(a) and (b);
ψ_1 is the $\mathscr{L}$-formalization of "If $x \in P$ then $\mathfrak{N}_x \models \varphi_1$";
ψ_2 is the first-order formalization of "If $x \notin P$ then $\mathfrak{N}_x \models \varphi_2$."

The latter two involve the appropriate substitutions and relativizations. Clearly (ii) too, holds, by our construction of $\mathrm{Tree}^i_P(\mathfrak{A}_1, \mathfrak{A}_2)$. And (i) holds because $\varphi_1 \wedge \varphi_2$ has no model. ☐

We shall use Lemma 4.1.7 in Section 4.4 to prove some abstract theorems.

4.2. Definability, Interpolation and Uniform Reduction

We first recall some definitions from Chapter II, Section 7.

4.2.1 Definitions. (i) A logic $\mathscr{L}$ has the *interpolation property*, and we write INT($\mathscr{L}$), if any two disjoint classes of τ-structures, which are RPC in $\mathscr{L}$, can be separated by some EC-class of $\mathscr{L}$.

(ii) A logic $\mathscr{L}$ has the Δ-*interpolation property*, and we write Δ–INT($\mathscr{L}$), if any class K of τ-structures, such that K and its complement are RPC in $\mathscr{L}$, then K is an EC-class of $\mathscr{L}$.

(iii) A logic $\mathscr{L}$ has the *weak Beth property*, and we write WBETH($\mathscr{L}$), if every *strong implicit definition* can be replaced by some explicit definition in $\mathscr{L}$.

(iv) A logic $\mathscr{L}$ has the *Beth property*, and we write BETH($\mathscr{L}$), if every *implicit definition* can be replaced by some explicit definition in $\mathscr{L}$.

(v) A logic $\mathscr{L}$ has the *projective weak Beth property*, and we write PWBETH($\mathscr{L}$), if every implicit definition which is RPC in $\mathscr{L}$, can be replaced by some explicit definition in $\mathscr{L}$.

The following summarizes the relationship between these properties.

4.2.2 Theorem. (i) *A logic $\mathscr{L}$ has the weak projective Beth property iff it has the Δ-interpolation property.*

(ii) *For a logic $\mathscr{L}$ the interpolation property implies, but is strictly stronger than, the Δ-interpolation property (and therefore the projective weak Beth property). This is true even for compact logics.*

(iii) *For a logic $\mathscr{L}$ the interpolation property implies, but is strictly stronger than, the Beth property. This is true even for compact logics.*

(iv) *For a logic $\mathscr{L}$ the Δ-interpolation property implies, but is strictly stronger than, the weak Beth property. In fact, the Δ-interpolation property does not imply the Beth property. This is true even for compact logics.*

(v) *For a logic $\mathscr{L}$ the Beth property implies, but is strictly stronger than, the weak Beth property, in fact the Beth property does not imply the Δ-interpolation property. This is even true for compact logics.*

Proof. The implications are all straightforward. (i) is Proposition 7.3.3 and (ii) is 7.2.7 in Chapter II. (iii) follows from (v). (iv) is Theorem 2.5 in Makowsky–Shelah [1979b] and (v) is proven in Makowsky–Shelah [1976] and will appear in Makowsky–Shelah [198?]. For compact logics (ii)–(v) follow from Theorems 4.6.12 and 4.6.13. ☐

4.2.3 Remark. For sublogics of $\mathscr{L}_{\omega_1\omega}$ of the form $\mathscr{L}_A$ with A primitive recursive closed, the Δ-interpolation property implies the interpolation property and therefore the Beth property. This is due to H. Friedman and proved in Makowsky–Shelah–Stavi [1976]. See also Chapter VIII, Theorem 6.3.1.

Next we investigate the relationship between the weak Beth property and recursive compactness. Of special interest here is that we need an additional assumption, namely either that the logic is finitely generated or the pair preservation property.

4.2.4 Definitions. (i) A logic $\mathscr{L}$ is *finitely generated*, if it is a Lindström logic over a finite set of new quantifier symbols.

(ii) A logic $\mathscr{L}$ is *recursively generated*, if it is a Lindström logic over a recursive set of new quantifier symbols.

(iii) A logic $\mathscr{L}$ is *recursively compact*, if $\mathscr{L}$ is recursively generated and if Σ is any recursive set of $\mathscr{L}$-sentences such that every finite subset of Σ has a model, so Σ has a model.

4.2.5 Remarks. (i) By Theorem 5.2.5 in Chapter II every logic, for which validity is recursively enumerable, is recursively compact.

(ii) A logic $\mathscr{L}$ is recursively compact iff no single sentence $\varphi \in \mathscr{L}[\tau]$, with τ containing a binary relation symbol denoted by $<$, characterizes the structure $\langle \omega, < \rangle$ up to isomorphism among (relativized) reducts of models of φ. Cf. also Chapter II, Section 5.2.

4.2.6 Theorem. (i) (Lindström). *Assume a logic $\mathscr{L}$ is finitely generated and has the weak Beth property, then $\mathscr{L}$ is recursively compact.*

(ii) *Assume a logic $\mathscr{L}$ is recursively generated and satisfies the weak Beth property and the pair preservation property. Then $\mathscr{L}$ is recursively compact.*

Proof. The proof of (i) is similar to the proof of Theorem 5.2.5 in Chapter II, cf. also Chapter III, Remark 2.1.5 or Chapter XVII, Section 4.

To prove (ii) we assume for contradiction that there is a $\varphi \in \mathscr{L}[\tau]$ as in the remark (ii) above. Since $\mathscr{L}$ is recursively generated we have at most 2^ω many theories over a countable vocabulary. Now consider the τ-structure

$$\mathfrak{A} = \langle A, P^n, Q, \in \rangle$$

where $A = \bigcup_{n \in \omega} P^n(\omega)$, P^n is the nth iteration of the power set operation, P^n are unary predicates with $P^n = P^n(\omega)$, $\in$ is the natural membership relation, and $Q \subset P^k$ where k is fixed and such that $\beth(k)$ is bigger than the number κ of inequivalent theories in $\mathscr{L}[\tau]$. Now consider the structure $[\mathfrak{A}, \mathfrak{A}]$ with universe of the first sort A_1 and universe of the second sort A_2 and let ψ be the formula in $\mathscr{L}$ which expresses:

(i) P^0 is standard ω. (Here we use φ.)
(ii) F is a partial map from A_1 to A_2, where F is a new function symbol.
(iii) F and F^{-1} preserve $\in$.
(iv) F is hereditary, i.e., if F is defined for x and $y \in x$ so F is defined for y.
(v) The domain of F is maximal with respect to (i)–(iv).

Clearly, ψ defines F strongly implicitly. Since there are at most $\kappa = 2^\omega$ many theories over τ, we can find two structures $\mathfrak{A}_1 = \langle A, P^n, Q_1, \in \rangle$, $\mathfrak{A}_2 = \langle A, P^n, Q_2, \in \rangle$, such that $\mathfrak{A}_1 \equiv_{\mathscr{L}} \mathfrak{A}_2$ but $Q_1 \neq Q_2$.

Let $\mathfrak{B}_1 = [\mathfrak{A}_1, \mathfrak{A}_2]$ and $\mathfrak{B}_2 = [\mathfrak{A}_1, \mathfrak{A}_1]$. Now we use PPP($\mathscr{L}$) to conclude that $\mathfrak{B}_1 \equiv_{\mathscr{L}} \mathfrak{B}_2$. Using the weak Beth property, let $\vartheta \in \mathscr{L}[\tau]$ define F explicitly. So ϑ defines on $\mathfrak{B}_i$ a partial map F_i with domain D_i. Clearly $Q_1 \subset D_1$, and since $\mathfrak{B}_1 \equiv_{\mathscr{L}} \mathfrak{B}_2$, also $Q_1 \subset D_2$. But then we can show by induction on $l \leq k$ that $Q_1 = Q_2$, contrary to our assumption. Note that, in this proof, we have only used a finite subset of the vocabulary τ. □

The same proof actually only requires that the number of theories for a countably vocabulary is smaller than $\beth(\omega_1^{\mathrm{CK}})$. This can be achieved by assuming either that the Löwenheim number is smaller than $\beth(\omega_1^{\mathrm{CK}})$ or directly, by assuming that there are not too many different formulas for a given countable vocabulary. One can vary the prove further for logics $\mathscr{L}$ such that $\mathrm{card}(\mathscr{L}[\tau]) < \beth(\alpha)$ for countable vocabulary τ. We state the corresponding results without proof:

4.2.7 Theorem. (i) *Assume a logic $\mathscr{L}$ satisfies the weak Beth property and the pair preservation property, and has a Lowenheim number $l(\mathscr{L}) < \beth(\omega_1^{\mathrm{CK}})$. Then no single sentence $\varphi \in \mathscr{L}[\tau]$, with τ containing a binary relation symbol denoted by $<$, characterizes the structure $\langle \omega, < \rangle$ up to isomorphism among reducts of models of φ. In other words, the well-ordering number $w_1(\mathscr{L})$ for single sentences of $\mathscr{L}$ is ω.*

(ii) *Assume a logic $\mathscr{L}$ satisfies the weak Beth property and the pair preservation property, and* $\mathrm{card}(\mathscr{L}[\tau]) < \beth(\alpha)$ *for countable vocabulary τ. Then no single sentence $\varphi \in \mathscr{L}[\tau]$, with τ containing a binary relation symbol denoted by $<$, characterizes the structure $\langle \omega + \alpha, < \rangle$ up to isomorphism among reducts of models of φ. In other words, the well-ordering number $w_1(\mathscr{L})$ for single sentences of $\mathscr{L}$ is $\omega + \alpha$.*

4.2.8 Corollary. *Let A be a countable admissible set with $\omega \in A$, or $A = \omega_1$. Then $\mathscr{L}_A$ does not satisfy the pair preservation property.*

Proof. Clearly $\langle \omega, < \rangle$ is characterizable in $\mathscr{L}_A$ and the interpolation property holds. □

We now want to look at a property introduced in Feferman [1974b] and further studied in Makowsky [1978], which is a generalization of both the interpolation property and some of the preservation properties.

4.2.9 Definition. Let $\mathscr{L}$ be a logic and $\mathfrak{A}_i$ be τ_i-structures ($i = 1, 2$) with τ the vocabulary for $[\mathfrak{A}_1, \mathfrak{A}_2]$. We say that $\mathscr{L}$ allows *uniform reduction for pairs*, or has the uniform reduction property for pairs, anc write URP($\mathscr{L}$), if for every $\varphi \in \mathscr{L}[\tau]$ there exists a pair of finite sequences of formulas $\psi_1^1, \ldots, \psi_{n_1}^1$ and $\psi_1^2, \ldots, \psi_{n_2}^2$ with $\psi_k^i \in \mathscr{L}[\tau_i]$ and a boolean function $B \in 2^{n_1+n_2}$ such that for every τ_i-structures $\mathfrak{A}_i$ $(i = 1, 2)$ $[\mathfrak{A}_1, \mathfrak{A}_2] \models \varphi$ iff $B(a_1^1, \ldots, a_{n_1}^1, a_1^2, \ldots, a_{n_2}^2) = 1$, where a_k^i is the truth value of $\mathfrak{A}_i \models \psi_k^i$.

4.2.10 Examples. (i) URP($\mathscr{L}$) holds for

$\mathscr{L} = \mathscr{L}_{\omega\omega}$ by Feferman–Vaught [1959].
$\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_\kappa)$ by Wojciechowska [1969].
$\mathscr{L} = \mathscr{L}_{\infty\infty}$ by Malitz [1971].

(ii) URP($\mathscr{L}$) does hold for $\mathscr{L} = \mathscr{L}_{\kappa\lambda}$ iff κ is strongly inaccessible, by Malitz [1971].

We want to generalize URP to constructions different from the simple pair.

4.2.11 Definitions. (i) Let $\tau_0, \tau_1, \ldots, \tau_n$ be disjoint vocabularies and let

$$R \subset \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1) \times \cdots \times \mathrm{Str}(\tau_n)$$

be an n-ary relation on structures. A sentence $\varphi \in \mathscr{L}[\tau_n]$ is said to be *invariant on the range of* R, if for all $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n, \mathfrak{A}'_n$ such that $R(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n)$ and $R(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}'_n)$ $\mathfrak{A}_n \models \varphi$ iff $\mathfrak{A}'_n \models \varphi$.

(ii) An n-tuple of sequences of sentences $\bar{\psi}_0, \bar{\psi}_1, \ldots, \bar{\psi}_{n-1}$ with

$$\bar{\psi}_k = (\psi^k_1, \ldots, \psi^k_{m_k})$$

and $\psi^k_i \in \mathscr{L}[\tau_k]$ together with a boolean function

$$B \in 2^{m_1 + \cdots + m_{n-1}}$$

is called an UR *n-tuple for φ on the domain of* R if for all $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n$ we have that $R(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n)$ implies that $\mathfrak{A}_n \models \varphi$ iff $B(a^1_1, \ldots, a^1_{m_1}, a^2_1, \ldots, a^{n-1}_{m_{n-1}}) = 1$ where a^i_j is defined as in Definition 4.2.9.

(iii) We say a logic $\mathscr{L}$ satisfies the *uniform reduction property for $(n+1)$-ary relations*, and we write UR$_n(\mathscr{L})$, if for every relation $R \subset \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1) \times \cdots \times \mathrm{Str}(\tau_n)$ which is PC in $\mathscr{L}$ and for every $\varphi \in \mathscr{L}[\tau_n]$ which is invariant in the range of R, there is an UR tuple for φ on the domain of R.

4.2.12 Remarks. (i) Clearly UR$_2(\mathscr{L})$ implies URP($\mathscr{L}$), since the construction of the pair $[\mathfrak{A}_1, \mathfrak{A}_2]$ is a PC$_{\mathscr{L}}$-operation, i.e., its graph is a PC$_{\mathscr{L}}$ relation.

(ii) Instead of the pair construct we could consider the cartesian product of a fixed finite number n of structures $\mathfrak{A}_i$ and define similarly *uniform reduction for n-fold cartesian products* (URProd$_n(\mathscr{L})$). Again UR$_n(\mathscr{L})$ implies URProd$_n(\mathscr{L})$.

The following clarifies the relationship between PPP and various uniform reduction properties.

4.2.13 Theorem. *Let $\mathscr{L}$ be a logic. Then*

(i) URP($\mathscr{L}$) *implies* PPP($\mathscr{L}$),
(ii) UR$_2(\mathscr{L})$ *implies* PPPO($\mathscr{L}$).

If additionally $\mathscr{L}$ *has an dependence number* $o(\mathscr{L}) = \kappa$ *and is* (μ, ω)*-compact, with* $\mu = \sup\{\mathrm{card}(\mathscr{L}[\tau])\colon \mathrm{card}(\tau) < \kappa\}$, *then*:

(iii) (Shelah [198?e]). PPP($\mathscr{L}$) *implies* URP($\mathscr{L}$); *and*
(iv) (Shelah [198?e]). PPPO($\mathscr{L}$) *implies* $\mathrm{UR}_n(\mathscr{L})$ *for every* $n \in \omega$.

Proof. (i) and (ii) are straightforward. To prove (iii) assume φ is a counterexample to URP. So for every pair of sequences of formulas $\bar{\psi}_1 = (\psi^1_1, \ldots, \psi^1_{n_1})$ and $\bar{\psi}_2 = (\psi^2_1, \ldots, \psi^2_{n_2})$ with $\psi^i_k \in \mathscr{L}[\tau_i]$ and every boolean function $B \in 2^{n_1+n_2}$ there are τ_i-structures $\mathfrak{A}^j_i$ such that $[\mathfrak{A}^1_1, \mathfrak{A}^1_2] \models \varphi$, $[\mathfrak{A}^2_1, \mathfrak{A}^2_2] \models \neg\varphi$, but

$$B(a(j)^1_1, \ldots, a(j)^1_{n_1}, a(j)^2_1, \ldots, a(j)^2_{n_2}) = 1,$$

where $a(j)^i_k$ is the truth value of $\mathfrak{A}^j_i \models \psi^i_k$.

Claim 1. *For every such pair of sequences of formulas* $\bar{\psi}_1$, $\bar{\psi}_2$ *there is a function* $h\colon \bar{\psi}_1 \cup \bar{\psi}_2 \to 2$ *such that*

$$\Sigma^1_h = \{\varphi\} \cup \{\vartheta \leftrightarrow h(\vartheta)\colon \vartheta \in \bar{\psi}_1 \cup \bar{\psi}_2\},$$

and

$$\Sigma^0_h = \{\neg\varphi\} \cup \{\vartheta \leftrightarrow h(\vartheta)\colon \vartheta \in \bar{\psi}_1 \cup \bar{\psi}_2\}$$

have both models.

If not, for every h as above either Σ^1_h or Σ^0_h is has no model. We then could construct a boolean function B as follows: Put

$$B_h = \bigwedge \{\vartheta\colon h(\vartheta) = 1\} \wedge \bigwedge \{\neg\vartheta\colon h(\vartheta) = 0\}.$$

Now we put

$$B = \bigvee \{B_h\colon \Sigma^1_h \text{ has a model}\}.$$

Subclaim. $[\mathfrak{A}_1, \mathfrak{A}_2] \models \varphi$ *iff* $B = B(a^1_1, \ldots, a^1_{n_1}, a^2_1, \ldots, a^2_{n_2}) = 1$, *where* a^i_k *is the truth value of* $\mathfrak{A}_i \models \psi^i_k$.

To see this, assume $[\mathfrak{A}_1, \mathfrak{A}_2] \models \varphi$. Now put $h_0(\psi^i_k) = a^i_k$. Clearly, $B = 1$. Conversely, if $B = 1$, there is h such that Σ^1_h has a model. So, by our assumption, Σ^0_h has no model. So $[\mathfrak{A}_1, \mathfrak{A}_2] \models \varphi$.

Using Claim 1, we define H to be the set of functions $h\colon \bar{\psi}_1 \cup \bar{\psi}_2 \to 2$ such that Σ^1_h and Σ^0_h have both models.

We define a filter F_0 on H with filter basis $U_\vartheta = \{h \in H\colon \vartheta \in \mathrm{dom}(h)\}$ where $\vartheta \in \mathscr{L}[\tau_1] \cup \mathscr{L}[\tau_2]$. Let F be an ultrafilter extending F_0. Now we define a function $g\colon \mathscr{L}[\tau_1] \cup \mathscr{L}[\tau_2] \to 2$ by $g(\vartheta) = 0$ iff $\{h \in H\colon h(\vartheta) = 0\} \in F$. Clearly, we have:

Claim 2. *For every pair of sequences $\bar{\psi}_1$, $\bar{\psi}_2$ there is a function $h \in H$ such that* $g \restriction \mathrm{dom}(h) = h$.

Now we define Σ_g^i $(i = 0, 1)$ like the Σ_h^i's. Using (μ, ω)-compactness and Claim 2 we get:

Claim 3. *There are* $\mathfrak{A}_j^i$ $(i = 0, 1, j = 1, 2)$ *such that* $[\mathfrak{A}_1^i, \mathfrak{A}_2^i] \models \Sigma_g^i$.

But the latter contradicts PPP($\mathscr{L}$), since, by the definition of Σ_g^i, $\mathfrak{A}_1^i \equiv_{\mathscr{L}} \mathfrak{A}_1^{1-i}$ $(i = 0, 1)$.

The proof of (iv) is essentially the same. □

Uniform reduction is closely related to the interpolation property. Feferman [1974] derived UR_1 from it and in Makowsky [1978] the converse was observed.

4.2.14 Theorem (Feferman, Makowsky). *Let $\mathscr{L}$ be a logic with finite dependence. Then* $\mathrm{UR}_1(\mathscr{L})$ *iff $\mathscr{L}$ has the interpolation property.*

Proof. (i) Assume $\mathrm{UR}_1(\mathscr{L})$ and let $\mathbf{K}_1, \mathbf{K}_2 \subset \mathrm{Str}(\tau_0)$ be two disjoint classes of τ_0-structures which are PC in $\mathscr{L}$. So there are vocabularies τ_i and sentences $\psi_i \in \mathscr{L}[\tau_i]$ such that $\mathbf{K}_i = \mathrm{Mod}(\psi_i) \restriction \tau_0$. Since $\mathscr{L}$ has finite dependence all the vocabularies can be assumed finite. We now define $R \subset \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1 \cup \tau_2)$ by $R(\mathfrak{A}, \mathfrak{B})$ iff $\mathfrak{A} \cong \mathfrak{B} \restriction \tau_0$ and $\mathfrak{B} \restriction \tau_1 \in \mathbf{K}_1$ or $\mathfrak{B} \restriction \tau_2 \in \mathbf{K}_2$. Clearly R is $\mathrm{PC}_{\mathscr{L}}$ using an additional predicate for the isomorphism and the fact that τ_0 is finite.

Claim. *Both ψ_1, ψ_2 are invariant in the range of R.*

This follows from the fact that $\mathbf{K}_1 \cap \mathbf{K}_2 = \varnothing$.

Now let ϑ_i be UR sentences for ψ_i, respectively. It is easy to check that $\vartheta_1 \wedge \neg\vartheta_2$ is the desired interpolating sentence.

(ii) Now assume that $\mathscr{L}$ has the interpolation property, R is a $\mathrm{PC}_{\mathscr{L}}$-relation on $\mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1)$ and $\varphi \in \mathscr{L}[\tau_1]$ is invariant on the range of R. Assume R is defined by $\psi \in \mathscr{L}[\tau]$. Now put

$$\mathbf{K}_1 = \mathrm{Mod}(\psi \wedge \varphi) \restriction \tau_0 \text{ and } \mathbf{K}_2 = \mathrm{Mod}(\psi \wedge \neg\varphi) \restriction \tau_0.$$

Claim. $\mathbf{K}_1 \cap \mathbf{K}_2 = \varnothing$.

This follows from the fact that φ is invariant on the range of R. So let $\vartheta \in \mathscr{L}[\tau_0]$ be an interpolating sentence. Therefore, whenever $R(\mathfrak{A}, \mathfrak{B})$ we have that $\mathfrak{A} \models \vartheta$ iff $\mathfrak{B} \models \varphi$, in other words, ϑ is an UR sentence for φ. □

Note that in Feferman [1974b] uniform reduction is defined for PC_δ, and Theorem 4.2.14(ii) is stated assuming some compactness properties.

4.2.15 Theorem. (i) *For a logic $\mathscr{L}$ the following are equivalent*:

(a) $\mathrm{UR}_2(\mathscr{L})$.
(b) $\mathrm{UR}_1(\mathscr{L})$ *(or equivalently the interpolation property) together with* URP($\mathscr{L}$).
(c) $\mathrm{UR}_n(\mathscr{L})$ *for* $n \geq 2$.

(ii) *For a compact logic* $\mathscr{L}$ *the following are equivalent*:

(a) $\mathrm{UR}_2(\mathscr{L})$.

(b) $\mathrm{UR}_1(\mathscr{L})$ (*or equivalently the interpolation property*) *together with* $\mathrm{PPP}(\mathscr{L})$.

(c) $\mathrm{UR}_n(\mathscr{L})$ *for* $n \geq 2$.

(d) $\mathrm{PPPO}(\mathscr{L})$.

(iii) $\mathrm{URP}(\mathscr{L})$ *does not imply* $\mathrm{UR}_1(\mathscr{L})$, *not even for compact logics.*

(iv) $\mathrm{UR}_1(\mathscr{L})$ *does not imply* $\mathrm{URP}(\mathscr{L})$. (*For compact logics this is open.*)

Proof. (i) (a) implies (b) by Theorem 4.2.14 and Remarks 4.2.12. (b) implies (c), since URP allows us to reduce n-ary relations to binary relations, and (c) implies (a) is trivial. To prove (ii) we combine (i) with Theorem 4.2.13.

To prove (iii) we observe that by Example 4.2.10(ii) $\mathscr{L}_{\omega\omega}(Q_\kappa)$ satisfies URP, but, as shown in Counterexamples II.7.1.3, it does not have the interpolation property. So the result follows from Theorem 4.2.14. For a compact counterexample see Remark 4.2.17 below.

To prove (iv) we note that $\mathscr{L}_{\omega_1\omega}$ satisfies the interpolation property, and therefore, by Theorem 4.2.14. $\mathrm{UR}_1(\mathscr{L}_{\omega_1\omega})$ holds. As noted in Example 4.2.10(ii) $\mathrm{URP}(\mathscr{L}_{\omega_1\omega})$ does not hold. ☐

The last proposition in this section gives us a connection between the tree preservation property and uniform reduction, but it is only interesting for logics which are not recursively generated, because the latter hypothesis together with UR_2 implies recursive compactness, by Theorem 4.2.6(ii).

4.2.16 Proposition. *Assume* $\mathscr{L}$ *is a logic in which* $\langle\omega, \in\rangle$ *is not characterizable by a single sentence with additional predicates and sorts* (*in particular* $\mathscr{L}$ *is not recursively compact*). *Then* $\mathrm{UR}_2(\mathscr{L})$ *implies* $\mathrm{TPP}(\mathscr{L})$.

Proof, Clearly, we can use the PC-definition of $\langle\omega, \in\rangle$ to get a PC-definition of the tree construction involved in the tree preservation property. See also Remark 4.1.6(iii). ☐

4.2.17 Remark. In Section 4.6 we shall present an example of a logic $\mathscr{L}$ which satisfies the Beth property, the pair preservation property, is compact, but does not satisfy the interpolation property.

4.3. The Finite Robinson Property

In Section 3.3 we have seen that the amalgamation property implies compactness therefore (Corollary 3.3.5) that the Robinson property implies compactness. These results depend on some assumptions on the dependence number of the logic. In Chapter XIX the Robinson property is further investigated and instead of the dependence number we have different smallness assumptions on the logic. Here we want to study two weakened version of the Robinson property. They were studied first in Makowsky–Shelah [1979b], and the assumptions on the logics also did not involve the dependence number.

4.3.1 Definition. Let $\mathscr{L}$ be a logic.

(i) $\mathscr{L}$ satisfies the *finite Robinson property* (FROB), if given a complete set Σ of $\mathscr{L}[\tau]$-sentences and two sentences φ_1 $(\varphi_2) \in \mathscr{L}[\tau_1]$ $(\mathscr{L}[\tau_2])$ with $\tau_1 \cap \tau_2 = \tau$ such that $\Sigma \cup \{\varphi_i\}$ has a τ_i-model then $\Sigma \cup \{\varphi_1, \varphi_2\}$ has a $\tau_1 \cup \tau_2$-model.

(ii) $\mathscr{L}$ satisfies the *weak finite Robinson property* (WFROB), if given a complete set Σ of $\mathscr{L}[\tau]$-sentences and two sentences φ_1 $(\varphi_2) \in \mathscr{L}[\tau_1]$ $(\mathscr{L}[\tau_2])$ with $\tau_1 \cap \tau_2 = \tau$ such that $\Sigma \cup \{\varphi_i\}$ has a τ_i-model then $\{\varphi_1, \varphi_2\}$ has a $\tau_1 \cup \tau_2$-model.

4.3.2 Proposition. (i) *Both* FROB *and* WFROB *are consequences of the Robinson property.*

(ii) *Clearly* FROB *implies* WFROB.

(iii) *The interpolation property implies* WFROB.

(iv) *If* $\mathscr{L}$ *is compact then the Robinson property is equivalent to both* FROB, WFROB *and the interpolation property.*

(v) WFROB *does not imply* FROB.

The proof of (i)–(iii) is left to the reader. For (iv) cf. Chapter II, Theorem 7.1.5. For (v) we note that $\mathscr{L}_{\omega_1\omega}$ has the interpolation property and therefore, by (iii) above the WFROB. That $\mathscr{L}_{\omega_1\omega}$ does not satisfy FROB is shown in Keisler [1971a, p. 22].

Our next aim is to study when the pair preservation property suffices to make FROB equivalent to the Robinson property. The answer is given in Theorem 4.3.8.

4.3.3 Definition. (i) We call a logic $\mathscr{L}$ *tiny*, if for every vocabulary τ with card(τ) smaller than the first uncountable measurable cardinal μ_0

$$\mathrm{card}(\mathscr{L}[\tau]) < \mu_0.$$

(ii) We call a logic $\mathscr{L}$ *small*, if for every vocabulary τ, which is a set, $\mathscr{L}[\tau]$ is a set. (Smallness was already introduced in Chapter II, Theorem 6.1.4). Clearly, if a logic $\mathscr{L}$ is tiny, it is also small, provided measurable cardinals exist. If no uncountable measurable cardinals exist, then tiny and small coincide. There are logics with dependence number $o(\mathscr{L}) = \omega$ which are not small, and it is not dfficult to construct logics which are small but have no dependence number. We leave this as an exercise to the reader. The logic defined in Example 2.2.5(ii) is tiny, but has an dependence number which is bigger than the first uncountable measurable cardinal.

(iii) If a logic $\mathscr{L}$ is small then there is function s on the cardinals such that for every vocabulary τ of cardinality λ, $\lambda \leq \mathrm{card}(\mathscr{L}[\tau]) < s(\lambda)$. We call this function the *size function of L*. If $\mathscr{L}$ is tiny then $\lambda < \mu_0$ implies $s(\lambda) < \mu_0$.

(iv) Recall that a logic $\mathscr{L}$ is said to be *ultimately compact*, if $\mathscr{L}$ is (∞, λ)-compact for some cardinal λ.

4.3.4 Theorem. *If $\mathscr{L}$ has the Robinson property and is tiny then:*

(i) *$\mathscr{L}$ is $[\omega]$-compact; and*
(ii) *$\mathscr{L}$ has the finite dependence property.*

This differs from Corollary 3.3.5 inasmuch as here we do not require that $\mathscr{L}$ has an dependence number, whereas in Corollary 3.3.5 we require that $o(\mathscr{L})$ exists and is smaller than the first uncountable measurable cardinal.

4.3.5 Theorem. *If $\mathscr{L}$ has the pair preservation property, the finite Robinson property and is tiny then:*

(i) *$\mathscr{L}$ is $[\omega]$-compact; and*
(ii) *$\mathscr{L}$ has the finite dependence property.*

Proof. Clearly in both theorems (ii) follows from (i) by Theorem 2.2.1. To prove (i) we proceed in parallel and point out the difference in the appropriate places.

Let B_1, B_2 be two infinite sets of different cardinality β_1, β_2 smaller than the first uncountable measurable cardinal μ_0. Now we fix $\kappa > \max\{\beta_1, \beta_2\}$ but $\kappa < \mu_0$ and put $\mathfrak{A}_\kappa = \langle \mathfrak{H}(\kappa^+), P_1, P_2 \rangle$ where $\mathfrak{H}(\kappa^+)$ is the complete expansion of $\langle \kappa^+, \in \rangle$ and P_1, P_2 are unary predicates of cardinality β_1, β_2, respectively. Let τ_κ be the vocabulary of $\mathfrak{A}_\kappa$ and Σ the complete $\mathscr{L}[\tau_\kappa]$-theory of $\mathfrak{A}_\kappa$. Assuming that $\mathscr{L}$ is not $[\omega]$-compact, we conclude, using the Rabin–Keisler theorem (1.2.3), that Σ is categorical. Let $\mathfrak{B}_i = [\mathfrak{A}_k, B_i]$ for $i = 1, 2$ be τ_i-structures with $\tau_1 \cap \tau_2 = \tau_\kappa$.

Assumption: *$\mathfrak{B}_1$ and $\mathfrak{B}_2$ are $\mathscr{L}$-equivalent (after appropriate name changing, so that both are τ_1-structures).*

We first finish the proof from the assumption. Let φ_i be the first-order formula which says that "f_i is a bijection from P_i onto the universe of the second sort." Clearly $\mathfrak{B}_i \models \Sigma \cup \{\varphi_i\}$, but $\Sigma \cup \{\varphi_1, \varphi_2\}$ has no model.

To satisfy the assumption the two proofs differ. In the case of Theorem 4.3.5 we use tinyness and an argument as in the proof of the existence of Hanf numbers (Section II.6.1) to find β_1, β_2 such that for $\tau = \{=\}$ B_1 *and* B_2 are $\mathscr{L}$-equivalent. Since τ is finite we may assume that $\beta_1, \beta_2 < \mu_0$. Now we can use the pair preservation property to conclude that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are $\mathscr{L}$-equivalent (after appropriate name changing).

In the case of Theorem 4.3.4 we fix a countable universal vocabulary τ_∞ which has countably many relation symbols for every arity. Using enough constants τ_C, we can think of Σ as being written over the vocabulary $\tau_\infty \cup \tau_C$. Let Σ_∞ be $\Sigma \upharpoonright \tau_\infty$. Using tinyness we find, as in the case of Theorem 4.3.4, κ, $\beta_1 \beta_2$ such that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are $\mathscr{L}[\tau_\infty]$-equivalent.

Let τ_1 and τ_2 be two disjoint copies of τ_C and put $\Sigma_i = \Sigma \cup \{\varphi_i\}$ written over $\tau_\infty \cup \tau_i$. Clearly $\Sigma_\infty \cup \Sigma_i$ has each a model, but $\Sigma_\infty \cup \Sigma_1 \cup \Sigma_2$ has not. $\square$

The following is an improvement of Theorem 4.3.4.

4.3.6 Theorem*. (i) *The Robinson property implies the joint embedding property.*

(ii) *If a logic $\mathscr{L}$ is small and has the joint embedding property then $\mathscr{L}$ is ultimately compact.*

(iii) *If a logic $\mathscr{L}$ is tiny and has the joint embedding property then $\mathscr{L}$ is $[\omega]$-compact.*

Proof. (i) is proved in a similar way to Theorem 3.1.14. (ii) is Theorem 1.1 from Chapter XIX and (iii) follows from (ii) and the fact that $\mathscr{L}$ was assumed to be tiny. ▯

4.3.7 Examples. (i) If a logic $\mathscr{L}$ has a Löwenheim number $l_1(\mathscr{L})$ then $\mathscr{L}$ is small.

(ii) In Chapter XIX, Theorem 1.1.1 it is shown that if $\mathscr{L}$ is small and satisfies the joint embedding property then $\mathscr{L}$ is ultimately compact.

(iii) If $\mathscr{L}$ has an dependence number and satisfies the amalgamation property then $\mathscr{L}$ is ultimately compact. This holds in particular, if $\mathscr{L}$ satisfies the Robinson property (Theorem 3.3.1).

Our next theorem shows that already the finite Robinson property implies ultimate compactness.

4.3.8 Theorem (Shelah). *Let $\mathscr{L}$ be a tiny logic which satisfies both the preservation property for pairs and the finite Robinson property. Then*

(i) *$\mathscr{L}$ is ultimately compact. In fact, if s is the size function of $\mathscr{L}$ and $2^{s(\omega)} < 2^{\omega_{\alpha+n}}$ then $\mathscr{L}$ is $[\infty, \omega_\alpha]$-compact.*

(ii) *If additionally $\mathscr{L}$ is countably generated or $s(\omega) < \omega_n$ for some $n \in \omega$, then $\mathscr{L}$ is compact and satisfies the uniform reduction properties* $\mathrm{UR}_n(\mathscr{L})$.

For the proof we need a lemma. Parts (ii) and (iii) the author has learned from S. Shelah, though others probably have observed them, too.

4.3.9 Lemma. (i) (Ulam). *Let κ be an infinite cardinal. If $S \subset \kappa^+$ is stationary, S may be decomposed into κ^+ disjoint stationary subsets.*

(ii) *There is a family* $\mathbf{S}$ *of 2^{κ^+} many stationary subsets of κ^+ such that for any $S_1, S_2 \in \mathbf{S}$ the symmetric difference $S_1 \Delta S_2$ is stationary as well.*

(iii) *There are 2^{κ^+} many stationary subsets of κ^+ such that any finite boolean combination of them is stationary as well.*

Proof. (i) is standard, e.g., Theorem 3.2 in Chapter B.3 of the *Handbook of Mathematical Logic* [Barwise, 1977].

To prove (ii) let $\{S_\alpha: \alpha < \kappa^+\}$ be the disjoint family of stationary sets from (i). Let $X \subset \kappa^+$, $X \neq \varnothing$. Define $T_X = \bigcup_{\alpha \in X} S_{2\alpha} \cup \bigcup_{\alpha \notin X} S_{2\alpha+1}$. Clearly each T_X is stationary and $X \neq Y$ implies that $T_X \Delta T_Y$ is stationary.

The proof of (iii) is similar, but uses a combinatorial result from Engelking–Karlowicz [1965]. ▯

Proof of Theorem 4.3.8. Let κ be as required. We can assume it is regular, by Theorem 1.5.16. Assume $\mathscr{L}$ is not $[\kappa]$-compact, so by Theorem 1.5.16 again, $\mathscr{L}$ is not $[\kappa^+]$-compact, and, by induction, we can assume that κ is such that $2^s(\omega) < 2^\kappa$. Let $C_\kappa = \{\beta: \beta \in \kappa^+ \text{ and } \mathrm{cf}(\beta) = \kappa\}$. For every $S \subset C_\kappa$ we define a structure $\mathfrak{M}_S = \langle \kappa^+, \in, S \rangle$. By Lemma 4.3.9(ii) there are 2^{κ^+} many stationary sets in C_κ with their symmetric difference stationary, too. So, by our assumption on the size function of $\mathscr{L}$, and by Proposition 2.1.3, there are $S_1, S_2 \in C_\kappa$, with $\mathfrak{M}_{S_1} \equiv_{\mathscr{L}} \mathfrak{M}_{S_2}$. We put now $\mathfrak{A} = \langle \kappa^+, \varepsilon, S_1, S_2, S_3, \mathrm{cf} \rangle$ with $S_3 = S_1 \Delta S_2$ and $\in$, cf membership and cofinality on κ^+. Let $\mathfrak{B}$ be the complete expansion of $\mathfrak{A}$. We note that in $\mathfrak{B}$ every ordinal of cofinality κ or κ^+ is cofinally characterized by the complete $\mathscr{L}$-theory of $\mathfrak{B}$. Using that $\mathscr{L}$ has the pair preservation property, we conclude that $[\mathfrak{B}, \mathfrak{M}_{S_1}] \equiv_{\mathscr{L}} [\mathfrak{B}, \mathfrak{M}_{S_2}]$. Let Σ be the complete theory of $[\mathfrak{B}, \mathfrak{M}_{S_1}]$. We want to build a counterexample to FROB. For this purpose let F_i $(i = 1, 2)$ be new unary function symbols and φ_i be the sentence which says that "F_i is an isomorphism between $\langle \kappa^+, \in, S_i^B \rangle$ and $\mathfrak{M}_{S_i}$. Clearly $\Sigma \cup \{\varphi_i\}$ is each satisfiable but it is not difficult to show that $\Sigma \cup \{\varphi_1, \varphi_2\}$ has no model. ☐

A complete proof may be found in Makowsky–Shelah [1979b].

A combination of the proofs of Theorem 4.3.8 and Proposition 4.3.2 gives us the following theorem:

4.3.10 Theorem. *Let $\mathscr{L}$ be a logic which is small and satisfies either the Robinson property or the finite Robinson property together with the pair preservation property. Then $\mathscr{L}$ is ultimately compact.*

Combining Theorem 4.3.10 with the hypothesis $A(\infty)$ from Section 1.5 we get:

4.3.11 Corollary (Makowsky–Shelah, Mundici). *For a logic as in Theorem* 4.3.10 *we have*:

(i) *If $A(\infty)$ holds then $\mathscr{L}$ is compact.*
(ii) *If $\mathscr{L}$ is tiny and there are no uncountable measurable cardinals, then $\mathscr{L}$ is compact.*

Proof. Assume $A(\infty)$, so there are no uncountable measurable cardinals, by Theorem 1.5.4(iii). Therefore, if a logic $\mathscr{L}$ is small, then it is tiny and by Theorems 4.3.4 or 4.3.5 $[\omega]$-compact. So Theorem 4.3.8 together with Theorem 1.5.7 give us that $\mathscr{L}$ is compact. This proves both (i) and (ii). ☐

Let us end this section with an open problem.

4.3.12 Problem. Is there a countable logic, different from first-order logic, which satisfies both the Robinson property and the uniform reduction property (as in Theorem 4.3.8)?

4.4. Constructing Counter Examples to the Beth Property

This last section is devoted to an abstract theorem (4.4.5) whose main use it is to direct us in the construction of possible counterexamples to the Beth property. For compact logics, it gives a sufficient condition, the tree preservation property, for the Beth and the interpolation property to be equivalent. As the example in Theorem 4.6.12 shows, the pair preservation property does not suffice. Experience shows that in many cases where we do not have the interpolation property, we actually can find a counterexample to the weak finite Robinson property. The following theorem gives some indication on how to transform such a counterexample into a counterexample of the Beth property.

4.4.1 Theorem. (i) *Let $\mathscr{L}$ be a logic which satisfies the Beth property and the tree preservation property. Then $\mathscr{L}$ also satisfies the weak finite Robinson property.*

(ii) *If additionally to the tree preservation property $\mathscr{L}$ is compact, then $\mathscr{L}$ has the Beth property iff it has the interpolation property.*

Stated in this form the theorem does not have many applications. But its proof still gives directions on how to construct counterexamples to the Beth property, provided the interpolation property fails. In Makowsky–Shelah [198?b] this approach lead to a proof that $\Delta(\mathscr{L}_{\infty\omega})$ does not have the Beth property. Another way of making Theorem 4.4.1 more useful, is to define all the properties involved for pairs of logics.

4.4.2 Definitions. (i) Let $\mathscr{L}_1$, $\mathscr{L}_2$ be two logics such that $\mathscr{L}_1 \leq \mathscr{L}_2$. We define the various Robinson properties ROB, FROB, WFROB for the pair $\mathscr{L}_1$, $\mathscr{L}_2$ and write ROB($\mathscr{L}_1, \mathscr{L}_2$), FROB($\mathscr{L}_1, \mathscr{L}_2$), WFROB($\mathscr{L}_1, \mathscr{L}_2$), respectively. For ROB this looks explicitly as follows: If Σ is a complete set of formulas in $\mathscr{L}_2(\tau_0)$, Σ_1, Σ_2 are in $\mathscr{L}_1(\tau_1)$, $\mathscr{L}_1(\tau_2)$, respectively, $\tau_1 \cap \tau_2 = \tau$ and $\Sigma \cup \Sigma_i$ $(i = 1, 2)$ have models each, then $\Sigma \cup \Sigma_1 \cup \Sigma_2$ has a model. We leave it to the reader to state the corresponding properties FROB, WFROB.

(ii) Similarly we define the various Beth and interpolation properties BETH, WBETH, INT, Δ-INT for the pair $\mathscr{L}_1$, $\mathscr{L}_2$ and write BETH($\mathscr{L}_1, \mathscr{L}_2$), WBETH($\mathscr{L}_1, \mathscr{L}_2$), INT($\mathscr{L}_1, \mathscr{L}_2$), Δ-INT($\mathscr{L}_1, \mathscr{L}_2$), respectively, if the implicit definition or the formulas to be interpolated are in $\mathscr{L}_1$ and the explicit definition or the interplant is in $\mathscr{L}_2$.

(iii) Similarly we define the various preservation properties PPP, TPP for the pair $\mathscr{L}_1$, $\mathscr{L}_2$ and write PPP($\mathscr{L}_1, \mathscr{L}_2$), TPP($\mathscr{L}_1, \mathscr{L}_2$), if the given structures are $\mathscr{L}_2$-equivalent and the resulting structures are $\mathscr{L}_1$-equivalent.

4.4.3 Examples. (i) While $\mathscr{L}_{\omega\omega}(Q_0)$ does not have the interpolation property by Counterexamples II.7.1.3, INT($\mathscr{L}_{\omega\omega}(Q_0), \mathscr{L}_{\omega_1\omega}$) does hold.

(ii) The logics $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ and $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ both do not satisfy the interpolation property (Makowsky–Shelah [1981, Proposition 6.6] and Counterexamples II.7.1.3) but, as we shall see in Propsotion 4.6.7,

$$\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)}), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$$

does hold.

(iii) $\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q_1), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$ does not hold, by Counterexamples II.7.1.3.

4.4.4 Proposition. *Let* PROPERTY *be any of the above defined definability properties, and let* $\mathscr{L}_{10} < \mathscr{L}_{11} < \mathscr{L}_{20} < \mathscr{L}_{21}$ *be logics. Then* $\mathrm{PROPERTY}(\mathscr{L}_{11}, \mathscr{L}_{20})$ *implies* $\mathrm{PROPERTY}(\mathscr{L}_{10}, \mathscr{L}_{21})$.

Proof. Obvious.

With these definitions we can state a slightly stronger theorem.

4.4.5 Theorem. (i) *Let* $\mathscr{L}_1 < \mathscr{L}_2 < \mathscr{L}_3$ *be three logics such that* $\mathrm{BETH}(\mathscr{L}_1, \mathscr{L}_2)$ *and* $\mathrm{TPP}(\mathscr{L}_2, \mathscr{L}_3)$ *hold. Then* $\mathrm{WFROB}(\mathscr{L}_1, \mathscr{L}_3)$ *holds.*

(ii) *If in addition* $\mathscr{L}_3$ *is compact, then* $\mathrm{INT}(\mathscr{L}_1, \mathscr{L}_3)$ *holds.*

Proof. Let φ_1, φ_2 be two formulas of $\mathscr{L}_1(\tau_i)$, respectively, with $\tau_i = \tau_0 \cup_{\mathrm{disjoint}} \{\mathbf{P}_i\}$, which form a counterexample to $\mathrm{WFROB}(\mathscr{L}_1, \mathscr{L}_3)$. Let $\mathfrak{A}_i$ be τ_i-structures such that $\mathfrak{A}_1 \upharpoonright \tau_0 \equiv_{\mathscr{L}_3} \mathfrak{A}_2 \upharpoonright \tau_0$. Without loss of generality we assume that both $\mathbf{P}_i$'s are of the same arity. In case they are unary, we apply Lemma 4.1.7 directly, otherwise we combine it with Remark 4.1.6. So we obtain a formula $\psi \in \mathscr{L}_1(\tau_0 \cup \tau_{\mathrm{tree}} \cup \{\mathbf{P}\})$ which defines $\mathbf{P}$ implicitly. So let $\vartheta \in \mathscr{L}_2(\tau_0 \cup \tau_{\mathrm{tree}})$ be an explicit definition of $\mathbf{P}$. So we get $\mathrm{Tree}_P^1(\mathfrak{A}_1, \mathfrak{A}_2) \models \vartheta(\mathbf{c})$ but $\mathrm{Tree}_P^0(\mathfrak{A}_1, \mathfrak{A}_2) \models \neg\vartheta(\mathbf{c})$ which contradicts

$$\mathrm{Tree}_P^1(\mathfrak{A}_1, \mathfrak{A}_2) \upharpoonright \tau_0 \cup \tau_{\mathrm{tree}} \equiv_{\mathscr{L}_2} \mathrm{Tree}_P^0(\mathfrak{A}_1, \mathfrak{A}_2) \upharpoonright \tau_0 \cup \tau_{\mathrm{tree}}.$$

as were required by $\mathrm{TPP}(\mathscr{L}_2, \mathscr{L}_3)$. $\square$

Stating definability and preservation properties for pairs of logics allows us to sharpen results which were previously proven for absolute logics (and therefore for Karp logics). The reader should also consult Chapter XVII.

4.4.6 Proposition (Barwise). *If* $\mathscr{L}$ *is a logic which satisfies* $\mathrm{WFROB}(\mathscr{L}, \mathscr{L}_{\infty\omega})$, *then it has Löwenheim number* ω.

Proof. Since $\mathscr{L} \subset \mathscr{L}_{\infty\omega}$, $\mathscr{L}$ is a Karp logic. Therefore, if $\mathscr{L}$ properly extends first-order logic, there is a sentence $\varphi \in \mathscr{L}[\tau_1]$ such that the relativized reducts of its models are all countably infinite, by Lemma 2.1.2 of Chapter 3. Assume, for contradiction that there is a sentence $\psi \in \mathscr{L}[\tau_2]$ with $\tau_1 \cap \tau_2 = \{=\}$, which has only uncountable models. Let Σ be the $\mathscr{L}_{\omega\omega}$ theory of infinite sets. So Σ, φ, ψ form a counterexample to $\mathrm{WFROB}(\mathscr{L}, \mathscr{L}_{\infty\omega})$. $\square$

4.4.7 Corollary. *Let $\mathscr{L}$ be a logic which satisfies* BETH($\mathscr{L}, \mathscr{L}_{\infty\omega}$). *Then it satisfies*

(i) WFROB($\mathscr{L}, \mathscr{L}_{\infty,\omega}$).
(ii) *The Löwenheim number $l_1(\mathscr{L})$ of $\mathscr{L}$ is ω.*

Proof. (i) follows from Theorem 4.4.5 and Proposition 4.2.16, and (ii) follows from (i) together with Proposition 4.4.6. □

We end this section with some more concrete examples:

4.4.8 Examples. (i) The logic $\mathscr{L}_{\omega\omega}(Q_1)$ from Chapters II or VII satisfies the tree preservation property, as one proves easily with a back-and-forth argument. By Counterexample II.7.1.3 it does not satisfy the interpolation property and therefore, since it is countably compact, not the weak finite Robinson property. So Theorem 4.4.5 gives us that it does not satisfy the Beth property.

(ii) The logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ is compact and does not satisfy the interpolation property by Counterexample II.7.1.3. It is not too difficult to check that TPP holds for this logic. So again by Theorem 4.4.5, the Beth property fails.

(iii) The logic $\mathscr{L}_{\omega\omega}$(aa) from Chapter IV does not satisfy the Beth property by Makowsky–Shelah [1981]. This is shown using the ideas in the proof of Theorem 4.4.5, though by Example 4.1.2(iv) $\mathscr{L}_{\omega\omega}$(aa) does not satisfy even the pair preservation property. To carry through the proof one has only to verify that it holds for specific structures.

(iv) We cannot replace TPP by PPP in Theorem 4.4.5, as the example in Section 4.6 shows.

4.5. *Definability and Existence of Models with Automorphisms*

The aim of this section is to explore further the consequences of the assumption that a logic $\mathscr{L}$ satisfies both PPP($\mathscr{L}$) and ROB($\mathscr{L}$). As stated in Problem 4.3.12, it is an open problem whether such logics exist which properly extend first-order logic. The results below may give us directions in solving that problem. Our main theorem is

4.5.1 Theorem (Shelah). *Let $\mathscr{L}$ be a small logic which has the pair preservation property and the Robinson property. Then every infinite τ-structure $\mathfrak{A}$ has $\mathscr{L}$-extensions with arbitrarily large τ-automorphism groups.*

For first-order logic this is a corollary to the celebrated theorem by Ehrenfeucht and Mostowski concerning indiscernibles. The reader may consult Chang–Keisler [1973, Chapter 3.3] for a detailed exposition. In the proof of Theorem 4.5.1 we discern various possibilities of defining abstract model theoretic properties centering around the existence of various automorphisms. Let us explore these first:

4.5.2 Definition. Let $\mathscr{L}_1 \subset \mathscr{L}_2$ be logics.

(i) We say that the pair of logics $\mathscr{L}_1$, $\mathscr{L}_2$ has the *homogeneity property* (*homogeneity property for finite vocabularies*), if for every τ-structure (τ finite) $\mathfrak{M}$ and $c_1, c_2 \in M$ such that $\langle \mathfrak{M}, c_1 \rangle \equiv_{\mathscr{L}_2} \langle \mathfrak{M}, c_2 \rangle$ there is model $\langle \mathfrak{N}, c_1^N, c_2^N \rangle$ of $\mathrm{Th}_{\mathscr{L}_1}(\langle \mathfrak{M}, c_1, c_2 \rangle)$ and a τ-automorphism g of $\mathfrak{N}$ such that $g(c_1^N) = c_2^N$. If $\mathscr{L}_1 = \mathscr{L}_2$ we just say that $\mathscr{L}_1$ has the *homogeneity property* (*homogeneity property for finite vocabularies*).

(ii) We say that the pair of logics $\mathscr{L}_1$, $\mathscr{L}_2$ has the *local homogeneity property*, if for every τ-structure $\mathfrak{M}$ and $c_1, c_2 \in M$ such that $\langle \mathfrak{M}, c_1 \rangle \equiv_{\mathscr{L}_2} \langle \mathfrak{M}, c_2 \rangle$ and every $\varphi \in \mathrm{Th}_{\mathscr{L}_1}(\langle \mathfrak{M}, c_1, c_2 \rangle)$ there is model $\langle \mathfrak{N}, c_1^N, c_2^N \rangle \models \varphi$ and a τ-automorphsim g of $\mathfrak{N}$ such that $g(c_1^N) = c_2^N$. If $\mathscr{L}_1 = \mathscr{L}_2$ we just say that $\mathscr{L}_1$ has the *local homogeneity property*.

(iii) We say that $\mathscr{L}$ has the (*local*) *automorphism property*, if for every τ-structure $\mathfrak{M}$ and infinite subset $P \subset M$, the theory (every sentence φ of the theory) $\mathrm{Th}_{\mathscr{L}}(\langle, P \rangle)$ has a model $\langle \mathfrak{N}, P \rangle$ which has an automorphism g of $\mathfrak{N}$ such that $g \restriction P \neq \mathrm{Id}$.

4.5.3 Remarks. (i) If $\mathscr{L}$ is compact, then the local homogeneity property and the homogeneity property coincide. The same holds for the automorphism property. We shall be mainly interested in the compact case. The local case may be of independent interest for further developments.

(ii) If a logic does not satisfy the Beth property, one may construct its Beth closure in the natural way. Unlike the Δ-closure, studied in Chapter II and Chapter XVII, the Beth closure cannot easily be proven to preserve compactness. In Shelah [1983, Manuscript] the properties of the Beth closure were studied extensively. It turns out that stronger forms of the homogeneity property yield a sufficient condition for the Beth closure to preserve compactness. In Theorem 4.6.12 an example of a compact logic satisfying PPP and the Beth property is presented, whose proof relies on this idea.

4.5.4 Proposition* (Makowsky). (i) *Let $\mathscr{L}$ be a logic which has the automorphism property. Then $\mathscr{L}$ satisfies* $\mathrm{REXT}(\mathscr{L})$ *and therefore is* $[\omega]$*-compact.*

(ii) *Let $\mathscr{L}$ be a logic which has the local automorphism property. Then $\mathscr{L}$ has well-ordering number $w_1(\mathscr{L}) = \omega$. In particular, if $\mathscr{L}$ is recursively generated then $\mathscr{L}$ is also recursively compact.*

Proof. (i) We show that $\mathrm{REXT}(\mathscr{L})$, which is equivalent to $[\omega]$-compactness by Theorem 3.2.1. Let $\langle \mathfrak{M}, P^M \rangle$ be a τ-structure with $\mathbf{P} \in \tau$ and P^M infinite. Let τ_1 be a vocabulary, extending τ, giving every element in P^M a different name and let $\mathfrak{M}_1$ be the corresponding expansion. Clearly $\langle \mathfrak{M}_1, P^M \rangle$ still satisfies the hypothesis of the automorphism property. So let $\langle \mathfrak{N}, P^N \rangle$ be a $\mathscr{L}[\tau_1]$-extension of $\langle \mathfrak{M}_1, P^M \rangle$ with the required automorphism. Clearly, $P^M \subsetneq P^N$.

(ii) Here we just use that the standard model of arithmetic is rigid. For the latter remark we apply Remarks 4.2.5. ☐

In general the homogeneity property does not imply compactness.

4.5.5 Example. Let κ be a compact cardinal. The pair of logics $\mathscr{L}_{\kappa\omega}$, $\mathscr{L}_{\kappa\kappa}$ has the homogeneity property. To see this one uses an ultralimit construction as in Hodges–Shelah [1981]. Clearly, for $\lambda < \kappa$, these logics are not $[\lambda]$-compact.

However, for compact logics we have:

4.5.6 Proposition. *If $\mathscr{L}$ is a small and compact logic, which has the homogeneity property, then $\mathscr{L}$ has the automorphism property.*

Proof. Let $\langle \mathfrak{M}, P\rangle$ be a structure with P infinite. Using compactness there are $\mathscr{L}$-extensions $\langle \mathfrak{N}, P\rangle$ with P of arbitrary large cardinality. Using smallness we can find such an extension with $c_1, c_2 \in P$, $c_1 \neq c_2$ satisfying the same $\mathscr{L}$-type. Now we apply the homogeneity property. □

Now we are in a position to prove the existence of models with many automorphisms.

4.5.7 Proposition. *Let $\mathscr{L}$ be a compact logic with the automorphism property. Then every $\mathscr{L}$-theory with infinite models has models with arbitrarily large automorphism groups.*

Proof. Let Σ be an $\mathscr{L}$ theory and $\mathfrak{A}$ be an infinite model of Σ. We want to define by induction vocabularies τ_α and theories Σ_α which are sets such that, if $\mathfrak{A} \models \Sigma_\alpha$, then $\mathfrak{A} \upharpoonright \tau \models \Sigma$ and that $\mathfrak{A} \upharpoonright \tau$ has at least card(α) many different automorphisms.

For $\alpha = 0$ we proceed as follows. Since $\mathscr{L}$ is small the complete $\mathscr{L}$-theory Σ_0 of $\mathfrak{A}$ is a set. Again using smallness together with compactness we can find a model $\mathfrak{B}$ and $b, b' \in B$ satisfying the same type. So there is a model $\mathfrak{M}_0$ with a non-trivial automorphism F_0. Now we put Σ_1 to be the complete $\mathscr{L}$-theory of $\langle \mathfrak{M}_0, F_0\rangle$. Clearly, this also works for α successor. For α limit we put $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta$. To show that Σ_α has a model we use compactness in the form of Proposition 1.1.1. □

4.5.8 Example (Shelah). We define a quantifier binding four variables and acting on two formulas (i.e., of type $\langle 2, 2\rangle$) in the following way: Let $\mathfrak{A}$ be a τ-structure.

$$\mathfrak{A} \models Q^{\text{ibool}}uvwx(\varphi(u, v, \bar{z}), \psi(w, x, \bar{z}))[\bar{a}]$$

if $\langle A_\varphi^{\bar{a}}, R_\varphi^{\bar{a}}\rangle$ and $\langle A_\psi^{\bar{a}}, R_\psi^{\bar{a}}\rangle$ are partially ordered structures, where the order satisfies the axioms of a boolean algebra and

$$\langle A_\varphi^{\bar{a}}, R_\varphi^{\bar{a}}\rangle \cong \langle A_\psi^{\bar{a}}, R_\psi^{\bar{a}}\rangle.$$

By $A_\varphi^{\bar{a}}$ we denote the set $\{b \in A : \mathfrak{A} \models \varphi[b, b, \bar{a}]\}$ and by $R_\varphi^{\bar{a}}$ the relation

$$\{(b, c) \in A^2 : \mathfrak{A} \models \varphi[b, c, \bar{a}]\},$$

and similarly for ψ.

4.5.9 Theorem (Shelah [1983d]). *Assume* GCH. *Then the logic $\mathscr{L}_{\omega\omega}(Q^{\text{ibool}})$ is compact.*

4.5.10 Proposition. *There is a sentence* $\Psi_{\mathrm{rigid}} \in \mathscr{L}_{\omega\omega}(Q^{\mathrm{ibool}})$ *such that:*

(i) *Every model of* Ψ_{rigid} *is rigid, i.e., has no non-trivial automorphisms.*
(ii) Ψ_{rigid} *has models of every infinite cardinality.*

Proof. Let P be a ternary predicate symbol. Define Ψ_{rigid} to be the conjunction of the following formulas:

$$\psi_1 = \forall z Q^{\mathrm{ibool}} xyx'y'(P(x, y, z), P(x', y', z))$$

and

$$\psi_2 = \forall zz'(z \neq z' \rightarrow \neg Q^{\mathrm{ibool}} xyx'y'(P(x, y, z), P(x', y', z'))$$

To prove (i) let $\mathfrak{A}$ be a model of Ψ_{rigid}, $a \in A$ and let h be an automorphism of $\mathfrak{A}$. Clearly, $\langle A_R^a, R_P^a \rangle$ is a boolean algebra by ψ_1. Since h is an automorphism, so is $\langle A_R^{h(a)}, R_P^{h(a)} \rangle$ and they are isomorphic. So, by ψ_2, $h(a) = a$.

To prove (ii), let λ an infinite cardinal and $\{\mathfrak{B}_i = \langle B_i, \leq_i \rangle : i < \lambda\}$ a family of λ many pairwise non-isomorphic boolean algebras of cardinality λ each. Without loss of generality $B_i = \lambda$. We define a model $\mathfrak{A} = \langle A, P^A \rangle$ of Ψ_{rigid} as follows: We put $A = \lambda$ and $P^A = \{(i, a, b) \in \lambda^3 : a \leq_i b\}$. Clearly, $\mathfrak{A} \models \Psi_{\mathrm{rigid}}$. Note that (ii) does not follow from the compactness of $\mathscr{L}_{\omega\omega}(Q^{\mathrm{ibool}})$. On the other hand (ii) does not require GCH, as the proof of compactness. ☐

4.5.11 Corollary (GCH). *The logic* $\mathscr{L}_{\omega\omega}(Q^{\mathrm{ibool}})$ *is compact but does not satisfy the homogeneity property.*

Proof. By Theorem 4.5.9 the logic is compact. Assume, for contradiction, the homogeneity property. So by Propositions 4.5.6 and 4.5.7 we get models with arbitrarily large automorphism groups, contradicting Proposition 4.5.10. ☐

4.5.12 Proposition (GCH). *There is a compact logic* $\mathscr{L}$ *which does not have the automorphism property.*

Proof. This follows from Proposition 4.5.10 and Corollary 4.5.11. ☐

4.5.13 Theorem (Shelah). *Let* $\mathscr{L}$ *be a logic.*

(i) *If* $\mathscr{L}$ *satisfies* PPP($\mathscr{L}$) *and* ROB($\mathscr{L}$), *then* $\mathscr{L}$ *has the homogeneity property.*
(ii) *If* $\mathscr{L}$ *satisfies* PPP($\mathscr{L}$) *and* FROB($\mathscr{L}$), *then* $\mathscr{L}$ *has the homogeneity property for finite vocabularies.*
(iii) *If* $\mathscr{L}$ *satisfies* PPP($\mathscr{L}$) *and* INT($\mathscr{L}$), *then* $\mathscr{L}$ *has the local homogeneity property.*

Proof. We prove only (i), the others being similar. Let $\mathfrak{M}$ and $c_1, c_2 \in M$ be as in the hypothesis of the homogeneity property.

Let $\mathfrak{M}'$, c_1', c_2' be disjoint copies. Put $\mathfrak{N} = [\mathfrak{M}, \mathfrak{M}']$. Put

$$T = \mathrm{Th}_{\mathscr{L}}(\langle \mathfrak{N}, c_1, c_2, c_1' \rangle) = \mathrm{Th}_{\mathscr{L}}(\langle \mathfrak{N}, c_1, c_2, c_2' \rangle).$$

The equality holds because of PPP($\mathscr{L}$). Let $\mathbf{c}_1, \mathbf{c}_2$ be constant symbols with interpretations c_1, c_2 and $\mathbf{c}$ be a constant symbol with interpretation c_1' or c_2', respectively. Let $\mathbf{F}$ be a new function symbol. Let $\psi_i (i = 1, 2)$ be the sentence which says that $\mathbf{F}$ is a τ-isomorphism (modulo name changing) mapping the first sort into the second sort which maps $\mathbf{c}_i$ into $\mathbf{c}$. If τ is infinite, we need a set of sentences Ψ_i defined similarly.

Clearly, $T \cup \{\psi_i\}$ has a model. So by ROB($\mathscr{L}$) or, if τ is finite, by WROB($\mathscr{L}$), $T \cup \{\psi_1, \psi_2\}$ has a model $[\mathfrak{M}_1, \mathfrak{M}_1']$ which gives as the required automorphism in $\mathfrak{M}_1$ □

4.5.14 Remarks. (i) In Proposition 4.5.13 above the three cases coincide for compact logics.

(ii) If we assume that the logics are tiny, the hypotheses in the cases 4.5.13(i) and (ii) imply that the logics are $[\omega]$-compact and ultimately compact. Assuming that $\mathscr{L}$ has an dependence number $o(\mathscr{L})$ which is smaller than the first uncountable measurable cardinal, the hypothesis in Theorem 4.5.13(i) actually implies compactness. In Theorem 4.5.13(ii) we need for this, that the logic $\mathscr{L}$ has size function $s(\omega) < \omega_n$ for some $n \in \omega$ (cf. Theorem 4.3.8 and Corollary 3.3.5).

4.5.15 Corollary. *Let $\mathscr{L}$ be a logic with dependence number $o(\mathscr{L})$ smaller than the first uncountable measurable cardinal (or, alternatively, with size function $s(\omega) < \omega_n$ for some $n \in \omega$). If $\mathscr{L}$ satisfies* PPP($\mathscr{L}$) *and* ROB($\mathscr{L}$), *then $\mathscr{L}$ has the automorphism property.*

Proof. We use Remark 4.5.14(ii) above and Proposition 4.5.13. □

This corollary, together with Theorem 4.5.7 gives us a proof of Theorem 4.5.1.

4.6. Some More Examples: Stationary Logic and Its Friends

In this last section we want to discuss, mostly without proofs, some more examples and consistency results, which all come from Shelah [198?e] and Mekler–Shelah [1983, 198?]. They are all concerned with preservation and definability properties of compact or (ω, ω)-compact logics. Our first example concerns extensions of $\mathscr{L}_{\omega\omega}(Q_1)$. Let us recall some facts:

4.6.1 Proposition. *The logic $\mathscr{L}_{\omega\omega}(Q_1)$ has the following properties:*

(i) *$\mathscr{L}_{\omega\omega}(Q_1)$ is (ω, ω)-compact, but not (ω_1, ω)-compact.*

(ii) *$\mathscr{L}_{\omega\omega}(Q_1)$ does satisfy the pair preservation property.*

(iii) *$\mathscr{L}_{\omega\omega}(Q_1)$ does not satisfy the Δ-interpolation property, and therefore neither the interpolation property.*

It remains open whether $\mathscr{L}_{\omega\omega}(Q_1)$ satisfies the weak Beth property. However, there is the following consistency result proved in Mekler–Shelah [198?].

4.6.2 Theorem (Shelah). *Every model* $\mathfrak{M}$ *of* ZFC *has a generic extension* $\mathfrak{M}[G]$ *in which* $\mathscr{L}_{\omega\omega}(Q_1)$ *satisfies the weak Beth property.*

For the stronger definability properties there is a consistency result in the other direction. We want to state, that it is consistent with ZFC, that no "reasonable" extension of $\mathscr{L}_{\omega\omega}(Q_1)$ satisfies both PPP and the interpolation property (or equivalently the uniform reduction property UR_2). For this we need a definition:

4.6.3 Definition (Definable Logics). (i) A logic $\mathscr{L}$ is *definable*, if the relations "$\varphi \in \mathscr{L}[\tau]$" ("$\varphi$ is a $\mathscr{L}[\tau]$-formula") and "$\mathfrak{M} \models \varphi$" ("$\mathfrak{M}$ is a model of φ") are definable by a formula of set theory *without parameters.*

(ii) A logic $\mathscr{L}$ is *λ-definable*, for λ a cardinal, if the relations "$\varphi \in \mathscr{L}[\tau]$" ("$\varphi$ is a $\mathscr{L}[\tau]$-formula") and "$\mathfrak{M} \models \varphi$" ("$\mathfrak{M}$ is a model of φ") are definable by a formula of set theory *with a parameter* $A \subset \lambda$.

4.6.4 Remark. In Chapter XVII *absolute logics* were introduced. This notion is not quite comparable with the above definition. For a logic to be absolute definability *with parameters* is allowed, but definability is restricted to Δ_1-definability.

4.6.5 Examples. (i) Logics of the form $\mathscr{L}_{\omega\omega}(Q^i)_{i \in n}$ are definable, provided the quantifiers are set presentable in the sense of Definition 1.5.8.

(ii) The logics $\mathscr{L}_{\kappa\lambda}$ are definable.

(iii) Not all logics are definable without parameters. Especially some of the fragments $\mathscr{L}_A \subset \mathscr{L}_{\omega_1\omega}$ are not definable, but they are ω_1-definable with parameter $A \subset \omega_1$. If A is a countable admissible fragment which has a code in ω then $\mathscr{L}_A$ is even ω-definable.

(iv) The logic $\mathscr{L}_{F\omega}$ from Section 1.6 is definable, provided the ultrafilter F is definable. The definability of this filter may very well depend on the set-theoretic assumptions under consideration.

4.6.6 Theorem (Shelah). *For every model* $\mathfrak{M}$ *of* ZFC *that there is generic extension* $\mathfrak{M}[G]$ *such that no definable logic* $\mathscr{L}$ *extending* $\mathscr{L}_{\omega\omega}(Q_1)$ *satisfies both* PPP($\mathscr{L}$) *and the interpolation property (or, equivalently, the uniform reduction property* UR_2) *in* $\mathfrak{M}[G]$.

It was widely believed that the Δ-closure of $\mathscr{L}_{\omega\omega}(Q_1)$ is a rather untackable logic. That this need not be the case is shown by the next consistency result from Mekler–Shelah [198?]. Let us first recall some facts about the logic $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ from Section IV.4 and Counterexample II.7.1.3.

4.6.7 Proposition. (i) *The logic* $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ *is* (ω, ω)*-compact, r.e. for validity, but does not satisfy the interpolation property.*
(ii) $\mathscr{L}_{\omega\omega}(Q_1)$ *is a sublogic of* $\mathscr{L}_{\omega\omega}(\mathrm{aa})$.
(iii) $\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q_1), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$ *does not hold.*

Inspired by Theorem 4.6.2 we can state the following problem:

4.6.8 Problem. (Shelah). Does every model $\mathfrak{M}$ of ZFC have a generic extension $\mathfrak{M}[G]$ in which $\Delta\text{-INT}(\mathscr{L}_{\omega\omega}(Q_1), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$ holds?

In Mekler–Shelah [1983] a positive answer is given for Δ-Interpolation on finitely determinate structures. In contrast to this it is shown in Counterexample II.7.1.3 that $\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q_1), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$ does not hold.

The next example involves the logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$.

4.6.9 Proposition. (i) *The logic* $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ *is compact, r.e. for validity, but does not satisfy the interpolation property.*
(ii) $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ *is a sublogic of* $\mathscr{L}_{\omega\omega}(\mathrm{aa})$.

Proof. From Section II.2.4, and Makowsky–Shelah [1981] we know (i). To see (ii) we axiomatize the class of orderings of cofinality ω by the $\mathscr{L}_{\omega\omega}(\mathrm{aa})$-sentence which says that the ordering has no last element, but that almost every countable set P is unbounded. □

The next theorem shows that $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ behaves more like second-order logic, than originally suspected, since it provides interpolating formulas for the logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$. Note that for Hanf number calculations $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ is as strong as the logic which allows unrestricted quantification over countable sets, as shown in Kaufmann–Shelah [198?].

4.6.10 Theorem (Shelah). $\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)}), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$.

The proof may be found in Mekler–Shelah [198?].

4.6.11 A Generalization. The pair of logics in Theorem 4.6.10 can be generalized to higher cardinals. For $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ this gives us the logics $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq\lambda})$ which requires the ordering to be of infinite cofinality less or equal to λ. As shown in Makowsky–Shelah [1981] this logic is still compact, but does not satisfy the interpolation property. For $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ we have to define a logic $\mathscr{L}_{\omega\omega}(\mathrm{aa}_\lambda)$ for an appropriate filter D_λ. A detailed exposition may be found in Mekler–Shelah [198?]. What is important here, is a theorem of Shelah which states that the pair $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq\lambda})$ and $\mathscr{L}_{\omega\omega}(\mathrm{aa}_\lambda)$ satisfies a strong form of the homogeneity property, as defined in Section 4.5. As mentioned in Section 4.5, such homogeneity properties can be used to prove that the Beth closure preserves PPP and compactness.

Using this line of thought Shelah proved the following theorem:

4.6.12 Theorem (Shelah). *The Beth closure $\mathscr{L}$ of the logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq 2^\omega})$ is a compact logic which satisfies:*

(i) PPP($\mathscr{L}$) (*and therefore, by compactness,* URP);
(ii) *has the Beth property; but*
(iii) *does not satisfy the interpolation property (and therefore, by compactness, none of the Robinson properties).*

This shows, that in Theorem 4.4.5 the tree preservation property cannot be weakened to the pair preservation property. For otherwise, since the logic is compact, the Beth property would imply the interpolation property. It also shows that the uniform reduction property for pairs does not imply even the uniform reduction property UR_1, which, by Theorem 4.2.12 is equivalent to the interpolation property.

This example is also the first example so far, which exhibits a compact logic satisfying the Beth property. Note that it is easy to construct compact logics, which satisfy the weak Beth property or the Δ-interpolation property by the construction of the Δ-closure or weak Beth closure, as described in Proposition II.7.2.5 and, in more detail, Makowsky–Shelah–Stavi [1976].

Also the Δ-closure of $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq 2^\omega})$ has remarkable properties:

4.6.13 Theorem* (Shelah). *The Δ-closure of $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq 2^\omega})$ does not have the Beth property.*

A proof will appear in Makowsky–Shelah [198?b].

The following is open:

4.6.14 Problem. Is there a logic $\mathscr{L}$ which satisfies both the Beth property and Δ-interpolation, is compact but does not satisfy the interpolation property? In particular, is the iterated Beth and Δ-closure of $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq 2^\omega})$ compact, and if yes, does it satisfy the interpolation property?

4.7. *Which Definability Property?*

The first definability property proven for $\mathscr{L}_{\omega\omega}$ was the Beth property (Beth [1953]). The interpolation property was introduced in Craig [1957b], and is sometimes also called Craig's interpolation property. Its main application was to give a simplified proof of the Beth property. Another proof of the Beth property for $\mathscr{L}_{\omega\omega}$ was given in Robinson [1956a] where the Robinson property, or rather the finite Robinson property, was introduced. The choice of these properties was not really questioned in this period. The weak Beth property was first discussed in Friedman [1973]. Friedman suggested also first that it was the weak Beth property which really mattered in the context of logics different from first-order logic. The first thorough discussion of definability properties for logics in general is in Feferman [1974a, b, 1975].

Feferman focuses the attention on the Δ-interpolation, pointing out its equivalence to the weak projective Beth property. His paper had great impact and the Δ-closure was studied extensively in Barwise [1974], Makowsky–Shelah–Stavi [1976], Hutchinson [1976], Väänanen [1977a, 1979a, 1983], Paulos [1976] and Makowsky–Shelah [198?]. From this it emerged that the Δ-closure may well be a "better" definability property than all the others studied so far. This is especially so, since the Δ-closure of a logic $\mathscr{L}$ preserves compactness and the recursive enumerability of the validities of its finitely generated sublogics.

It was also in Feferman [1974a, b] and in Feferman [1972] that preservation properties were first discussed in the general setting. $\mathrm{UR}_n(\mathscr{L})$ was introduced to unify known preservation theorems and interpolation theorems. In Makowsky [1978] the equivalence of $\mathrm{UR}_1(\mathscr{L})$ and the interpolation property was established. From this one was led to think that the next "reasonable" strengthening of the Δ-interpolation property would be uniform reduction $\mathrm{UR}_2(\mathscr{L})$. Note that the equivalence of non-uniform and uniform reduction for pairs PPP and URP for compact logics, due to Shelah, appears here for the first time.

The finite Robinson property was first discussed in the general setting in Makowsky–Shelah [1976] and the Robinson property in Mundici [1979a]. Mundici suggested that the Robinson property is a "natural" property of logics, since it is equivalent, for finitely generated logics, to compactness and the interpolation property. But, as it emerges in this chapter, it seems to us that it is the Robinson property together with PPP which has more merits: In the case of compact logics they are together again equivalent to $\mathrm{UR}_2(\mathscr{L})$ or to the preservation property for projective operations PPPO.

It should be pointed out here that this comparison of definability properties has still a severe drawback: The lack of an abundance of examples. There are, by now, many compact, and therefore many compact and Δ-closed logics, mostly constructed by Shelah. But there are no interesting examples satisfying any strengthening of the interpolation property, such as uniform reduction or the Robinson property.

TABLES

Table 1. Transfer of Compactness Properties

From	To	Condition	Reference
$\mathrm{cf}(\kappa)$	κ		1.1.6
κ^+	κ	κ singular κ regular	1.3.11(i) 1.5.4 1.3.11(ii) 1.5.4
κ	ω	$\kappa < \mu_0$ μ_0 first uncountable measurable cardinal	1.5.2

Table 2. The Compactness Spectrum

Form	Condition	Reference
Comp($\mathscr{L}$) is initial segment	$A(\infty)$	1.5.7(i)
Comp($\mathscr{L}$) contains final segment	Vopenka's principle	1.5.16(iv)
First element measurable		1.5.2
Gaps in spectrum		1.6

Table 3. Transfer of Dependence Properties

From	To	Condition	Reference
κ	ω	compactness	5.1.3 in Chapter II
κ	ω	$[\omega]$-compactness $\kappa < \mu_0$ μ_0 first uncountable measurable cardinal	2.2.1
κ	Finite dependence structure	$[\omega]$-compactness	2.4.3

Table 4. Compactness and Extensions

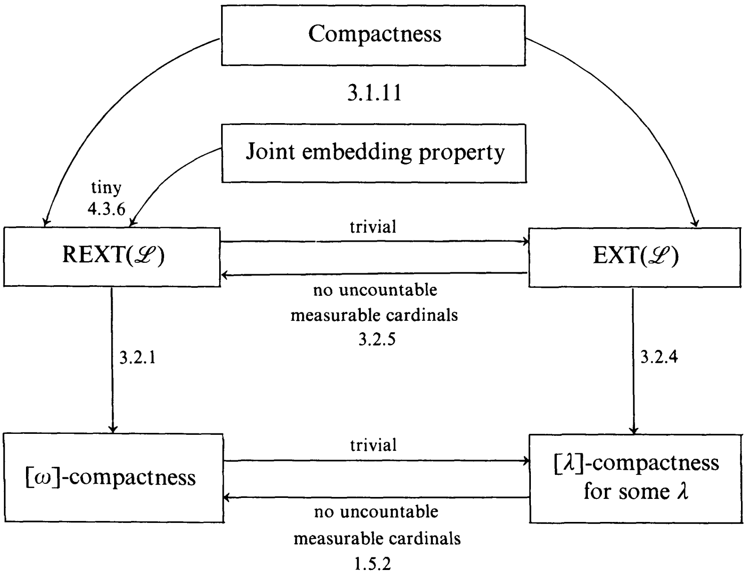

Table 5. Amalgamation, Joint Embeddings, and Compactness

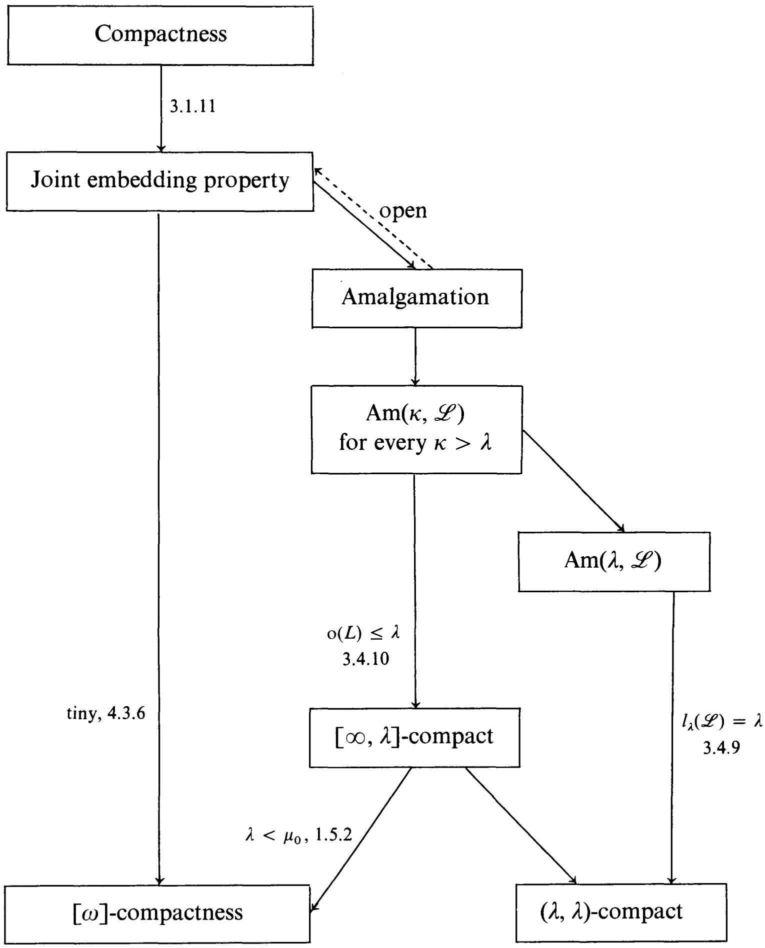

Table 6. Compactness, Definability, and Automorphisms (for logics with finite dependence)

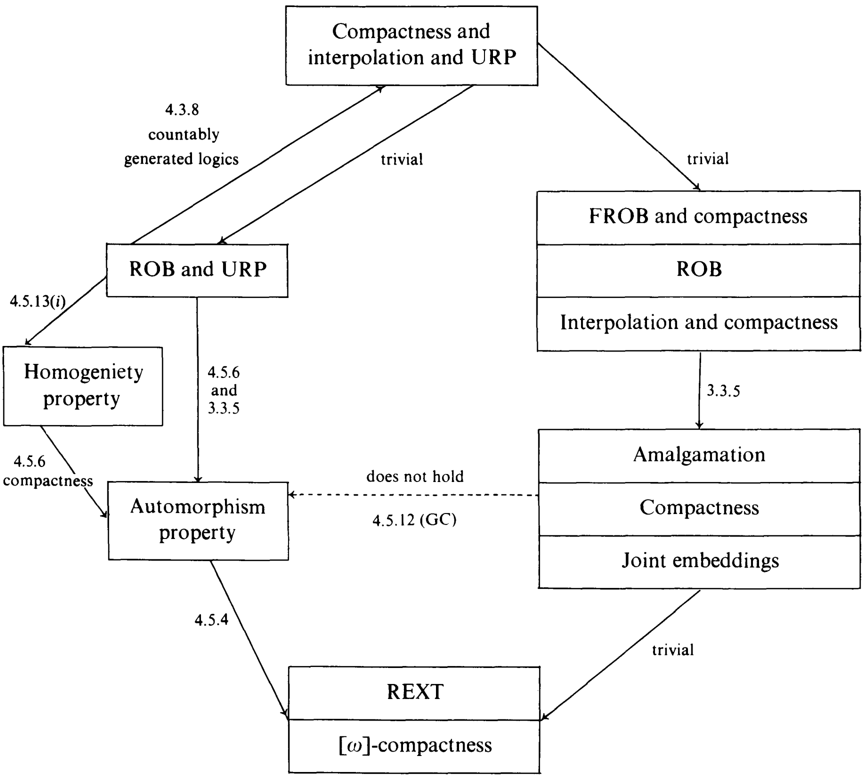

Table 7. Definability Properties

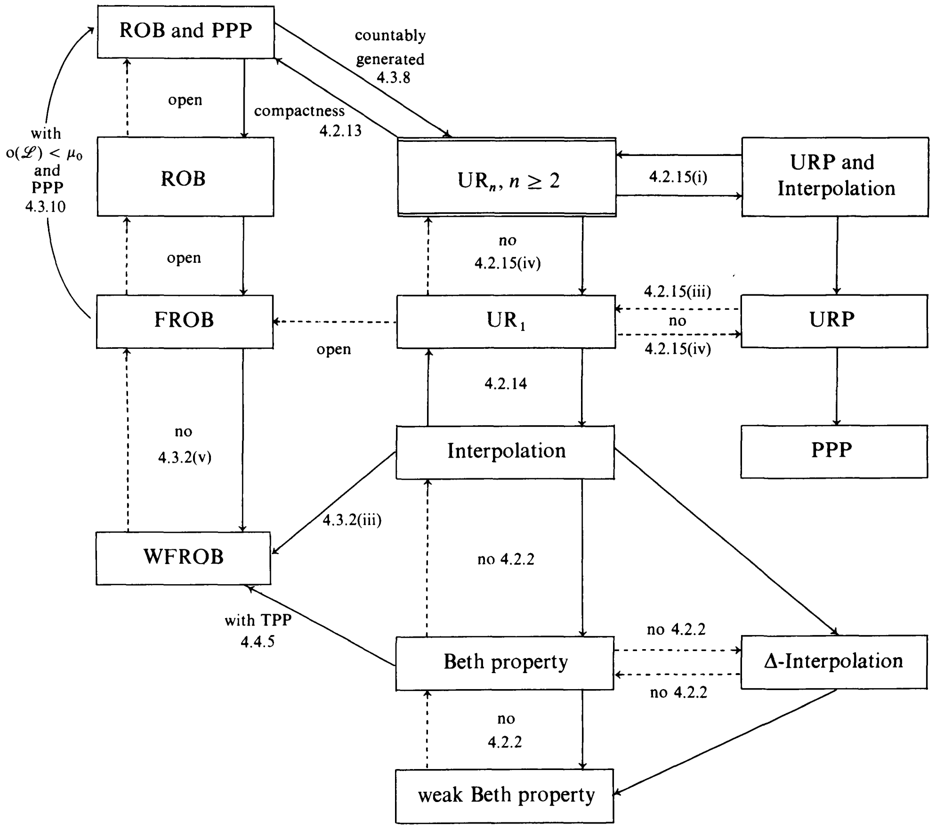

Table 8. Definability for Compact Logics